TRANSACTIONS

OF THE

AMERICAN MATHEMATICAL SOCIETY

EDITED BY

ROBERT D. CARMICHAEL

DUNHAM JACKSON

FRANCIS R. SHARPE

VOLUME 33
1931

PUBLISHED BY THE SOCIETY
MENASHA, WIS., AND NEW YORK
1931

General

Bound Jan 1932

COMPOSED, PRINTED AND BOUND BY
The Collegiate Press
GEORGE BANTA PUBLISHING COMPANY
MENASHA, WISCONSIN

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VOLUME 33, 1931

A METHOD FOR FINDING UNITS IN CUBIC ORDERS OF A NEGATIVE DISCRIMINANT*

BY
J. V. USPENSKY

Among other extremely remarkable things in Hermite's famous letters to Jacobi† we find a very brief outline of a method which can be used for the actual discovery of units in cubic orders. Much, however, remained to be done in order that Hermite's ideas could be embodied in a really practical method easily applicable to numerical examples. Zolotareff was the first to develop Hermite's suggestion in the case of a negative discriminant. In his little known but remarkable thesis *On an indeterminate equation of the third degree* published in Russian in 1869, Zolotareff developed a method for finding units in the order $x+y\theta+z\theta^2$ where θ is a root of the irreducible equation $\theta^3=A$, based on Hermite's principle of continuous variables; that is, on the study of successive minima of a certain positive ternary form containing a continuous parameter.

Zolotareff's most important contribution consisted in the peculiar manner of reducing the study of successive minima of a ternary quadratic form to a similar problem concerning binary forms. In itself Zolotareff's method is remarkable, but it requires further complements in order to give *all* the successive minima, as is strictly required by Hermite's principle, and these complements unfortunately detract much from its practical value. When studying Zolotareff's paper I noticed, however, that, by retaining his basic idea, but departing from Hermite's requirement to consider *minima* of a variable ternary form, one can build up a new method for finding units in cubic orders of a negative discriminant which can be applied to numerical examples with comparative ease. The best method hitherto known for that purpose is one given by Voronoi in 1896‡.

I do not venture to say that the new method explained in this paper is more expedient. That may be decided only by application of both to numerous examples.

1. An order (or ring) in an algebraic field is a system of integers of that

* Presented to the Society, August 29, 1929; received by the editors in August, 1930.

† See Hermite's Collected Works, vol. 1, pp. 131–135.

‡ G. Voronoi, *On a Generalization of the Algorithm of Continued Fractions*, Warsaw, 1896 (in Russian).

This remarkable thesis contains also a method of finding two fundamental units in the case of a positive discriminant.

field containing the rational unit and reproducing itself by addition, subtraction, and multiplication of its numbers. An order in a cubic field can be built up by repeated addition and subtraction of three fundamental numbers, $1, \alpha, \beta$, forming its *basis*. Hence the numbers of an order are obtained from its *basic form*

$$\phi = x + \alpha y + \beta z$$

by giving to x, y, z all rational integral values. As the numbers $\alpha^2, \alpha\beta, \beta^2$ by definition belong to the order they must be contained in the preceding linear form for suitable integral values of the variables. Together with $1, \alpha, \beta$, the numbers $1, \alpha+c, \beta+d$, where c and d are arbitrary integers, form a basis of the same order. By selecting c and d in a proper way the product

$$(\alpha + c)(\beta + d)$$

will be a rational integer. Hence we can always select a basis $1, \alpha, \beta$ in such a way that the product $\alpha\beta$ is a rational integer. We shall call a basis of this kind a normal basis. Supposing that the basis is normal we must have the equations

$$\alpha\alpha = e + f\alpha + g\beta,$$
$$\alpha\beta = c,$$

where e, f, g and c are integers. By elimination of β we find the equation which α satisfies,

$$\alpha^3 - f\alpha^2 - e\alpha - cg = 0.$$

Similarly β is a root of the equation

$$\beta^3 + (e/g)\beta^2 + (cf/g)\beta - c^2/g = 0$$

and

$$\beta^2 = - cf/g + (c/g)\alpha - (e/g)\beta.$$

But as β^2 belongs to the order, $k=c/g$ and $l=e/g$ are integers. Thus every order has a basis $1, \alpha, \beta$, where α and β are roots of the respective equations

$$\alpha^3 - f\alpha^2 - gl\alpha - kg^2 = 0,$$
$$\beta^3 + l\beta^2 + kf\beta - k^2g = 0,$$

f, g, k, l being certain integers. Moreover

$$\begin{aligned} \alpha\alpha &= lg + f\alpha + g\beta, \\ \alpha\beta &= kg, \\ \beta\beta &= - kf + k\alpha - l\beta. \end{aligned} \tag{1}$$

2. Denoting by α', α'' conjugate numbers to α and using similar notation for β, the discriminant of the order is given by

$$\begin{vmatrix} 1 & \alpha & \beta \\ 1 & \alpha' & \beta' \\ 1 & \alpha'' & \beta'' \end{vmatrix}^2 = -D,$$

and from now on we shall suppose it to be negative so that $D>0$.

By α we denote a number of the real cubic field; then its conjugates α', α'' are conjugate imaginary numbers. It is easy to see that the discriminant of α is

$$\mathcal{D}(\alpha) = -Dg^2$$

and likewise

$$\mathcal{D}(\beta) = -Dk^2.$$

3. For practical purposes it is advisable to select numbers α, β of a normal basis in a special way. Namely, we shall take for α a number of the order for which the expression

$$-(\alpha' - \alpha'')^2 + 2(\alpha - \alpha')(\alpha - \alpha'')$$

is minimum; that is, can only increase if we substitute for α any other number of the order. Let 1, A, B be any normal basis of the order, so that we have

$$AA = LG + FA + GB,$$
$$AB = KG.$$

By taking

$$\alpha = P + AX + BY,$$

the preceding expression becomes

$$\psi = -(A' - A'')^2(X - (A/G)Y)^2$$
$$+ (A - A')(A - A'')(X - (A'/G)Y)(X - (A''/G)Y),$$

which is a positive quadratic form with the determinant $-3D$. Let the minimum of this form be attained for $X=\lambda$, $Y=\mu$, these being relatively prime numbers. Then we can take for α the number

$$\alpha = P + \lambda A + \mu B,$$

where the integer P remains arbitrary. Denoting by λ', μ' two integers satisfying the equation

$$\lambda\mu' - \lambda'\mu = 1,$$

we can take for the second number of the basis

$$\beta = Q + \lambda' A + \mu' B,$$

and finally dispose of P and Q so as to make the basis 1, α, β normal.

Thus we finally reach the following conclusion: for any given cubic order a normal basis $1, \alpha, \beta$ can be found such that

$$-(\alpha' - \alpha'')^2 + 2(\alpha - \alpha')(\alpha - \alpha'')$$

is minimum; the value of this minimum never exceeds $2D^{1/2}$. On the other hand, as

$$-(\alpha' - \alpha'')^2(\alpha - \alpha')^2(\alpha - \alpha'')^2 = Dg^2,$$

it is easy to see that

$$-(\alpha' - \alpha'')^2 + 2(\alpha - \alpha')(\alpha - \alpha'') \geqq 3(Dg^2)^{1/3}.$$

In the following we shall suppose that the normal basis of the order has been chosen as explained in this paragraph.

4. Again let ϕ represent the basic form of the order

$$\phi = x + \alpha y + \beta z$$

and ϕ', ϕ'' its conjugates. For the sake of simplicity introducing the notation

$$\Phi = \phi'\phi'',$$

we shall consider a positive ternary quadratic form

$$f = 2\Phi + \phi^2/\Delta$$

containing a continuously varying positive parameter Δ. The discriminant of this form is D/Δ. Besides the form f we shall consider its adjoint form F whose expression is

$$F = -(\alpha'' - \alpha')^2\omega^2 + 2(\alpha - \alpha')(\alpha - \alpha'')\Omega/\Delta,$$

where ω is a linear form of the contragredient variables X, Y, Z defined by

$$(\alpha'' - \alpha')\omega = Z(\alpha'' - \alpha') + Y(\beta' - \beta'') + X(\alpha'\beta'' - \alpha''\beta'),$$

while

$$\Omega = \omega'\omega''.$$

Referring to the equations (1) we easily find

$$\omega = Z + (\alpha/g)Y + (\beta + l)X.$$

Numbers ω for integral values of X, Y, Z constitute a modulus with the basis $1, \alpha/g, \beta$. It can be readily verified that the product of any number of this modulus by any number of the order again belongs to the same modulus. According to Dedekind's terminology the cubic order $[1, \alpha, \beta]$ is the order of the modulus $[1, \alpha/g, \beta]$. They coincide if and only if $g=1$. Needless to say that the discriminant of F is D^2/Δ^2.

We can express x, y, z linearly in ϕ, ϕ', ϕ'' and similarly X, Y, Z are expressible linearly in ω, ω', ω''. By substituting these expressions in the bilinear form

$$Xx + Yy + Zz,$$

we can find a remarkable expression of the latter. But such a straightforward way would lead to complicated calculations. The best and most elegant way is to proceed as follows.

Let i, i_1, i_2 be three mutually perpendicular unit vectors, so that performing the scalar multiplication we have

$$\begin{aligned} ii &= 1, & ii_1 &= i_1 i = 0, \\ i_1 i_1 &= 1, & ii_2 &= i_2 i = 0, \\ i_2 i_2 &= 1, & i_1 i_2 &= i_2 i_1 = 0. \end{aligned}$$

Now take the scalar product of the vector

$$A = i(\alpha'' - \alpha')\omega + i_1(\alpha - \alpha'')\omega' + i_2(\alpha' - \alpha)\omega''$$

by the vector

$$B = i\phi + i_1\phi' + i_2\phi''.$$

We have first

$$A \cdot B = (\alpha'' - \alpha')\phi\omega + (\alpha - \alpha'')\phi'\omega' + (\alpha' - \alpha)\phi''\omega''.$$

On the other hand, if we express A, B as linear functions of x, y, z and X, Y, Z with vectorial coefficients, we easily find

$$A \cdot B = \begin{vmatrix} 1 & \alpha & \beta \\ 1 & \alpha' & \beta' \\ 1 & \alpha'' & \beta'' \end{vmatrix} (Xx + Yy + Zz) = (-D)^{1/2}(Xx + Yy + Zz)$$

and thus

$$(2)\quad (-D)^{1/2}(Xx + Yy + Zz) = (\alpha'' - \alpha')\phi\omega + (\alpha - \alpha'')\phi'\omega' + (\alpha' - \alpha)\phi''\omega''.$$

This is the required expression.

5. The following simple proposition is of fundamental importance for what follows.

Let $f(x, y, z)$ be a positive ternary quadratic form with the discriminant D, and $F(x, y, z)$ its adjoint form with discriminant D^2. There exist two triads of integers x, y, z; X, Y, Z without common factor connected by the bilinear relation

$$Xx + Yy + Zz = 0$$

and satisfying the inequalities

$$f(x, y, z) < \frac{4}{3} D^{1/3}, \quad F(X, Y, Z) < \frac{4}{3} D^{2/3}.$$

For we can first find a triad x, y, z without common factor, so as to make

$$f(x, y, z) < \frac{4}{3} D^{1/3}.$$

On the other hand, the bilinear relation

$$Xx + Yy + Zz = 0$$

is satisfied in the most general manner by

$$X = \lambda\alpha + \mu\alpha',$$
$$Y = \lambda\beta + \mu\beta',$$
$$Z = \lambda\gamma + \mu\gamma',$$

where λ, μ are arbitrary integers and $\alpha, \beta, \gamma; \alpha', \beta', \gamma'$ are two independent and fundamental solutions of the indeterminate equation

$$\xi x + \eta y + \zeta z = 0.$$

Introducing these expressions for X, Y, Z into F we obtain a binary form in λ, μ whose determinant is $-Df(x, y, z)$. Hence, it is possible to take for λ, μ relative prime numbers so as to have

$$F(X, Y, Z) \leqq \left[\frac{4}{3} Df(x, y, z)\right]^{1/2},$$

whence

$$F(X, Y, Z) < \frac{4}{3} D^{2/3}.$$

The numbers X, Y, Z have no common divisor if λ and μ are relatively prime.

6. Whatever positive value we attribute to Δ it is possible therefore to find two triads $x, y, z; X, Y, Z$ consisting of numbers without common divisor and satisfying the inequalities

$$f = 2\Phi + \frac{\phi^2}{\Delta} < \frac{4}{3}\left(\frac{D}{\Delta}\right)^{1/3},$$

$$F = -(\alpha'' - \alpha')^2\omega^2 + 2(\alpha - \alpha')(\alpha - \alpha'')\frac{\Omega}{\Delta} < \frac{4}{3}\left(\frac{D}{\Delta}\right)^{2/3},$$

together with the bilinear relation

$$Xx + Yy + Zz = 0.$$

Let us seek further information concerning these triads when $\Delta = 1$. When $\Delta = 1$ it is easy to present f and F in the form

$$f = 3(x + ly + mz)^2 + \tfrac{1}{3}\psi(y, z),$$
$$F = \psi(Y', Z') + (D/3)X^2,$$

where $\psi(y, z)$ is the form

$$\psi(y, z) = -(\alpha'' - \alpha')^2(y - (\alpha/g)z)^2 + 2(\alpha - \alpha')(\alpha - \alpha'')(y - (\alpha'/g)z)(y - (\alpha''/g)z),$$

and

$$Y' = -Z + mX,$$
$$Z' = Y - lX,$$
$$l = \tfrac{1}{3}f,\quad m = -\tfrac{1}{3}l,\quad n = \frac{-\tfrac{2}{3}lf + l\alpha - f\beta}{gl + f\alpha + 3g\beta - \tfrac{1}{3}f^2}.$$

If we want to make

$$F < (4/3)D^{2/3},$$

we have necessarily

$$X^2 < 4D^{-1/3}$$

and hence $X = 0$ if only $D \geqq 64$. On the other hand, the minimum of $\tfrac{1}{3}\psi(y, z)$ is attained for $y = 1$, $z = 0$, if the basis was selected according to the explanations in §3. Now, if this minimum is $\geqq (4/3)D^{1/3}$, which is certainly the case if $g > 1$, it is obvious that the inequality

$$f < (4/3)D^{1/3}$$

cannot be satisfied except by taking $x = \pm 1$, $y = 0$, $z = 0$. But it may happen that this minimum is $< (4/3)D^{1/3}$. The smallest value of $\tfrac{1}{3}\psi(y, z)$ corresponding to $z \neq 0$ is then $> \tfrac{1}{4}D^{2/3}$ and this is surely $> (4/3)D^{1/3}$ if $D > 151$. Hence, if $D > 151$ we have necessarily $z = 0$.

The excluded cases deserve special attention. There are only a finite number of cubic orders with $D \leqq 151$. They are*

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* See B. Delaunay's beautiful paper in Mathematische Zeitschrift, vol. 30 (1930).

$$D = 140;\ [1, \theta, \theta^2];\ \theta^3 + 2\theta - 2 = 0;$$
$$D = 139;\ [1, \theta, \theta^2];\ \theta^3 - 8\theta - 9 = 0;$$
$$D = 135;\ [1, \theta, \theta^2];\ \theta^3 - 3\theta - 3 = 0;$$
$$D = 116;\ [1, \theta, \theta^2];\ \theta^3 - \theta^2 - 2 = 0;$$
$$D = 108;\ [1, \theta, \theta^2];\ \theta^3 - 2 = 0;$$
$$D = 107;\ [1, \theta, \theta^2];\ \theta^3 - 2\theta^2 + 4\theta - 1 = 0;$$
$$D = 104;\ [1, \theta, \theta^2];\ \theta^3 - \theta - 2 = 0;$$
$$D = 87;\ [1, \theta, \theta^2];\ \theta^3 + \theta^2 + 2\theta - 1 = 0;$$
$$D = 83;\ [1, \theta, \theta^2];\ \theta^3 + 2\theta^2 + 2\theta - 1 = 0;$$
$$D = 76;\ [1, \theta, \theta^2];\ \theta^3 - \theta^2 + 3\theta - 1 = 0;$$
$$D = 59;\ [1, \theta, \theta^2];\ \theta^3 + 2\theta - 1 = 0;$$
$$D = 44;\ [1, \theta, \theta^2];\ \theta^3 + \theta^2 + \theta - 1 = 0;$$
$$D = 31;\ [1, \theta, \theta^2];\ \theta^3 + \theta - 1 = 0;$$
$$D = 23;\ [1, \theta, \theta^2];\ \theta^3 - \theta - 1 = 0.$$

It can be verified directly that for $D \geqq 76$ it always turns out that $z=0$ while in the remaining cases the fundamental unit is obvious. Thus if $D \geqq 76$ we have $X=0$ and $z=0$ and the bilinear relation reduces to

$$Yy = 0.$$

Hence, if $y \neq 0$ we have $Y=0$, $X=0$, $Z=\pm 1$ and if $Y \neq 0$ we have $x=\pm 1$, $y=0$, $z=0$. If, therefore, for $\Delta=1$ we try to satisfy the inequalities

$$f < (4/3)D^{1/3},$$
$$F < (4/3)D^{2/3},$$

together with the relation

$$Xx + Yy + Zz = 0,$$

it turns out that necessarily either $\phi = \pm 1$ or $\omega = \pm 1$ provided $D \geqq 76$. As

$$-(\alpha' - \alpha'')^2 + 2(\alpha - \alpha')(\alpha - \alpha'') \leqq 2D^{1/2}$$

it is obvious that for $\Delta = 1$ the triads 1, 0, 0 and 0, 0, 1 satisfy all the requirements.

7. As we have seen, for any particular value of Δ there exist two triads x_0, y_0, z_0; X_0, Y_0, Z_0 consisting of numbers without common divisor, connected by the bilinear relation

$$X_0x_0 + Y_0y_0 + Z_0z_0 = 0$$

and satisfying the inequalities

$$f < \frac{4}{3}\left(\frac{D}{\Delta}\right)^{1/3},$$

$$F < \frac{4}{3}\left(\frac{D}{\Delta}\right)^{2/3}.$$

When Δ starts to increase these inequalities continue to hold, but that cannot go on indefinitely. The triad x_0, y_0, z_0 has to be replaced by another triad when the following equality is reached:

$$\text{(a)} \qquad f = \frac{4}{3}\left(\frac{D}{\Delta}\right)^{1/3}.$$

Likewise, the triad X_0, Y_0, Z_0 has to be replaced when Δ attains such a value that

$$\text{(b)} \qquad F = \frac{4}{3}\left(\frac{D}{\Delta}\right)^{2/3}.$$

Equations (a) and (b) have each two positive roots and their greatest roots which interest us can be found as follows.

Let

$$\Gamma = \frac{8}{27}D\left(\frac{k}{\tau_1}\right)^3,$$

where

$$k = \frac{1}{4}N(\phi_0)\left(\frac{D}{27}\right)^{-1/2}; \ \phi_0 = x_0 + \alpha y_0 + \beta z_0,$$

and τ_1 is the smallest positive root of the equation

$$\tau^3 - \tau + k = 0.$$

Then the value of Δ beyond which the triad x_0, y_0, z_0 cannot be retained is given by

$$\Delta_0 = \Gamma\Phi_0^{-3} = \Gamma(\phi_0/N(\phi_0))^3.$$

Similarly, putting

$$\Gamma' = \frac{8}{27^{1/2}}D\,|\,\alpha'' - \alpha'\,|^{-3}\tau_2^3,$$

where τ_2 is the greatest root of the equation

$$\tau^3 - \tau + k = 0,$$

and

$$k = \frac{g}{4} N(\omega_0) \left(\frac{D}{27}\right)^{-1/2},$$

we have for the greatest root of the equation (b) the following expression:

$$\Delta_1 = \Gamma' \omega_0^{-3} = \Gamma'(\Omega_0 / N(\omega_0))^3.$$

Suppose now that $\Delta_0 < \Delta_1$. Then we have to change the triad x_0, y_0, z_0 when the variable parameter Δ reaches the value Δ_0, the second triad remaining the same in a certain interval beyond Δ_0. The new triad following x_0, y_0, z_0 can be determined by the process described in §5. For practical purposes it is advantageous to proceed in the following manner.

Determine three integers l, m, n from the equations

$$\begin{aligned} ny_0 - mz_0 &= X_0, \\ lz_0 - nx_0 &= Y_0, \\ mx_0 - ly_0 &= Z_0, \end{aligned}$$

and set

$$\begin{aligned} x &= \lambda x_0 + \mu l, \\ y &= \lambda y_0 + \mu m, \\ z &= \lambda z_0 + \mu n. \end{aligned} \tag{A}$$

By substituting these expressions into the form ϕ it becomes

$$\phi = \phi_0 \lambda + (l + m\alpha + n\beta)\mu.$$

Multiplying it by Φ_0 we obtain another linear form,

$$g = N(\phi_0)\lambda + A\mu,$$

where A is a determined number of the order. Now the binary quadratic form

$$T = 2G + N(\phi_0) g^2 / \Gamma,$$

where

$$G = g' g'' N(\phi_0)^{-1},$$

differs but by a constant factor from f. By reduction of the form T we obtain integers λ, μ making f minimum. Substituting the values of λ, μ thus found, into (A), we obtain a triad x_1, y_1, z_1 taking the place of x_0, y_0, z_0.

The advantage of considering the form T instead of f resides in the fact that coefficients of g and G are moderate numbers limited by D.

If $\Delta_1 < \Delta_0$ we have to change the triad X_0, Y_0, Z_0 and this can be done as follows.

Determine three integers L, M, N from the equations

$$NY_0 - MZ_0 = x_0,$$
$$LZ_0 - NX_0 = y_0,$$
$$MX_0 - LY_0 = z_0,$$

and set

$$\text{(B)} \qquad \begin{aligned} X &= \lambda X_0 + \mu L, \\ Y &= \lambda Y_0 + \mu M, \\ Z &= \lambda Z_0 + \mu N. \end{aligned}$$

By this substitution the form ω becomes

$$\omega = \omega_0\lambda + (N + M\alpha/g + L(\beta + l))\mu$$

and, multiplying it by Ω_0, we obtain another linear form in λ, μ,

$$N(\omega_0)\lambda + B\mu = N(\omega_0)h.$$

The binary quadratic form

$$V = h^2 + 2H/\Gamma_1,$$

where

$$H = N(\omega_0)h'h''$$

and

$$\Gamma_1 = 8(D/27)^{1/2}\tau_2^3 g^{-1},$$

differs but by a constant factor from F. By the process of reduction we find integers λ, μ giving a minimum of V or F and substituting them into (B) we have a triad X_1, Y_1, Z_1 following X_0, Y_0, Z_0.

8. The process described in the preceding paragraph is simple and the only objection that could be raised is that Γ and Γ' are determined by the solution of the cubic equation

$$\tau^3 - \tau + k = 0$$

which may appear burdensome. However, this equation having three real roots can easily be solved by known and handy formulas. On the other hand, because of the existence of three real roots we have necessarily

$$k < \frac{2\cdot 3^{1/2}}{9} < 0.385,$$

and it is easy to form a small table giving $\log \tau_2$ and $\log (k/\tau_1)$ to three decimal places, which is amply sufficient in most cases. If such a table cannot be used, which can happen only under extremely exceptional circumstances, then direct solution becomes necessary. However, the labor involved in this

solution is quite insignificant compared with that required by numerous other operations. From the fact that

$$k < \frac{2 \cdot 3^{1/2}}{9},$$

it follows readily that numbers ϕ and ω obtained in the described manner have their norms limited by D; we have in fact

$$N(\phi) < (8/27)D^{1/2}, \quad N(\omega) < (8/27)D^{1/2}g^{-1}.$$

To facilitate the application of our method to numerical examples, we give here a small table mentioned above.

TABLE I

k	$\log \tau_2$	k	$\log \tau_2$	k	$\log \tau_2$	k	$\log \tau_2$	k	$\log \tau_2$
0.01	9.998–10	0.10	9.976	0.19	9.948	0.28	9.908	0.360	9.8410
0.02	9.996	0.11	9.973	0.20	9.944	0.29	9.902	0.362	9.8382
0.03	9.993	0.12	9.970	0.21	9.940	0.30	9.896	0.364	9.8350
0.04	9.991	0.13	9.967	0.22	9.936	0.31	9.889	0.366	9.8318
0.05	9.989	0.14	9.964	0.23	9.932	0.32	9.882	0.368	9.8283
0.06	9.986	0.15	9.961	0.24	9.928	0.33	9.874	0.370	9.8247
0.07	9.984	0.16	9.958	0.25	9.923	0.34	9.864	0.372	9.8206
0.08	9.981	0.17	9.955	0.26	9.918	0.35	9.854	0.374	9.8162
0.09	9.979	0.18	9.951	0.27	9.913	0.36	9.841	0.376	9.8113

TABLE II

k	$\log (k/\tau_1)$	k	$\log (k/\tau_1)$	k	$\log (k/\tau_1)$	k	$\log (k/\tau_1)$	k	$\log (k/\tau_1)$
0.01	0.000	0.10	9.996	0.19	9.983	0.28	9.956	0.360	9.9004
0.02	0.000	0.11	9.994	0.20	9.981	0.29	9.952	0.362	9.8977
0.03	0.000	0.12	9.994	0.21	9.978	0.30	9.947	0.364	9.8949
0.04	9.999–10	0.13	9.992	0.22	9.976	0.31	9.942	0.366	9.8919
0.05	9.999	0.14	9.991	0.23	9.973	0.32	9.936	0.368	9.8887
0.06	9.998	0.15	9.990	0.24	9.970	0.33	9.929	0.370	9.8883
0.07	9.998	0.16	9.988	0.25	9.967	0.34	9.921	0.372	9.8815
0.08	9.997	0.17	9.986	0.26	9.964	0.35	9.912	0.374	9.8773
0.09	9.996	0.18	9.985	0.27	9.960	0.36	9.900	0.376	9.8727

9. Starting with the triads $x_0=1$, $y_0=0$, $z_0=0$; $X_0=0$, $Y_0=0$, $Z_0=1$ corresponding to $\Delta=1$ and letting Δ increase, we come to $\Delta=\Delta_0$ determined according to the explanation in (7) where new triads $x_1, y_1, z_1; X_1, Y_1, Z_1$ replace the old ones and these triads hold until Δ reaches the value $\Delta_1>\Delta_0$ where new triads $x_2, y_2, z_2; X_2, Y_2, Z_2$ appear in place of $x_1, y_1, z_1; X_1, Y_1, Z_1$ and so on. Thus we can form an infinite chain of triads

$$\left\{\begin{matrix} x_0, & y_0, & z_0 \\ X_0, & Y_0, & Z_0 \end{matrix}\right\}; \quad \left\{\begin{matrix} x_1, & y_1, & z_1 \\ X_1, & Y_1, & Z_1 \end{matrix}\right\}; \quad \left\{\begin{matrix} x_2, & y_2, & z_2 \\ X_2, & Y_2, & Z_2 \end{matrix}\right\}; \cdots,$$

and two series of corresponding values of Φ and ω,

$$\text{(A)} \qquad \Phi_0 = 1,\ \Phi_1,\ \Phi_2,\ \Phi_3,\ \cdots,$$

$$\text{(B)} \qquad \omega_0 = 1,\ \omega_1,\ \omega_2,\ \omega_3,\ \cdots.$$

Let $E>1$ be the direct fundamental unit of the order and $\epsilon=E^{-1}$ the inverse unit. If series (A) and (B) are extended sufficiently we necessarily find ϵ either in (A) or in (B). To prove this statement we observe first that numbers in the sequence

$$\Delta_0 < \Delta_1 < \Delta_2 < \Delta_3 < \cdots$$

finally surpass any given limit so that the unit E falls into the interval between two consecutive numbers of this sequence. Let $x, y, z; X, Y, Z$ be two triads corresponding to $\Delta=E^3$. We have

$$2\Phi + \phi^2/E^3 < \frac{4}{3}\left(\frac{D}{E^3}\right)^{1/3},$$

$$-(\alpha''-\alpha')^2\omega^2 + 2(\alpha-\alpha')(\alpha-\alpha'')\frac{\Omega}{E^3} < \frac{4}{3}\left(\frac{D}{E^3}\right)^{2/3},$$

and

$$Xx + Yy + Zz = 0,$$

or, by formula (2),

$$(3) \qquad (\alpha''-\alpha')\phi\omega + (\alpha-\alpha'')\phi'\omega' + (\alpha'-\alpha)\phi''\omega'' = 0.$$

Setting

$$E^{-1}\phi = \bar{\phi} = \bar{x} + \alpha\bar{y} + \beta\bar{z},$$

$$E\omega = \bar{\omega} = \overline{Z} + (\alpha/g)\overline{Y} + (\beta+l)\overline{X},$$

where $\bar{x}, \bar{y}, \bar{z}; \overline{X}, \overline{Y}, \overline{Z}$ are integers, the preceding inequalities may also be written as follows:

$$
(4)\qquad \begin{aligned} &2\bar{\Phi} + \bar{\phi}^2/1 < (4/3)D^{1/3},\\ &-(\alpha'' - \alpha')^2\bar{\omega}^2 + 2(\alpha - \alpha')(\alpha - \alpha'')\bar{\Omega}/1 < (4/3)D^{2/3}. \end{aligned}
$$

Now

$$\bar{\phi}\bar{\omega} = \phi\omega,$$

and by virtue of (3) and (2)

$$\bar{X}\bar{x} + \bar{Y}\bar{y} + \bar{Z}\bar{z} = 0.$$

It follows from the inequalities (4) that either $\bar{\Phi}=1$ or $\bar{\omega}=1$ provided $D \geqq 76$. In the former case we have $\Phi=\epsilon$ and in the latter case $\omega=\epsilon$.

10. We are sure, therefore, to find the unit ϵ in (A) or in (B) as soon as the variable parameter Δ reaches the value $\Delta=E^3$. But it may happen that a unit appears in (A) or in (B) before that, and then the question arises whether this is a fundamental unit or not. To settle this question we observe that any unit which may appear in (A) or (B) is <1. Let it be, for example, Φ_k. If $\bar{\Delta}$ is the parameter value for which the equality

$$2\Phi_0 + \frac{\phi_0^2}{\Delta} = \frac{4}{3}\left(\frac{D}{\Delta}\right)^{1/3}$$

is reached, the parameter value corresponding to the equality

$$2\Phi_k + \frac{\phi_k^2}{\Delta} = \frac{4}{3}\left(\frac{D}{\Delta}\right)^{1/3}$$

is $\bar{\Delta}\Phi_k^{-3}$, and evidently this must be $>\bar{\Delta}$; that is, $\Phi_k<1$.

In a similar manner it can be proved that any unit ω_k in (B) must be <1.

Suppose now that Φ_k is the first unit we meet in the series (A) and (B). It must be a positive power of ϵ, say ϵ^n. Accepting $\bar{\Delta}$ in the same meaning as before, we see that the inequality

$$(\mathrm{C})\qquad 2\epsilon^n + \frac{E^{2n}}{\Delta} < \frac{4}{3}\left(\frac{D}{\Delta}\right)^{1/3}$$

holds up to

$$\Delta = \bar{\Delta}E^{3n} > E^{3n}.$$

On the other hand, if $n>1$ the first parameter value where Φ_k appears must be $<E^3$, so that (C) holds good for

$$\Delta = E^3, E^6, \cdots, E^{3n}.$$

If $n \geqq 3$ we take in (C) $\Delta = E^{3n-3}$ and $\Delta = E^{3n-6}$, and the resulting inequalities are equivalent to

$$2\epsilon + E^2 < (4/3)D^{1/3}; \;\; 2\epsilon^2 + E^4 < (4/3)D^{1/3},$$

and, supposing $D \geqq 76$, we have, by §6,

$$E = a + b\alpha, \;\; E^2 = a' + b'\alpha,$$

which is impossible. Hence $n \leqq 2$. If $n = 2$, in a similar manner we see that the fundamental unit must be binary,

$$E = a + b\alpha,$$

and it is very easy to discover whether the unit ϕ_k, given by our process, can be a square of the binary unit.

If ω_k is the first unit we meet in (A) or (B), in exactly the same manner it may be shown that it is the fundamental unit ϵ unless it is a square of the binary (fundamental) unit, provided again $D \geqq 76$. Thus our operations may be stopped as soon as we find in (A) or (B) a unit, and if this is not a fundamental unit, the latter can easily be found. For the remaining discriminants $\leqq 59$ the method undergoes slight modifications and easily leads to fundamental units.

Note. Although the discovery of the fundamental unit presents no difficulty, it can be shown in certain cases that the first unit found in (A) or (B) is necessarily a fundamental unit. For instance, that is true in the case of the order with basic form

$$\phi = x + y\theta + z\theta^2,$$

where θ is a root of the irreducible equation

$$\theta^3 = A.$$

11. Before passing to numerical examples designed to illustrate the foregoing method, it is well to collect certain formulas which greatly simplify the march of operations. The first necessary step is to find an appropriate basis of the order as explained in §2. For that purpose one may have a ready expression of the form denoted there by ψ. Written in the usual manner this form is

$$(5)\; \psi = 3(gl - f^2/3 + f\alpha + 3g\beta, - (2/3)lf + l\alpha - f\beta, - kf - l^2/3 + 3k\alpha - l\beta).$$

When an appropriate basis has been found, we know the corresponding coefficients f, g, k, l and have the following equations which are constantly used:

$$\begin{aligned} \alpha\alpha &= gl + f\alpha + g\beta, \\ \alpha\beta &= kg, \\ \beta\beta &= - kf + k\alpha - l\beta; \end{aligned} \tag{6}$$

$$\begin{gathered}\alpha' + \alpha'' = f - \alpha, \quad \beta' + \beta'' = -l - \beta,\\ \alpha'\alpha'' = g\beta, \qquad \beta'\beta'' = k\alpha,\\ \alpha'\beta'' + \alpha''\beta' = -kg - lf + l\alpha - f\beta.\end{gathered} \tag{7}$$

Also, it is useful to have before one's eyes the expressions of forms T and V. We have

$$\begin{aligned} T &= 2G + N(\phi_0)g^2/\Gamma,\\ g &= \Phi_0(\lambda\phi_0 + \mu(l + m\alpha + n\beta)) = N(\phi_0)\lambda + A\mu,\\ G &= g'g''N(\phi_0)^{-1},\\ \Gamma &= (8/27)D(k/\tau_1)^3, \quad k = \tfrac{1}{4}N(\phi_0)(D/27)^{-1/2}.\end{aligned}$$

Similarly

$$\begin{aligned} V &= h^2 + 2H/\Gamma_1,\\ \Omega_0[\lambda\omega_0 + \{N + M(\alpha/g) + L(\beta + l)\}\mu] &= N(\omega_0)h,\\ H &= N(\omega_0)h'h'',\\ \Gamma_1 &= 8(D/27)^{1/2}\tau_2^3 g^{-1}.\end{aligned}$$

As to parameter values where either the triad x_0, y_0, z_0 or X_0, Y_0, Z_0 has to be changed, they are given by

$$\Delta = \Gamma\Phi_0^{-3}, \quad \Delta_1 = \Gamma_1 D^{1/2} g\omega_0^{-3} | \alpha'' - \alpha' |^{-3}.$$

Finally, to obtain $|\alpha'' - \alpha'|$ we have the equation

$$| \alpha'' - \alpha' |^2 = -gl - f^2 + f\alpha + 3g\beta.$$

In finding approximate values of different numbers, it is hardly ever necessary to retain more than three or even two decimals. It is advisable once and for all to have logarithms of

$$8D/27, \quad 8(D/27)^{1/2}g^{-1}, \quad gD^{1/2} | \alpha'' - \alpha' |^{-3}$$

and also the value of

$$4(D/27)^{1/2}.$$

As to τ_2 and k/τ_1, their logarithms are supplied by the auxiliary tables in §8. The different numerical operations required by the method are uniform and go quickly and smoothly after a certain amount of practice.

12. For the first example, let us consider the order $[1, \theta, \theta^2]$ where θ is a root of the equation

$$\theta^3 - \theta^2 + 3\theta - 5 = 0.$$

The discriminant of this order is -524, so that $D=524$. The basis 1, θ, θ^2 is not normal, but the equivalent normal basis is easily found to be

$$1, A = -1 + \theta, \quad B = 3 + \theta^2.$$

Corresponding to this basis we have

$$f = -2, \quad g = 1, \quad k = 2, \quad l = -4.$$

From (5) we find

$$\psi = 3(-16/3 - 2A + 2B, \quad -16/3 - 4A + 2B, \quad -4/3 + 6A + 4B).$$

On the other hand

$$A = 0.403; \quad B = 4.968,$$

approximately, and

$$\tfrac{1}{3}\psi = (8.77; \ 2.99; \ 20.96).$$

This is a reduced form and therefore the basis 1, A, B satisfies all the requirements. Denoting by α, β the last two numbers of this basis, we have

$$\alpha = 0.403; \quad \beta = 4.968,$$

and by (6) and (7)

$$\alpha\alpha = -4 - 2\alpha + \beta, \quad \beta\beta = 4 + 2\alpha + 4\beta, \quad \alpha\beta = 2,$$

$$\alpha' + \alpha'' = -2 - \alpha, \quad \beta' - \beta'' = 4 - \beta,$$

$$\alpha'\alpha'' = \beta, \qquad \beta'\beta'' = 2\alpha,$$

$$\alpha'\beta'' + \alpha''\beta' = -10 - 4\alpha + 2\beta.$$

Next we find

$$\log(8D/27) = 2.191; \ \log[8(D/27)^{1/2}] = 1.547; \ 4(D/27)^{1/2} = 17.62;$$

$$|\alpha'' - \alpha'|^2 = 14.10; \ \log[D^{1/2}|\alpha'' - \alpha'|^{-3}] = \bar{1}.636.$$

Having found these numbers we can proceed to our operations. We start with triads

$$x_0 = 1, \quad y_0 = 0, \quad z_0 = 0,$$

$$X_0 = 0, \quad Y_0 = 0, \quad Z_0 = 1,$$

corresponding to $\Delta = 1$, and seek the next following triad.

First operation. As $\phi_0 = 1$, $\Phi_0 = 1$, $N(\phi_0) = 1$, we have

$$k = 0.057$$

and from the tables on p. 12,

$$\log \tau_2 = \bar{1}.987;\ \log (k/\tau_1) = \bar{1}.998;$$

$$\begin{array}{rlrl} & 2.191 & & 1.547 \\ 3\log(k/\tau_1) = & \bar{1}.994 & 3\log\tau_2 = & \bar{1}.961 \\ \hline \log\Gamma = & 2.185; & \log\Gamma_1 = & 1.508\ ;\ \Gamma = 153.1;\ \Gamma_1 = 32.21; \end{array}$$

$$\begin{array}{rr} & 1.508 \\ & \bar{1}.636 \\ \hline \log\Delta = 2.185; & 1.144 = \log\Delta_1. \end{array}$$

As $\Delta > \Delta_1$ the second triad has to be changed first. Following the indications in §7 we must find integers L, M, N satisfying the system of equations

$$N\cdot 0 - M\cdot 1 = 1;\quad M = -1;$$
$$L\cdot 1 - N\cdot 0 = 0;\quad L = 0;$$
$$M\cdot 0 - L\cdot 0 = 0;\quad N \text{ arbitrary}.$$

Next we find the binary form in λ, μ

$$\omega = \lambda + \mu(N - \alpha),$$

and choose N so as to make $N - \alpha$ as small as possible. We must take $N = 0$ and then

$$h = \omega = \lambda - \mu\alpha,$$
$$h^2 = (1,\ -\alpha,\ -4 - 2\alpha + \beta) = (1;\ -0.403;\ 0.162),$$
$$2H = 2h'h'' = (2,\ 2 + \alpha,\ 2\beta) = (2;\ 2.403;\ 9.936),$$
$$2H/\Gamma_1 = (0.062;\ 0.075;\ 0.308),$$
$$1000V = (1062,\ -328,\ 470).$$

Hence, the minimum of V is attained for $\lambda = 0$, $\mu = \pm 1$. We take $\lambda = 0$, $\mu = -1$ to make ω positive. Thus

$$\omega_1 = \alpha,$$

and

$$\Omega_1 = \alpha'\alpha'' = \beta,\ N(\omega_1) = 2,$$
$$k = 0.114;\qquad \log\tau_2 = \bar{1}.972;$$

$$\begin{array}{rrr} & & 1.463 \\ & & \bar{1}.636 \\ \cline{3-3} & 1.547 & 1.099 \\ & \bar{1}.916 & \bar{2}.816 \\ \cline{2-3} \log\Gamma_1 = & 1.463;\ \Gamma_1 = 29.04; & \bar{2}.283 = \log\Delta_1. \end{array}$$

Second operation. As the new value for Δ_1 is greater than the previously found Δ, now we have to change the first triad. From the equations

$$n\cdot 0-m\cdot 0=0,\ l\cdot 0-n\cdot 1=1,\ m\cdot 1-l\cdot 0=0,$$

we find $m=0$, $n=-1$, l arbitrary, and consequently

$$\phi = \lambda + (l-\beta)\mu.$$

For the sake of simplicity we shall take $l=0$ and then

$$\begin{aligned}
\phi &= g = \lambda - \beta\mu,\\
g^2 &= (1,\ -\beta,\ 4+2\alpha+4\beta) = (1;\ -4.968;\ 24.678),\\
N(\phi_0)g^2/\Gamma &= (0.006;\ -0.032;\ 0.161),\\
2G &= (2,\ -4+\beta,\ 4\alpha) = (2;\ 0.968;\ 1.612),\\
1000T &= (2006,\ 936,\ 1773).
\end{aligned}$$

Hence $\lambda=0$, $\mu=-1$, and

$$\phi_1 = \beta;\ \Phi_1 = 2\alpha;\ N(\phi_1) = 4;$$

$$k = 0.227;\ \log(k/\tau_1) = \bar{1}.974;$$

$$\begin{array}{rcl}
 & \underline{\begin{array}{r}2.191\\ \bar{1}.922\end{array}} & \qquad \underline{\begin{array}{r}2.113\\ \bar{1}.719\end{array}}\\
\log\Gamma = & 2.113; & \Gamma = 129.7;\ 2.394 = \log\Delta.
\end{array}$$

Third operation. Now we have to change the second triad. Solving the equations

$$N\cdot 1 - M\cdot 0 = 0;\ L\cdot 0 - N\cdot 0 = 0;\ M\cdot 0 - L\cdot 1 = 1,$$

we find $L=-1$, $N=0$ and take $M=2$. Then

$$\begin{aligned}
\omega &= \lambda\alpha + (4+2\alpha-\beta)\mu, \quad h = \lambda - \alpha\mu,\\
h^2 &= (1;\ -0.403;\ 0.162),\\
2H &= (4;\ 4.806;\ 19.872),\\
2H/\Gamma_1 &= (0.138;\ 0.165;\ 0.684),\\
1000V &= (1138,\ -238,\ 846),
\end{aligned}$$

whence $\lambda=0$, $\mu=1$ and

$$\omega_2 = -4-2\alpha+\beta,\ \Omega_2 = 4+2\alpha+4\beta;\ N(\omega_2) = 4;$$

$$k = 0.227; \qquad \log\tau_2 = \bar{1}.933;$$

$$\begin{array}{rcl}
 & & \underline{\begin{array}{r}1.346\\ \bar{1}.636\end{array}}\\
 & \underline{\begin{array}{r}1.547\\ \bar{1}.799\end{array}} & \qquad \underline{\begin{array}{r}0.982\\ \bar{3}.629\end{array}}\\
\log\Gamma_1 = & 1.346; & \Gamma_1 = 22.18;\ 3.343 = \log\Delta_1.
\end{array}$$

Fourth operation. Now we have to change the first triad and accordingly find l, m, n from the equations

$$n\cdot 0 - m\cdot 1 = 1;\ l\cdot 1 - n\cdot 0 = -2;\ m\cdot 0 - l\cdot 0 = 0.$$

We find $l=-2$, $m=-1$, while n remains arbitrary. Taking $n=0$ we have

$$\phi = \lambda\beta - (2+\alpha)\mu;\ g = 4\lambda + (8-2\beta)\mu;$$

$$G = 4(2,\ 2+\beta/2,\ \alpha+2\beta) = 4(2;\ 4.484;\ 10.339);$$

$$N(\phi_1)g^2 = 4(16,\ 32-8\beta,\ 80+8\alpha-16\beta) = 4(16;\ -7.744;\ 3.736);$$

$$N(\phi_1)g^2/\Gamma = 4(0.124;\ -0.060;\ 0.029);$$

$$250T = (2124,\ 4424,\ 10368)\begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix}(2124,\ 176,\ 1168).$$

Hence $\lambda=2$, $\mu=-1$ and

$$\phi_2 = 2+\alpha+2\beta,\ \Phi_2 = -4-2\alpha+\beta;\ N(\phi_2) = 2;$$

$$k = 0.114;\ \log(k/\tau_1) = \bar{1}.994;$$

$$\begin{array}{rlr} & \underline{2}.191 & \underline{2}.173 \\ & \bar{1}.982 & \bar{3}.629 \\ \hline \log\Gamma = & 2.173;\quad \Gamma = 149.0; & 4.544 = \log\Delta. \end{array}$$

Fifth operation. Again the second triad has to be changed. From

$$-2N = 2,\quad -N = 1,\quad M+2L = 2,$$

we find $N=-1$, $M=2-2L$ and take $L=1$. Thus

$$\omega = (-4-2\alpha+\beta)\lambda + (-5+\beta)\mu;$$

$$h = \lambda - \tfrac{1}{2}\alpha\mu;$$

$$H = (4,\ 4+2\alpha,\ 2\beta);$$

$$2H/\Gamma_1 = (0.181;\ 0.217;\ 0.448);$$

$$h^2 = (1;\ -0.202;\ 0.040);$$

$$1000V = (1181,\ 15,\ 448),$$

whence $\lambda=0$, $\mu=-1$,

$$\omega_3 = 5-\beta,\ \Omega_3 = 5+2\alpha+5\beta,\ N(\omega_3) = 1.$$

Thus ω_3 is an algebraic unit and as it is not a square of the binary unit of the form $a+b\alpha$ it must be the required fundamental unit ϵ:

$$\epsilon = 5-\beta$$

or

$$\epsilon = 2-\theta^2,\ E = 18+2\theta+5\theta^2.$$

13. For the last application of the method we shall find the fundamental unit in the field determined by the root of the equation

$$\theta^3 = 19.$$

A basis of integers of this field is

$$1, \theta, \frac{1+\theta+\theta^2}{3},$$

and we obtain a suitable normal basis by taking

$$\alpha = \theta - 1, \ \beta = \frac{1+\theta+\theta^2}{3}.$$

For this basis we have

$$f = -3, \ g = 3, \ k = 2, \ l = -1,$$

and

$$\alpha\alpha = -3 - 3\alpha + 3\beta, \ \beta\beta = 6 + 2\alpha + \beta,$$

$$\alpha\beta = 6, \ \alpha'\alpha'' = 3\beta, \ \beta'\beta'' = 2\alpha,$$

$$\alpha' + \alpha'' = -3 - \alpha, \ \beta' + \beta'' = 1 - \beta,$$

$$\alpha'\beta'' + \alpha''\beta' = -9 - \alpha + 3\beta.$$

Moreover,

$$\alpha = 1.668; \ \beta = 3.596; \ D = 3.19^2;$$

$$4(D/27)^{1/2} = 25.33; \ \log(8D/27) = 2.506; \ \log\left[8(D/27)^{1/2}g^{-1}\right] = 1.228;$$

$$\log\left[D^{1/2}g\,|\alpha'' - \alpha'|^{-3}\right] = 0.$$

Starting with $\phi_0 = 1, \omega_0 = 1$, we have respectively

$$\begin{aligned} k &= 0.04; \quad 3\log(k/\tau_1) = \bar{1}.997 \\ &\qquad\qquad\qquad\qquad\quad\ \ 2.506 \\ &\qquad\qquad\qquad \log\Gamma = 2.503; \ \Gamma = \Delta = 318.4; \\ k &= 0.12; \qquad 3\log\tau_2 = \bar{1}.910 \\ &\qquad\qquad\qquad\qquad\quad\ \ 1.228 \\ &\qquad\qquad\qquad \log\Gamma_1 = 1.138; \ \Gamma_1 = \Delta_1 = 13.74. \end{aligned}$$

First operation. We take $M = -1, L = 0, N = 1$, so that

$$h = \omega = \lambda + \mu(1 - \alpha/3),$$

$$2H = (2, 3 + \alpha/3, 4 + \tfrac{2}{3}\alpha + \tfrac{2}{3}\beta),$$

$$100V = (115, 70, 75)\begin{pmatrix}1 & -1\\ 0 & 1\end{pmatrix}(115, -45, 50).$$

Hence

$$\lambda = 1,\ \mu = -1;\ \omega_1 = \alpha/3,\ \Omega_1 = \beta/3,\ N(\omega_1) = \tfrac{2}{3};$$

$$\begin{aligned} k = 0.08;\ 3\log\tau_2 &= \bar{1}.943 \\ &\ \underline{1.228} \\ \log\Gamma_1 &= 1.171;\ \Gamma_1 = 14.83; \\ &\ \underline{\bar{1}.245} \\ &\ 1.926 = \log\Delta_1. \end{aligned}$$

Second operation. Here we have $N=1$, $L=0$, $M=0$, and

$$\omega = \tfrac{1}{3}\lambda\alpha + \mu;$$

$$h = \lambda + (\beta/2)\mu;\ 2H = (4/3,\ (1-\beta)/3,\ 2\alpha/3);$$

$$100V = (109,\ 174,\ 331)\begin{pmatrix}1 & -2\\ 0 & 1\end{pmatrix}(109,\ -44,\ 71).$$

Hence $\lambda=2$, $\mu=-1$ and

$$\omega_2 = -1 + 2\alpha/3,\ \Omega_2 = 3 + 2\alpha/3 + 4\beta/3,\ N(\omega_2) = 1;$$

$$\begin{aligned} k = 0.12;\ 3\log\tau_2 &= \bar{1}.910 \\ &\ \underline{1.228} \\ \log\Gamma_1 &= 1.138;\ \Gamma_1 = 13.74; \\ &\ \underline{2.849} \\ &\ 3.987 = \log\Delta_1. \end{aligned}$$

Third operation. Now the first triad is to be changed. Here we find

$$l = 0,\ m = -1,\ n = -2,$$

and

$$\phi = \lambda - (\alpha + 2\beta)\mu = g;$$

$$G = (1,\ \tfrac{1}{2} + \tfrac{1}{2}\alpha + \beta,\ -18 + 6\alpha + 9\beta);$$

$$100T = (200,\ 983,\ 4899)\begin{pmatrix}1 & -5\\ 0 & 1\end{pmatrix}(200,\ -17,\ 69),$$

whence $\lambda=5$, $\mu=-1$ and

$$\phi_1 = 5 + \alpha + 2\beta,\quad \Phi_1 = 2 + \alpha - \beta,\quad N(\phi_1) = 1.$$

Here, for the first time, we find an algebraic unit. As ϕ_1 is not a square of a binary unit it must be the direct fundamental unit of the field. Thus

$$E = 5 + \alpha + 2\beta = \tfrac{1}{3}(14 + 5\theta + \theta^2),$$

$$\epsilon = 2 + \alpha - \beta = \tfrac{1}{3}(2 + 2\theta - \theta^2).$$

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ON THE ASYMPTOTIC SOLUTIONS OF ORDINARY DIFFERENTIAL EQUATIONS, WITH AN APPLICATION TO THE BESSEL FUNCTIONS OF LARGE ORDER*

BY

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1. **Introduction.** The investigations which have hitherto been made of the solutions of the ordinary linear differential equation

$$u''(x) + p(x)u'(x) + \{\rho^2\phi^2(x) + q(x)\}u(x) = 0,\dagger \tag{1}$$

with respect to their asymptotic dependence upon the complex parameter ρ, have almost without exception been restricted to the case in which $\phi^2(x)$, the coefficient of the parameter, remains positive (or negative) over the entire assigned interval of the variable x. The results in that case are familiar.‡ A determination of coefficients $\alpha_{ij}(x)$ in a pair of expressions of the type

$$\begin{aligned}&\phi^{-1/2}(x)e^{i\xi}\left\{1+\frac{\alpha_{11}(x)}{\rho}+\frac{\alpha_{12}(x)}{\rho^2}+\cdots\right\},\\ &\phi^{-1/2}(x)e^{-i\xi}\left\{1+\frac{\alpha_{21}(x)}{\rho}+\frac{\alpha_{22}(x)}{\rho^2}+\cdots\right\}, \qquad \xi=\rho\int_{x_0}^{x}\phi(x)dx,\end{aligned} \tag{2}$$

is possible, so that the resulting forms represent asymptotically a pair of solutions of the equation given. Moreover, each of these forms is then known to represent one and the same solution so long as ρ remains in a region of the complex plane in which its pure imaginary part is either invariably greater than or invariably less than some constant.

The forms (2) are evidently of oscillatory or exponential type according as $\phi^2(x)$ is positive or negative on the x interval under account, and on different intervals in which $\phi^2(x)$ is of opposite sign the forms are respectively of the two opposing types. Between two such intervals a change in the character of any solution must, therefore, take place. It is evident, however, that the manner of this change is not deducible from the forms (2), for whatever the magnitude of ρ these forms fail to remain significant at a zero of $\phi^2(x)$.

* Presented to the Society, April 18, 1930; received by the editors April 11, 1930.

† It is merely as a matter of convenience that $\rho^2\phi^2(x)$ is written rather than $\rho\phi(x)$.

‡ Horn, Mathematische Annalen, vol. 52 (1899), pp. 340–362; Birkhoff, these Transactions vol. 9 (1908), p. 219.

Hence there always exists some interval about such a zero in which the forms do not represent a solution. Because of this failure in the mode of representation, the relation between the solutions respectively represented by the forms (2) on opposite sides of a zero of $\phi^2(x)$ is obscured, and the law for its determination does not appear to have been given. Upon this law depends the complete determination of the form of any specific solution of the equation in different x intervals.

Directly connected with this problem of the association of solutions represented by the forms (2) for different values of x, is that of the relation between the solutions so represented for any fixed value of x but differently restricted values of ρ. Indeed, it will be shown that it is a single problem to determine completely the asymptotic forms in question for unrestricted values of both x and ρ. It is this larger problem which is taken as the subject of the present discussion. It contains the more restricted problems already mentioned as integral parts, and may perhaps lay claim to some degree of interest and importance. Aside from its utility as illustrated by the application to be discussed below, it may be remarked, in particular, that a solution of the problem is a requisite to the general extension of the method of asymptotic forms to the treatment of such boundary problems as result when an equation of type (1) is subjected to boundary conditions which apply at points between which $\phi^2(x)$ becomes zero.

A special boundary problem of this latter type has on previous occasion* been considered by the author. While a solution of the general problem described above was not found necessary in that case, due to the special structure of the differential system considered, the discussion and results of that paper have been largely suggestive for the present considerations. In this connection it should be noted that the problem at hand as applied to a specialized equation of the type (1) was raised and briefly discussed by H. Jeffreys† in 1924, in connection with an application to the solutions of the Mathieu equation. A resemblance of thought will be noticeable in a comparison of certain of the considerations of the present paper with those of the paper cited.

The procedure of the discussion may be briefly outlined as follows. It is shown to begin with, by the derivation of explicit formulas, that there may be associated with any given equation of the type (1) a certain "related" equation. This related equation on the one hand approximates the equation given in a specific sense and on the other hand is explicitly solvable with the use of Bessel functions of particular order and argument. It is shown then,

* These Transactions, vol. 31 (1930), pp. 1–24.

† Proceedings of the London Mathematical Society, (2), vol. 23 (1925), p. 428.

at first for restricted values of x and ρ, and subsequently for general values, that the solutions of the given equation are represented asymptotically by the known solutions of the related equation. The asymptotic form so obtained for any particular solution of the equation given is subject to discontinuous changes as x passes from one to the other side of a zero of $\phi^2(x)$, or as the value of $\arg \rho$ passes certain specific bounds.* The law which governs this change is determined and is shown to depend upon the degree to which $\phi^2(x)$ vanishes at the zero under consideration. These deductions occupy Part I of the paper.

In Part II of the paper the theory of the preceding part is applied to a special equation which is known to be solved by the Bessel functions of order ρ and argument ρe^x. As a result the formulas for the functions J_ρ, Y_ρ and H_ρ with arguments ρ sech α and $\rho \sec \beta$ are obtained for all real values of α and β and for large positive values of ρ. These formulas are in the main those which are familiarly known as representing the functions in question, and which are given, for instance, in Watson's treatise. For the so-called intermediate values of the arguments, however, the formulas here obtained show certain differences and possibly hold certain advantages over those heretofore given. In any event, it may be not without interest that the various formulas are derived here practically as a group and through the means of a special application of more or less general results, rather than, as has heretofore been the case, by methods which were especially developed and adapted to the peculiar ends in view and which vary considerably from one set of formulas to the next.

Part I

The general theory of the asymptotic forms

2. **The given differential equation.** A familiar change of dependent variable reduces the equation (1) to the normal form

$$u''(x) + \{\rho^2\phi^2(x) - \chi(x)\}u(x) = 0, \tag{3}$$

in which $\chi(x)$ is a function simply determinable from the coefficients $p(x)$, and $q(x)$. This form of the equation is conveniently adapted to the considerations which are to be made. The transformation will, therefore, be supposed to have been carried out, and throughout the discussion the differential equation will be supposed given in the form (3).

The primary assumption which is to be made, and which principally

* It will be recalled that such changes of asymptotic form occur also in the theory of the Bessel functions, and that the phenomenon has been designated in that connection as the Stokes' phenomenon.

characterizes the equations to which the discussion is devoted, is that the interval of the argument x may include a zero of the coefficient $\phi^2(x)$. This and the remaining assumptions are made specific in the following statement of hypotheses.

(i) *The function* $\phi^2(x)$ *vanishes at a point* $x=x_0$ *to the degree* ν, *where* ν *is any real positive constant or zero.*

(ii) *On some interval* I_1 *which contains the point* $x=x_0$, *the function* $\phi^2(x)$ *has no zero other than that at* x_0.*

(iii) *On some interval* I_2 *which contains the point* $x=x_0$, *the function* $(x-x_0)^{-\nu}\phi^2(x)$ *possesses a continuous second derivative and is real and positive except possibly for a constant complex factor.*

(iv) *The function* $\chi(x)$ *is defined on some interval* I_3 *which contains the point* $x=x_0$, *and is bounded on any finite portion of this interval.*

In the case of any specifically given equation (3) the intervals I_1, I_2, and I_3, on which the respective hypotheses above are satisfied, may be finite or may extend to infinity in either or both directions. Inasmuch as all the hypotheses are fulfilled only for values of x which are common to all three of these intervals, the variable x will be taken throughout the following discussion to lie on an interval I, which is closed at such finite end points as it may have, which includes the point $x=x_0$, and which contains only points common to the three intervals I_1, I_2, and I_3. Subject to these specifications the sub-intervals into which I is divided by the point $x=x_0$ may be either finite or infinite.

The hypotheses (i) to (iv) are concerned primarily with the character of the given equation in the proximity of the point $x=x_0$. In case the interval I is infinite, it is to be expected that the character of the equation for values of x remote from x_0 is likewise of significance. This is in fact so, and necessitates a further hypothesis which is to be found below in §7.

Neither the form of the differential equation (3) nor the validity of the hypotheses made is affected either by the change of independent variable $x'=x-x_0$, or by any transfer of a constant factor from the function $\phi^2(x)$ to the parameter ρ^2. Hence it may be assumed without loss of generality, firstly, that the origin is located at the zero of $\phi^2(x)$, i.e., that $x_0=0$, and, secondly, that the function $x^{-\nu}\phi^2(x)$ is real and positive.

Since the function $x^{-\nu}\phi^2(x)$ has by hypothesis a continuous second derivative, an application of Taylor's theorem yields the formula

$$\phi^2(x) = x^{\nu}\{\alpha_0 + \alpha_1 x + \alpha_2(x)x^2\}, \tag{4}$$

* The case of several, or any number of isolated zeros of $\phi^2(x)$ would, of course, be treated by sub-dividing the interval and considering separately the sub-intervals containing just one zero.

in which the coefficients are real, α_0 and α_1 being constants, and $\alpha_2(x)$ a function continuous on the interval I. For positive values of x the function $\phi^2(x)$ is real and positive. For negative values of x its specification is made definitive by the relation

$$\arg \phi^2(x) = \arg x^\nu = \nu\pi.$$

The function $\phi(x)$ will be defined as that root of $\phi^2(x)$ which is positive for positive values of x.

A material simplification of the manipulations which follow is attained by the use of suitable abbreviations, and by the introduction of certain functions determined by the equation at hand. To this end let the functions $\Phi(x)$, $\Omega(x)$, and $\Psi(x)$ be defined respectively by the formulas

$$\Phi(x) = \int_0^x \phi(x)dx,$$

$$(5) \qquad \Omega(x) = \frac{\Phi(x)}{\phi(x)},$$

$$\Psi(x) = \frac{\{\Phi(x)\}^{1/2-\mu}}{\{\phi(x)\}^{1/2}},$$

in which the constant μ is given by the equation

$$(6) \qquad \mu = \frac{1}{\nu + 2} \cdot$$

The further relations

$$(7) \qquad \xi = \rho\Phi(x), \quad \tau = \rho\Phi(t)$$

may be looked upon as defining the abbreviations ξ and τ.

It is a matter of simple verification that the functions

$$x^{-1/(2\mu)}\Phi(x), \quad x^{-1}\Omega(x), \quad \Psi(x)$$

are each real and positive, and have each on the interval I a continuous second derivative. Their Taylor's developments lead, therefore, to the formulas

$$\Phi(x) = x^{1/(2\mu)}\{\beta_0 + \beta_1 x + \beta_2(x)x^2\}, \quad \beta_0 \neq 0,$$

$$(5a) \qquad \Omega(x) = x\{\gamma_0 + \gamma_1 x + \gamma_2(x)x^2\}, \quad \gamma_0 = 2\mu,$$

$$\Psi(x) = \{\delta_0 + \delta_1 x + \delta_2(x)x^2\}, \quad \delta_0 \neq 0,$$

in which the coefficients with subscript 2 are continuous functions, while those with subscripts 0 and 1 are constants easily computable from those in formula (4). The evaluation of γ_0 has been especially noted for a subsequent purpose.

It is to be particularly remarked that the constant μ as given by formula (6) is in every case less than or at most equal to $\frac{1}{2}$. This constant will occur prominently in many subsequent formulas.

3. **The related differential equation.** The relation

$$y(x) = \rho^k\{\Omega(x)\}^{1/2}C(\xi) \tag{8}$$

defines the function $y(x)$ in terms of a constant k and a function C which will for the moment be left unspecified. Upon suitable differentiation of this relation, and an application of the equality

$$\Omega^{1/2}\phi' + \frac{\Omega'\phi}{\Omega^{1/2}} = \frac{\phi}{\Omega^{1/2}},$$

which follows readily from the formulas (5), it is found that

$$y''(x) = \left\{\frac{\Omega''}{2\Omega} - \frac{(\Omega')^2}{4\Omega^2}\right\}y(x) + \rho^{2+k}\phi^2(x)\Omega^{1/2}\left\{C''(\xi) + \frac{1}{\xi}C'(\xi)\right\}. \tag{9a}$$

This result suggests the choice of the function C as a cylinder function, for if C is so chosen, say as a cylinder function of order k, then

$$C''(\xi) + \frac{1}{\xi}C'(\xi) = \left\{-1 + \frac{k^2}{\xi^2}\right\}C(\xi),$$

and the relation (9a) reduces in consequence to the form

$$y''(x) + \rho^2\phi^2 y(x) = \left[\frac{(\Omega^2)'' - 3(\Omega')^2 + 4k^2}{4\Omega^2}\right]y(x), \tag{9b}$$

namely to a differential equation for the function $y(x)$.

Superficially this equation (9b) is of the form of equation (3). However, since $\Omega(x)$ vanishes at $x=0$ a consideration of the coefficient enclosed within brackets in the right-hand member is not dispensable. This consideration is simply made. If the function within the brace of the second of formulas (5a) is denoted by the symbol $\Omega_1(x)$, i.e.,

$$\Omega(x) = x\Omega_1(x),$$

it is readily found that the coefficient in question may be written in the form

$$\left[\frac{4k^2 - \{\Omega_1^2 - x(\Omega_1^2)'\} + x^2\{2\Omega_1\Omega_1'' - \Omega_1'^2\}}{4x^2\Omega_1^2}\right].$$

Now on the one hand $\Omega_1^2(x) \neq 0$, and on the other hand the function

$$\Omega_1^2(x) - x\{\Omega_1^2(x)\}'$$

differs from the value $\Omega_1^2(0)$ by a quantity of which x^2 is a factor. It follows from this that the coefficient in question is continuous at $x=0$ if and only if $4k^2-\Omega_1^2(0)=0$, i.e., from (5a) if $k^2=\mu^2$. The motivation for the choice of C in formula (8) as a cylinder function of order $\pm\mu$, i.e., $C(\xi)=C_{\pm\mu}(\xi)$, is clear. With this choice and with the choice $k=\mu$, the formula (8) may be written

$$y(x) = \Psi(x)\xi^{\mu}C_{\pm\mu}(\xi), \tag{10}$$

the function $\Psi(x)$ here involved being that given in the third of formulas (5). By the deductions thus concluded the formula (10) solves explicitly the differential equation

$$y''(x) + \{\rho^2\phi^2(x) - \omega(x)\}y(x) = 0, \tag{11}$$

in which

$$\omega(x) = \frac{1}{4\Omega^2}\{(\Omega^2)'' - 3(\Omega')^2 + 4\mu^2\}. \tag{12}$$

In virtue of the continuity of $\omega(x)$ on the interval I, it is immediately evident that the differential equation (11) is possessed of every property hypothesized for the given equation (3). It is found convenient on this account to designate the equation (11) as *related* to the equation (3), or simply as the *related equation*. The result thus obtained is embodied in the following theorem:

THEOREM 1. *To every equation of type* (3) *which satisfies the hypotheses* (i) *to* (iv), *there corresponds a related equation* (11) *which is of the same type and involves the same coefficient* $\phi^2(x)$, *and which is explicitly solved by the formula* (10).

4. **The solutions of the related equation.** By Theorem 1 a complete set of solutions of the related equation (11) may be obtained by substituting successively in formula (10) two linearly independent cylinder functions of order $\pm\mu$. As a first choice it is proposed to use for this purpose the Bessel functions of the third kind* with suitably chosen arguments. The choice of arguments is to be dictated by the desired asymptotic forms, and will depend, therefore, upon the ranges of the variables x and ρ. The considerations involved are the following.

For values of the complex variable z which are of large modulus, the Bessel functions $H_\mu^{(1)}(z)$ and $H_\mu^{(2)}(z)$ are of known asymptotic forms. These forms remain invariant, i.e., each of the functions is represented by one and the same corresponding analytic expression, so long as z remains in a (any)

* Cf. G. N. Watson, *A Treatise on the Theory of Bessel Functions*, Cambridge University Press, 1922, p. 73. This reference will be indicated in the text by [W].

specific right or left-hand half of the complex z plane. However, if z varies unrestrictedly the original expressions may cease to represent the functions in question, and the rôle may be filled in different half-planes by different analytic forms. This phenomenon will assume essential prominence in the later portions of the discussion. It is desirable at the moment, however, to evade the complications which it involves, and this may be simply done by a restriction of the variable z to a specific half-plane. For the purpose immediately at hand, therefore, it is sufficient to note here the forms [W, p. 198]

$$(13)\qquad \begin{aligned} H_\mu^{(j)}(z) &\sim A_j^\mu z^{-1/2} e^{\pm iz}\left[1+\sum_{m=1}^{\infty}\frac{(\mu, m)}{\left(\pm\frac{i}{2}z\right)^m}\right], \quad (\mu, m) \text{ constants},\\ A_j^\mu &= \left(\frac{2}{\pi}\right)^{1/2} e^{\mp(\mu+1/2)\pi i/2},^* \end{aligned}$$

valid for $-\pi/2 \leqq \arg z \leqq \pi/2$. These formulas will serve as a basis for the derivation of the asymptotic forms of those solutions of the related equation which are to be chosen.

The variable x is real, but may be either positive or negative. The parameter ρ and the quantity ξ on the other hand are complex and may range over the entire respective complex planes. The manner and extent to which restrictions are imposed upon the locations of x and ρ by corresponding restrictions upon the location of ξ, and vice versa, is easily determined as follows.

Let the complex ξ plane be divided into quadrants $\Xi_{k,l}$, $k=0, \pm 1, \pm 2, \cdots$; $l=1, 2$, by the relations

$$(14)\qquad \Xi_{k,1}: (k-\tfrac{1}{2})\pi < \arg\xi \leqq k\pi, \qquad \Xi_{k,2}: k\pi < \arg\xi \leqq (k+\tfrac{1}{2})\pi.$$

For $x>0$ the function $\Phi(x)$ is real and positive and therefore $\arg\xi=\arg\rho$. On the other hand, for $x<0$, $\arg\Phi(x)=\pi/(2\mu)$, and hence $\arg\xi=\arg\rho+\pi/(2\mu)$. It is evident from this that

ξ in $\Xi_{k,1}$ corresponds to

$$(15)\qquad \begin{cases} \text{either } x>0, \text{ and } \left(k-\dfrac{1}{2}\right)\pi < \arg\rho \leqq k\pi,\\ \text{or } x<0, \text{ and } \left(k-\dfrac{1}{2}-\dfrac{1}{2\mu}\right)\pi < \arg\rho \leqq \left(k-\dfrac{1}{2\mu}\right)\pi; \end{cases}$$

* The notation requires explanation. It will frequently be convenient as in the present case to write two formulas in one by the use of double signs together with an index j. It will be understood in every such case that the upper signs are to be associated with the value $j=1$ and the lower signs with $j=2$.

ξ in $\Xi_{k,2}$ corresponds to

$$\begin{cases} \text{either } x > 0, \text{ and } k\pi < \arg\rho \leqq \left(k + \dfrac{1}{2}\right)\pi, \\ \text{or } x < 0, \text{ and } \left(k - \dfrac{1}{2\mu}\right)\pi < \arg\rho \leqq \left(k + \dfrac{1}{2} - \dfrac{1}{2\mu}\right)\pi. \end{cases}$$

Any condition expressed in terms of ξ and x is, therefore, easily translated into terms of ρ and x, and the mode of expression may be considered immaterial. The pair of adjacent quadrants $\Xi_{k,1}$ and $\Xi_{k,2}$ will be referred to as the half-plane Ξ_k.

If k is given any integral value, the formulas

$$(16)\qquad y_{k,j}(x) = \begin{cases} \dfrac{\Psi(x)}{i^k A_j^{\mu}} \xi^{\mu} H_{\mu}^{(j)}(\xi e^{-k\pi i}), \text{ if } k \text{ is even}, \\ \dfrac{\Psi(x)}{i^k A_{3-j}^{\mu}} \xi^{\mu} H_{\mu}^{(3-j)}(\xi e^{-k\pi i}), \text{ if } k \text{ is odd}, \ j = 1, 2, \end{cases}$$

define, by Theorem 1, a pair of linearly independent solutions $y_{k,1}(x)$, $y_{k,2}(x)$, for the equation (11). These solutions and their derivatives are of importance in the deductions which follow. The computation of the derivatives is simple. Thus, since [W, p. 74]

$$\frac{d}{dz}\left\{ z^{\mu} H_{\mu}^{(j)}(z) \right\} = e^{\pm(1-\mu)\pi i} z^{\mu} H_{1-\mu}^{(j)}(z), \ j = 1, 2,$$

while from formulas (7) and (5) it is found that

$$(17)\qquad \frac{d\xi}{dx} = \frac{\rho^{2\mu}\xi^{1-2\mu}}{\Psi^2(x)},$$

it follows that

$$(18)\quad y_{k,j}'(x) = \frac{\Psi'(x)}{\Psi(x)} y_{k,j}(x) + \begin{cases} \dfrac{\rho^{2\mu} e^{\pm(1-\mu)\pi i}}{i^k \Psi(x) A_j^{\mu}} \xi^{1-\mu} H_{1-\mu}^{(j)}(\xi e^{-k\pi i}), \text{ if } k \text{ is even}, \\ \dfrac{\rho^{2\mu} e^{\mp(1-\mu)\pi i}}{i^k \Psi(x) A_{3-j}^{\mu}} \xi^{1-\mu} H_{1-\mu}^{(3-j)}(\xi e^{-k\pi i}), \text{ if } k \text{ is odd}. \end{cases}$$

It is readily verified that the formulas (18) may be written in terms of the quantities

$$(19)\qquad \bar{y}_{k,j}(x) = \left\{ y_{k,j}'(x) - \frac{\Psi'(x)}{\Psi(x)} y_{k,j}(x) \right\} \frac{\Psi^2(x)}{i\rho^{2\mu}}, \quad j = 1, 2,$$

in the form

$$\text{(16a)}\qquad \pm\,\bar{y}_{k,j}(x) = \begin{cases} \dfrac{\Psi(x)}{i^k A_j^{1-\mu}}\xi^{1-\mu}H_{1-\mu}^{(j)}(\xi e^{-k\pi i}), & \text{if } k \text{ is even,} \\[2ex] \dfrac{\Psi(x)}{i^k A_{3-j}^{1-\mu}}\xi^{1-\mu}H_{1-\mu}^{(3-j)}(\xi e^{-k\pi i}), & \text{if } k \text{ is odd, } j = 1, 2. \end{cases}$$

This suggests an advantage in the consideration of the functions (16a) in place of the derivatives (18), for on the one hand the values $y_{k,j}'(x)$ are directly obtainable from the values $\bar{y}_{k,j}(x)$ and $y_{k,j}(x)$, while on the other hand the formulas (16a) are obtainable from the formulas (16) by the mere substitution of $1-\mu$ in place of μ. Many of the following deductions concerned with the quantities (16) can, therefore, be adapted by slight and obvious changes to apply to the quantities (16a), and the pertinent results for the derivatives (18) may thus be cheaply obtained.

The asymptotic forms of the quantities (16) and (16a) for $|\xi|$ large and ξ in the half-plane Ξ_k may be found from the formulas (13). Thus when ξ lies in Ξ_k the value $\xi_0=\xi e^{-k\pi i}$ lies in the half-plane Ξ_0, namely, in the region of validity of the relations (13). The forms desired are, therefore, available in terms of ξ_0 and may, of course, be simply expressed in terms of ξ. It is thus found that

$$\text{(20)}\qquad \begin{aligned} y_{k,j}(x) &\sim \Psi(x)\xi^{\mu-1/2}e^{\pm i\xi}\left\{1+\sum_{m=1}^{\infty}\frac{c_m^{k,j}}{\xi^m}\right\}, \quad \text{for } \xi \text{ in } \Xi_k, \\ \pm\,\bar{y}_{k,j}(x) &\sim \Psi(x)\xi^{1/2-\mu}e^{\pm i\xi}\left\{1+\sum_{m=1}^{\infty}\frac{\bar{c}_m^{k,j}}{\xi^m}\right\}, \quad j = 1, 2. \end{aligned}$$

The coefficients c in these formulas are constants which are easily computed. A record of their explicit values is, however, not essential to the purpose at hand.

For subsequent use certain further facts concerning the quantities (16) and (16a) are to be noted. Inasmuch as all considerations are to be made on the assumption that $|\rho|$ is bounded from zero, any relation $|\xi|<N$, in which N is a positive constant, restricts the values of x to a finite portion of the interval I. For such values of x, however, the function $\Psi(x)$ is bounded, and since the quantity

$$z^q H_q^{(j)}(z), \qquad j = 1, 2; \qquad q > 0,$$

remains bounded with $|z|$, and approaches a limit not zero as $z\to 0$, it follows that

(21) $$| y_{k,j}(x) | \leqq M, \quad | \bar{y}_{k,j}(x) | \leqq M, \quad \text{for } | \xi | \leqq N,^*$$

and that the values $y_{k,j}(0)$ and $\bar{y}_{k,j}(0)$ are constants different from zero.

From the form of equation (11) it is observable that the Wronskian of the solutions $y_{k,j}(x)$ is independent of x. Its value, as computed from the formulas (16) and (17), is found to be

$$W(y_{k,1}, y_{k,2}) = (-1)^k \frac{\rho^{2\mu}\xi}{A_1^{\mu} A_2^{\mu}} W(H_\mu^{(1)}(\xi e^{-k\pi i}), H_\mu^{(2)}(\xi e^{-k\pi i})),$$

and since the Wronskian on the right of this relation has the value [W, p. 76]

$$-\frac{4i}{\pi \xi e^{-k\pi i}},$$

the formula reduces to

(22) $$W(y_{k,1}, y_{k,2}) \equiv -2i\rho^{2\mu}.$$

For subsequent use it is desirable to note also the related formula

(22a) $$D(y_{k,1}, y_{k,2}) \equiv \begin{vmatrix} y_{k,1}(x) & y_{k,2}(x) \\ \bar{y}_{k,1}(x) & \bar{y}_{k,2}(x) \end{vmatrix} = -2\Psi^2(x),$$

which follows directly from the equalities (19) and (22). The results of this section may be summarized as follows:

THEOREM 2. *The formulas* (16) *define for the related differential equation* (11) *a pair of solutions* $y_{k,1}(x)$, $y_{k,2}(x)$, *corresponding to any integer* k. *These solutions satisfy the relations* (21) *and* (22), *and are asymptotically given by the formulas* (20) *for the values of* x *and* ρ *which correspond to values of* ξ *in the half-plane* Ξ_k.

5. **The formal solution of the given equation.** Let the given differential equation (3) be written in the form

(3a) $$u''(x) + \{\rho^2\phi^2(x) - \omega(x)\} u(x) = \theta(x)u(x),$$

in which

(23) $$\theta(x) = \chi(x) - \omega(x),$$

and $\omega(x)$ is the function so designated in equation (11). The structure of the left members of equations (3a) and (11) is then the same, and familiar reasoning, in which the relation (3a) is looked upon as non-homogeneous, leads to the equation

* The symbols M and N are used here, and will be used in the work to follow, merely to designate "some positive constant." It is to be understood, therefore, that the constants so represented in different formulas are not necessarily the same.

$$(24)\qquad u(x) = c_1 y_1(x) + c_2 y_2(x) - \int_{x_0}^{x} \frac{y_1(x)y_2(t) - y_2(x)y_1(t)}{W(y_1, y_2)}\theta(t)u(t)dt,$$

the values c_1, c_2, and x_0 being constants, and $y_1(x)$, $y_2(x)$ any linearly independent solutions of equation (11). By actual substitution, it is easily verified that the equation (3a) is solved by any solution of the equation (24), howsoever the constants and solutions y involved in the latter may be chosen. It is proposed to determine the form of certain solutions of equation (3a) through the means of establishing the existence and corresponding form of certain solutions of equation (24), with specific choices of the elements involved. These choices will be governed by the configuration of the values x and ρ.

For any pair of values x, ρ, the corresponding value ξ lies in some quadrant $\Xi_{k,l}$, and ξ will remain in this quadrant if x is restricted to maintain its sign and ρ is confined to a corresponding quadrant of the ρ plane as shown by the relations (15). It will be supposed that x and ρ are so restricted throughout the discussion immediately at hand. From the relations (15) it is clear that two distinct configurations of x and ρ correspond to ξ in the quadrant $\Xi_{k,l}$. While a distinction between them is not necessary at this point, it will nevertheless be supposed that one and the same configuration persists throughout the discussion.

Under the assumption that ξ is confined to the quadrant $\Xi_{k,l}$, the solutions $y_{k,j}(x)$, $j=1, 2$, are known to have the forms deduced for them in the preceding section. These solutions may in particular be chosen to serve as the functions y involved in the formula (24). If the designation $u_{k,j}(x)$ is given to the solution u corresponding to the choice of values $y_1 = y_{k,1}$, $y_2 = y_{k,2}$, $c_j = 1$, $c_{3-j} = 0$, the equation (24) becomes

$$(24\text{a})\qquad u_{k,j}(x) = y_{k,j}(x) + \frac{1}{2i\rho^{2\mu}}\int_{x_0}^{x}\{y_{k,1}(x)y_{k,2}(t) - y_{k,2}(x)y_{k,1}(t)\}\theta(t)u_{k,j}(t)dt,$$
$$j = 1, 2,$$

A determination of the constant x_0 remains to be made, but will for the moment be deferred.

Let the quantities $U_j(x)$ and $Y_j(x)$ be defined by the formulas

$$(25)\qquad U_j(x) = \frac{u_{k,j}(x)}{\Psi(x)}e^{\mp i\xi}, \quad Y_j(x) = \frac{y_{k,j}(x)}{\Psi(x)}e^{\mp i\xi}, \quad j = 1, 2.$$

With the use of these functions the equation (24a) may be written in the form

$$U_j(x) = Y_j(x) + \frac{1}{\rho^{2\mu}}\int_{x_0}^{x} K_j(x, t, \rho)U_j(t)dt, \quad j = 1, 2, \tag{24b}$$

with

$$K_j(x, t, \rho) = \pm\frac{\theta(t)\Psi^2(t)}{2i}\{Y_j(x)Y_{3-j}(t) - Y_{3-j}(x)Y_j(t)e^{\mp 2i(\xi-\tau)}\}. \tag{26}$$

It will be recalled that the symbol τ here involved was defined in formulas (7).

The equation (24b) is in form an integral equation for $U_j(x)$, and as such may be iterated in familiar fashion, the quantity U_j under the sign of integration being replaced by the entire right-hand member of the equation. A continued repetition of this process leads ultimately to the formal equality

$$U_j(x) = Y_j(x) + \sum_{n=1}^{\infty}\frac{Y_j^{(n)}(x)}{\rho^{n\sigma}}, \tag{27}$$

in which

$$\begin{aligned} Y_j^{(n)}(x) &= \rho^{\sigma-2\mu}\int_{x_0}^{x} K_j(x, t, \rho)Y_j^{(n-1)}(t)dt \quad (n = 1, 2, \cdots; j = 1, 2), \\ Y_j^{(0)}(x) &= Y_j(x). \end{aligned} \tag{28}$$

The relations (27) and (28) when taken together are independent of the constant σ which has been introduced into them. It will be seen later that a specific choice of this constant is a source of convenience for the discussion.

The infinite series on the right of the relation (27) formally satisfies the equation (24b). It will be shown in the following sections that a choice of the undetermined elements involved may be made so that the series converges uniformly and the relation (27) in consequence represents a true solution.

Let the functions $\bar{u}_{k,j}(x)$ be defined in a manner analogous to (19) by the formula

$$\bar{u}_{k,j}(x) = \left(u_{k,j}'(x) - \frac{\Psi'(x)}{\Psi(x)}u_{k,j}(x)\right)\frac{\Psi^2(x)}{i\rho^{2\mu}}, \qquad j = 1, 2. \tag{29}$$

If it is observed that the equation (24a) may be formally differentiated by the mere substitution of $u_{k,j}'$ and $y_{k,j}'$ respectively for the quantities $u_{k,j}$ and $y_{k,j}$ where they occur with the argument x, then it is readily seen also that the similar substitution of $\bar{u}_{k,j}(x)$ and $\bar{y}_{k,j}(x)$ for $u_{k,j}(x)$ and $y_{k,j}(x)$ in (24a) leads to a valid formula. It follows that the formal considerations made above for the functions $u_{k,j}(x)$ may be made equally well and with similar result for the functions $\bar{u}_{k,j}(x)$.

6. **Lemmas.** The proof of the existence of a set of solutions $U_j(x)$ of equation (24b), and the associated investigation of their structure, is to be based upon the relation (27) obtained formally in §5. It is requisite to this end that the infinite series involved in the relation be shown convergent. This is conveniently done by means of the facts which are framed below in the form of a set of simple lemmas.

Let N be an arbitrarily chosen positive constant and consider the relation

$$|\rho\Phi(\bar{x})| = N. \tag{30}$$

The function $\Phi(x)$ is continuous on the interval I. From its definition it is clear, moreover, that $|\Phi(x)|$ increases monotonically with $|x|$ on each of the sub-intervals into which I is divided by the point $x=0$. If I^* is used to designate that one of these sub-intervals which contains x under the configuration of values which was assumed in the preceding section, it follows that a unique point $\bar{x}$ on the interval I^* is determined by the relation (30) for each value of ρ. The dependence of $\bar{x}$ upon ρ is evident. In virtue of the formulas (5a) it is seen that this dependence satisfies quantitatively a relation of the type

$$m_1|\rho|^{-2\mu} \leqq |\bar{x}| \leqq m_2|\rho|^{-2\mu}, \tag{31}$$

in which m_1 and m_2 are suitably determined positive constants.

The lemmas which follow are concerned with the evaluation of an integral

$$I(\alpha, \beta) = \int_\alpha^\beta H(x, t, \rho)dt, \tag{32}$$

subject to the following specifications:

(a) The points α and β lie on the interval I^*;

(b) The configuration of values x, ρ is that assumed in §5;

(c) The integrand satisfies a pair of relations

$$\begin{aligned} &|H(x, t, \rho)| < Mh(t), \quad \text{for } t \text{ on } I^* \text{ and } |\tau| \leqq N, \\ &|\tau^\delta H(x, t, \rho)| < Mh(t), \quad \text{for } t \text{ on } I^* \text{ and } |\tau| > N, \end{aligned} \tag{33}$$

in which the exponent δ is a specific positive constant, and $h(t)$ is a function continuous on the interval I^*.

LEMMA 1. *If* $0 \leqq |\alpha| \leqq |\beta| \leqq |\bar{x}|$, *then*

$$|I(\alpha, \beta)| \leqq M_1 |\rho|^{-2\mu}.$$

The proof is simple. On the interval of integration the first of relations (33) is satisfied. Since $h(t)$ is bounded, the same is true of the entire inte-

grand, and by relation (31) the length of the interval of integration is of the order of $|\rho|^{-2\mu}$.

LEMMA 2. *If x_c is any fixed point (i.e., independent of ρ) on the interval I^*, and if $|\bar{x}| \leqq |\alpha| \leqq |\beta| \leqq |x_c|$, then*

$$|I(\alpha,\beta)| \leqq \begin{cases} M_2|\rho|^{-\delta}, & \text{if } \mu > \delta/2, \\ M_2|\rho|^{-\delta}\log|\rho|, & \text{if } \mu = \delta/2, \\ M_2|\rho|^{-2\mu}, & \text{if } \mu < \delta/2. \end{cases}$$

On the interval of integration the second of formulas (33) is valid, and inasmuch as this interval is finite the function $h(t)$ is bounded. It is clear, therefore, that the integrand is of the order of $\tau^{-\delta}$. Since by formula (5a) τ is of the order of $\rho t^{1/(2\mu)}$, it follows that

$$|I(\alpha,\beta)| \leqq M|\rho|^{-\delta}\int_{|\bar{x}|}^{|x_c|}\frac{dt}{t^{\delta/(2\mu)}}\,.$$

The conclusion is at hand because of the relation (31).

If the interval I^* is finite, the discussion of cases arising from possible choices of α and β is completely covered by the Lemmas 1 and 2, for the point x_c may in particular be chosen as the end point of I^*. If the interval extends to infinity, however, a complete discussion must include also the further lemma which follows.

LEMMA 3. *If $|x_c| \leqq |\alpha| \leqq |\beta| \leqq \infty$ and if the integral*

$$\int_{x_c}^{\infty}\frac{h(t)}{\{\phi(t)\}^{\delta}}dt,$$

extended over the interval I^*, is convergent, then

$$|I(\alpha,\beta)| \leqq M_3|\rho|^{-\delta}.$$

From the second of relations (33) the integrand of $I(\alpha, \beta)$ is of the order of $h(t)/\tau^{\delta}$. Since this quantity is positive on the interval I^*, except possibly for a constant complex factor, the substitution of the value of τ from formula (7) yields the relation

$$|I(\alpha,\beta)| \leqq M|\rho|^{-\delta}\left|\int_{\alpha}^{\beta}\frac{h(t)}{\{\Phi(t)\}^{\delta}}\,dt\right|.$$

The conclusion follows.

7. **The dominant solution of the given equation for ξ in a quadrant $\Xi_{k,l}$.** From the definition of the quadrants $\Xi_{k,l}$ as given in the relations (14), it is

directly evident that when ξ remains on such a quadrant the real part of $i\xi$ is either always positive oral ways negative. Hence one and the same one of the two solutions $y_{k,j}(x)$ remains dominant for all such values of ξ. Whether the subscript value associated with this dominant solution is $j=1$, or $j=2$, will depend, of course, upon the particular quadrant $\Xi_{k,l}$ under consideration, namely, upon the values k and l. To avoid an unessential differentiation of cases the value j in question will be designated simply by $j=j'$.

The subject of immediate attention in the present section is the derivation of a solution of the equation (24b) from the relation (27) for the value $j=j'$. This involves a determination of the conditions under which the relation is convergent, and centers, therefore, upon a consideration of the quantities $Y_j^{(n)}(x)$ defined by the formula (28).

The constant x_0 was introduced into the formulas of §5, but was referred for later specification. This specification for the case in hand will now be made as follows, namely, *when* $j=j'$, *then* $x_0=0$. It will be understood throughout this section that this value of x_0 has been fixed upon, and it will be understood likewise without repeated specific mention of the fact that j temporarily takes the single value j' in the various formulas to be written.

The definitions (25) together with the formulas (20) and (21) yield readily the inequalities

$$|Y_j(x)| < M, \text{ for } |\xi| \leqq N,$$
$$|\xi^{1/2-\mu}Y_j(x)| < M, \text{ for } |\xi| > N.$$

These two relations may be combined into the single one

$$|\xi^{(1/2-\mu)s}Y_j(x)| < M, \tag{34}$$

if it is agreed to assign to the symbol s the value

$$s = \begin{cases} 0, \text{ if } |\xi| \leqq N, \\ 1, \text{ if } |\xi| > N. \end{cases} \tag{35}$$

This agreement of notation will be adopted and the symbol s will be used in accordance with it whenever a multiplicity of written formulas can thereby be avoided. For the sake of clarity the use of the letter s in any sense other than that defined by the relation (35) will be avoided.

The functions $Y_j^{(n)}(x)$ depend by the relations (28) upon the kernel $K_j(x, t, \rho)$ which is defined in the formula (26). This formula involves on the one hand the functions Y_1 and Y_2 which are bounded in virtue of the relations (34), and on the other hand the factor $\exp\{\mp 2i(\xi-\tau)\}$ which is now to be considered. As a result of the choice of x_0 the variable of integration, t, in formulas (24b) and (28) ranges on the interval I^* between 0 and x. Clearly

then $|\tau| \leqq |\xi|$. Since the value $\arg t$ is constant on the interval I^* it follows that for the values of t in hand

$$\arg [i(\xi - \tau)] = \arg i\xi.$$

The exponent $\mp 2i(\xi-\tau)$ in formula (26) is, therefore, of real part opposite in sign to that of the corresponding exponent in the formula (20). Inasmuch as the latter is positive for $j=j'$, in virtue of the determination of j', the exponential in $K_j(x, t, \rho)$ is bounded when $|t|$ does not exceed $|x|$. This conclusion, together with the inequalities (34), establishes, for $|t| \leqq |x|$, the relations

$$(36)\qquad \begin{aligned} |\xi^{(1/2-\mu)s}K_j(x, t, \rho)| &\leqq M\Psi^2(t)\,|\theta(t)|, \text{ for } |\tau| \leqq N,\\ |\tau^{(1/2-\mu)}\xi^{(1/2-\mu)s}K_j(x, t, \rho)| &\leqq M\Psi^2(t)\,|\theta(t)|, \text{ for } |\tau| > N. \end{aligned}$$

It is proposed to utilize this result to show that the quantities $Y_j^{(n)}(x)$ of formula (28) will satisfy the inequalities

$$(37)\qquad |\xi^{(1/2-\mu)s}Y_j^{(n)}(x)| \leqq M^{n+1} \qquad (n = 0, 1, 2, \cdots),$$

provided firstly that the positive constant M is suitably determined, and secondly that the hitherto unspecified constant σ in formulas (28) is properly chosen.

The relation (37) is valid for $n=0$ since it reduces for that value to the relation (34). The proof of the general validity of (37) by induction will accordingly be complete if it is shown that the assumption of its validity with n replaced by $(n-1)$ is sufficient to establish it as written. This may be done as follows.

If in (37) the value $(n-1)$ is substituted for n and t is written in place of x, the hypothesis tentatively adopted becomes

$$(37\text{a})\qquad \begin{aligned} |Y_j^{(n-1)}(t)M^{-n}| &< 1, \text{ when } |\tau| \leqq N,\\ |\tau^{1/2-\mu}Y_j^{(n-1)}(t)M^{-n}| &< 1, \text{ when } |\tau| > N. \end{aligned}$$

These inequalities together with the relations (36) show that the function

$$(38)\qquad \xi^{(1/2-\mu)s}K_j(x, t, \rho)Y_j^{(n-1)}(t)M^{-n}$$

satisfies, with the value $\delta=1-2\mu$, the hypotheses imposed in §6 upon the function $H(x, t, \rho)$. This is true, moreover, independently of the value assumed for n. Hence the formula (28) may be expressed in terms of integrals with the structure of those discussed in the lemmas of §6. The attendant considerations depend upon the range of values ascribed to the variable x and are the following.

Case 1. $|x| \leqq |\bar{x}|$. In this case $|\xi| \leqq N$. The function (38) with $s=0$ may be designated, therefore, by $H(x, t, \rho)$, and as a result the formula (28) becomes

$$Y_j^{(n)}(x) = M^n \rho^{\sigma-2\mu} I(0, x).$$

Lemma 1 is applicable to the right member of this equation and yields the inequality

$$| Y_j^{(n)}(x) | \leqq M^n | \rho |^{\sigma-4\mu} M_1,$$

in which the value M_1 is independent of n since the hypotheses on $H(x, t, \rho)$ are satisfied by the function (38) uniformly with respect to n. If M is chosen at least as great as the specifiable value M_1, the relation (37) follows for the values of x momentarily under consideration, provided σ is chosen not to exceed the value 4μ.

Case 2. $|\bar{x}| < |x| \leqq |x_c|$. In this case $|\xi| > N$, and the rôle of the function $H(x, t, \rho)$ may be assumed by the function (38) with $s=1$. Hence the formula (28) is expressible, after multiplication by $\xi^{1/2-\mu}$, in the form

$$\xi^{1/2-\mu} Y_j^{(n)}(x) = M^n \rho^{\sigma-2\mu} \{ I(0, \bar{x}) + I(\bar{x}, x) \}.$$

Lemmas 1 and 2 are applicable to the respective terms on the right of this relation, and serve to establish the inequality

$$| \xi^{1/2-\mu} Y_j^{(n)}(x) | \leqq M^n \{ M_1 | \rho |^{\sigma-4\mu} + M_2 | \rho |^{\sigma-\sigma_1} \},$$

the constant σ_1 which is involved being defined by the appropriate formula

$$\sigma_1 = \begin{cases} 1, & \text{if } \mu > \frac{1}{4}, \\ 1 - \epsilon, \text{ with } \epsilon > 0 \text{ but arbitrarily small,} & \text{if } \mu = \frac{1}{4}, \\ 4\mu, & \text{if } \mu < \frac{1}{4}. \end{cases} \tag{39}$$

Since M may be chosen to exceed both M_1 and M_2, it is clear that the relation (37) will follow in this case also provided σ is chosen not to exceed σ_1.

If the interval I^* is finite, the point x_c in Case 2 may in particular be chosen as the end point of this interval. The cases considered then exhaust the discussion. In the case of an infinite interval I^*, however, the discussion must be extended by a consideration also of the following additional case.

Case 3. $|x_c| < |x|$. The procedure of Case 2 is applicable without change to the case in hand, and by means of it the formula

$$\xi^{1/2-\mu} Y_j^{(n)}(x) = M^n \rho^{\sigma-2\mu} \{ I(0, \bar{x}) + I(\bar{x}, x_c) + I(x_c, x) \}$$

is readily obtained. The application of Lemmas 1 and 2 to the respective terms on the right is direct. If it is tentatively assumed that Lemma 3 is applicable in similar fashion to the final term it may be concluded that

$$|\xi^{1/2-\mu}Y_j^{(n)}(x)| \leqq M^n\{M_1|\rho|^{\sigma-4\mu} + M_2|\rho|^{\sigma-\sigma_1} + M_3|\rho|^{\sigma-1}\},$$

and hence that the relation (37) is valid provided, as in Case 2, the chosen value of σ does not exceed σ_1.

The application of the Lemma 3 in the manner provisionally assumed above is dependent, of course, upon the fulfillment of the hypotheses upon which the lemma is based, namely, for the case in hand, upon the convergence of the integral of the function $\Psi^2(t)|\theta(t)|/\{\Phi(t)\}^{1-2\mu}$. This function reduces in virtue of the formulas (5) to the form $|\theta(t)|/\phi(t)$, and it is clear, therefore, that the conclusions reached above are substantiable if, when the interval I is infinite, the list of hypotheses of §2 is augmented by the following addition.

(v) *The coefficients of the given differential equation are such that the integral*

$$\int \frac{\theta(t)}{\phi(t)}dt$$

converges absolutely when extended over those portions of the interval I on which $|t|$ exceeds some positive constant.

In summary, then, the relations (37) have been established firstly for $|\xi| \leqq N$ if $\sigma = 4\mu$, and secondly for $|\xi| > N$ if $\sigma = \sigma_1$. It follows directly, of course, that the infinite series in the relation (27) converges uniformly as to x when $|\rho|$ is sufficiently large, and that the relation in question accordingly represents a solution of the equation (24b). Specifically, therefore, there exists a solution of this equation which satisfies the relations

$$(40)\quad \begin{aligned} U_j(x) &= Y_j(x) + \sum_{n=1}^{\infty} \frac{E_n(x,\rho)}{\rho^{4n\mu}}, \text{ when } |\xi| \leqq N, \\ \xi^{1/2-\mu}U_j(x) &= \xi^{1/2-\mu}Y_j(x) + \sum_{n=1}^{\infty} \frac{E_n(x,\rho)}{\rho^{n\sigma_1}}, \text{ when } |\xi| > N, \end{aligned}$$

the symbols E being of the significance explained as follows.

The terms of the infinite series in the relation (27) are successively expressed by the formula (28) in terms of functions either initially known or previously determined. They are, therefore, known in the sense that they are expressible by specifiable formulas, and as such are, at least theoretically, computable. These observed facts together with the boundedness of the functions in question constitute the essential features with respect to which the series in the relation (27) and those series later to be derived from (27)

are of interest from the viewpoint of the discussion of the present paper. An economy of thought and formulas may, therefore, be achieved by the use of the letter E as a generic symbol in the following sense. The symbol E shall designate merely some computable function which is bounded for the ranges of its arguments under consideration. In a given formula different functions E will be distinguished by the use of subscripts. There is to be no presumption, however, that the same symbol in different formulas designates the same function. As a case in point it is evident that the functions similarly designated in the two equations (40) are not the same.

By means of the substitutions (25) the relations (40) are now readily made to yield the formulas for the solution $u_{k,j}(x)$ of the given differential equation. It may be observed to begin with that when $|\xi| \leqq N$ the functions $\Psi(x)$ and $e^{\pm i\xi}$ are bounded. Hence the first of the relations (40) reduces to the formula

$$(41a) \quad u_{k,j}(x) = y_{k,j}(x) + \sum_{n=1}^{\infty} \frac{E_n(x, \rho)}{\rho^{4n\mu}}, \text{ for } |\xi| \leqq N, \text{ and } \xi \text{ in } \Xi_{k,l}.$$

The second of the relations (40) yields in the same way the formula

$$(41b) \qquad u_{k,j}(x) = y_{k,j}(x) + \Psi(x)\xi^{\mu-1/2}e^{\pm i\xi} \sum_{n=1}^{\infty} \frac{E_n(x, \rho)}{\rho^{n\sigma_1}},$$
$$\text{for } |\xi| > N, \text{ and } \xi \text{ in } \Xi_{k,l}$$

and if the asymptotic expression (20) is substituted for $y_{k,j}(x)$ on the right of this equation there results the asymptotic relation

$$(41c) \; u_{k,j}(x) \sim \Psi(x)\xi^{\mu-1/2}e^{\pm i\xi}\left\{1 + \sum_{n=1}^{\infty} \frac{E_n(x, \rho)}{\rho^{n\sigma_1}} + \frac{c_n^{k,j}}{\xi^n}\right\}, \text{ for } \xi \text{ in } \Xi_{k,l}.$$

The formulas (41a), (41b), and (41c) have been established, it will be recalled, under the supposition that $j = j'$. They embody the results for which the deductions of the present section were primarily made.

It was observed in §5 that the formulas there derived remain valid if the quantities $u_{k,j}$ and $y_{k,j}$, where they occur with the argument x, are replaced throughout by $\bar{u}_{k,j}(x)$ and $\bar{y}_{k,j}(x)$ respectively. The persisting validity of the relation (34) under this change is contingent upon the simultaneous substitution of the factor $\xi^{(\mu-1/2)s}$ in place of $\xi^{(1/2-\mu)s}$. Inasmuch as all conclusions of the present section are based on the relation (34) and the formulas of §5, it is therefore clear that a repetition of the arguments made with the substitutions of quantities noted will lead to formulas for the quantity $\bar{u}_{k,j}(x)$ analogous to those for $u_{k,j}(x)$ explicitly derived above. It will

suffice, therefore, merely to note the formulas which are obtained in the manner indicated, namely

$$\text{(42a)}\quad \bar{u}_{k,j}(x) = \bar{y}_{k,j}(x) + \sum_{n=1}^{\infty} \frac{E_n(x,\rho)}{\rho^{4n\mu}}, \quad \text{for } |\xi| \leqq N, \text{ and } \xi \text{ in } \Xi_{k,l},$$

$$\text{(42b)}\quad \bar{u}_{k,j}(x) = \bar{y}_{k,j}(x) + \Psi(x)\xi^{1/2-\mu}e^{\pm i\xi} \sum_{n=1}^{\infty} \frac{E_n(x,\rho)}{\rho^{n\sigma_1}}, \quad \text{for } |\xi| > N, \text{ and } \xi \text{ in } \Xi_{k,l},$$

$$\text{(42c)}\quad \pm\bar{u}_{k,j}(x) \sim \Psi(x)\xi^{1/2-\mu}e^{\pm i\xi}\left\{1 + \sum_{n=1}^{\infty} \frac{E_n(x,\rho)}{\rho^{n\sigma_1}} + \frac{\bar{c}_n^{k,j}}{\xi^n}\right\} \quad \text{for } \xi \text{ in } \Xi_{k,l}.$$

THEOREM 3. *If x and ξ are confined respectively to an interval I^* and a quadrant $\Xi_{k,l}$, and if for these values $y_{k,j'}(x)$ is the dominant solution of the related equation, then there exists a solution $u_{k,j'}(x)$ of the given differential equation which is described for the values in question by the various formulas* (41) *and* (42). *If the interval I^* is infinite, the hypothesis* (v) *must be fulfilled.*

8. The sub-dominant solution for ξ in a quadrant $\Xi_{k,l}$. Theorem 3 asserts in brief the existence of a solution of the given differential equation which is asymptotically represented by the dominant solution of the related equation. It is to be shown in this section that the sub-dominant solution of the related equation likewise serves as the asymptotic representative of certain solutions of the equation given. As in the case of the preceding section, the considerations are to be based upon the relations formally obtained in §5. These relations will be considered now, however, for the subscript value to be designated by $j=j''$, and associated with the sub-dominant one of the solutions $y_{k,j}(x)$.

The value of the constant x_0, unspecified in the formulas of §5, was chosen for the case $j=j'$ in §7. For the case $j=j''$ now in hand no single choice of this constant is of exclusive advantage, and it is primarily due to this fact that a difference will be observable in the results of this section and those of the preceding one. The various choices of x_0 are governed, however, by certain general considerations which will now be made.

The inequalities (34) are valid for either value of j, and hence in particular for $j=j''$. The boundedness of the quantity within the brace of formula (26) depends, therefore, upon the function $\exp\{\mp 2i(\xi-\tau)\}$, the sign involved to be chosen as that associated with the value $j=j''$. Now for this value of j the corresponding exponent in the formula (20) has a negative real part. If x and t are both taken to lie on the interval I^*, it follows that the real part of the exponent in formula (26) will also be negative provided

$$\arg\,[i(\xi-\tau)] = \arg\,[-\,i\xi],$$

a condition which is fulfilled if and only if $|\tau| \geqq |\xi|$, i.e., if $|t| \geqq |x|$. Inasmuch as the variable of integration t in formulas (24b) and (28) ranges over the interval bounded by x and x_0, it is clear that when x_0 is chosen the considerations must be confined to those values of x on the interval I^* for which $0 \leqq |x| \leqq |x_0|$. With this restriction the inequalities (36) are readily found to be valid for $j = j''$ provided $|t| \geqq |x|$.

With any choice of x_0 on the interval I^*, and x restricted as determined above, it follows from the assumption (37a), as in the preceding section, that the function (38) satisfies the hypotheses imposed in §6 upon the function $H(x, t, \rho)$. Hence the expression of the formula (28) by means of integrals with the structure of those considered in the lemmas of §6 is possible, and with appropriate choices of the value σ the relations (37) may be established as follows.

Case 1. $x_0 = \bar{x}$. With this choice of x_0, $|\xi| \leqq N$. The function (38) with $s = 0$ may be employed, therefore, in the part of $H(x, t, \rho)$ and as a result the relation (28) may be written in the form

$$Y_j^{(n)}(x) = M^n \rho^{\sigma - 2\mu} I(x, \bar{x}).$$

By Lemma 1 this leads to the inequalities (37) provided σ is chosen not greater than 4μ. It may be concluded, then, that the infinite series in the relation (27) represents a solution of the equation (24b), and that it may be written in the form of the first of the relations (40). It is thus found that there exists a solution of the given differential equation which is described by the formulas (41a) and (42a) for the value $j = j''$.

THEOREM 4a. *If x and ξ are confined respectively to the interval I^* and the quadrant $\Xi_{k,l}$ and if for these values $y_{k,j''}(x)$ is the sub-dominant solution of the related equation, then there exists a solution $u_{k,j''}(x)$ of the given differential equation which is described by the formulas* (41a) *and* (42a).

It should be remarked that Theorem 4a makes no assertion as to the form of the solution concerned for values of x such that $|\xi| > N$.

Case 2. $x_0 = x_e$, where x_e is any fixed point of the interval I^*. The procedure is now familiar. The formula (28) may be written for x on the respective ranges shown in the forms

$$\xi^{1/2-\mu} Y_j^{(n)}(x) = M^n \rho^{\sigma - 2\mu} I(x, x_e), \text{ when } |\bar{x}| < |x| \leqq |x_e|,$$

$$Y_j^{(n)}(x) = M^n \rho^{\sigma - 2\mu}\{I(x, \bar{x}) + I(\bar{x}, x_e)\}, \text{ when } 0 \leqq |x| \leqq |\bar{x}|.$$

The conclusion (37) follows from Lemmas 1 and 2 provided $\sigma \leqq \sigma_1$. If the interval I^* is finite all values of x may be included in Case 2 by choosing x_e as the end point of I^*.

It is to be especially noted that by formula (28) the quantities $Y_{j''}^{(n)}(x)$ are essentially dependent upon the value x_0, and that these quantities in Case 2 are, therefore, not identical with those in Case 1. Despite this fact it will not be found necessary to resort to a distinguishing notation for the solutions concerned in the respective cases.

If the interval I^* is infinite, it will be supposed as in §7 that the hypothesis (v) is fulfilled. The considerations may be extended, then, also to the following case.

Case 3. $|x_0| = \infty$. In this case the formula (28) may be written for x on the respective ranges shown in the forms

$$\xi^{1/2-\mu} Y_j^{(n)}(x) = M^n \rho^{\sigma-2\mu} I(x, \infty), \text{ when } |x_c| < |x| < \infty,$$
$$\xi^{1/2-\mu} Y_j^{(n)}(x) = M^n \rho^{\sigma-2\mu} \{I(x, x_c) + I(x_c, \infty)\}, \text{ when } |\bar{x}| < |x| \leqq |x_c|,$$
$$Y_j^{(n)}(x) = M^n \rho^{\sigma-2\mu} \{I(x, \bar{x}) + I(\bar{x}, x_c) + I(x_c, \infty)\}, \text{ when } 0 < |x| \leqq |\bar{x}|.$$

An application of the lemmas establishes the relations (37) under the assumption that $\sigma \leqq \sigma_1$.

The uniform convergence of the infinite series in the relation (27) follows in both Cases 2 and 3 when $|\rho|$ is sufficiently large, and the solution $U_j(x)$ is found to satisfy the second of the relations (40). The first of the relations (40) must, however, be replaced in these cases by the equation

$$U_j(x) = Y_j(x) + \sum_{n=1}^{\infty} \frac{E_n(x, \rho)}{\rho^{n\sigma_1}}, \text{ for } |\xi| \leqq N, \text{ and } \xi \text{ in } \Xi_{k,l}.$$

Hence it may be concluded that the corresponding solution of the equation (3a) satisfies the relations (41b), (41c) and (42b), (42c), whereas the relations (41a) and (42a) must be replaced by the equations

$$\text{(41d)} \quad u_{k,j}(x) = y_{k,j}(x) + \sum_{n=1}^{\infty} \frac{E_n(x, \rho)}{\rho^{n\sigma_1}}, \text{ for } |\xi| \leqq N, \text{ and } \xi \text{ in } \Xi_{k,l},$$

$$\text{(42d)} \quad \bar{u}_{k,j}(x) = \bar{y}_{k,j}(x) + \sum_{n=1}^{\infty} \frac{E_n(x, \rho)}{\rho^{n\sigma_1}}, \text{ for } |\xi| \leqq N, \text{ and } \xi \text{ in } \Xi_{k,l}.$$

THEOREM 4b. *If x and ξ are respectively restricted to the interval I^* and the quadrant $\Xi_{k,l}$, and if for these values $y_{k,j''}(x)$ is the sub-dominant solution of the related equation, then there exists a solution $u_{k,j''}(x)$ of the given equation which is described by the formulas* (41b), (41c), (41d) *and* (42b), (42c), (42d).

9. **The general solution of the given equation.** The discussion has been concerned hitherto with the form of certain particular solutions of the given and related equations for values of x and ξ peculiarly restricted. It is pro-

posed now to extend the considerations to the general solutions of these equations and to general values of the parameter and variable.

In the foregoing sections it was convenient to base the deductions upon the particular sets of solutions $y_{k,j}(x)$, because of the especial simplicity of the corresponding asymptotic forms for the values of ξ on the restricted ranges considered. The advantage in this choice naturally fails to persist when general values of ξ are drawn into account. It is for this reason convenient to introduce at this point as a basis for the continuing discussion a set of solutions distinct from any of those hitherto used. These solutions $y_j(x)$ and their associated quantities $\bar{y}_j(x)$ will be defined by the formulas

$$(43)\qquad \begin{aligned} y_j(x) &= \Psi(x)\xi^{\mu}J_{\mp\mu}(\xi), \\ \bar{y}_j(x) &= \pm\, i\Psi(x)\xi^{1-\mu}J_{\pm(1-\mu)}(\xi), \quad j = 1, 2, \end{aligned}$$

in which the symbols J designate the familiar Bessel functions of the first kind. The suitability of this definition is warranted by Theorem 1. It is, moreover, directly verified that the quantities $\bar{y}_j(x)$ as given in (43) are related to the respective solutions $y_j(x)$, by the formula obtainable from (19) by deletion of the subscripts k. The significance and purpose of the quantities $\bar{y}_j(x)$ and their relation to the derivatives $y_j'(x)$, is familiar from the rôle of the quantities (19) in the earlier discussion.

The point $x=0$ is an ordinary point for both the given and related equations. Hence the values $y_j(0)$, $\bar{y}_j(0)$, $j=1, 2$, are completely determinate, and a pair of solutions $u_1(x)$, $u_2(x)$ of the given differential equation is uniquely defined by the conditions

$$(44)\qquad u_j(0) = y_j(0), \quad \bar{u}_j(0) = \bar{y}_j(0), \quad j = 1, 2,$$

the quantities $\bar{u}_j(x)$ being obtainable from the formula (29) by suppression of the subscripts k. In virtue of the familiar relation [W, p. 40]

$$(45)\qquad \lim_{z\to 0} z^{\eta}J_{-\eta}(z) = \frac{2^{\eta}}{\Gamma(1-\eta)},$$

the specific values which thus fix the solutions $u_j(x)$ are readily found to be

$$(46)\qquad \begin{aligned} u_1(0) &= \frac{2^{\mu}\Psi(0)}{\Gamma(1-\mu)}, & \bar{u}_1(0) &= 0, \\ u_2(0) &= 0, & \bar{u}_2(0) &= \frac{2^{1-\mu}\Psi(0)}{i\Gamma(\mu)}. \end{aligned}$$

It is evident that the solutions $u_1(x)$ and $u_2(x)$ are linearly independent. Hence the general solution, $u(x)$, of the equation given can be expressed

linearly in terms of them. The explicit formula is in fact easily obtained, and is the following: if the solution $u(x)$ takes at $x=0$ the values

$$u(0) = C, \quad \bar{u}(0) = \overline{C}, \tag{47a}$$

then

$$u(x) \equiv \frac{\Gamma(1-\mu)C}{2^{\mu}\Psi(0)}u_1(x) + \frac{i\Gamma(\mu)\overline{C}}{2^{1-\mu}\Psi(0)}u_2(x). \tag{47b}$$

With the formula (47b) at hand it is clear that a determination of form for the solutions $u_1(x)$ and $u_2(x)$ suffices in every respect for a corresponding determination of form for the general solution of the equation given.

Let any value x on the interval I be chosen, and let ρ be located at pleasure in the complex plane subject to the condition that $|\rho|$ be sufficiently large. The corresponding value of ξ lies in some one of the quadrants (14), and the designation of this quadrant by $\Xi_{k,l}$ has merely the significance of an assignment of values to k and l. For the value of k thus fixed upon and for the value of ξ in question the solutions $u_{k,j}(x)$ given by the Theorems 3 and either 4a, or 4b, have the forms respectively deduced for them in §§7, and 8, and are given accordingly by the appropriate formulas (41) and (42). Between any pair of these solutions and the solutions $u_j(x)$ considered above there exists, of course, a set of identical relations

$$\begin{aligned} u_l(x) &= \overset{(k)}{a}_{l,1}u_{k,1}(x) + \overset{(k)}{a}_{l,2}u_{k,2}(x), \\ \bar{u}_l(x) &= \overset{(k)}{a}_{l,1}\bar{u}_{k,1}(x) + \overset{(k)}{a}_{l,2}\bar{u}_{k,2}(x), \qquad l = 1, 2, \end{aligned} \tag{48a}$$

with coefficients $a_{l,j}^{(k)}$ which are free from x but which naturally depend upon the solutions $u_{k,j}(x)$ involved. The existence of an analogous set of relations involving the corresponding solutions of the related equation, i.e.,

$$\begin{aligned} y_l(x) &= \overset{(k)}{c}_{l,1}y_{k,1}(x) + \overset{(k)}{c}_{l,2}y_{k,2}(x), \\ \bar{y}_l(x) &= \overset{(k)}{c}_{l,1}\bar{y}_{k,1}(x) + \overset{(k)}{c}_{l,2}\bar{y}_{k,2}(x), \qquad l = 1, 2, \end{aligned} \tag{48b}$$

is evident.

The coefficients in the identities (48a) and (48b) are easily calculated by setting $x=0$ and solving the resulting system of algebraic equations. In virtue of the relations (44) the resulting values may be written

$$\overset{(k)}{a}_{l,3-j} = \frac{\mp\left\{y_l(0)\bar{u}_{k,j}(0) - \bar{y}_l(0)u_{k,j}(0)\right\}}{D(u_{k,1}(0),\, u_{k,2}(0))}, \tag{49a}$$

$$\overset{(k)}{c}_{l,3-j} = \frac{\mp\left\{y_l(0)\bar{y}_{k,j}(0) - \bar{y}_l(0)y_{k,j}(0)\right\}}{D(y_{k,1}(0),\, y_{k,2}(0))}, \qquad j = 1, 2, \tag{49b}$$

the symbol D having the significance given it in formula (22a). The evident similarity of structure of the right hand members of formulas (49a) and (49b) will be used as a basis for deducing also a similarity of values of the coefficients which they represent.

Let the attention be given first to a consideration of those values of x and ρ for which $|\xi| \leqq N$. The solutions $u_{k,j}(x)$ in formulas (48a) and (49a) are in this case conveniently taken as those described by the Theorems 3 and 4a. The formulas (41a) and (42a) may accordingly be drawn upon for an evaluation of the quantities $u_{k,j}(0)$ and $\bar{u}_{k,j}(0)$, and if these values are substituted into the relations (49a) and the resulting formulas are compared with the relations (49b), it is found that

$$\overset{(k)}{a_{l,j}} = \overset{(k)}{c_{l,j}} + \sum_{n=1}^{\infty} \frac{E_n(\rho)}{\rho^{4n\mu}}, \qquad j, l = 1, 2.$$

With these coefficient values, and with the values of $u_{k,j}(x)$ and $\bar{u}_{k,j}(x)$ given by the formulas (41a) and (42a), the identities (48a) are found upon comparison with (48b) to reduce to the form

$$u_l(x) = y_l(x) + \sum_{n=1}^{\infty} \frac{E_n(x, \rho)}{\rho^{4n\mu}},$$

$$\bar{u}_l(x) = \bar{y}_l(x) + \sum_{n=1}^{\infty} \frac{E_n(x, \rho)}{\rho^{4n\mu}}, \qquad l = 1, 2.$$

Inasmuch as the quantities y in these equations are those of the formulas (43), the results obtained may be summarized as follows:

THEOREM 5. *The solutions $u_1(x)$, $u_2(x)$, of the given differential equation, which at $x=0$ take on the values* (46), *are described by the formulas*

$$(50) \qquad \begin{aligned} u_j(x) &= \Psi(x)\xi^{\mu} J_{\mp\mu}(\xi) + \sum_{n=1}^{\infty} \frac{E_n(x, \rho)}{\rho^{4n\mu}}, \\ \bar{u}_j(x) &= \pm\, i\Psi(x)\xi^{1-\mu} J_{\pm(1-\mu)}(\xi) + \sum_{n=1}^{\infty} \frac{E_n(x, \rho)}{\rho^{4n\mu}}, \qquad j = 1, 2, \end{aligned}$$

for all values of x and ρ subject to the relation $|\xi| \leqq N$.

Let the attention be turned now to a consideration of the values of x and ρ for which $|\xi| > N$. The procedure to be followed is in its initial stage similar to that followed above, with the difference, however, that the solutions $u_{k,j}(x)$ involved in the formulas (48a) and (49a) are to be looked upon as those given by the Theorems 3 and 4b rather than by Theorems 3 and 4a as heretofore. This change in the point of view is seen directly to be permissible,

for the solutions on the left of the identities (48a) are uniquely determined by the values (46) and are, therefore, in particular independent of the solutions used on the right. A change in this set of solutions requires, of course, a compensating change in the corresponding coefficients. This, however, is entirely accounted for if the solutions used in formulas (48a) and (49a) are the same.

The solutions $u_{k,j}(x)$ described by Theorems 3 and 4b are given when $|\xi| \leqq N$, and hence, in particular when $x=0$, by the relations (41d). A substitution of the values thus obtained into the formulas (49a) and a comparison of the resulting relations with the formulas (49b) show in a manner already familiar that in this case

$$\overset{(k)}{a}_{l,j} = \overset{(k)}{c}_{l,j} + \sum_{n=1}^{\infty} \frac{E_n(\rho)}{\rho^{n\sigma_1}}, \qquad j, l = 1, 2. \tag{51}$$

The explicit values of the coefficients c on the right of the equations (51) are deducible without difficulty from a comparison of the identities (48b) with certain standard and well known formulas from the theory of the Bessel functions. Thus if the forms (43) and (20) are substituted respectively on the left and right of the relations (48b) the formulas which result may be written

$$J_{\mp\mu}(\xi) \sim \frac{1}{\xi^{1/2}}\left\{ \overset{(k)}{c}_{j,1} e^{i\xi}\left[1 + \sum_{n=1}^{\infty} \frac{\overset{k,1}{c_n}}{\xi^n}\right] + \overset{(k)}{c}_{j,2} e^{-i\xi}\left[1 + \sum_{n=1}^{\infty} \frac{\overset{k,2}{c_n}}{\xi^n}\right]\right\}, \qquad j = 1, 2. \tag{52a}$$

Inasmuch as the asymptotic relations here represented are explicitly familiar [W, p. 202] an identification of the coefficients c is possible. Since ξ lies in the half-plane Ξ_k, it is found in this way that

$$\frac{1}{(2\pi)^{1/2}} e^{(k+1/2)(1/2\mp\mu)\pi i} = \begin{cases} \overset{(k)}{c}_{j,1} = \overset{(k)}{c}_{j,2}\, e^{(1/2\mp\mu)\pi i}, & \text{when } k \text{ is odd,} \\ \overset{(k)}{c}_{j,2} = \overset{(k)}{c}_{j,1}\, e^{(1/2\mp\mu)\pi i}, & \text{when } k \text{ is even.} \end{cases} \tag{52b}$$

These are the values of the coefficients which occur on the right of the equations (51).

The deduction of the asymptotic forms of the solutions $u_1(x)$ and $u_2(x)$ for the arbitrarily chosen configuration of values x and ρ is now at hand. The quantities on the right of the identities (48a) are given when $|\xi| > N$ by the forms (41c) and (42c), and these forms may now be substituted. Since the corresponding coefficients are evaluated by the set of relations (51) and (52b) it is thus found finally that the solutions in question are given for all values of x and ρ subject to $|\xi| > N$ by the asymptotic formulas

$$
\begin{aligned}
u_j(x) &\sim \Psi(x)\xi^{\mu-1/2}\left\{a_{j,1}e^{i\xi}\left[1+\sum_{n=1}^{\infty}\frac{E_{n1}(x,\rho)}{\rho^{n\sigma_1}}+\frac{E_{n2}}{\xi^n}\right]\right. \\
&\qquad \left. + a_{j,2}e^{-i\xi}\left[1+\sum_{n=1}^{\infty}\frac{E_{n3}(x,\rho)}{\rho^{n\sigma_1}}+\frac{E_{n4}}{\xi^n}\right]\right\}, \\
\bar{u}_j(x) &\sim \Psi(x)\xi^{1/2-\mu}\left\{a_{j,1}e^{i\xi}\left[1+\sum_{n=1}^{\infty}\frac{\overline{E}_{n1}(x,\rho)}{\rho^{n\sigma_1}}+\frac{\overline{E}_{n2}}{\xi^n}\right]\right. \\
&\qquad \left. - a_{j,2}e^{-i\xi}\left[1+\sum_{n=1}^{\infty}\frac{\overline{E}_{n3}(x,\rho)}{\rho^{n\sigma_1}}+\frac{\overline{E}_{n4}}{\xi^n}\right]\right\},
\end{aligned} \tag{53a}
$$

with $a_{j,i}=a_{j,i}^{(k)}$ when ξ lies in the half-plane Ξ_k, and

$$
\begin{aligned}
a_{j,1}^{(2p)} &= \frac{1}{(2\pi)^{1/2}}e^{(2p-1/2)(1/2\mp\mu)\pi i}+\sum_{n=1}^{\infty}\frac{E_n(\rho)}{\rho^{n\sigma_1}}, \\
a_{j,2}^{(2p)} &= \frac{1}{(2\pi)^{1/2}}e^{(2p+1/2)(1/2\mp\mu)\pi i}+\sum_{n=1}^{\infty}\frac{E_n(\rho)}{\rho^{n\sigma_1}}, \\
a_{j,1}^{(2p+1)} &= \frac{1}{(2\pi)^{1/2}}e^{(2p+3/2)(1/2\mp\mu)\pi i}+\sum_{n=1}^{\infty}\frac{E_n(\rho)}{\rho^{n\sigma_1}}, \\
a_{j,2}^{(2p+1)} &= \frac{1}{(2\pi)^{1/2}}e^{(2p+1/2)(1/2\mp\mu)\pi i}+\sum_{n=1}^{\infty}\frac{E_n(\rho)}{\rho^{n\sigma_1}}, \qquad j=1,2.
\end{aligned} \tag{53b}
$$

The results thus obtained may be summarized in the following and final theorem.

THEOREM 6. *The solutions* $u_1(x)$, $u_2(x)$ *of the given differential equation, which at* $x=0$ *take the values* (46), *are asymptotically described by the formulas* (53a), (53b), *for all values of* x *and* ρ *subject to the relation* $|\xi|>N$.

Theorems 5 and 6 are to be considered as culminating the general theoretical deductions to which this paper has been given. A brief critique only of the concluding formulas (53) is still in order. The asymptotic formulas for the Bessel functions given above in (52a) are familiarly known to be subject to abrupt changes of the coefficients when the complex argument ξ passes from any one to any other half-plane Ξ_k. This characteristic, discovered by Stokes in 1857 [W, p. 201] and designated as the Stokes' phenomenon, is qualitatively displayed and quantitatively determined by the dependence of the formulas (52b) upon the value of k. It is now evident from the formulas (53a) and (53b) that the asymptotic forms of the solutions $u_j(x)$ of any given differential equation (3) are likewise and in similar manner dependent upon the location of ξ in the complex plane, namely, upon the configuration of the values of x and ρ. The Stokes' phenomenon, far from

being peculiar to the Bessel functions, is thus revealed as the special manifestation of a characteristic common to the solutions of all differential equations of the types (1) or (3).

Aside from the dependence of the coefficients (53b) upon the value of p as noted, the further dependence of these coefficients upon the value of μ should be observed. The value μ, it will be recalled, is determined by the degree to which $\phi^2(x)$, the coefficient of the parameter in the given differential equation, vanishes at its zero on the interval I. The character of this zero is, therefore, seen to be determinative of the law by which the change in asymptotic forms specific of the Stokes' phenomenon takes place with changing values of x and ρ.

The border line case $\mu = \frac{1}{2}$ is of sufficient peculiarity to deserve a word of mention. This value of μ is associated with the case in which the coefficient $\phi^2(x)$ vanishes to the degree zero, namely, does not vanish at all, on the interval under consideration. It is easily verified that when $\mu = \frac{1}{2}$ the dependence of the coefficients (53b) upon the value of p is only apparent, and hence in this case, and in this case only, does the Stokes' phenomenon fail to arise.*

Part II

An application to the theory of the Bessel functions of large order

10. **Introduction.** The theory constructed for the general differential equation of type (3) in the preceding sections leans heavily for its results upon the classical theory of the Bessel functions of the small real orders μ and $1-\mu$. It is almost in the manner of compensation, therefore, that it finds an extensive application in the theory of the Bessel functions of orders real and large. It is to this application that the discussion of the present part of the paper will be devoted.

The theory of the Bessel functions, it will be recalled, customarily deals with the derivation of asymptotic formulas in two distinct ramifications. To quote on this point from Watson's treatise [W, p. 194], "There are really two aspects of the problem to be considered; the investigation when [the order] is large is very different from the investigation when [the order] is not large. The former investigation is, in every respect, of a more recondite character than the latter · · · ."

Despite this partition of the theory and the associated differentiation in methods of investigation, it is nevertheless found that certain of the resulting formulas relate intimately the functions of the two categories separately

* This is, of course, in harmony with the elementary fact that if ϕ^2 is a constant and the equation is $u''+\rho^2\phi^2u=0$, then the expressions $u_i=\exp(\pm i\rho\phi x)$ are explicit solutions, and are in consequence of the same form for all values of x and ρ.

considered. This fact is accounted for in an entirely natural manner in virtue of the application of the theory of the present paper to the problem at hand. The derivation of the results through the medium of this application should, moreover, hardly be characterized as "recondite." The theory itself, as will have been found, is essentially elementary in character, and the application is correspondingly simple.

Consistent with the policy pursued in the deductions hitherto, no effort has in general been spent toward carrying the explicit computation of the formulas beyond that of the leading terms. The principal point of interest would seem to lie in the fact that the various formulas, which frequently depend upon widely diverse methods for their derivation, spring here, as from a single and unified source, from the formulas of §9. Certain differences in the formulas here obtained from those usually given for the functions in question will be noted as they arise.

11. **The transformed Bessel's equation.** The differential equation

(54) $$u''(x) + \rho^2\{e^{2x} - 1\}u(x) = 0$$

is transformed by the substitution $z = \rho e^x$ into the Bessel's equation with parameter ρ and argument z. Its solutions, therefore, include in particular the functions

(55) $$J_\rho(\rho e^x), \quad Y_\rho(\rho e^x), \quad H_\rho^{(1)}(\rho e^x),$$

respectively Bessel functions of the first, second, and third kinds. Any two of the functions (55) are linearly independent, and hence any Bessel function of like order and argument can be expressed in terms of them. Between the three functions (55) there accordingly exists an identical relation, and this is known to be

(56) $$J_\rho(z) + iY_\rho(z) - H_\rho^{(1)}(z) = 0.$$

The equation (54) is obviously of the type (3). Inasmuch as the coefficient $\phi^2(x)$ is in this case the function $e^{2x}-1$, which vanishes to the degree 1 at $x=0$, the following formulas are readily found for the explicit identification of the respective quantities, all of which are of familiar import in the general theory:

(57a)
$$\begin{aligned}
\mu &= \frac{1}{3}, \quad \sigma_1 = 1,\\
\phi(x) &= (e^{2x} - 1)^{1/2},\\
\Phi(x) &= \phi(x) - \tan^{-1}\phi(x),\\
\Psi(x) &= \frac{\{\Phi(x)\}^{1/6}}{\{\phi(x)\}^{1/2}},\\
\chi(x) &\equiv 0.
\end{aligned}$$

The hypotheses (i) to (iv) of §2 are found at a glance to be satisfied with the interval I chosen as the entire axis of x, i.e., $-\infty < x < \infty$. The admissibility of this choice of an infinite interval and the corresponding availability of the general results for all values of x depend further, therefore, only upon the applicability of the hypothesis (v) of §7 to the case in hand. This applicability may be investigated without difficulty by the procedure outlined as follows. For values of x which are numerically large and respectively positive or negative the function $\Omega(x)$ defined in formula (5) may be expanded in negative or positive powers of e^x. It is found in this way that for $x>0$,

$$\Omega(x) = 1 - \frac{\tan^{-1}\phi(x)}{\phi(x)} = 1 - \frac{\pi}{2}e^{-x} + e^{-2x} + \cdots,$$

while for $x<0$,

$$\Omega(x) = 1 + \frac{x - \log\left[1 + i\phi(x)\right]}{i\phi(x)}$$

$$= x\left\{1 + \frac{1}{2}e^{2x} + \cdots\right\} + (1 - \log 2)\left\{1 + \frac{1}{2}e^{2x} + \cdots\right\}.$$

By means of the formulas (12) and (23) it is now readily computed that for positive values of x, $\theta(x) = O(1)$, and in consequence $\theta(x)/\phi(x) = O(e^{-x})$, whereas when x is negative $\theta(x) = O(x^{-2})$, and $\theta(x)/\phi(x) = O(x^{-2})$. The hypothesis (v) accordingly demands the convergence of a pair of integrals of the type

$$\int_c^\infty O(e^{-x})dx, \quad \int_{-\infty}^{-c} O\left(\frac{1}{x^2}\right)dx, \text{ with } c > 0,$$

and this requirement is clearly fulfilled.

It will be understood throughout the following discussion that the parameter ρ is taken to be large, real, and positive. The value of e^x is greater than or less than unity according as x is positive or negative. It is convenient to introduce in place of x on these two ranges the familiar real variables β and α defined by the relations

$$e^x = \sec\beta, \text{ for } x \geqq 0; \quad e^x = \operatorname{sech}\alpha, \text{ for } x < 0.$$

In terms of these variables it is readily found that

$$\begin{aligned} \phi(x) &= \tan\beta, \\ \xi &= \rho\left[\tan\beta - \beta\right], \qquad (57\text{b}) \\ \Psi(x) &= \frac{\left[\tan\beta - \beta\right]^{1/6}}{\left[\tan\beta\right]^{1/2}}, \text{ for } x \geqq 0, \end{aligned}$$

and

$$
\begin{aligned}
\phi(x) &= i \tanh \alpha, \\
\xi &= i^3\rho[\alpha - \tanh \alpha], \\
\Psi(x) &= \frac{[\alpha - \tanh \alpha]^{1/6}}{[\tanh \alpha]^{1/2}}, \quad \text{for } x < 0.
\end{aligned} \tag{57c}
$$

The formulas of §9 would readily yield the description of the general solution of the equation (54). This, however, would serve little toward the purpose in view. The problem proposed is the derivation of formulas for the specifically defined solutions to which the symbols J, Y, and H have by custom been assigned, and to this end a knowledge of the general solution of the equation in question does not suffice. It is over and above that essential that the solutions given be identified in order that the formulas obtained may be specific and involve no undetermined or arbitrary elements.

It was shown in §9 how, with the use of the relations (47), the formulas for any solution specified in terms of its values at $x=0$ may be obtained from the formulas explicitly given. The mode of procedure there in question is, however, of no avail for the case in hand, for the values taken by the solutions (55) at $x=0$, far from being available for use in this way, are precisely among the quantities which it is proposed to derive. The method, therefore, which will be adopted to resolve the difficulty is distinct from any of those of the preceding sections. It will be clear in view of these remarks that the discussion which follows may naturally be expected to consist of two essentially distinct parts, and this division will in fact be readily observable. In the first part the immediate problem is the identification of the solutions (55) through the medium of certain of their elementary properties, and following this identification the later part of the discussion will be concerned with the derivation of the desired formulas from the general results of §9.

12. **The identification of the solutions.** In virtue of the identical relation (56) it is obviously sufficient to identify any two of the solutions (55). The solutions to be chosen for this purpose will be those of the first and third kinds. The identification will be made in terms of the solutions $u_1(x)$ and $u_2(x)$ of equation (54) which are determined by the specifications of §9. Inasmuch as these solutions are linearly independent there exists, of course, a pair of identical relations

$$
\begin{aligned}
&\text{(a)} \qquad J_\rho(\rho e^x) = A_1 u_1(x) + A_2 u_2(x), \\
&\text{(b)} \qquad H_\rho^{(1)}(\rho e^x) = B_1 u_1(x) + B_2 u_2(x),
\end{aligned} \tag{58}
$$

in which the coefficients A_i, B_i are free from x but are not necessarily independent of ρ. An identification of the solutions on the left of the equalities (58) will be effected through a determination of the coefficients on the right.

Consider to begin with the first of the relations (58). For negative values of x the corresponding value of ξ lies on the negative axis of imaginaries, i.e., in the half-plane Ξ_1. With the use of the relations (48a), therefore, the identity in question may be written

$$(59)\quad J_\rho(\rho \operatorname{sech} \alpha) \equiv \{A_1 a_{11}^{(1)} + A_2 a_{21}^{(1)}\} u_{11}(x) + \{A_1 a_{12}^{(1)} + A_2 a_{22}^{(1)}\} u_{12}(x).$$

Now when $|x|$ is large the asymptotic forms (41c) may be substituted in the right hand member of (59). As a result it becomes evident that the relation

$$(60a)\qquad A_1 a_{11}^{(1)} + A_2 a_{21}^{(1)} = 0$$

must obtain, for in the alternative the first term on the right of the identity (59) would become infinite as $x \to -\infty$, whereas each remaining term of the identity approaches zero. Hence the relation at hand leads to the formula

$$J_\rho(\rho \operatorname{sech} \alpha) \sim \{A_1 a_{12}^{(1)} + A_2 a_{22}^{(1)}\} \frac{e^{-\rho(\alpha - \tanh \alpha)}}{\rho^{1/6}(i \tanh \alpha)^{1/2}} \left\{1 + \sum_{n=1}^{\infty} \frac{E_n(x, \rho)}{\rho^n}\right\}.$$

Let each member of this formula be divided by the quantity $(\rho \operatorname{sech} \alpha)^\rho$. The limiting form of the relation which results, as $\operatorname{sech} \alpha \to 0$, i.e., as $x \to -\infty$, is found then in virtue of the formula (45) to be

$$\frac{1}{2^\rho \Gamma(\rho + 1)} = \{A_1 a_{12}^{(1)} + A_2 a_{22}^{(1)}\} \frac{e^\rho}{\rho^{1/6} i^{1/2} \rho^\rho 2^\rho} \left\{1 + \sum_{n=1}^{\infty} \frac{E_n(\rho)}{\rho^n}\right\}.$$

It follows with the use of the familiar Stirling's formula,

$$\Gamma(\rho+1) = \rho^\rho e^{-\rho} (2\pi\rho)^{1/2} \left\{1 + \sum_{n=1}^{\infty} \frac{E_n(\rho)}{\rho^n}\right\},^*$$

that

$$(60b)\qquad A_1 a_{12}^{(1)} + A_2 a_{22}^{(1)} = \left(\frac{i}{2\pi}\right)^{1/2} \rho^{-1/3} \left\{1 + \sum_{n=1}^{\infty} \frac{E_n(\rho)}{\rho^n}\right\}.$$

---

* Cf. Bromwich, *An Introduction to the Theory of Infinite Series*, 2d edition, London, 1926, p. 330

The values of the coefficients $a_{ji}^{(1)}$ are given by the formulas (53b). With these values substituted into the equations (60a) and (60b) there results an algebraic system which is easily solved and which gives the values

$$A_1 = \frac{1}{3^{1/2}\rho^{1/3}}\left\{1 + \sum_{n=1}^{\infty} \frac{E_n(\rho)}{\rho^n}\right\},$$

$$A_2 = \frac{1}{3^{1/2}\rho^{1/3}}\left\{1 + \sum_{n=1}^{\infty} \frac{E_n(\rho)}{\rho^n}\right\}.$$

The explicit form of the relations (58a) and (59) is thus found to be

$$(61)\quad \begin{aligned} &\text{(a)}\ J_\rho(\rho e^x) = \frac{1}{3^{1/2}\rho^{1/3}}\left\{u_1(x)\left[1 + \sum_{n=1}^{\infty} \frac{E_n(\rho)}{\rho^n}\right] + u_2(x)\left[1 + \sum_{n=1}^{\infty} \frac{E_n(\rho)}{\rho^n}\right]\right\}, && \text{for all } x,\\ &\text{(b)}\ J_\rho(\rho \operatorname{sech} \alpha) \sim \left(\frac{i}{2\pi}\right)^{1/2} \rho^{-1/3} u_{12}(x)\left[1 + \sum_{n=1}^{\infty} \frac{E_n(\rho)}{\rho^n}\right], && \text{for } x < 0,\end{aligned}$$

and the identification of the first of the solutions (55) is thus accomplished.

The principle thus employed in the derivation of formulas (61) may be used also to determine the coefficients in the identity (58b). The details, however, are somewhat different. For positive values of x the corresponding value ξ lies on the positive axis of reals, namely, in the half-plane Ξ_0, and hence the relation (58b) may be written in the form

$$H_\rho^{(1)}(\rho \sec \beta) \equiv \left\{B_1 a_{11}^{(0)} + B_2 a_{21}^{(0)}\right\} u_{01}(x) + \left\{B_1 a_{12}^{(0)} + B_2 a_{22}^{(0)}\right\} u_{02}(x).$$

Let this relation be multiplied, now, by the quantity

$$\rho^{1/6}(\tan \beta)^{1/2} e^{-i\rho(\tan\beta - \beta)},$$

and let the asymptotic forms (13) and (41c) be substituted respectively in the left and right hand members. The relation which thus results is found to be

$$\begin{aligned} &\left(\frac{2 \sin \beta}{\pi i}\right)^{1/2} \rho^{-1/3} e^{-\rho i(\pi/2 - \beta + \tan\beta - \sec\beta)}\left\{1 + \sum_{n=1}^{\infty} \frac{E_n(\rho)}{(\rho \sec \beta)^n}\right\} \\ &\quad \sim \left\{B_1 a_{11}^{(0)} + B_2 a_{21}^{(0)}\right\}\left\{1 + \sum_{n=1}^{\infty} \frac{E_{n1}(\rho)}{\rho^n} + \frac{E_{n2}}{\xi^n}\right\} \\ &\quad\quad + \left\{B_1 a_{12}^{(0)} + B_2 a_{22}^{(0)}\right\} e^{-2i\rho(\tan\beta - \beta)}\left\{1 + \sum_{n=1}^{\infty} \frac{E_{n3}(\rho)}{\rho^n} + \frac{E_{n4}}{\xi^n}\right\}.\end{aligned}$$

Now the coefficient of the final term in this relation must evidently be zero, for in the alternative this term would remain oscillatory as $\beta\to\pi/2$, i.e., as $x\to\infty$, whereas the remaining terms approach definite and easily determined limits. Because of this fact and by means of an evaluation of the limits in question, it is found then that

$$B_1a_{11}^{(0)} + B_2a_{21}^{(0)} = \left(\frac{2}{\pi i}\right)^{1/2}\rho^{-1/3}\left\{1 + \sum_{n=1}^{\infty}\frac{E_n(\rho)}{\rho^n}\right\},$$

$$B_1a_{12}^{(0)} + B_2a_{22}^{(0)} = 0.$$

This system of equations, with the values $a_{ji}^{(0)}$ given by the formulas (53b), is readily solved and gives the values

$$B_1 = \frac{2}{3^{1/2}\rho^{1/3}}e^{-\pi i/3}\left\{1 + \sum_{n=1}^{\infty}\frac{E_n(\rho)}{\rho^n}\right\},$$

$$B_2 = \frac{2}{3^{1/2}\rho^{1/3}}e^{\pi i/3}\left\{1 + \sum_{n=1}^{\infty}\frac{E_n(\rho)}{\rho^n}\right\}.$$

The explicit form of the relation (58b) is thus found to be

$$\text{(a)}\quad H_\rho^{(1)}(\rho\sec\beta) \sim \left(\frac{2}{\pi i}\right)^{1/2}\rho^{-1/3}u_{01}(x)\left[1 + \sum_{n=1}^{\infty}\frac{E_n(\rho)}{\rho^n}\right], \quad \text{for } x > 0,$$

(62)

$$\text{(b)}\quad H_\rho^{(1)}(\rho e^x) = \frac{2}{3^{1/2}\rho^{1/3}}\left\{u_1(x)e^{-\pi i/3}\left[1 + \sum_{n=1}^{\infty}\frac{E_n(\rho)}{\rho^n}\right]\right.$$

$$\left. + \; u_2(x)e^{\pi i/3}\left[1 + \sum_{n=1}^{\infty}\frac{E_n(\rho)}{\rho^n}\right]\right\}, \quad \text{for all } x,$$

and this completes the identification of the solutions (55).

13. **The asymptotic formulas when $|\xi|$ is large.** When the quantity ξ is large in absolute value the asymptotic forms for the various solutions u are obtainable from the formulas (41c) and (53), by substituting in them the values given by the several relations (57). To facilitate this substitution it may be noted that since $1/\xi = \phi^{-1}(x)/\rho$, while for the case in point $\sigma_1 = 1$, any product of the type

$$\left\{1 + \sum_{n=1}^{\infty}\frac{E_{n1}(x,\rho)}{\rho^{n\sigma_1}} + \frac{E_{n2}}{\xi^n}\right\}\left\{1 + \sum_{n=1}^{\infty}\frac{E_n(\rho)}{\rho^n}\right\}$$

is expressible in the form

$$\left\{1 + \sum_{n=1}^{\infty}\frac{P_n(\Phi^{-1}(x))}{\rho^n}\right\},$$

the symbol P_n representing a polynomial of the degree n in the argument shown, with coefficients which are bounded functions of x and ρ. The symbol P_n will be used in the following as a generic designation for such a polynomial function. As with the symbol E, so also with the symbol P_n, two functions similarly designated in different formulas are not to be considered as in general the same.

For positive values of x the substitution of the form (41c) into the relation (62a) yields the asymptotic formula for the function $H_\rho^{(1)}$. The corresponding formula for the function J_ρ is similarly obtained by the substitution of the forms (53) into the relation (61a), and the two formulas thus derived may be made to yield also that for the function Y_ρ in virtue of the identity (56). Certain simple and obvious manipulations reduce the formulas in question to the forms

$$
\begin{aligned}
H_\rho^{(1)}(\rho \sec\beta) &= \left(\frac{2}{\pi\rho\tan\beta}\right)^{1/2} e^{i\rho(\tan\beta-\beta)-\pi i/4}\left[1+\sum_{n=1}^{\infty}\frac{P_n\left(\dfrac{1}{\tan\beta-\beta}\right)}{\rho^n}\right],\\
J_\rho(\rho\sec\beta) &= \left(\frac{2}{\pi\rho\tan\beta}\right)^{1/2}\left[\cos\left(\rho\tan\beta-\rho\beta-\frac{\pi}{4}\right)\right.\\
&\qquad\cdot\left[1+\sum_{n=1}^{\infty}\frac{P_n\left(\dfrac{1}{\tan\beta-\beta}\right)}{\rho^n}\right]\\
&\qquad\left.+\sin\left(\rho\tan\beta-\rho\beta-\frac{\pi}{4}\right)\sum_{n=1}^{\infty}\frac{P_n\left(\dfrac{1}{\tan\beta-\beta}\right)}{\rho^n}\right],\\
Y_\rho(\rho\sec\beta) &= \left(\frac{2}{\pi\rho\tan\beta}\right)^{1/2}\left[\sin\left(\rho\tan\beta-\rho\beta-\frac{\pi}{4}\right)\right.\\
&\qquad\cdot\left[1+\sum_{n=1}^{\infty}\frac{P_n\left(\dfrac{1}{\tan\beta-\beta}\right)}{\rho^n}\right]\\
&\qquad\left.+\cos\left(\rho\tan\beta-\rho\beta-\frac{\pi}{4}\right)\sum_{n=1}^{\infty}\frac{P_n\left(\dfrac{1}{\tan\beta-\beta}\right)}{\rho^n}\right].
\end{aligned}
\tag{63}
$$

These relations are (to the extent to which the computation is explicit) essentially those by which the functions represented are customarily described [W, pp. 244, 245].

For negative values of x such that $|\xi|$ is large the procedure is similar, being based now, however, upon the formulas (61b) and (62b). Inasmuch as any term involving a factor $e^{-\rho[\alpha-\tanh\alpha]}$ is asymptotically of negligible magnitude in the presence of any term involving a factor $e^{\rho[\alpha-\tanh\alpha]}$, the asymptotic formulas which are obtained in the manner indicated are found to be expressible in the form

$$J_\rho(\rho\,\mathrm{sech}\,\alpha) \sim \frac{e^{-\rho(\alpha-\tanh\alpha)}}{(2\pi\rho\tanh\alpha)^{1/2}}\left[1+\sum_{n=1}^{\infty}\frac{P_n\left(\dfrac{1}{\alpha-\tanh\alpha}\right)}{\rho^n}\right],$$

$$(64)\quad H_\rho^{(1)}(\rho\,\mathrm{sech}\,\alpha) \sim \frac{-\,ie^{\rho(\alpha-\tanh\alpha)}}{\left(\dfrac{\pi}{2}\rho\tanh\alpha\right)^{1/2}}\left[1+\sum_{n=1}^{\infty}\frac{P_n\left(\dfrac{1}{\alpha-\tanh\alpha}\right)}{\rho^n}\right],$$

$$Y_\rho(\rho\,\mathrm{sech}\,\alpha) \sim \frac{-\,e^{\rho(\alpha-\tanh\alpha)}}{\left(\dfrac{\pi}{2}\rho\tanh\alpha\right)^{1/2}}\left[1+\sum_{n=1}^{\infty}\frac{P_n\left(\dfrac{1}{\alpha-\tanh\alpha}\right)}{\rho^n}\right].$$

These relations again are essentially those which are familiar for the description of the functions in question and for the range of argument considered [W, p. 243].

14. The asymptotic forms for intermediate values of ξ. For values of x such that $|\xi|$ is bounded, i.e., such that ξ is either of moderate magnitude or small, the appropriate formulas for the description of the solutions $u_j(x)$, $j=1, 2$, are those embodied in the relations (50). With these values substituted it is found that the formulas (61a) and (62b) lead to the forms

$$J_\rho(\rho e^x) = \frac{\Psi(x)\xi^{1/3}}{3^{1/2}\rho^{1/3}}\{J_{-1/3}(\xi)+J_{1/3}(\xi)\}+\frac{E(x,\rho)}{\rho^{4/3}},$$

$$(65)\qquad H_\rho^{(1)}(\rho e^x) = \frac{2\Psi(x)\xi^{1/3}}{3^{1/2}\rho^{1/3}}\{e^{-\pi i/3}J_{-1/3}(\xi)+e^{\pi i/3}J_{1/3}(\xi)\}+\frac{E(x,\rho)}{\rho^{4/3}},$$

$$Y_\rho(\rho e^x) = \frac{-\,\Psi(x)\xi^{1/3}}{\rho^{1/3}}\{J_{-1/3}(\xi)-J_{1/3}(\xi)\}+\frac{E(x,\rho)}{\rho^{4/3}},\ \text{for } |\xi|\leqq N.$$

Let the attention be given first to those values of x which are positive and such that the magnitude of ξ is moderate. In virtue of the values (57b) the relations (65) become in this case the formulas

$$
(66)\quad
\begin{aligned}
&\text{(a)}\quad J_\rho(\rho\sec\beta) = \left(\frac{\tan\beta-\beta}{3\tan\beta}\right)^{1/2}\left\{J_{-1/3}(\rho\tan\beta-\rho\beta)+J_{1/3}(\rho\tan\beta-\rho\beta)\right\}\\
&\qquad\qquad + \frac{E(x,\rho)}{\rho^{4/3}},\\
&\text{(b)}\quad H_\rho^{(1)}(\rho\sec\beta) = 2\left(\frac{\tan\beta-\beta}{3\tan\beta}\right)^{1/2}\left\{e^{-\pi i/3}J_{-1/3}(\rho\tan\beta-\rho\beta)\right.\\
&\qquad\qquad \left.+\, e^{\pi i/3}J_{1/3}(\rho\tan\beta-\rho\beta)\right\} + \frac{E(x,\rho)}{\rho^{4/3}},\\
&\text{(c)}\quad Y_\rho(\rho\sec\beta) = -\left(\frac{\tan\beta-\beta}{\tan\beta}\right)^{1/2}\left\{J_{-1/3}(\rho\tan\beta-\rho\beta)\right.\\
&\qquad\qquad \left.-\, J_{1/3}(\rho\tan\beta-\rho\beta)\right\} + \frac{E(x,\rho)}{\rho^{4/3}}\cdot
\end{aligned}
$$

These forms it would seem have not been heretofore given, and a word of critique may, therefore, not be inappropriate. Formulas for the functions represented by the relations (66), and for the range of argument at present under consideration, have, of course, been previously known. Their derivation is due to Watson [W, p. 249]. Incidentally the procedure by which they were obtained is described by its author as "a method which is theoretically simple (though actually it is very laborious)."

For purposes of comparison just one of Watson's formulas may here be noted, namely,

$$
(67)\quad
\begin{aligned}
J_\rho(\rho\sec\beta) &= \frac{1}{3}\tan\beta\cos\left(\rho\left[\tan\beta-\frac{1}{3}\tan^3\beta-\beta\right]\right)\left\{J_{-1/3}\left(\frac{1}{3}\rho\tan^3\beta\right)\right.\\
&\quad \left.+\, J_{1/3}\left(\frac{1}{3}\rho\tan^3\beta\right)\right\} + \frac{1}{3^{1/2}}\tan\beta\sin\left(\rho\left[\tan\beta-\frac{1}{3}\tan^3\beta-\beta\right]\right)\\
&\quad \cdot\left\{J_{-1/3}\left(\frac{1}{3}\rho\tan^3\beta\right) - J_{-1/3}\left(\frac{1}{3}\rho\tan^3\beta\right)\right\} + \frac{24\theta_2}{\rho}, \qquad |\theta_2| < 1.
\end{aligned}
$$

The complexity of this formula need hardly be remarked upon. It is a feature also of Watson's corresponding formula for the function Y_ρ. It will be evident that in comparison the formulas (66) are very pronouncedly simpler. Aside from this it should be observed that the formulas (66) are explicit to terms of the order $O(\rho^{-4/3})$ while the degree of explicitness of formula (67) and of its companion formula for Y_ρ extends only to the order $O(\rho^{-1})$.

There is, of course, no conflict between the formulas (66) and those of which (67) is representative. The formulas in question, it will be recalled, are all designed for use when ξ is of moderate magnitude. This is readily translated into the condition that the value of β be of the order $O(\rho^{-1/3})$, and for such values of β the quantities $(\tan\beta-\beta)$ and $\frac{1}{3}\tan^3\beta$ differ by an amount which is of the order $O(\rho^{-5/3})$. With the use of this fact, the agreement of the formulas (66) with those represented by (67) may be established to the extent to which both sets of formulas are explicit.

Let the attention be turned now to the negative values of x for which $|\xi|$ is moderate. In this case ξ is pure imaginary with arg $\xi=3\pi/2$, and it is accordingly convenient to utilize the familiar Bessel functions $I_{\pm 1/3}$, and $K_{1/3}$, which may be taken as defined by the relations [W, pp. 77, 78]

$$J_{\mp 1/3}(z) = \mp\, iI_{\mp 1/3}(ze^{-3\pi i/2}),$$

$$K_{1/3}(z) = \frac{3^{1/2}}{\pi}\left\{I_{-1/3}(z) - I_{1/3}(z)\right\}.$$

The substitution of these functions together with the values (57c) into the formulas (65) gives to the latter the forms

$$\begin{aligned}
J_\rho(\rho \operatorname{sech}\alpha) &= \frac{1}{\pi}\left(\frac{\alpha-\tanh\alpha}{\tanh\alpha}\right)^{1/2} K_{1/3}(\rho\alpha - \rho\tanh\alpha) + \frac{E(x,\rho)}{\rho^{4/3}},\\
H_\rho^{(1)}(\rho \operatorname{sech}\alpha) &= 2\left(\frac{\alpha-\tanh\alpha}{3\tanh\alpha}\right)^{1/2}\left\{e^{-\pi i/3}I_{-1/3}(\rho\alpha-\rho\tanh\alpha)\right.\\
&\qquad \left. - e^{\pi i/3}I_{1/3}(\rho\alpha-\rho\tanh\alpha)\right\} + \frac{E(x,\rho)}{\rho^{4/3}},\\
Y_\rho(\rho \operatorname{sech}\alpha) &= -\left(\frac{\alpha-\tanh\alpha}{\tanh\alpha}\right)^{1/2}\left\{I_{-1/3}(\rho\alpha-\rho\tanh\alpha)\right.\\
&\qquad \left. + I_{1/3}(\rho\alpha-\rho\tanh\alpha)\right\} + \frac{E(x,\rho)}{\rho^{4/3}}.
\end{aligned} \tag{68}$$

These formulas as well as those in (66) appear to be new. Watson has given for the function J_ρ and the range of argument under consideration the formula [W, p. 250]

$$\begin{aligned}
J_\rho(\rho \operatorname{sech}\alpha) &= \frac{\tanh\alpha}{\pi 3^{1/2}} e^{\rho(\tanh\alpha - (1/3)\tanh^3\alpha - \alpha)} K_{1/3}\left(\frac{1}{3}\rho\tanh^3\alpha\right)\\
&\quad + \frac{3\theta_1}{\rho} e^{\rho(\tanh\alpha-\alpha)}, \quad |\theta_1| < 1.
\end{aligned} \tag{69}$$

In comparison with this the first of the formulas (68) has an evident advantage of simplicity. Its additional explicitness to a term in $\rho^{-4/3}$ may be contrasted with the explicitness of (69) which extends only to the order of ρ^{-1}. That agreement exists between the formula (69) and the corresponding formula of the set (68) may be established on the basis of considerations entirely similar to those specifically pointed out in the discussion of the formulas (66) above.

15. **The formulas when argument and order are nearly equal.** The values of x for which $|\xi|$ is small, i.e., $\xi=o(1)$, are by formula (5a) the values characteristic of the relation $x=o(\rho^{-2/3})$. It is convenient for this range of values to introduce the quantities z and ϵ by means of the relations

$$z = \rho e^{x}, \quad \rho = z(1-\epsilon).$$

It is found then that $\epsilon z=o(\rho^{-1/3})$. Moreover, in terms of these variables z, ϵ the relations (57a) are readily shown to be expressible in the manner

$$\begin{aligned} \phi(x) &= (2\epsilon)^{1/2}\left\{1+\frac{3\epsilon z}{4z}+\cdots\right\}, \\ (70)\qquad \xi &= \frac{(2\epsilon z)^{3/2}}{3z^{1/2}}\left\{1+\frac{\epsilon z}{20z}+\cdots\right\}, \\ \frac{\Psi(x)}{\rho^{1/3}} &= \frac{1}{3^{1/6}z^{1/3}}\left\{1+\frac{2\epsilon z}{15z}+\cdots\right\}. \end{aligned}$$

Since the value of ξ is small the Bessel functions involved in the right hand members of the formulas (65) are essentially evaluated by the initial terms of their power series expansions, i.e.,

$$\begin{aligned} \xi^{1/3}J_{-1/3}(\xi) &= \frac{2^{1/3}}{\Gamma\left(\frac{2}{3}\right)} - \frac{\xi^2}{2^{5/3}\Gamma\left(\frac{5}{3}\right)} + \cdots, \\ \xi^{1/3}J_{1/3}(\xi) &= \frac{\xi^{2/3}}{2^{1/3}\Gamma\left(\frac{4}{3}\right)} + \cdots. \end{aligned}$$

These evaluations may be expressed in terms of the quantities z and ϵ in virtue of the relations (70). With the use of the familiar formula

$$3^{1/2}\,\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = 2\pi,$$

it is found then that

$$
(71)\qquad
\begin{aligned}
\xi^{1/3}J_{-1/3}(\xi) &= \frac{3^{1/2}\Gamma\left(\frac{1}{3}\right)}{2^{2/3}\pi} - \frac{3^{1/2}\Gamma\left(\frac{4}{3}\right)(\epsilon z)^3}{2^{2/3}\pi z} + \cdots,\\
\xi^{1/3}J_{1/3}(\xi) &= \frac{3^{5/6}\Gamma\left(\frac{2}{3}\right)}{2^{1/3}\pi z^{1/3}} + \cdots .
\end{aligned}
$$

The values (70) and (71) may now be substituted into the relations (65) and give to the latter the forms

$$
\begin{aligned}
J_\rho(z) &= \frac{1}{2^{2/3}3^{1/6}\pi}\left[\frac{\Gamma\left(\frac{1}{3}\right)}{z^{1/3}} + \frac{6^{1/3}\Gamma\left(\frac{2}{3}\right)\epsilon z}{z^{2/3}}\right.\\
&\qquad \left. - \frac{6\Gamma\left(\frac{4}{3}\right)\left\{\frac{(\epsilon z)^3}{6} - \frac{\epsilon z}{15}\right\}}{z^{4/3}} + \cdots\right] + \frac{E(x,\rho)}{\rho^{4/3}},
\end{aligned}
$$

$$
(72)\qquad
\begin{aligned}
H_\rho^{(1)}(z) &= \frac{2^{1/3}}{3^{1/6}\pi}\left[\frac{\Gamma\left(\frac{1}{3}\right)e^{-\pi i/3}}{z^{1/3}} + \frac{6^{1/3}\Gamma\left(\frac{2}{3}\right)e^{\pi i/3}\epsilon z}{z^{2/3}}\right.\\
&\qquad \left. - \frac{6\Gamma\left(\frac{4}{3}\right)e^{-\pi i/3}\left\{\frac{(\epsilon z)^3}{6} - \frac{\epsilon z}{15}\right\}}{z^{4/3}} + \cdots\right] + \frac{E(x,\rho)}{\rho^{4/3}},
\end{aligned}
$$

$$
\begin{aligned}
Y_\rho(z) &= \frac{-3^{1/3}}{2^{2/3}\pi}\left[\frac{\Gamma\left(\frac{1}{3}\right)}{z^{1/3}} - \frac{6^{1/3}\Gamma\left(\frac{2}{3}\right)\epsilon z}{z^{2/3}}\right.\\
&\qquad \left. - \frac{6\Gamma\left(\frac{4}{3}\right)\left\{\frac{(\epsilon z)^3}{6} - \frac{\epsilon z}{15}\right\}}{z^{4/3}} + \cdots\right] + \frac{E(x,\rho)}{\rho^{4/3}}.
\end{aligned}
$$

To the extent to which these formulas are explicit they are precisely those which are customarily given for the description of the functions in question when the argument z is nearly equal to the order ρ [W, p. 247].

Lastly the value of ϵ may be set equal to zero, i.e., z may be set equal to ρ. The resultant reduction in the relations (72) yields then the familiar formulas [W, p. 232]

$$J_\rho(\rho) = \frac{\Gamma\left(\frac{1}{3}\right)}{2^{2/3}3^{1/6}\pi\rho^{1/3}} + O(\rho^{-4/3}),$$

(73)

$$H_\rho^{(1)}(\rho) = \frac{2^{1/3}\Gamma\left(\frac{1}{3}\right)}{3^{1/6}\pi\rho^{1/3}} + O(\rho^{-4/3}), \quad Y_\rho(\rho) = \frac{-3^{1/3}\Gamma\left(\frac{1}{3}\right)}{2^{2/3}\pi\rho^{1/3}} + O(\rho^{-4/3}),$$

and with this result the discussion will be concluded.

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SINGULAR RELATIONS BETWEEN CERTAIN ARITHMETICAL FUNCTIONS*

BY

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I. Introduction and summaries

1. Introduction. We shall say that $f(x)$ is an arithmetical function of x if $f(x)$ is uniform and finite for all finite integer values >0 of x. The arithmetical functions $f(x)$, $g(x)$ are said to be distinct if $f(x)\neq g(x)$ for at least one integer $x>0$.

Let $a_1, \cdots, a_r$ be numerical constants all different from zero, and let $f_1(x), \cdots, f_r(x)$ be r distinct arithmetical functions of x. Then if and only if

$$a_1f_1(x) + \cdots + a_rf_r(x) = 0 \tag{1.1}$$

has only a finite number of integer solutions $x>0$, we shall call (1.1) a singular relation between $f_1(x), \cdots, f_r(x)$. In (1.1) precisely one of $f_1(x), \cdots, f_r(x)$ may be the arithmetical function whose value is 1 for all integers $x>0$. Having obtained a singular relation in any instance we shall require also the statement of all integers $x>0$ which satisfy it. The number of completely solved singular relations known is very small, probably not more than a dozen. In this paper, we add about 50 more, concerning the functions next defined.

The following notation will be used throughout the paper without further reference.

n, m, p denote integers >0; n is arbitrary, m odd, and p is an odd prime; 1 is considered as being composite.

$\sigma(n)$ = the sum of the divisors of n.†

$\tau(n)$ = the excess of the number of divisors of n of the forms $8t+1$, $8t+3$ over the number of divisors of the forms $8t+5$, $8t+7$.

$\omega(n)$ = the excess of the number of divisors of n of the form $3t+1$ over the number of divisors of the form $3t+2$.

$\xi(n)$ = the excess of the number of divisors of n of the form $4t+1$ over the number of divisors of the form $4t+3$.

* Presented to the Society, November 29, 1930; received by the editors in August, 1930.

† The rudimentary state of the subject is seen from the following. Let $\phi(n)=n$ for all integers $n>0$. The relation $\sigma(n)-2\phi(n)=0$ is as simple in appearance as any in this paper, yet it is not known whether it is singular. It is not known whether $\sigma(m)-2\phi(m)=0$ is solvable, although $\sigma(2n)-4\phi(n)=0$ is.

$\epsilon(x)=1$ if x is the square of an integer >0, and $\epsilon(x)=0$ otherwise.

$12E(n)$ = the number of representations of n as a sum of 3 squares; $E(n)$ is the usual notation for a class number function.

$F(n)$ = the number of odd classes of binary quadratic (Gauss) forms of determinant $-n$, with all the usual conventions (as in H. J. S. Smith's *Report*, and similarly for $E(n)$). For example, $F(1)=1/2$, $F(2)=1, \cdots$, $F(5)=2, \cdots$, $F(9)=5/2, \cdots$, $F(25)=5/2, \cdots$, $F(100)=5$. There is a table of $F(n)$, $n=1, \cdots, 100$, useful in numerical verifications, in the Tôhoku Mathematical Journal, vol. 19 (1921), p. 116.

Writing $f=f(x_1, \cdots, x_s)=a_1x_1^2+\cdots+a_sx_s^2$, where $a_1, \cdots, a_s$ are integers >0, we shall denote by $N(n=f)$ the total number of representations of n in f (the roots of the squares $x_1^2, \cdots, x_s^2$ being $\gtreqless 0$); by $R(n=f)$ the number of representations of n in f in which $x_1, \cdots, x_s$ are restricted to be $\geqq 0$; and by $R'(n=f)$ the number of representations of n in f in which $x_1, \cdots, x_s$ are restricted to be all distinct and $\geqq 0$.

In a paper which will be published elsewhere, Dr. Gordon Pall has determined for all $s \geqq 3$ the explicit solutions n of

$$R'(n=x_1^2+\cdots+x_s^2)=0. \tag{1.2}$$

We shall assume his results for $s=4$.

(1.3) THEOREM. *All n for which* (1.2) *is solvable with $s=4$ are the Pall numbers $2^{2h}a$, $h \geqq 0$, where*

$$a = 2, 6, 10, 18, 22, 34, 58, 82; 1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 23, 25, 27, 31, 33,$$
$$37, 43, 47, 55, 67, 73, 97, 103.$$

These then are the only n which are not sums of 4 unequal squares $\geqq 0$.

From (1.3) we shall deduce all the singular relations which it implies. These relations concern the functions σ, τ, ω, ξ, ϵ, F above defined.

2. Summary of lemmas. This section states a number of lemmas.

(2.1) LEMMA. $2^sR(n=a_1x_1^2+a_2x_2^2+\cdots+a_sx_s^2)=\sum_i N(n=a_ix_i^2)+\sum_{i,j}N(n=a_ix_i^2+a_jx_j^2)+\cdots+N(n=a_1x_1^2+\cdots+a_sx_s^2)$, *the notation being as in* §1, *and* $\sum_{i,j,\ldots,k}$ *referring to all choices of t distinct indices $i, j, \cdots, k$, where t is the number of indices, chosen from* $1, 2, \cdots, s$; $t=1, 2, \cdots, s$.

This is proved in §5.

(2.2) LEMMAS. $16R(n=x^2+y^2+z^2+w^2)=4N(n=x^2)+6N(n=x^2+y^2)+4N(n=x^2+y^2+z^2)+N(n=x^2+y^2+z^2+w^2)$.

This is an immediate consequence of (2.1), as also are the next.

$$8R(n = x^2 + y^2 + 2z^2) = 2N(n = x^2) + N(n = 2x^2) + N(n = x^2 + y^2)$$
$$+ 2N(n=x^2 + 2y^2) + N(n = x^2 + y^2 + 2z^2);$$
$$4R(n = x^2 + 3y^2) = N(n = x^2) + N(n = 3x^2) + N(n = x^2 + 3y^2);$$
$$4R(n = 2x^2 + 2y^2) = 2N(n = 2x^2) + N(n = 2x^2 + 2y^2);$$
$$2R(n = 4x^2) = N(n = 4x^2).$$

(2.3) LEMMA. $R(n=x^2+y^2+z^2+w^2)=R'(n=x^2+y^2+z^2+w^2)+6R'(n=x^2+y^2+2z^2)+3R'(n=2x^2+2y^2)+4R'(n=x^2+3y^2)+R'(n=4x^2)$.

For proof, see §6.

Directly from the definitions of R, R', we have

(2.4) LEMMAS. $R'(n=4x^2)=R(n=4x^2)$;

$$R'(n = x^2 + 3y^2) = R(n = x^2 + 3y^2) - R(n = 4x^2);$$
$$R'(n = 2x^2 + 2y^2) = R(n = 2x^2 + 2y^2) - R(n = 4x^2);$$
$$R'(n = x^2 + y^2 + 2z^2) = R(N = x^2 + y^2 + 2z^2) - 2R(n = x^2 + 3y^2)$$
$$- R(n = 2x^2 + 2y^2) + 2R(n = 4x^2).$$

In the last of these the preceding results were used to reduce the identity given at once by the definitions,

$$R(n = x^2 + y^2 + 2z^2) = R'(n = x^2 + y^2 + 2z^2) + 2R'(n = x^2 + 3y^2)$$
$$+ R'(n = 2x^2 + 2y^2) + R'(n = 4x^2).$$

Combining (2.2)–(2.4) in an obvious way we reach

(2.5) LEMMA. $16R'(n=x^2+y^2+z^2+w^2)=12N(n=x^2)+12N(n=2x^2)+32N(n=3x^2)-48N(n=4x^2)-6N(n=x^2+y^2)-24N(n=x^2+2y^2)+32N(n=x^2+3y^2)+12N(n=2x^2+2y^2)+4N(n=x^2+y^2+z^2)-12N(n=x^2+y^2+2z^2)+N(n=x^2+y^2+z^2+w^2)$.

For easy reference we collect some well known results in the next two.

(2.6) LEMMAS. *If h is an integer $\geqq 0$, and $n=2^h m$, then*

$$N(n = x^2 + y^2) = 4\xi(m),\ N(n = x^2 + 2y^2) = 2\tau(m),\ N(n = x^2 + 3y^2) = C\omega(m),$$

where $C=2$, 0, *or* 6 *according as* $h=0$, *h is odd, or h is even and* >0; $N(n=x^2+y^2+z^2+w^2)=b\sigma(m)$, $b=8$, *or* 24 *according as* $h=0$ *or* $h>0$.

(2.7) LEMMAS. $N(n=x^2+y^2+z^2)=12E(n)$;

$$N(m = x^2 + y^2 + 2z^2) = 4F(2m),\quad N(2n = x^2 + y^2 + 2z^2) = 12E(n).$$

Proofs for the last two are given in the American Mathematical Monthly, vol. 31 (1924), p. 128. The following well known reduction formulas will be found useful, $t \geqq 0$:

$$3E(8t+3) = 2F(8t+3), \quad E(8t+7) = 0, \quad E(4n) = E(n),$$
$$E(4t+1) = F(4t+1), \quad E(4t+2) = F(4t+2), \quad F(4n) = 2F(n).$$

Further reductions, such as $\xi(4t+3)=0$, $\epsilon(2^{2t}m)=\epsilon(m)$, etc., which are obvious, will be used in stating final forms of theorems without reference.

From (1.3), (2.5) we have the next two.

(2.8) LEMMA. *If and only if $2n$ is a Pall number (as in (1.3)),*

$$\begin{aligned}12\,N(2n = x^2+y^2+2z^2) - 4N(2n = x^2+y^2+z^2)\\ = 12N(2n=x^2) + 12N(n=x^2) + 32N(2n=3x^2) - 48N(n=2x^2)\\ - 6N(2n=x^2+y^2) - 24N(2n=x^2+2y^2) + 32N(2n=x^2+3y^2)\\ + 12N(n=x^2+y^2) + N(2n=x^2+y^2+z^2+w^2).\end{aligned}$$

(2.9) LEMMA. *If and only if m is one of the odd numbers in* (1.3),

$$\begin{aligned}12N(m = x^2+y^2+2z^2) - 4N(m = x^2+y^2+z^2)\\ = 12N(m=x^2) + 32N(m=3y^2) - 6N(m=x^2+y^2)\\ - 24N(m=x^2+2y^2) + 32N(m=x^2+3y^2) + N(m=x^2+y^2+z^2+w^2).\end{aligned}$$

3. **Singular relations.** From (2.6), (2.7), (2.9) we have the following.

(3.1) THEOREM. *The only $m \equiv 7$ mod 8 for which*

$$6F(2m) = 8\omega(m) + \sigma(m)$$

are $m = 7, 15, 23, 31, 47, 55, 103$.

(3.2) THEOREM. *The only $m \equiv 3$ mod 8 for which*

$$6F(2m) - 4F(m) = 8\epsilon(m/3) - 6\tau(m) + 8\omega(m) + \sigma(m)$$

are $m = 3, 11, 19, 27, 43, 67$.

(3.3) THEOREM. *The only $m \equiv 1$ mod 8 for which*

$$6F(2m) - 6F(m) = 3\epsilon(m) - 3\xi(m) - 6\tau(m) + 8\omega(m) + \sigma(m)$$

are $m = 1, 9, 17, 25, 33, 73, 97$.

(3.4) THEOREM. *The only $m \equiv 5$ mod 8 for which*

$$6F(2m) - 6F(m) = -3\xi(m) + 8\omega(m) + \sigma(m)$$

are $m = 5, 13, 37$.

(3.5) THEOREM. (3.1)–(3.4) *contain all the singular relations implied by the odd solutions of* (1.2) *with* $s=4$.

The next are obtained similarly from (2.8) with n therein of the form $2^{2h}m (h \geqq 0)$. Several simple reductions by (2.6), (2.7), which need not be preserved, have been used.

(3.6) THEOREM. *There is no* $m \equiv 7 \bmod 8$ *such that*

$$2F(2m) = 2\tau(m) - \sigma(m).$$

(3.7) THEOREM. *The only* $m \equiv 3 \bmod 8$ *such that*

$$4F(m) - 2F(2m) = \sigma(m) - 2\tau(m)$$

are $m=3, 11$.

(3.8) THEOREM. *The only* $m \equiv 1 \bmod 4$ *such that*

$$6F(m) - 2F(2m) = \epsilon(m) + \xi(m) + \sigma(m) - 2\tau(m)$$

are $m=1, 5, 9, 17, 29, 41$.

The following come in the same way from (2.8) with n therein of the form $2^{h+1}m (h \geqq 0)$.

(3.9) THEOREM. *The only* $m \equiv 7 \bmod 8$ *for which*

$$6F(2m) = 8\omega(m) - 2\tau(m) + \sigma(m)$$

are $m=7, 15, 23, 31, 47, 55, 103$.

(3.10) THEOREM. *The only* $m \equiv 3 \bmod 8$ *for which*

$$18F(2m) - 4F(m) = 8\epsilon(m/3) - 6\tau(m) + 24\omega(m) + 3\sigma(m)$$

are $m=3, 11, 19, 27, 43, 67$.

(3.11) THEOREM. *The only* $m \equiv 1 \bmod 4$ *for which*

$$6F(2m) - 2F(m) = -3\epsilon(m) + \xi(m) - 2\tau(m) + 8\omega(m) + \sigma(m)$$

are $m=1, 9, 13, 17, 25, 33, 37, 73, 97$.

(3.12) THEOREM. (3.6)–(3.11) *contain all the singular relations implied by the even solutions of* (1.2) *with* $s=4$.

Hence all singular relations implied by (1.2) with $s=4$ have been obtained.

4. Singular relations with prime argument p. In §3, taking $m=p$ (p prime, as always), and reducing the results by the definitions of the functions involved, we get the following, which are numbered correspondingly to §3.

(4.1) THEOREM. *The only* $p\equiv 7 \bmod 24$ *for which* $6F(2p)=p+17$ *are* $p=7$, 31, 103; *the only* $p\equiv 23 \bmod 24$ *for which* $6F(2p)=p+1$ *are* $p=23$, 47.

(4.2) THEOREM. *The only* $p\equiv 11 \bmod 24$ *such that* $6F(2p)-4F(p)=p-11$ *is* $p=11$; *the only* $p\equiv 19 \bmod 24$ *such that* $6F(2p)-4F(p)=p+5$ *are* $p=19$, 43, 67.

(4.3) THEOREM. *The only* $p\equiv 1 \bmod 24$ *such that* $6F(2p)-6F(p)=p-1$ *are* $p=73$, 97; *there is no* $p\equiv 17 \bmod 24$ *such that* $6F(2p)-6F(p)=p-17$.

(4.4) THEOREM. *The only* $p\equiv 5 \bmod 24$ *such that* $6F(2p)-6F(p)=p-5$ *is* $p=5$; *the only* $p\equiv 13 \bmod 24$ *such that* $6F(2p)-6F(p)=p+11$ *are* $p=13$, 37.

(4.7) THEOREM. *The only* $p\equiv 3 \bmod 8$ *such that* $4F(p)-2F(2p)=p-3$ *are* $p=3$, 11.

(4.8) THEOREM. *The only* $p\equiv 1 \bmod 8$ *such that* $6F(p)-2F(2p)=p-1$ *are* $p=17$, 41; *the only* $p\equiv 5 \bmod 8$ *such that* $6F(p)-2F(2p)=p+3$ *are* $p=5$, 29.

The theorem (4.9) (obtained from (3.9)) is identical with (4.1).

(4.10) THEOREM. *The only* $p\equiv 3 \bmod 8$ *such that* $18F(2p)-4F(p)=24\omega(p)+3p-1$ *is* $p=3$; *the only* $p\equiv 11 \bmod 24$ *such that* $18F(2p)-4F(p)=3p-9$ *is* $p=11$; *the only* $p\equiv 19 \bmod 24$ *such that* $18F(2p)-4F(p)=3p+39$ *are* $p=19$, 43, 67.

(4.11) THEOREM. *The only* $p\equiv 13 \bmod 24$ *such that* $6F(2p)-2F(p)=p+19$ *are* $p=13$, 37; *the only* $p\equiv 1 \bmod 24$ *such that* $6F(2p)-2F(p)=p+15$ *are* $p=73$, 97; *the only* $p\equiv 17 \bmod 24$ *such that* $6F(2p)-2F(p)=p-1$ *is* $p=17$; *there is no* $p\equiv 5 \bmod 24$ *such that* $6F(2p)-2F(p)=p+3$.

(4.12) THEOREM. (4.1)–(4.11) *contain all the singular relations with prime arguments implied by* (1.2) *with* $s=4$.

II. PROOFS

5. Proof of (2.1). This is practically obvious. However, if a formal proof be desired, a simple one is given by the identity

$$2^{-s}\prod_{i=1}^{s}[1+\theta_3(q^{a_i})] = \prod_{i=1}^{s}[1+\tfrac{1}{2}(\theta_3(q^{a_i})-1)],$$

where

$$\theta_3(q) = \sum_{t=-\infty}^{\infty} q^{t^2} = 1 + 2\sum_{n=1}^{\infty} q^{n^2}, \quad |q| < 1,$$

on the right of which the coefficient of q^n is the R function in (2.1). On the left the product is distributed before collecting the coefficient of q^n.

6. **Proof of (2.3).** From the definitions of R, R', the left of (2.3) enumerates the same representations of n as the right. For, on the left, all of x^2, y^2, z^2, w^2 may be unequal, or two may be equal and distinct from the remaining two which may be either equal or unequal, or three may be equal and distinct from the fourth, or all four may be equal, and these cases are exhaustive and mutually exclusive. To account for the numerical factors 1, 6, 3, 4, 1, consider one, say 6. We have

$$R'(n = x^2 + y^2 + 2z^2) = R'(n = x^2 + y^2 + z^2 + z^2).$$

From a particular representation x^2, y^2, z^2, z^2, since x^2, y^2, z^2 are unequal, there are only 2 representations, obtained by permuting x^2, y^2 contributed to the left of (2.3). But on the left of (2.3) the equal squares (if any) in the representations enumerated are free in position. If in the particular representation the z^2's are free, the contribution to the left is $4!/(1!1!2!), =12$. Hence $R'(n=x^2+y^2+2z^2)$ must be multiplied by $12/2, =6$.

All proofs are now completed.

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THE CRITICAL POINTS OF A FUNCTION OF n VARIABLES*

BY

MARSTON MORSE

1. **Introduction.** This paper contains among other results a treatment of the critical† points of a real analytic function without restriction as to the nature of the critical points. Together with the results stated by the author elsewhere‡ on the removal of the boundary conditions it constitutes a complete treatment of the problem, the first of its kind.

In most of the paper the function considered is of class C'', and may have critical loci not even complexes, in fact an infinite set of such loci. Moreover even in the analytic case it is not assumed that the critical loci are complexes, a considerable advantage in any case, and the more so because no adequate proof exists that they are complexes.

Starting with a topological definition of type numbers in terms of ordinary neighborhoods of the critical sets, it ends with a most precise determination of these type numbers *in terms of regions bounded by closed analytic manifolds without singularity*. At no point is it necessary to break up regions more complicated than these into complexes.

All of the results on critical points known to the author, with one exception,§ follow as special cases. The results on isolated critical points obtained by Brown‖ in his Harvard Thesis are the simplest of corollaries. The author's¶ previous results on non-degenerate critical points are obtained with more difficulty. It is shown that the definitions of type numbers given are justified by a kind of invariance under slight analytic deformations of the function.

The treatment will carry over to regular n-spreads in $(n+r)$-space. In it deformations predominate. It is essentially a generalization of the methods

* Presented to the Society, October 25, 1930; received by the editors October 7, 1930.

† A critical point of a function is a point at which all of the first partial derivatives of the function vanish. The value of the function at such a point is called a critical value.

‡ Morse II, Proceedings of the National Academy of Sciences, vol. 13 (1927), p. 813.

§ Whyburn, W. M., Bulletin of the American Mathematical Society, vol. 35 (1929), p. 701. Here the critical values are not necessarily finite in number.

‖ Brown, American Journal of Mathematics, vol. 52 (1930), p. 251. See also Annals of Mathematics, vol. 31 (1930), p. 449.

¶ Morse I, these Transactions, vol. 27 (1925), p. 345.

found necessary in a treatment of n-dimensional critical loci in the calculus of variations* in the large where a reduction to complexes was not possible.

2. **The functions.** Let $(x)=(x_1, \cdots, x_n)$ be a point in euclidean n-space in a finite region Σ, bounded by a closed point set M consisting of a finite number of connected, regular,† non-intersecting $(n-1)$-spreads of class C'''.

Let $f(x)=f(x_1, \cdots, x_n)$ be a real function of class C'' defined on a region including Σ in its interior. On M we suppose that the directional derivative of f in the sense of the exterior normal to M is positive. As in Morse I§20 we can then alter the definition of f neighboring M so that the resulting function, which we will again call f, will take on an absolute maximum on M relative to its values on Σ. This can be done without introducing any new critical points.

We *assume* that the critical values of f are *finite* in number. This hypothesis is always fulfilled if f is analytic.

If a and b are any two ordinary (not critical) values of f, with no critical values between them, the domains $f \leqq a$ and $f \leqq b$ are homeomorphic (Morse I §7). When there are critical values between a and b this will not in general be so. We are concerned in what follows with the topological differences between the domains $f \leqq a$ and $f \leqq b$, and the manner in which these differences depend on the critical points of f.

We shall begin by supposing that there is just one critical value of f between a and b, and shall denote the domains $f \leqq a$ and $f \leqq b$ by A and B respectively, $a < b$.

3. **The neighborhoods of critical sets** g. By a *critical set* g will be understood any closed set of critical points on which f is constant, which is at a positive distance from other critical points. It may or may not be connected. In general it will not be a complex.

By a neighborhood N of g will be meant an open set of points which includes all points within a small positive distance of g. We admit only neighborhoods which lie on $B-A$ and are at a positive distance from other critical points of f. A neighborhood N^1 will be called smaller than N if it is on N and the distance between the boundaries of N and N^1 is positive. We always suppose N^1 smaller than N and in particular so small that any point on N^1 can be connected to g on N. This is always possible.

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* Morse III, these Transactions, vol. 36 (1930), p. 599. See also Birkhoff, these Transactions, vol. 18 (1917), p. 240. Poincaré, Liouville's Journal, (4), vol. 1 (1885), pp. 167–244. Kronecker, Werke, vol. I, pp. 175–226, and vol. II, pp. 71–82.

† An $(n-1)$-spread is called regular and of class C''', if in the neighborhood of any one of its points it can be represented by giving one of its coördinates as a function of class C''' of the remaining coördinates.

Suppose $f=0$ on g. By $\overline{N}$ and $\overline{N}^1$ we shall mean those points of N and N^1 at which $f<0$.

We assume for the present that g is of such sort that for a proper choice of N and corresponding sufficiently small choice of N^1 the following sets of cycles exist.

(α) *A complete set* $(a)_k$ *of k-cycles on* $\overline{N}^1$, *independent on* $\overline{N}$, *dependent on* N^1.

(β) *A complete set* $(c)_k$ *of k-cycles on* N^1 *independent on N of the k-cycles on* $\overline{N}$.

We say that the set $(a)_k$ [substitute (c_k)] is *independent* of the choice of admissible neighborhoods N and N^1 if there exists a fixed neighborhood N^* with the following properties. The set $(a)_k[(c_k)]$ determined for any N smaller than N^* and sufficiently small N^1 corresponding to N is equivalent† on $\overline{N}[N]$ to the set $(a)_k[(c)_k]$ determined for any smaller N and corresponding sufficiently small N^1.

We assume that the sets $(a)_k$ *and* $(c)_k$ *are independent of the choice of N and* N^1 *in the preceding sense, and admit only neighborhoods N smaller than* N.*

We shall show in this paper that our assumptions are always fulfilled in the analytic case and in certain other particular cases. In a later paper we shall show that these assumptions are always fulfilled for the most general critical set as defined above.

Let (N^0N) and (NN^1) be two admissible choices of N and N^1. Let $(a)_k$ and $(c)_k$ be the cycles described in (α) and (β) determined for the neighborhoods N and N^1.

4. **Classification of cycles.** Suppose now that g is the set of *all* critical points at which $f=0$.

We replace $(a)_k$ by an algebraically‡ equivalent set made up of two sets of k-cycles

$$(b)_k,\quad (l)_k, \tag{4.1}$$

so chosen that each cycle of $(l)_k$ bounds on $f<0$, and no linear combination of the cycles $(b)_k$ not null so bounds.

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† We mean here that each cycle of the first set is dependent on cycles of the second set and vice versa. Dependent and independent are terms always understood with respect to bounding. Cycles and chains are here taken in the absolute sense or, with obvious changes, mod m with $m>1$ and prime. The phrase "a complete set of k-cycles etc." may be replaced by "a set containing the maximum number of k-cycles etc." Terms in analysis situs will in general be used in the senses defined by Alexander, *Combinatorial analysis situs*, these Transactions, vol. 28 (1926), p. 301. Chains will be understood, however, to be symbolic expressions for oriented complexes, singular or non-singular in the sense of Veblen, The Cambridge Colloquium, Part II, *Analysis Situs*. The Colloquium Lectures by Lefschetz were not out at the time of the writing of this article but the author knows that they will be helpful to the reader.

‡ By algebraically equivalent we mean that a non-zero multiple of each member of the first set algebraically equals a linear combination of members of the second set and vice versa. Various of these sets may be null.

Let l_{k-1} be any linear combination of cycles of the set $(l)_{k-1}$. We have

$$L_k \to l_{k-1} \text{ on } f < 0,$$

where L_k is a k-chain. Suitable multiples of the cycles (4.1) bound on N^1. Without loss of generality, we can suppose the cycles (4.1) themselves bound on N^1. Thus

$$u_k \to - l_{k-1} \text{ on } N^1$$

where u_k is a k-chain on N^1.

We now introduce the k-cycle

$$L_k + u_k = \lambda_k$$

which we will say *links* l_{k-1}. We shall term u_k and L_k the *upper* and *lower* parts respectively of λ_k. We shall denote by $(\lambda)_k$ the set of k-cycles which link the respective $(k-1)$-cycles of $(l)_{k-1}$.

On the cycles $(b)_k$, $f<0$, and hence is less than some negative constant e. These cycles are thus on the domain

$$f \leqq e. \tag{4.2}$$

They must form a subset of a complete set of k-cycles on (4.2). There then exist other k-cycles $(i)_k$ on (4.2) such that the two sets

$$(b)_k, \quad (i)_k$$

form a complete set of k-cycles on $f \leqq e$.

We can, without loss of generality, suppose the set $(i)_k$ lies on A, since it is homologous on (4.2) to such a set.

We introduce the following table of complete sets of cycles with appropriate terms:

$(c)_k$: *critical cycles*;
$(l)_k$: *linkable cycles*; $(\lambda)_k$: *linking cycles*;
$(i)_k$: *invariant cycles*; $(b)_k$: *newly bounding cycles*.

Any linear combination of k-cycles of any one of these sets will be called by the same name.

With each linkable cycle l_{k-1} we associate a linking k-cycle λ_k so that λ_k and l_{k-1} arise from the same linear combination of corresponding cycles of the sets $(\lambda)_k$ and $(l)_{k-1}$. We note that λ_k cannot be null unless l_{k-1} is null. For if λ_k were null, closed k-cells with points at which $f \geqq 0$ on u_k, the upper part of λ_k, would have to cancel among themselves and we would have l_{k-1} bounding on $\overline{N}^1$ contrary to the nature of the set $(a)_{k-1}$ upon which l_{k-1} depends.

5. A complete set of k-cycles on B. Let the orthogonal trajectories of the contour manifolds f constant (Morse I) be represented in the form $\dot{x}_i = -f_{x_i}$ where $\dot{x}_i$ stands for the derivative of x_i with respect to the time. At each critical point p we understand that there exists a trajectory coincident with p for all time.

The following lemma is fundamental.

DEFORMATION LEMMA. *Any cycle on z_k on B satisfies an homology $z_k \sim z_k' + z_k'' = z_k^*$ on B in which z_k' is a k-chain on N^1 and z_k'' a k-chain on $f<0$ while the boundary z_{k-1}' of z_k' is on $\overline{N}^1$.*

Let each point p of z_k be deformed along the orthogonal trajectory of f through p, starting at p when the time $t=0$, and moving along this trajectory until t equals a time t_0. Denote the resulting k-cycle by z_k^*. For t_0 sufficiently large z_k^* must consist of points either on N^1, or $f<0$, as one readily proves.

Let z_k' be the chain of k-cells of z_k^*, which, closed, are wholly on N^1, and z_k'' the chain of remaining k-cells of z_k^*. We see that z_k' lies on N^1. We see also that if z_k^* had been sufficiently finely subdivided, z_k'' would lie on $f<0$.

The lemma follows directly.

We call z_k' and z_k'' respectively the *upper* and *lower* parts of z^*.

We shall prove the following theorem.

THEOREM 1. *A complete set of k-cycles for B is formed by the sets $(i)_k$, $(\lambda)_k$, $(c)_k$.*

In other words a complete set of k-cycles for B is obtained by deleting the k-cycles of A which are newly bounding on B, keeping the invariant k-cycles of A, and adding the linking and critical k-cycles.

We shall prove this theorem with the aid of two lemmas.

LEMMA 1. *Any k-cycle z_k on B is dependent on linking, critical, and invariant k-cycles. (We admit any integer $n \neq 0$ as coefficient of z_k.)*

We first deform z_k into z_k^* in accordance with the Deformation Lemma, obtaining thereby the cycle z_{k-1}' of the lemma. Since z_{k-1}' is on $\overline{N}^1$, and bounds on $f<0$, it satisfies an homology

$$nz_{k-1}' - l_{k-1} \sim 0 \quad \text{on} \quad \overline{N}, \quad n \neq 0, \tag{5.1}$$

where l_{k-1} is a linkable $(k-1)$-cycle.

Let λ_k be the k-cycle linking l_{k-1}. The boundary of the lower part of the k-cycle,

$$nz_k^* - \lambda_k, \tag{5.2}$$

bounds on $\overline{N}$ according to (5.1), and its upper part is on N^1. Hence† the k-cycle (5.2) may be written as a sum of k-cycles on $f<0$ and k-cycles on N.

† The proof holds as stated even when $k=0$, regarding cycles with negative subscripts as null.

But on the one hand k-cycles on $f<0$ are dependent on B on invariant cycles. On the other hand k-cycles on N are dependent on N^0 on critical k-cycles and k-cycles on $\overline{N}^0$, that is, on $f<0$. But k-cycles on $f<0$ are dependent on B on invariant k-cycles.

Thus the k-cycle (5.2) is dependent on B on invariant and critical k-cycles, and the lemma is proved.

LEMMA 2. *The k-cycles of the theorem are independent on B.*

Suppose we had a relation of the form

$$D_{k+1} \to \lambda_k + c_k + i_k = D_k$$

where i_k, c_k, and λ_k are respectively invariant, critical, and linking k-cycles, and D_{k+1} a chain on B, and D_k its boundary.

We shall prove successively that λ_k, c_k, i_k are null.

We could deform D_{k+1} as in the Deformation Lemma into a set of points either on $f<0$ or N^1. Moreover, we could obtain the same result holding D_k fast, altering the deformation T of the lemma as follows.

Without loss of generality we can suppose c_k and λ_k are replaced by equivalent cycles null with the given cycles, and such that the new c_k, and u_k, the upper part of the new λ_k, are so near g that T will deform points within a sufficiently small positive distance r of c_k and u_k only through points on N^1.

Regard T as a movement depending continuously on the time t as t varies from 0 to 1. Let ρ be the initial distance of any point from D_k. We now perform T stopping the movement of a point initially within a distance r of D_k at a time $t=\rho/r$. We obtain thereby the desired deformation.

We can suppose that D_k bounds a chain D_{k+1} whose points lie on $f<0$ or else N^1.

Let d_{k+1} be a chain of all $(k+1)$-cells of D_{k+1} which, when closed, lie wholly on N^1. Let d_k be the boundary of d_{k+1}.

The points on D_{k+1} at which $f\geqq 0$ form a closed set on N^1. Accordingly if we suppose D_{k+1} sufficiently finely divided any closed k-cell of D_{k+1} possessing a point at which $f\geqq 0$ would be a k-cell of d_{k+1}, and would be on the boundary d_k if and only if it is on the boundary D_k of D_{k+1}. But boundary k-cells of D_k with points at which $f\geqq 0$ are found at most on c_k+u_k where u_k is the upper part of λ_k.

We see then that on the k-chain

$$d_k - c_k - u_k = e_k \tag{5.3}$$

the cells with points at which $f\geqq 0$ all cancel. Moreover e_k is on N^1, since d_k, c_k, and u_k are each on N^1. Thus e_k is on $\overline{N}^1$.

But the boundary of e_k is the boundary l_{k-1} of $-u_k$, since d_k and c_k are cycles. But a linkable cycle l_{k-1} does not bound on $\overline{N}$ unless null. Thus l_{k-1} is null. Accordingly λ_k is null as well as u_k.

We see then that

$$d_k = c_k + e_k.$$

But this is impossible unless c_k is null, for otherwise the critical cycle c_k would be homologous on N^1 to a cycle e_k on $\overline{N}^1$, since d_k bounds on N^1. Hence c_k is null.

Thus d_k is on $\overline{N}^1$. According to its origin it is homologous to i_k on $f<0$. Moreover, for some integer n not zero,

$$nd_k \sim l_k + b_k$$

on $\overline{N}$, where l_k and b_k are linkable and newly bounding cycles respectively. But l_k bounds on $f<0$. Hence $b_k \sim nd_k \sim ni_k$ on $f<0$. But this is impossible unless both b_k and i_k are null. Thus i_k is null.

The lemma is thereby proved.

The theorem follows at once from Lemmas 1 and 2.

6. **The associated ideal critical points.** With any critical set g we now associate a set of ideal critical points of *type k*. The number of points in this set will be denoted by M_k and called the *kth type number* of g.

The kth type number M_k of g shall be defined as the number of cycles in the sets $(a)_{k-1}$, and $(c)_k$ of §3.

This type number depends only on f and the topological properties of the neighborhoods of g. Accordingly, if $\bar{g}$ is a critical set composed of the sum of a finite number of distinct critical sets, the corresponding kth type number will be the sum of the kth type numbers of the component sets.

Let α and β be any two ordinary values of f $(\alpha<\beta)$. On the domain $f<\alpha$ there will be a complete set of $(k-1)$-cycles, independent on $f<\alpha$, but bounding on $f<\beta$. These we call *newly bounding relative* to the change from α to β. On $f<\beta$ there will be a complete set of k-cycles, independent on $f<\beta$, and independent on $f<\beta$ of k-cycles on $f<\alpha$. These we call *new k-cycles* relative to the change from α to β.

We shall evaluate the type number M_k.

The number of $(k-1)$-cycles in $(a)_{k-1}$ equals the number of newly bounding $(k-1)$-cycles and linking k-cycles in complete sets, as follows from the decomposition (4.1). The critical cycles $(c)_k$ are new k-cycles, and taken with the linking k-cycles form a complete set of new k-cycles as follows from Theorem 1.

We thus have the following theorem.

THEOREM 2. *If a and b are two ordinary values of f, $a<b$, between which lies just one critical value, the kth type number of the corresponding critical set will equal the number of newly bounding $(k-1)$-cycles plus the number of new k-cycles in complete sets, taking these sets relative to a change from $f\leqq a$ to $f\leqq b$.*

Let g be the critical set of the theorem. We shall divide the M_k ideal critical points of type k associated with g, into two sets of points in number m_k^+ and m_k^-, called respectively critical points of *increasing* or *decreasing* type.

The number m_k^+ shall be the number of linking k-cycles and critical k-cycles in complete sets, and the number m_k^- shall be the number of newly bounding $(k-1)$-cycles in a complete set.

We now have the following corollary of Theorems 1 and 2. See Morse I §18.

COROLLARY. *The kth Betti number of $f\leqq b$ minus the kth Betti number of $f\leqq a$ affords a difference given by the formulas*

$$\begin{aligned} \Delta R_k &= \overset{+}{m}_k - \overset{-}{m}_{k+1}, \\ M_k &= \overset{+}{m}_k + \overset{-}{m}_k \qquad (k = 0, \cdots, n), \end{aligned} \tag{6.1}$$

where m_k^+ and m_k^- are respectively the numbers of ideal critical points associated with the critical set g, of increasing and decreasing k-type respectively. Here $m_0^-=m_{n+1}^-=0$.

The preceding corollary also clearly holds if a and b are any two ordinary values of f. If then we eliminate either the integers m_k^- or m_k^+ from (6.1) we obtain the following theorem.

THEOREM 3. *Let a and b be any non-critical constants, $a<b$. Then between the changes in the Betti numbers as we pass from the domain $f\leqq a$ to the domain $f\leqq b$, and the sums of the type numbers of the critical sets with critical values between a and b the following relations hold:*

$$M_0 - M_1 + \cdots + (-1)^i M_i = \Delta(R_0 - R_1 + \cdots + (-1)^i R_i) + (-1)^i \overset{-}{m}_{i+1},$$
$$M_0 - M_1 + \cdots + (-1)^i M_i = \Delta(R_0 - R_1 + \cdots + (-1)^{i-1} R_{i-1}) + (-1)^i \overset{+}{m}_i,$$

where $i=0, \cdots, n$ and $m_{n+1}^-=0$.

Many inequalities and other equalities can be deduced from these relations. See, for example, Morse III §14 and §15, also Morse I §19. Note the interesting relation obtained from (6.1)

$$M_k = \Delta R_k + \overset{-}{m}_k + \overset{-}{m}_{k+1}.$$

A similar formula can be obtained involving the increasing type numbers.

It will be convenient for the next proof to replace the words "the number of k-cycles in a complete set of k-cycles" by the words "the *count* of k-cycles."

We now state a generalization of Theorem 2.

THEOREM 4. *If a and b are any two ordinary values of f the sum M_k of the kth type numbers of the critical sets with critical values between a and b will exceed or equal the count u of new k-cycles plus the count v of newly bounding $(k-1)$-cycles relative to a change from $f \leqq a$ to $f \leqq b$, $a < b$.*

Let c change from a to b taking on successively between a and b a set of ordinary values, separating the critical values. Let c_1 and c_2 be two such successive values.

Let h be the count of $(k-1)$-cycles on $f \leqq c_1$, independent on $f \leqq c_1$, bounding on $f \leqq c_2$. Let h^1 be the count of the subset of such cycles dependent on $f \leqq c_1$ on cycles of $f \leqq a$. We have $h^1 \leqq h$. Summing for all such changes of c,

$$v = \sum h^1 \leqq \sum h.$$

Now let m be the count of k-cycles on $f \leqq c_2$, independent on $f \leqq c_2$ of cycles on $f \leqq c_1$. Let m^1 be the count of the subset of such cycles independent on $f \leqq b$ of cycles on $f \leqq c_1$. Summing for all changes of c we have

$$u = \sum m^1 \leqq \sum m.$$

Combining these results we have

$$u + v \leqq \sum h + \sum m = M_k,$$

and the theorem is proved.

7. **The (ϕf) trajectories.** Let ϕ be a function of (x) of class C'' in the neighborhood of a point p. Suppose p is an ordinary point of both f and ϕ and that the gradients of f and ϕ at p are not parallel. By the (ϕf) *vector field* we mean the set of vectors, at each point q near p, obtained by projecting the gradient of ϕ on the tangent $(n-1)$-plane of the manifold $f = c$ at q.

By the (ϕf) *trajectories* we mean regular curves of class C' tangent at each point to vectors of the (ϕf) vector field.

The condition that the gradients of f and ϕ be not parallel is the following:

$$A(x) = \sum_{ik} [\phi_i f_k - \phi_k f_i]^2 \neq 0 \qquad (i, k = 1, \cdots, n) \tag{7.1}$$

where the subscripts i and k indicate partial differentiation with respect to the variables x_i and x_k.

The differential equations of the (ϕf) trajectories have the form*

* The summation convention of tensor analysis is used throughout.

$$\frac{dx_i}{dt} = \rho(f_k f_k \phi_i - \phi_k f_k f_i) = X_i(x), \qquad \rho \neq 0, \tag{7.2}$$

where ρ is a function of (x) of class C' near p. The right hand members of (7.2) do not all vanish near p as follows from (7.1).

On the (ϕf) trajectories f is constant.

For along such a trajectory we have

$$\frac{df}{dt} = f_i \frac{dx_i}{dt} = f_i X_i \equiv 0. \tag{7.3}$$

If we choose ρ as the reciprocal of

$$f_k f_k \phi_i \phi_i - \phi_k f_k f_i \phi_i = A(x), \tag{7.4}$$

then $d\phi/dt$ will be one along each trajectory. The function (7.4) is not zero since it equals the function $A(x)$ of (7.1) as indicated.

We can then so choose the parameter t on the (ϕf) trajectories that at each point we have $\phi = t$, and this choice we suppose made.

There is a (ϕf) trajectory through each point near p, and one through each point of the contour manifolds $\phi = c$ near p. The intersections of these trajectories and manifolds will vary continuously with the constants c and the trajectories.

We shall term the ordinary orthogonal trajectories of the contour manifolds, ϕ constant, the ϕ trajectories. We take their differential equations in the form

$$\frac{dx_i}{dt} = \frac{\phi_i}{\phi_k \phi_k} = Y_i(x). \tag{7.5}$$

So taken we may suppose $t = \phi$ at each point of a trajectory.

8. **Neighborhood functions.** The existence of neighborhood functions ϕ, as we shall define them, will enable us to express the type numbers in the simplest possible form.

Let g be any connected critical set of f on which $f = 0$. Relative to g we shall call a function $\phi(x)$ a *neighborhood function* if it satisfies the following conditions.

(a) It is of class C'' in the neighborhood of g.

(b) It takes on a relative minimum zero on g.

(c) At points near g but not on g it is ordinary.

(d) At points near g but not on g at which $f = 0$ the gradients of f and ϕ are not parallel

We shall exhibit neighborhood functions ϕ in certain important cases beginning with the analytic case.

THEOREM 5. *If the function f is analytic, the function $\phi = f_i f_i$ is an admissible neighborhood function ϕ relative to the critical set g of f.*

The function ϕ clearly satisfies (a) and (b). We shall finish by proving the following lemma.

LEMMA. *If f is analytic, any analytic function which takes on a proper relative minimum zero on g, is an admissible neighborhood function ϕ.*

Such a function ϕ satisfies (a) and (b). It must then satisfy (c). For g is a set of critical points of ϕ, and if ϕ were not ordinary near g the critical set g would be a subset of a larger connected critical set. But on all connected critical loci an analytic function is constant. Thus ϕ would be zero at some points near g not on g, contrary to the nature of a proper minimum. Thus (c) holds.

Now (d) could fail only at points not on g at which

$$A(x) = 0, \quad f = 0, \tag{8.1}$$

when $A(x)$ is given by (7.1).

But (8.1) is satisfied on g. Suppose it were satisfied on a larger analytic locus γ connected with g. Let h be any regular curve along which (8.1) is satisfied. On $h, f=0$ so that

$$f_i \frac{dx_i}{dt} = 0. \tag{8.2}$$

I say that on h

$$\phi_i \frac{dx_i}{dt} = 0.$$

This is certainly true on g and follows from $A(x)=0$ for points on h not on g. Thus ϕ is constant on h and accordingly on γ. It must then be zero on γ. From (b) we see that $\gamma = g$. Thus (d) holds.

The lemma and theorem are thereby proved.

We return now to the non-analytic case.

We term a critical set at which f takes on a relative maximum or minimum a *maximizing* or *minimizing* set respectively. We then state the following theorem, an immediate consequence of our definition of a neighborhood function ϕ.

THEOREM 6. *For a minimizing or maximizing set at which $f=0$, the functions f and $-f$ are respectively admissible neighborhood functions.*

In this connection we note the following.

If the critical values are isolated there are at most a finite number of minimizing or maximizing critical sets.

In fact the number of distinct contour manifolds on $f=c$ cannot vary as c approaches a critical value from either side. But the number of minimizing sets or maximizing sets cannot exceed the total number of these manifolds, and hence is finite.

There may however be an infinite number of distinct critical sets not minimizing or maximizing.

A neighborhood function ϕ always exists in the non-degenerate case as stated in the following theorem.

THEOREM 7. *If $(x)=(a)$ is a non-degenerate critical point of f the function*

$$(x_i - a_i)(x_i - a_i) = \phi$$

is an admissible neighborhood function.

The function clearly satisfies all the requirements, possibly excepting the one regarding gradients.

Suppose $(a)=(0)$. The relations of gradients will be unaltered if we use an orthogonal transformation to bring f to the form

$$f = \lambda_k x_k x_k + \eta \qquad (k = 1, \cdots, n),$$

where λ_k is a constant not zero, and η is of at least the second order with respect to the distance ρ to the origin.

If we can show that the function (7.4) does not vanish for real points on $f=0$ and for $\rho \neq 0$ in some neighborhood of the origin the proof will be complete.

Omitting terms of at least the fifth order this function (7.4) is seen to be

$$16\left[\lambda_k^2 x_k x_k x_i x_i - \lambda_k x_k x_k \lambda_i x_i x_i\right].$$

But on $f=0$ this becomes, up to the terms of at least the fifth order,

$$16\lambda_k^2 x_k x_k x_i x_i.$$

The ratio of the last expression to ρ^4 is positive and bounded away from zero for $\rho \neq 0$.

The function (7.4) is accordingly positive everywhere desired and the theorem is proved.

THEOREM 8. *If f is analytic and $(x)=(0)$ is an isolated critical point, the function $\phi = x_i x_i$ is an admissible neighborhood function.*

The theorem follows from the lemma under Theorem 5.

9. **The radial trajectories.** We shall now prove the existence of a set of trajectories termed *radial* trajectories. They lead away from g somewhat after the fashion of rays emanating from a point. The theorem is

THEOREM 9. *If ϕ is a neighborhood function for g, then on the domain*

$$R: \qquad 0 < \phi \leqq r, \; f \leqq 0, \tag{9.1}$$

where r is a small positive constant, there exists a radial field of trajectories, one through each point of R, satisfying differential equations of the form

$$\frac{dx_i}{dt} = B_i(x), \quad B_iB_i \neq 0,$$

where the functions $B_i(x)$ are of class C' on R. These trajectories reduce to (ϕf) trajectories on $f=0$. On them t may be taken equal to ϕ. As t increases they pass out of R only by reaching $\phi=r$.

The ϕ trajectories themselves would do except for the fact that they cross $f=0$ in general. We shall alter the ϕ trajectories neighboring $f=0$ so that they will suffice.

The $(f\phi)$-trajectories ζ emanating from $f=0$ on R in general form a field only for a short distance from $f=0$ depending upon how near ϕ is to zero. We shall be more precise and say that we can determine a negative function $h(\alpha)$ of class C', for $0<\alpha\leqq r$, such that the field persists on a trajectory ζ on which $\phi=\alpha$, and on which f decreases from zero to $h(\alpha)$.

We can in fact define $h(\alpha)$ successively on the intervals with end points

$$r, \; r/2, \; r/4, \; \cdots,$$

and so define $h(\alpha)$ on its entire interval.

Now let $M(z)$ be a function of (z) of class C' for $z\geqq 0$, identically one for $z>1$ and zero for z zero, otherwise positive.

Our radial trajectories will now be defined by (7.5), except for the above points on ζ where f decreases from zero to $h(\phi)$. At these points the differential equations of the radial trajectories shall have the form

$$\frac{dx_i}{dt} = X_i(x) + M\left[\frac{f(x)}{h(\phi(x))}\right][Y_i(x) - X_i(x)] \tag{9.2}$$

where X_i and Y_i are the functions of (7.2) and (7.5) respectively.

On $f=0$ the radial trajectories reduce to the (ϕf) trajectories (7.2). For $f=h(\phi)$ they take the form (7.5). Moreover on them

$$\frac{d\phi}{dt} = \phi_iX_i(1 - M) - Y_i\phi_iM = 1 - M + M = 1.$$

This shows that we can take $t=\phi$ on the radial trajectories. It also shows

that the right hand members of (9.2) are not all zero at any one point on R.

The theorem follows at once.

10. **The type numbers in terms of neighborhood functions.** By a *radial* deformation we shall mean one in which each point moves on a radial trajectory, and points for which ϕ is constant are deformed into points for which ϕ is constant.

The domains of points satisfying $\phi = e$ or $0 < \phi \leqq e$, where e is a small positive constant, less than the constant r of Theorem 9, will be respectively denoted by ϕ_e and ϕ_e^0. The points on these same domains at which $f < 0$ will be denoted by $\bar{\phi}_e$ and $\bar{\phi}_e^0$.

From the existence of the radial trajectories we infer the following.

(1) For any two constants e and η less than r the domains $\bar{\phi}_e$ and $\bar{\phi}_\eta$ are homeomorphic.

(2) If $e < \eta$ the domain $\bar{\phi}_\eta^0$ can be radially deformed onto the domain $\bar{\phi}_e^0$ leaving ϕ_e^0 fixed and never increasing ϕ.

(3) For any closed point set on $\bar{\phi}_\eta^0$ there exists a radial deformation of $\bar{\phi}_\eta^0$ that leaves $\bar{\phi}_\eta$ fixed and deforms the point set onto $\bar{\phi}_\eta$.

We can now prove the following theorem.

THEOREM 10. *If a neighborhood function ϕ exists, the sets of cycles $(a)_k$ and $(c)_k$ of §3 exist, and are independent of the choices of admissible neighborhoods N and N^1.*

As a choice of the fixed neighborhood N^* of the definition of independence of §3 we take the domain $\phi \leqq r$. If N be any neighborhood on $\phi \leqq r$ let η be a positive constant so small that the points on $\phi \leqq \eta$ lie on N. Corresponding to N a sufficiently small choice of the neighborhood N^1, as we shall see, will be any neighborhood N^1 on $\phi \leqq \eta$. Let e be a positive constant so small that the domain $\phi \leqq e$ consists of points on N^1.

It appears, then, that relative to N and N^1, the cycles $(a)_k$ may be taken as a complete set on $\bar{\phi}_e$ independent on $\bar{\phi}_e$, bounding on $\phi \leqq e$, while the k-cycles $(c)_k$ may be taken as a complete set on $\phi \leqq e$ independent on $\phi \leqq e$ of the cycles on $\bar{\phi}_e$. We have then the following theorem.

THEOREM 11. *The type number M_k of a critical set is the number of cycles in the following two sets:*

(a) *a complete set of $(k-1)$-cycles on $\bar{\phi}_e$ independent on $\bar{\phi}_e$, bounding on $\phi \leqq e$;*

(b) *a complete set of k-cycles on $\phi \leqq e$ independent on $\phi \leqq e$ of the cycles on $\bar{\phi}_e$.*

The number of cycles in these sets is independent of the choice of the constant e for e positive and sufficiently small.

A cycle of any set (a) has been termed linkable if it bounds a chain L_k on $f<0$. If it so bounds it must bound outside of $\phi<e$ as well. For the part of L_k on $\bar{\phi}_e^0$ can be radially deformed on $\bar{\phi}_e^0$ onto $\bar{\phi}_e$.

We summarize and complete these results as follows.

The linkable $(k-1)$-cycles are those on $\bar{\phi}_e$, independent on $\bar{\phi}_e$, bounding on $\phi\leqq e$, and bounding on $f<0$ outside of $\phi<e$.

The newly bounding $(k-1)$-cycles are those on $\bar{\phi}_e$ independent on $\bar{\phi}_e$, independent on $f<0$ outside of $\phi<e$, but bounding on $\phi\leqq e$.

The critical k-cycles are the cycles (b).

The number M_0 equals the number of critical 0-cycles. It is one for connected minimizing sets, and null for all other connected sets. It is of increasing type. The number M_n is the number of newly bounding $(n-1)$-cycles. This is true if we are operating in euclidean n-space or on a portion, not all, of a connected regular n-spread. For there are then no non-bounding n-cycles, in particular no linking or critical n-cycles. Unless a connected critical set is a maximizing set, M_n is null, for there are then no $(n-1)$-cycles in (a). It is obviously one for connected maximizing sets.

The following corollary of Theorems 8 and 11 brings all of Brown's results on isolated critical points under the results of the present paper.

Corollary. *Suppose $(x)=(0)$ is an isolated critical point in the analytic case. Then if we set $\phi=x_ix_i$ the kth type number M_k, for k not zero, is given by the formula*

$$M_k = R_{k-1} - \delta_1{}^k$$

where R_k is the kth Betti number of the region on the $(n-1)$-sphere $\phi=e$ on which f is negative. In the case of a minimum $M_0=1$. Otherwise M_0 is null.

This follows from Theorem 11, noting that in (a) of Theorem 11 all cycles on the $(n-1)$-sphere $\phi=e$ are bounding on the interior $\phi\leqq e$ excepting a point 0-cycle. From this exception the Kronecker delta $\delta_1{}^k$ arises. In (b) the complete set is null except in the case of a minimum, and then the origin may serve as a complete 0-set.

Because of its signal importance we state the following as a separate theorem.

Theorem 12. *In the analytic case Theorem* 11 *holds without exception with $\phi=f_if_i$.*

From Theorems 6 and 11 we have the following theorem, true in the most general non-analytic case. In it we suppose $f=0$ on g.

THEOREM 13. *For a connected minimizing or maximizing set g, the type numbers of g always exist.*

For a minimizing set, M_k is the kth Betti number of the domain $f \leqq e$, neighboring g.

For a maximizing set, M_k is the number of cycles in the following two sets:

(a) *a complete set of $(k-1)$-cycles on $f=-e$, independent on $f=-e$, bounding on $f \geqq -e$, neighboring g;*

(b) *a complete set of k-cycles on $f \geqq -e$ independent on the same domain of the k-cycles on its boundary.*

Consider a connected maximizing set.

In euclidean n-space the set (b) is empty. Let the domain $f \geqq -e$ neighboring g be denoted by S. If R_i and β_i denote Betti numbers of S and its boundary β, respectively, we have*

$$R_{i-1} + R_{n-i} = \beta_{i-1} \qquad (i = 1, \cdots, n).$$

Now there are R_{n-i} independent $(i-1)$-cycles on the residue of S in n-space. To add S to its residue is then to diminish the $(i-1)$st Betti number by R_{n-i}. There must then be at least R_{n-i} $(i-1)$-cycles independent on β, and newly bounding. But there cannot be more than R_{n-i} such cycles independent on β bounding on S, since R_{i-1} cycles on β do not bound on S. Thus $M_i = R_{n-i}$.

11. **The type number of a non-degenerate critical point.** This case is the most important in the many geometric applications.

Suppose $(x)=(0)$ is a non-degenerate critical point at which $f=0$. Let the quadratic form $f_{ij}x_ix_i$, in which the partial derivatives are evaluated at the origin, be carried by a linear transformation into a form with squared terms only. The number of terms with negative coefficients thereby resulting is called the *index* of the critical point.

We wish to establish anew the results of Morse I. Our problem is primarily to prove the following theorem.

THEOREM 14. *If k is the index of a non-degenerate critical point, the kth type number $M_k=1$, while all other type numbers are zero.*

It follows from our initial definition of type numbers that such a number, if it exists, will be independent of any one-to-one transformation of the neighborhood of the critical points of class C''. That the type number exists in this case is a consequence of the existence of the neighborhood function $x_ix_i=\phi$, as affirmed in Theorem 7.

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* Apply Alexander's duality relations to β (these Transactions, vol. 23 (1922), p. 348).

Now we have shown in Morse I that in the neighborhood of the origin the function can be carried by a transformation of the above sort into the form

$$f = -y_1^2 - \cdots - y_k^2 + y_{k+1}^2 + \cdots + y_n^2. \tag{11.1}$$

The function $y_i y_i$ is again an admissible neighborhood function. We see from Theorem 13 that the theorem is true if $k=0$ or n.

Suppose then that $0<k<n$. From Theorem 11 we have the following lemma.

LEMMA. *The ith type number M_i equals the $(i-1)$st Betti number R_{i-1} of the domain*

$$y_j y_j = 1, \ f < 0 \qquad (i, j = 1, \cdots, n), \tag{11.2}$$

minus one if $i=1$.

Our problem is then to determine the Betti numbers of the spherical region (11.2).

We shall prove that the domain (11.2) can be deformed on itself into the $(k-1)$-sphere

$$y_1^2 + \cdots + y_k^2 = 1, \ y_{k+1}^2 + \cdots + y_n^2 = 0, \tag{11.3}$$

by a deformation T that leaves (11.3) fixed. Accordingly the Betti numbers of (11.2) will be those of the $(k-1)$-sphere, and the theorem will follow from the lemma.

The deformation T will now be given.

Corresponding to any point $(x)=(a)$ on (11.2) there is a unique angle α such that

$$a_1^2 + \cdots + a_k^2 = \cos^2\alpha, \ a_{k+1}^2 + \cdots + a_n^2 = \sin^2\alpha, \ 0 \leqq \alpha < \frac{\pi}{4}, \tag{11.4}$$

and every point (a) satisfying (11.4) is on (11.2).

Now hold each such point (a) and corresponding α fast. In the required deformation the point (y) shall not move if it is initially on (11.3). If (y) is initially at a point (a) not on (11.3) it shall move as follows:

$$\begin{aligned} y_i &= \frac{\cos(\alpha t)}{\cos\alpha} a_i && (i = 1, \cdots, k), \\ y_j &= \frac{\sin(\alpha t)}{\sin\alpha} a_j && (j = k+1, \cdots n), \ \alpha \neq 0, \end{aligned} \tag{11.5}$$

as t varies from 1 to zero.

It is easily seen that this affords the required deformation except possibly for continuity of movement of points near (11.3). But for these points α is near zero, and we have from (11.5) that

$$y_{k+1}^2 + \cdots + y_n^2 = \sin^2(\alpha t),$$

so that for α near zero the variables $y_{k+1}, \cdots, y_n$ are uniformly near zero.

Thus the deformation is continuous.

The theorem now follows from the lemma.

The principal theorems on type number relations in Morse I now follow.

12. **A justification of the definitions of type numbers.** We shall investigate how the type numbers of a function change with variation of the function.

Let $F(x, \mu)$ be a function of (x) and a set of parameters (μ), analytic in (x) and (μ) for (x) on Σ and (μ) neighboring (0), and such that

$$f(x) \equiv F(x, 0).$$

For (μ) sufficiently near (0), $F(x, \mu)$ will satisfy our boundary conditions and possess critical points lying only in arbitrarily small neighborhoods of the critical sets of g.

We state the following lemma.

LEMMA 1. *If a and b are any ordinary values of f with $a<b$, then for (μ) sufficiently near* (0) *the domains $f \leqq b$ and $F \leqq b$ are homeomorphic under a transformation that makes $f \leqq a$ and $F \leqq a$ correspond.*

The lemma is easily established by using the orthogonal trajectories of f. A deformation can be set up along these trajectories which affords the homeomorphism, moving only those points which are very near $f=a$ and $f=b$. See Morse I §7.

We note that to prove this lemma we need only to have $F(x, \mu)$ of class C''.

Let g be a connected critical set of f. Recall that $\phi = f_i f_i$ is a neighborhood function for g. We now give a lemma which enables us to avoid critical sets with the same critical value.

LEMMA 2. *Corresponding to the critical set g of $F(x, 0)$ there exists a function $\psi(x)$ of class C'' throughout Σ with the following properties:*

(1) *Except when $\phi < e$, $\psi(x) \equiv 0$, where e is a small positive constant.*

(2) *When $\phi < e_1$, $\psi(x) \equiv \rho$, where ρ is an arbitrarily small positive constant and e_1 is a positive constant less than e.*

(3) *For (μ) sufficiently near* (0), *$F+\psi$ has no other critical points than those of F.*

Let $H(z)$ be a function of z of class C'' for $z \geqq 0$, one for $z < e_1$ and zero for $z > e$. The function $\psi(x)$ will now be defined as zero except when $\phi < e$, and when $\phi < e$ it will be defined by the equation

$$\psi(x) = \rho h[\phi(x)].$$

One sees that ψ has all the required properties, except possibly (3). But (3) could fail only when

$$e_1 < \phi < e. \tag{12.1}$$

For $\rho = 0$ and $(\mu) = (0)$ we have $F + \psi = f$. Moreover $F + \psi$ is of class C'' in (x), (μ), and ρ. On the domain (12.1) the gradient of f is not null. Accordingly for ρ and (μ) sufficiently near $\rho = 0$ and $(\mu) = (0)$ respectively, the gradient of $F + \psi$ is not null.

Thus ψ satisfies (3) and the lemma is proved.

Our definition of type numbers is justified by the following theorem.

THEOREM 15. *If (μ) be sufficiently near $(\mu) = (0)$, the critical points of $F(x, \mu)$ will appear only in sets arbitrarily near the critical sets of f. Corresponding to each critical set g of f the critical points of F which lie in g's neighborhood have a kth type number sum at least as great as the kth type number of g.*

By virtue of Lemma 2 we will lose no generality if we suppose the critical values at the different critical sets of f are all different. The addition of ψ in Lemma 2 did not change the position or type numbers of the critical points of F for (μ) sufficiently near (0).

Let g be any critical set of f whose critical value is separated from the other critical values of f by constants a and b. Let γ be the set of critical points of $F(x, \mu)$ which lie in the neighborhood of g. Let (μ) be taken so near (0) that the critical values of F on γ lie between a and b, and so that the homeomorphism of Lemma 1 holds.

The type number M_k of the set g relative to f will equal the number N_k of newly bounding $(k-1)$-cycles and new k-cycles in complete sets, relative to a change from the domain $f \leqq a$ to $f \leqq b$. By virtue of the homeomorphism of Lemma 1 the number N_k will be the same relative to F. But according to Theorem 4 the kth type number sum of the critical points γ of F will exceed or equal N_k, that is, the type number M_k of g.

The theorem is thereby proved.

Our work is further justified by the following theorem.

THEOREM 16. *If f is analytic there exists a set of constants (μ) arbitrarily near (0) such that the critical points of the function*

$$F(x, \mu) = f(x) + \mu_i x_i \qquad (i = 1, \cdots, n)$$

are non-degenerate and lie in arbitrarily small neighborhoods of the critical points of f.

Corresponding to each connected critical set g of f the critical points of $F(x, \mu)$ which lie in g's neighborhood have a kth type number sum at least as great as the kth type number of g.

The condition that $F(x, \mu)$ have no degenerate critical points is that the equations

$$f_{x_i} + \mu_i = 0$$

have no solution at which the Hessian of f vanishes. That a choice of the constants (μ) can be so made arbitrarily near $(\mu)=(0)$ follows from a general theorem formulated by Kellogg.*

The second statement in the theorem now follows from Theorem 15.

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* Kellogg, *Singular manifolds among those of an analytic family*, Bulletin of the American Mathematical Society, vol. 35 (1929), p. 711.

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A NEW DEFINITION OF GENUS FOR TERNARY QUADRATIC FORMS*

BY

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1. Introduction. The adjoint of the form $f=ax^2+by^2+cz^2+2ryz+2sxz+2txy$ is $\mathfrak{F}=Ax^2+By^2+Cz^2+2Ryz+2Sxz+2Txy$ where A is the cofactor of a, etc., in the Hessian of f, the Hessian being the determinant of the halves of the second derivatives of f. The primitive contravariant or reciprocal form F of f is $\mathfrak{F}/\Omega$ where Ω is the greatest common divisor of the literal coefficients of $\mathfrak{F}$. The Hessian $H=\Omega^2\Delta$.

The genus of the form f is defined by H. J. S. Smith‡ in terms of the quadratic character of the integers represented by f and F with respect to the odd prime factors of the Hessian, the congruences mod 8 satisfied by the odds represented by f and F and, in certain cases, certain so-called "simultaneous characters."

The "simultaneous characters" are redundant and L. E. Dickson§ omits them in his definition of genus. However, since these additional characters used by Smith and known by him to be redundant (see p. 470 of his article) are closely linked with the arithmetic progressions associated with a form, the author has retained them in his discussion.

In this paper, Smith's definition is shown to be equivalent to a *new* definition of genus expressed in terms of the integers represented by the form f alone without reference to F. (A more precise phrasing of this statement will be found in the theorem of this paper.) Furthermore, f may be indefinite or positive and the character of the integers represented is shown to determine the order (as well as Ω and Δ) of the form.

The new definition proved in this paper has immediate application in proving the important theorem shortly to be published that the integers represented by all the forms of a given genus fall exclusively in certain arithmetic progressions. An example of the ease of application of this definition to finding the genus of a form is given at the end of the paper.

Also, it should be noted that this new definition is peculiar to ternary quadratic forms, for the genus of a binary form depends on the character

* Presented to the Society, October 25, 1930; received by the editors in May, 1930.

† National Research Fellow.

‡ Collected Mathematical Papers, vol. 1, pp. 455–509. This article is constantly being referred to.

§ *Studies in the Theory of Numbers*, p. 52.

of the odd integers represented prime to the determinant and many quaternary forms are universal, i.e. represent all integers, positive if the form is positive.

2. **Notations.** $\phi=\alpha x^2+\beta y^2$, p denotes an odd prime, q a prime, k indicates the range of values $0, 1, 2, \cdots$, and $n=0, \pm 1, \pm 2, \cdots$. The form f of the introduction is abbreviated by enclosing the coefficients in parentheses: $(a, b, c, 2r, 2s, 2t)$; in case $r=s=t=0$ we use the notation $f=(a, b, c)$. For any integers a and b, μ_{ab} is that least positive integer for which a/μ_{ab} is an integer prime to b. Unless otherwise indicated, all letters denote integers. The letters f and g with subscripts and superscripts are used to denote ternary quadratic forms and $(f|p)=(a|p)$ means that the integers prime to p represented by f are all of the same quadratic character mod p as a. We use the usual notation $[a/b]$ for the greatest integer in a/b. The other notations used in this paper are explained in H. J. S. Smith's article previously referred to.

3. **Lemmas.** In this section we shall prove a number of lemmas.

Lemma 1. *$\phi\equiv a \pmod p$, a prime to p, is solvable for all and only the a's for which $(a|p)=(\alpha|p)$ or for all a's according as $\alpha\not\equiv 0\equiv\beta \pmod p$ or $\alpha\beta\not\equiv 0 \pmod p$.*

If $\beta\equiv 0\not\equiv\alpha \pmod p$ the proof is obvious. If $\alpha\beta\not\equiv 0 \pmod p$ note* that αx^2 and $a-\beta y^2$ each take $(p+1)/2$ incongruent values mod p as the values of x and y range over all integers and that therefore, for some value of x and y, we have $\alpha x^2\equiv a-\beta y^2 \pmod p$.

Lemma 2. *$\phi\equiv a \pmod{p^2}$ is solvable for every a prime to p if $\alpha\beta\not\equiv 0 \pmod p$.*

This follows from Lemma 1 by an elementary proof similar to that used in the Annals article referred to above.

Corollary 1. *$f=\alpha x^2+\beta y^2+\gamma z^2\equiv ap^r \pmod{p^{r+1}}$, $r=0, 1$, is solvable if $a\alpha\beta\gamma\not\equiv 0 \pmod p$ for we may then take $z=1$ or p according as $r=1$ or 0 and $\alpha x^2+\beta y^2\equiv ap^r-\gamma z^2 \pmod{p^2}$ is solvable.*

Corollary 2. *The above corollary holds for all r for $f\equiv ap^r \pmod{p^{r+1}}$ implies, multiplying the variables by p, that $f\equiv ap^{r+2} \pmod{p^{r+3}}$ solvable, etc.*

Lemma 3. *$\phi\equiv ap \pmod{p^2}$ is solvable for every a prime to p or for none according as $(-\alpha\beta|p)=1$ or -1 and in the latter case $\phi\equiv 0 \pmod p$ implies $x\equiv y\equiv 0 \pmod p$.*

Suppose $\alpha x^2+\beta y^2\equiv 0 \pmod p$ with both x and y prime to p. Choose x' so that $xx'\equiv 1 \pmod p$ and have $\alpha\equiv -\beta(yx')^2 \pmod p$ which implies $(-\alpha\beta|p)=1$.

* Cf. Annals of Mathematics, (2), vol. 28 (1927), p. 333.

Otherwise one and therefore both of x and y are divisible by p. On the other hand if $(-\alpha\beta|p)=1$ there exists a solution (x_1, y_1) of $\alpha x^2+\beta y^2\equiv 0 \pmod{p}$ with $x_1y_1\not\equiv 0 \pmod{p}$. Consider the set (x_1, y_i) where $i=1, 2, \cdots, p$ and $y_{i+1}^2\equiv y_i^2+p \pmod{p^2}$. Note $y_i^2\not\equiv y_j^2 \pmod{p^2}$ if $i\neq j$ and see that $\alpha x_1^2+\beta y_i^2\equiv 0 \pmod{p}$ takes p incongruent values mod p^2 one of which must be $\equiv ap \pmod{p^2}$.

COROLLARY. *$\phi\equiv ap^r \pmod{p^{r+1}}$, $r=0, 1$, is solvable for every a prime to p if $(-\alpha\beta|p)=1$.*

LEMMA 4. *$f=\alpha x^2+\beta y^2+\gamma pz^2\equiv a \pmod{p\mu_{ap}}$ is solvable for every a if $(-\alpha\beta|p)=1$.*

This is true for $a\not\equiv 0 \pmod{p^2}$ with $z=0$ from the above corollary and thus is true, multiplying each variable by p, for $ap^2\not\equiv 0 \pmod{p^4}$. This may be continued to prove the lemma.

COROLLARY. *If $(-\beta\gamma|p)=1$ and $\alpha\not\equiv 0 \pmod{p}$, $f\equiv\alpha x^2+\beta py^2+\gamma pz^2\equiv a \pmod{p\mu_{ap}}$ is solvable if and only if a is not of the form $pn+\alpha_{-1}$ where $(\alpha_{-1}|p)=-(\alpha|p)$.*

This follows from Lemmas 1 and 4 if we note that $f\equiv 0 \pmod{p}$ implies $x=px_1$.

LEMMA 5. *$f=\alpha x^2+\beta y^2+\gamma pz^2\equiv a \pmod{p\mu_{ap}}$, where $\gamma\not\equiv 0 \pmod{p}$ and $(-\alpha\beta|p)=-1$, is solvable if and only if a is not of the form $p^{2k}(p^2n+p\gamma_{-1})$ where $(\gamma_{-1}|p)=-(\gamma|p)$.*

If a is prime to p, the proof follows from Lemma 1. Thus Lemma 5 holds for all $ap^{2r}\not\equiv 0 \pmod{p^{2r+1}}$. On the other hand $f\equiv 0 \pmod{p}$ implies, by Lemma 3, $x=px_1$, $y=py_1$, and $f/p\equiv\alpha px_1^2+\beta py_1^2+\gamma z^2\equiv a_1 \pmod{p}$ solvable for a_1 prime to p if and only if $(a_1|p)=(\gamma|p)$. This proves the lemma for $a=a_1p\not\equiv 0 \pmod{p^2}$. The congruence $f\equiv 0 \pmod{p^3}$ implies $x\equiv y\equiv pz\equiv 0 \pmod{p^2}$ and $f/p^3=\alpha px_2^2+\beta py_2^2+\gamma z_1^2\equiv a_2 \pmod{p}$ which is solvable for a_2 prime to p if and only if $(a_2|p)=(\gamma|p)$ and so the process may be continued.

COROLLARY. *If $(-\beta\gamma|p)=-1$ and $\alpha\not\equiv 0 \pmod{p}$, $f=\alpha x^2+\beta py^2+\gamma pz^2\equiv a \pmod{p\mu_{ap}}$ is solvable if and only if a is not of the form $p^{2k}(pn+\alpha_{-1})$ where $(\alpha_{-1}|p)=-(\alpha|p)$.*

LEMMA 6. *If $\alpha\beta\equiv 1 \pmod{4}$, $\phi\equiv 2a \pmod{16}$, with a odd, is solvable if and only if $2a\equiv\alpha+\beta \pmod{8}$.*

$\alpha+\beta\equiv\alpha+9\beta \pmod{8}$ but they are not congruent mod 16. Thus, if $2a\equiv\alpha+\beta \pmod{8}$, one of $\alpha+\beta$, $\alpha+9\beta$ is $\equiv 2a \pmod{16}$. It is obvious that $\alpha x^2+\beta y^2\equiv\alpha+\beta+4 \pmod{8}$ is not solvable.

LEMMA 7. *If* $\alpha\beta\equiv 3 \pmod 8$, $\phi\equiv 4a \pmod{32}$ *is solvable for every odd* a *and* $\phi\equiv 0 \pmod 8$ *implies* $x\equiv y\equiv 0 \pmod 2$.

The proof is similar to that above.

LEMMA 8. *If* $\alpha\beta\equiv 7 \pmod 8$, $\phi\equiv 2^{r+2}a \pmod{2^{r+5}}$, $r\geqq 0$, *is solvable for every odd* a.

For $r=1$, the proof is similar to the above. $\phi\equiv 4 \pmod 8$ implies $x=2x_1$, $y=2y_1$ and $\phi/4=\alpha x_1^2+\beta y_1^2\equiv a \pmod 8$ is solvable for every odd a. Thus $\phi\equiv 4a \pmod{32}$ is solvable for every odd a.

LEMMA 9. $f=\alpha x^2+\beta y^2+\gamma z^2\equiv a \pmod{8\mu_{a2}}$ *with* $\alpha\beta\gamma\equiv 1 \pmod 8$ *is solvable if and only if* a *is not of the form* $4^k(8n+7)$ *or for all* a *according as* $\alpha\equiv\beta\equiv\gamma\equiv 1 \pmod 4$ *or not.*

If $a\equiv 2 \pmod 4$ note that two of α, β, γ are congruent (mod 4) and, permuting coefficients if necessary, take $\alpha\equiv\beta \pmod 4$. Then $a-16\gamma$ or $a-4\gamma\equiv\alpha+\beta \pmod 8$ and Lemma 6 applies to complete the proof. If a is odd, $f\equiv a \pmod 8$ is solvable unless $\alpha\equiv\beta\equiv\gamma \pmod 4$. Then $\alpha\beta\equiv\gamma \pmod 8$ implies $1\equiv\alpha\equiv\beta\equiv\gamma \pmod 4$ and $f\equiv a \pmod 8$ is solvable if and only if $a\not\equiv\alpha+\beta+\gamma+4\equiv\alpha+\beta+\alpha\beta+4\equiv 7 \pmod 8$. Furthermore, if $\alpha\equiv\beta\equiv\gamma \pmod 4$, $f\equiv 0 \pmod 4$ implies $x\equiv y\equiv z\equiv 0 \pmod 2$ and f represents an integer $4n$ if and only if it represents n.

LEMMA 10. $f=\alpha x^2+\beta y^2+2\gamma z^2\equiv a \pmod{8\mu_{a2}}$ *is solvable for every* a *if* $\alpha\beta\gamma\equiv 1 \pmod 8$ *and* $\alpha+\beta\equiv 6$ *or* $0 \pmod 8$.

This is obvious if μ_{a2} is an even power of 2. Consider $a\equiv 2 \pmod 4$. Then $f\equiv a \pmod{16}$ is solvable from Lemma 8 taking $z=1$ or Lemma 6 taking $z=0$ unless $\alpha+\beta\equiv 6 \pmod 8$ and $a\equiv\alpha+\beta+4\equiv 2 \pmod 8$. Now $f\equiv 2 \pmod 8$ implies $x=2x_1$, $y=2y_1$, and $f\equiv a \pmod{16}$ is solvable for $a\equiv 2 \pmod 8$ if and only if $f/2=2\alpha x_1^2+2\beta y_1^2+\gamma z^2\equiv a/2 \pmod 8$ is solvable. Now $\alpha+\beta\equiv 6 \pmod 8$ implies $\alpha\beta\equiv\gamma\equiv 1 \pmod 4$ and thus γ and $\gamma+2\alpha+2\beta$ $\equiv\gamma+4 \pmod 8$ are $\equiv 1$ and $5 \pmod 8$ in some order. Thus $f/2\equiv a/2 \pmod 8$ is solvable, $f\equiv a \pmod{16}$ is solvable for all $a\equiv 2 \pmod 4$, and therefore $f\equiv 4^k a \pmod{4^{k+2}}$ is solvable for all $a\equiv 2 \pmod 4$.

COROLLARY. $f=2\alpha x^2+2\beta y^2+\gamma z^2\equiv a \pmod{8\mu_{a2}}$, *where* $\alpha\beta\gamma\equiv 1 \pmod 8$ *and* $\alpha+\beta\equiv 6$ *or* $0 \pmod 8$, *is solvable if and only if* a *is not of the form* $8n+3$.

If a is odd, $f\equiv\gamma$, $\gamma+2\beta$, $\gamma+2\alpha$, $\gamma+2\alpha+2\beta \pmod 8$ is solvable and $f\equiv$ no other odd (mod 8) is solvable. If $\alpha+\beta\equiv 0 \pmod 8$, $\alpha\beta\equiv\gamma\equiv 7 \pmod 8$ and $f\not\equiv 7+4\equiv 3 \pmod 8$. If $\alpha+\beta\equiv 6 \pmod 8$, $\alpha\beta\equiv\gamma\equiv 1 \pmod 4$ and $f\not\equiv\gamma+6\alpha\equiv\alpha(\beta+6)\equiv\alpha(4-\alpha)\equiv 3 \pmod 8$. $f\equiv 1, 5, 7 \pmod 8$ is solvable in both cases. $f\equiv 0 \pmod 2$ implies $z=2z_1$ and the rest follows from Lemma 10.

LEMMA 11. *If* $\alpha\beta\gamma\equiv 1$ (mod 8) *and* $\alpha+\beta\equiv 2$ *or* 4 (mod 8), $f=\alpha x^2+\beta y^2+2\gamma z^2\equiv a$ (mod $8\mu_{a2}$) *is solvable if and only if a is not of the form* $4^k(16n+14)$.

As above we consider $a\equiv 2$ (mod 4). Then $f\equiv a$ (mod 16) is solvable from Lemmas 6 and 7 taking $z=0$ and 1 respectively unless $a\equiv 6$ (mod 8), for $\alpha+\beta\equiv 4$ (mod 8) implies $\alpha\beta\equiv\gamma\equiv 3$ (mod 8) and $\alpha+\beta+2\gamma\equiv 2$ (mod 8). Now $f\equiv 6$ (mod 8) implies $x=2x_1$, $y=2y_1$ and $f/2=2\alpha x_1^2+2\beta y_1^2+\gamma z^2$. If $\alpha+\beta\equiv 2$ (mod 8), then $\alpha\beta\equiv 1\equiv\gamma$ (mod 4), $f/2\equiv\gamma+2\alpha\equiv\alpha(\beta+2)\equiv 3$ (mod 8), but $f/2\not\equiv\gamma+6\alpha\equiv 7$ (mod 8). If $\alpha+\beta\equiv 4$ (mod 8), $\alpha\beta\equiv 3\equiv\gamma$ (mod 8) implies $f/2\equiv 3$ (mod 8) but $\not\equiv 7$ (mod 8). Thus $f\equiv 2a$ (mod 16) for an odd a if and only if a is not of the form $8n+7$. Noting $f\equiv 0$ (mod 8) implies $x\equiv y\equiv z\equiv 0$ (mod 2), we see $f\equiv 8a$ (mod 64) is solvable for every odd $a\not\equiv 7$ (mod 8). So the process may be continued.

COROLLARY. *If* $\alpha\beta\gamma\equiv 1$ (mod 8) *and* $\alpha+\beta\equiv 2$ *or* 4 (mod 8), $f=2\alpha x^2+2\beta y^2+\gamma z^2\equiv a$ (mod $8\mu_{a2}$) *is solvable if and only if a is not of the form* $4^k(8n+7)$.

The proof is similar to that for the corollary to Lemma 10.

LEMMA 12. *For any* ∇, *f and its reciprocal form F are equivalent to a pair of forms* ϕ *and* Φ *satisfying the congruences*

$$\phi\equiv\alpha x^2+\beta\Omega y^2+\gamma\Omega\Delta z^2 \qquad (\text{mod } \nabla),$$

$$\Phi\equiv\beta\gamma\Omega\Delta x^2+\alpha\gamma\Delta y^2+\alpha\beta z^2 \qquad (\text{mod } \nabla),$$

$$\alpha\beta\gamma\equiv 1 \qquad (\text{mod } \nabla),$$

except that ∇ *must be taken odd if f or F is improperly primitive.*

The proof is given by H. J. S. Smith in the article previously referred to, pages 461 and 462.

COROLLARY. *For any* ∇, $f=(\alpha, b, c, 2r, 2s, 2t)$, *with* α *prime to* ∇, *and its reciprocal form F are equivalent to a pair of forms* ϕ *and* Φ *above except that* ∇ *must be taken odd if F is improperly primitive.*

This follows from Smith's proof if we note α prime to ∇ implies ∇ odd if f is improperly primitive.

LEMMA 13. *For an improperly primitive form f, its reciprocal form F, and* ∇ *arbitrary* (*it may be taken even*), *there exist two forms*

$$f_2\equiv\alpha x^2+\beta\Omega y^2+4\gamma\Omega\Delta_1 z^2 \qquad (\text{mod } \nabla),$$

$$F_2\equiv\beta\gamma\Omega\Delta_1 x^2+\alpha\gamma\Delta_1 y^2+\alpha\beta z^2 \qquad (\text{mod } \nabla),$$

where $\alpha\beta\gamma\equiv 1$ (mod ∇), $\alpha\beta\Omega\equiv 3$ (mod 4), $\Delta_1=\Delta/2$, *the integers represented by f are the halves of the integers represented by* f_2 *with* $x\equiv y$ (mod 2), *i.e. the*

halves of the evens represented by f_2, *and the integers represented by* F *are those represented by* F_2 *with* $x\equiv y \pmod 2$.

Smith's proof of Lemma 13 carries through for this lemma to the middle of page 462, ibid., since f improperly primitive implies F is properly primitive and if $f=(a, a', a'', 2b, 2b', 2b'')$ we have $f\sim f_1\equiv ax^2+2b''xy+a'y^2+\gamma\Omega\Delta z^2 \pmod{\nabla'}$, ∇' being $\nabla\Omega^2\Delta$. Now f_1 is equivalent to a similar form with $a\equiv 2 \pmod 4$, for if $a\equiv 0\equiv a'+2 \pmod 4$ we interchange x and y, while if $a\equiv a'\equiv 0 \pmod 4$ the replacement of y by $x+y$ yields a form with leading coefficient double an odd, since b'' is odd. Similar reasoning shows we may consider $a/2$ to be prime to ∇' since f_1 is primitive. Thus let $a=2\alpha\equiv 2 \pmod 4$, $a'=2\alpha'$, and choose α_1 so that $\alpha\alpha_1\equiv 1 \pmod{\nabla'}$. Then $2f\equiv\alpha(2x+b''\alpha_1 y)^2+\alpha(4\alpha'\alpha_1-b''^2\alpha_1^2)y^2+2\gamma\Omega\Delta z^2 \pmod{2\nabla'}$. Now $4\alpha\alpha'-b''^2=\Omega A''\equiv 3 \pmod 4$ where A'' is defined in Smith's paper. Set $\beta\equiv A''\alpha_1 \pmod{\nabla'}$ and

$$2f_1 \equiv \alpha(2x + b''\alpha_1 y)^2 + \beta\Omega y^2 + 2\gamma\Omega\Delta z^2 \qquad \pmod{\nabla'},$$

and double the integers represented by f_1 are the integers represented by f_2 with $x\equiv y \pmod 2$. Then $\alpha\beta\Omega\equiv 3 \pmod 4$ and $\alpha\beta\gamma\equiv 1 \pmod{\nabla}$. Now

$$\Omega F_1 \equiv 2\alpha'\gamma\Omega\Delta x^2 + 2\alpha\gamma\Omega\Delta y^2 + \Omega A''z^2 - 2\gamma\Omega\Delta b''xy \qquad \pmod{\nabla'},$$

$$F_1 \equiv \beta\gamma\Delta_1\Omega x^2 + \alpha\gamma\Delta_1(2y - b''\alpha_1 x)^2 + \alpha\beta z^2 \qquad \pmod{\nabla'/\Omega},$$

and the integers represented by F_1 are those integers represented by F_2 with $x\equiv y \pmod 2$.

LEMMA 14. *Every properly primitive form* f *for which* $\Omega\equiv 2^{t_1} \pmod{2^{t_1+1}}$, $t_1\geqq 1$, *is equivalent to a form* $f_1\equiv ax^2+2^{t_1}by^2+2^{t_1}cz^2+2^{t_1+1}ryz \pmod{2^n}$ *where* n *is arbitrary,* a *is odd and* b *is odd or double an odd according as* F *is properly or improperly primitive. In the latter case* r *is odd and* c *even.*

This is a corollary of Lemma 12 if F is properly primitive. f represents primitively an odd integer a. Transform f by an equivalent transformation so that a is the leading coefficient. Then replace x by $x+\tau y+\sigma z$, choosing τ and σ so the new coefficients of xy and xz are $\equiv 0 \pmod{2^n}$. We have $f\sim f_2\equiv ax^2+b_1y^2+c_1z^2+2r_1yz \pmod{2^n}$ and $\Omega F_2\equiv(b_1c_1-r_1^2)x^2+ac_1y^2+ab_1z^2-2r_1ayz \pmod{2^n}$. Thus $c_1\equiv b_1\equiv r_1\equiv 0 \pmod{2^{t_1}}$. Now $b_1c_1-r_1^2\equiv 0 \pmod{2^{2t_1}}$ implies that not all of b_1, c_1 and r_1 are $\equiv 0 \pmod{2^{t_1+1}}$. Furthermore, both of $b_1/2^{t_1}=b$ and $c_1/2^{t_1}=c$ are even if and only if F_2 is improperly primitive. In this case $r_1/2^{t_1}=r\equiv 1 \pmod 2$. If $b\equiv 2 \pmod 4$ the lemma is proved. If $c\equiv 2\equiv b+2 \pmod 4$ interchange y and z to prove the lemma. If $b\equiv c\equiv 0 \pmod 4$ replace z by $y+z$ to prove the lemma.

LEMMA 15. *If* $f\equiv ap^r(p^{r+1})$ *is solvable, where* a *is prime to* p, *then, for* n *arbitrary,* $f\equiv ap^r(p^n)$ *is solvable* $(r\geqq 0)$.

By Lemma 12 we may take $f\equiv\alpha_1x^2+\beta_1y^2+\gamma_1z^2$ (mod p^n). Let $\alpha_1\xi^2+\beta_1\eta^2+\gamma_1\zeta^2=ap^r+kp^m$ where $m\geqq r+1$. We prove there exists a solution $x=\xi+X$, $y=\eta+Y$, $z=\zeta+Z$ of $f\equiv ap^r$ (mod p^{m+1}) if $m+1\leqq n$. Not all of $\alpha_1\xi^2$, $\beta_1\eta^2$ and $\gamma_1\zeta^2$ are divisible by p^{r+1}. Permute $\alpha_1\xi^2$, $\beta_1\eta^2$ and $\gamma_1\zeta^2$ if necessary to take $\alpha_1\xi^2\not\equiv 0$ (mod p^{r+1}). Let $Y=Z=0$ and $\alpha_1=\alpha_2p^t$, $\xi=\xi_1p^s$ where $\alpha_2\xi_1$ is prime to p and $t+2s\leqq r$. Take $X=p^{m-t-s}X_1$ and have

$$f(x, y, z) \equiv ap^r + p^m(k + 2\alpha_2\xi_1X_1) + \alpha_2p^{2m-2s-t}X_1^2 \equiv ap^r + p^m(k + 2\alpha_2\xi_1X_1) \pmod{p^{m+1}}$$

and X_1 may be chosen so that $k+2\alpha_2\xi_1X_1\equiv 0$ (mod p). Thus, by induction, we prove the lemma.

LEMMA 16. *If $f\equiv 2^ra(2^{r+3})$ is solvable, where a is odd, then, for n arbitrary, $f\equiv 2^ra$ (mod 2^n) is solvable.*

If f and F are both properly primitive we may by Lemma 12 consider $f\equiv\alpha_1x^2+\beta_1y^2+\gamma_1z^2$ (mod 2^n) and proceed as above except that $m\geqq r+3$ and we take $X=2^{m-t-s-1}X_1$.

If f is improperly primitive we have, by Lemma 13, a form $f_2\equiv\alpha_1x^2+\beta_1y^2+4\gamma_1z^2$ (mod 2^{n+1}), $\alpha_1\beta_1\equiv 3$ (mod 4) such that the evens represented by f_2 are twice the integers represented by f. The primitive contravariant of f_2 is properly primitive and thus the reasoning of the first paragraph of this proof applies to prove $f_2\equiv 2^{r+1}a$ (mod 2^n) and therefore $f\equiv 2^ra$ (mod 2^n) is solvable.

It remains to consider f properly primitive and F improperly primitive. By Lemma 14, we may consider $f\equiv\alpha x^2+2^{t+2}\beta y^2+2^{t+2}\gamma z^2+2^{t+2}\rho yz$ (mod 2^n) where $\Omega\equiv 2^{t+1}(2^{t+2})$, $\beta\rho\alpha\equiv 1$ (mod 2) and $t\geqq -1$. Choose β_1 so that $\beta\beta_1\equiv 1$ (mod 2^n).

First, if $t\geqq 0$, $f_1\equiv\alpha x^2+2^t\beta w^2+2^t\beta_1\delta z^2$ (mod 2^n) where $w=2y+\rho\beta_1z$, $4\beta\gamma-\rho^2=\delta$ and the integers represented by f are those integers represented by f_1 with $w\equiv z$ (mod 2). We have $f_1(\xi,\omega,\zeta)=2^ra+2^mk$, $m\geqq r+3$ and $\omega\equiv\zeta$ (mod 2). Let $\xi=2^\nu\xi_1$, $\omega=2^\sigma\omega_1$, $\zeta=2^\mu\zeta_1$ where $\xi_1\omega_1\zeta_1\equiv 1$ (mod 2), ν, σ and $\mu\geqq 0$. Note that not all of 2ν, $2\sigma+t$, $2\mu+t$ are $>r$ and $\alpha\beta\beta_1\delta\equiv 1$ (mod 2). If $2\nu\leqq r$ take $x=\xi+2^{m-\nu-1}X$, $w=\omega$, $z=\zeta$ and $f_1(x,w,z)=f_1(\xi,\omega,\zeta)+\alpha(2^{m-\nu}\xi X+2^{2m-2\nu-2}X^2)\equiv 2^ra+2^m(k+\alpha\xi_1X)$ (mod 2^{m+1}), since $2m-2\nu-2\geqq m+1$ and X may be chosen so that $f_1(x,w,z)\equiv f(x,y,z)\equiv 2^ra$ (mod 2^{m+1}). If $2\nu>r$ and $2\sigma+t\leqq r$ take $x=\xi$, $z=\zeta$, $w=\omega+2^{m-t-\sigma-1}W\equiv\omega$ (mod 2) and $f_1(x,w,z)\equiv 2^ra+2^m(k+\beta\omega_1W)$ (mod 2^{m+1}) and W may be chosen so that $f_1(x,w,z)\equiv f(x,y,z)\equiv 2^ra$ (mod 2^{m+1}). Proceed similarly if $2\nu>r$, $2\sigma+t>r$ and $2\mu+t\leqq r$.

Second, if $t=-1$, $f\equiv\alpha x^2+2\beta y^2+2\gamma z^2+2\rho yz$ (mod 2^n) and $f_2=2f\equiv 2\alpha x^2+\beta w^2+\beta_1\delta z^2$ (mod 2^{n+1}) where w and δ have the same values as for $t\geqq 0$,

$\beta\beta_1\delta\equiv 3 \pmod 4$ and the evens represented by f_2 are twice the integers represented by f. The proof proceeds in a manner similar to that above.

LEMMA 17. *$f\equiv a \pmod{8h\mu}$ implies $f\equiv a \pmod N$ is solvable for N arbitrary, where h is the product of the first powers of the odd prime divisors of H and μ is the least integer for which a/μ is an integer prime to $2H$.*

Let p_i be the odd prime factors common to h and N, and r_i their respective multiplicities in N. Then, by Lemma 15, there exist solutions (x_i, y_i, z_i) of $f\equiv a \pmod{p_i^{r_i}}$. Also, by Lemma 16, there exists a solution (x_0, y_0, z_0) of $f\equiv a \pmod{2^s}$ where $N\equiv 2^s \pmod{2^{s+1}}$. Let p_i' be the odd prime factors of N which are prime to h and s_i their respective multiplicities in N. By Lemma 12 we may take $f\equiv\alpha x^2+\beta\Omega y^2+\gamma\Omega\Delta z^2 \pmod{p_i'^{s_i}}$, where $\alpha\beta\gamma\Delta\Omega^2$ is prime to $p_i'^{s_i}$. Thus, by Corollary 2 of Lemma 2 and Lemma 15, there exist solutions x_i', y_i', z_i' of $f\equiv a \pmod{p_i'^{s_i}}$. There exists, by the Chinese Remainder Theorem, an x, y and z such that $x\equiv x_i, y\equiv y_i, z\equiv z_i \pmod{p_i}$, $x\equiv x_0$, $y\equiv y_0$, $z\equiv z_0 \pmod{2^s}$ and $x\equiv x_i', y\equiv y_i', z\equiv z_i' \pmod{p_i'^{s_i}}$. Such an (x, y, z) is a solution of $f\equiv a \pmod N$.

4. THEOREM. *With every ternary quadratic form of Hessian H there is associated a set of arithmetic progressions:*

$$2^r(8n + a_j')p^{r_i}(p_in + a_{ij}) \qquad (n = 0, \pm 1, \pm 2, \cdots) \tag{1}$$

such that no integer falling in any one of them is represented by f, and for every integer a not falling in any of (1) *it is true that $f\equiv a \pmod{8h\mu}$, and therefore* $\pmod N$ *for N arbitrary, is solvable, where p_i are odd prime factors of H, a_{ij} are some or all the members of a complete residue system* mod p_i, *r and r_i range over some or all of the positive integers and zero, a_j' are some, none or all of* 1, 3, 5 *and* 7, *h is the product of the first powers of the odd prime factors of H and μ is the smallest positive integer for which a/μ is an integer prime to $2h$.*

The Hessian H and the progressions (1) *serve to define the genus* and to determine the order and invariants Ω and Δ of the form: i.e., all forms of the same Hessian and having the same progressions (1) associated with them are of the same genus and order and have the same invariants Ω and Δ and, conversely, all forms of the same genus and order and having the same Hessian, Ω and Δ, have the same progressions (1). The forms of (1) are more precisely given below in the course of the proof.

I. We first prove, for a given form f, the existence of certain progressions (1) having the desired properties and which, together with H, determine by their nature the invariants Δ and Ω and whether F and f are properly or improperly primitive.

For an odd prime factor p of H we use Lemma 12 to see that we may take $f\equiv\alpha x^2+\beta\Omega y^2+\gamma\Omega\Delta z^2$ (mod p^t) where t is the highest power of p in H. We shall use the following abbreviations: fNa denotes: f represents no integer of the form a; fCa denotes: for every integer of the form a, $f\equiv a$ (mod $p\mu_{ap}$) is solvable. The above congruential form of f shows, using Lemma 1,

$$fC(pn+a),\text{ i.e. } FCp^{2k}(pn+a);\ a \text{ prime to } p \text{ if } \Omega\not\equiv 0 \pmod{p},$$

$$fN(pn+\alpha_{-1}),\ fC(pn+\alpha_1),\ fCp^{2k}(pn+\alpha_1),\ \text{where}$$

$$(\alpha_{-1}\,|\,p) = -(\alpha\,|\,p) \text{ and } (\alpha_1\,|\,p) = (\alpha\,|\,p),\ \text{if } \Omega\equiv 0 \pmod{p}.$$

In the latter case to find the power to which p occurs in Ω, we note $f\equiv 0$ (mod p) implies $x=px_1$ and $f/p=p\alpha x_1^2+\beta y^2\Omega/p+\gamma\Delta z^2\Omega/p$ (mod p^{t-1}) which represents an integer prime to p if and only if $\Omega\not\equiv 0$ (mod p^2). If $\Omega\equiv 0$ (mod p^2), $f/p^2\equiv\alpha x_1^2+\beta y^2\Omega/p^2+\gamma\Delta z^2\Omega/p^2$ (mod p^{t-2}), and $f/p^2\equiv\alpha_{-1}$ (mod p) if and only if $\Omega\not\equiv 0$ (mod p^3). So continuing we have, taking t_1 to be the highest power of p in Ω,

$$fN(p^{2r+1}a),\qquad r=0,1,\cdots,[(t_1-2)/2];$$

$$fNp^{2s}(pn+\alpha_{-1}),\qquad s=0,1,\cdots,[(t_1-1)/2];$$

$$fC(\beta\Omega) \text{ and, if } t_1 \text{ is even, } fCp^{t_1}(pn+\alpha_{-1}).$$

This shows the existence of certain progressions (1). Then, given these progressions, we can determine t_1 as follows: find the least odd power $2r_1+1$ of p such that for some integer b prime to p, $p^{2r_1+1}b$ does not occur in (1) and the least even power $2s_1$ of p for which $p^{2s_1}(pn+\alpha_{-1})$ does not occur in (1). Either $2r_1+1$ or $2s_1$ or both are finite, for if t_1 is odd, $fC(\beta\Omega)$ and $\beta\Omega/p^{t_1}$ is an integer prime to p. t_1 is the lesser of $2r_1+1$ and $2s_1$ for $fCp^{t_1}(\beta\Omega/p^{t_1})$ if t_1 is odd.

To determine the greatest power t_1 of 2 occurring in Ω as a factor if f is properly primitive, we use Lemma 14 and consider $f\equiv ax^2+2^{t_1}by^2+2^{t_1}cz^2+2^{t_1+1}ryz$ (mod 2^{3+t}). If $t_1\geqq 2$, $f\equiv 0$ (mod 2) implies $x=2x_1$ and $f/4\equiv ax_1^2+2^{t_1-2}by^2+2^{t_1-2}cz^2+2^{t_1-1}ryz$ (mod 2^{1+t}). So proceeding we have

$$\text{for } t_1 \text{ even},\quad f/2^{t_1-4}\equiv ax^2+16by^2+16cz^2+32ryz \pmod{128} \text{ if } t_1\geqq 4,$$

$$f/2^{t_1-2}\equiv ax^2+4by^2+4cz^2+8ryz \pmod{32} \text{ if } t_1\geqq 2,$$

$$f/2^{t_1}\equiv ax^2+by^2+cz^2+2ryz \pmod{8};$$

$$\text{for } t_1 \text{ odd},\quad f/2^{t_1-3}\equiv ax^2+8by^2+8cz^2+16ryz \pmod{64} \text{ if } t_1\geqq 3,$$

$$f/2^{t_1-1}\equiv ax^2+2by^2+2cz^2+4ryz \pmod{16} \text{ if } t_1\geqq 1.$$

Let s_1 be the least value of s for which $f\equiv 4^s(a+4) \pmod{8\cdot 4^s}$ is solvable and s_2 the least value of s for which $f\equiv 4^s(a+2) \pmod{8\cdot 4^s}$ is solvable for some integer a represented by f. Inspection of the above, taking into account the conditions of Lemma 14, gives the following table:

t_1 even, F p.p.	t_1 even, F i.p.	t_1 odd, F p.p.	t_1 odd, F i.p.
$2s_1=t_1-2$ or $t_1=2s_1=0$	$\geqq t_1$	$\geqq t_1-1$	$=t_1-1$
$2s_2\geqq t_1$	$=t_1$	$=t_1-1$	$>t_1-1$

p.p. and i.p. are abbreviations of "properly primitive" and "improperly primitive" respectively.

The above table shows the existence of certain progressions (1). Given the progressions we note that F can be improperly primitive only if Δ is odd, i.e. only if t (the highest power of 2 in H) is even and $t_1=t/2$. The numbers $2s_1$ and $2s_2$ can be found from progressions (1).

If $0\neq 2s_1\geqq 2s_2$ we have only the second and third columns of the table to consider and see that $2s_2=t_1$ or t_1-1. If $t=4s_2$ then $t_1=2s_2$ and F is improperly primitive, for if $2s_2=t_1-1$, $4s_2=2t_1-2<t$. If $t\neq 4s_2$, F is properly primitive and $t_1-1=2s_2$.

If $0\neq 2s_1<2s_2$, we see from the first and fourth columns that $2s_1=t_1-2$ or t_1-1. If $t=4s_1+2$ then $t_1=2s_1+1$ and F is improperly primitive, for $t_1-2=2s_1$ implies $4s_1+2=2t_1-2<t$. If $t\neq 4s_1+2$, F is properly primitive and $2s_1=t_1-2$.

If $2s_1=0$ note that $H\not\equiv 0 \pmod 4$ implies $t_1=0$. We consider below only $H\equiv 0 \pmod 4$.

If $2s_1=0=2s_2$ and $H\equiv 0 \pmod 4$ we have to consider the first and third columns, for, if the second held, $0=t_1$, $H\equiv 0 \pmod 2$ and F would be improperly primitive, which is impossible. Thus F is properly primitive, $0=t_1$, t_1-1, or t_1-2 and, using Lemma 12, we may take $f\equiv\alpha x^2+\beta\Omega y^2+\gamma\Omega\Delta z^2 \pmod 8$ and note that $t_1=0$ and $H\equiv 0 \pmod 4$ implies $\Delta\equiv 0 \pmod 4$. Inspection of the three cases gives the following table:

t_1	Progressions (1) contain
0	$4n+2$ or $4n+3\nu$ but not both
1	neither $4n+2$ nor $4n+3\nu$
2	$4n+2$ and $4n+3\nu$

where ν is an integer such that $f\equiv\nu \pmod 4$ is solvable. Thus the value of t_1 may be determined from the form of progressions (1).

If $2s_1=0<2s_2$ and $H\equiv 0 \pmod 4$ we have to consider the first and fourth columns and see that $t_1=0$ or 2 and F is properly primitive or $t_1=1$ and F is improperly primitive. If $t_1=2$, $H\equiv 0 \pmod{16}$ while, if $t_1=1$ and F is improperly primitive, Δ is odd and thus $H\equiv 4 \pmod 8$. We have the following table:

t_1	F	H	Progressions (1) contain
0	p.p.	$\equiv 0 \pmod 4$	$4n+2$ or $4n+3\nu$ but not both
2	p.p.	$\equiv 0 \pmod{16}$	$4n+2$ and $4n+3\nu$
1	i.p.	$\equiv 4 \pmod 8$	$4n+2$ and $4n+3\nu$

where ν is an integer such that $f\equiv\nu \pmod 4$ is solvable. Thus, in this case, the value of t_1 and whether F is properly or improperly primitive may be determined from H and the progressions (1).

f is improperly primitive if and only if $2n+1$ occurs in progressions (1). Here $t_1=0$.

Thus progressions (1) determine not only t_1, the highest power of 2 contained in Ω, but whether F and f are properly or improperly primitive.

II. After Ω and Δ have been determined by the above process we must exhibit the progressions (1) and show that the genus is determined by them. Since, for any odd prime factor p of H, the progressions (1) determine the quadratic character with respect to p of the integers represented by the form, it remains to show that the progressions (1) involving a particular p determine the quadratic character with respect to p of the integers represented by F. We exhibit the progressions (1) and show their relation to the character of the integers represented by F.

Consider f to be of the form of ϕ in Lemma 12 with $\nabla=p^{t+1}$ where t is the highest power of p in H. For any α we use α_{-1} to denote any α_{-1} for which $(\alpha_{-1}|p)=-(\alpha|p)$ and a is any integer prime to p.

A. If p is prime to Ω, it must divide Δ. Then from the form of f and Lemma 1 we see $f\equiv a \pmod p$ is solvable for every a prime to p and therefore $f\equiv ap^{2r} \pmod{p^{2r+1}}$ is solvable. Let $\Delta/p^t=\Delta'\not\equiv 0 \pmod p$. First if $(-\alpha\beta\Omega|p)=1$ the first row in the table below results from Lemma 4 and $(F|p)=(\alpha\beta|p)$. Second, if $(-\alpha\beta\Omega|p)=-1$ we see from Lemma 3 that $f\equiv 0 \pmod p$ implies $x=px_1$, $y=py_1$ and $f/p\equiv\alpha p x_1^2+\beta p y_1^2+\gamma\Omega\Delta z^2/p$ $\pmod{p^t}$, which represents integers prime to p if and only if $\Delta\not\equiv 0 \pmod{p^2}$ when we use Lemma 5. If, on the other hand, $\Delta\equiv 0 \pmod{p^2}$ we have $f/p^2\equiv\alpha x_1^2+\beta\Omega y_1^2+\gamma\Omega\Delta z^2/p^2 \pmod{p^{t-1}}$ and $f/p^2\equiv pa \pmod{p^2}$ is solvable for every a prime to p if and only if $\Delta\not\equiv 0 \pmod{p^3}$ from Corollary 1 of Lemma 2. This process may be repeated until we have the results below:

$(-\alpha\beta\Omega\mid p)$	Progressions (1) involving p	$(F\mid p)$
1	none	$(-\Omega\mid p)$
-1	$p^{2r+1}a$, $r=0, 1, \cdots, [(t-2)/2]$ if $t\geqq 2$ and, if t is odd, $p^{2k+1}(pn+\alpha_1)$	$-(-\Omega\mid p)$

where $(\alpha_1\mid p)=-(\gamma\Omega\Delta'\mid p)=-(\alpha\beta\Omega\Delta'\mid p)=(-\Delta'\mid p)$ and a is prime to p. The character of the progressions thus determines $(F\mid p)$.

B. If $\Omega\equiv 0 \pmod{p^{t_1}}$, $t_1>0$ and $\Omega/p^{t_1}=\Omega'\not\equiv 0 \pmod p$, we have $\Delta\equiv 0 \pmod{p^{t-2t_1}}$ and $\Delta/p^{t-2t_1}=\Delta'\not\equiv 0 \pmod p$ where $t-2t_1\geqq 0$. Now $f\equiv 0 \pmod p$ implies $x=px_1$ and $f/p\equiv p\alpha x_1^2+\beta y^2\Omega/p+\gamma\Delta z^2\Omega/p \pmod{p^t}$ and f/p represents integers prime to p if and only if $\Omega\not\equiv 0 \pmod{p^2}$. If $\Omega\equiv 0 \pmod{p^2}$, $f/p^2\equiv\alpha x_1^2+\beta\Omega y^2/p^2+\gamma\Delta z^2\Omega/p^2 \pmod{p^{t-1}}$. So continuing we have

$$\text{if } t_1 \text{ is even, } f/p^{t_1}\equiv\alpha x'^2+\beta\Omega' y^2+\gamma\Omega'\Delta z^2 \pmod{p^{t-t_1}},$$

and

$$\text{if } t_1 \text{ is odd, } f/p^{t_1}\equiv p\alpha x'^2+\beta\Omega' y^2+\gamma\Omega'\Delta z^2 \pmod{p^{t-t_1}}.$$

First, if $t=2t_1$, $(F\mid p)=\pm 1$ and the progressions (1) are $p^{2r}(pn+\alpha_{-1})$, $p^{2s+1}a$, where the second progression occurs only if $t_1>1$, where $s=0, 1, \cdots, [(t_1-2)/2]$ and, if t_1 is odd and $(-\beta\gamma\Delta\mid p)=(-\alpha\Delta\mid p)=-1$, $r=0, 1, 2, \cdots$. Otherwise $r=0, 1, 2, \cdots, [(t_1-1)/2]$.

Second, if $t>2t_1$ set $t_2=t-2t_1>0$. Note that, if t_1 is even and $(-\alpha\beta\Omega'\mid p)=-1$, it is true that $f/p^{t_1}\equiv 0 \pmod p$ implies $x'=px_2$, $y=py_2$ and $f/p^{t_1+1}\equiv p\alpha x_2^2+\beta\Omega' p y_2^2+\gamma\Delta\Omega' z^2/p \pmod{p^{t-t_1-1}}$. Also if t_1 is odd, $f/p^{t_1}\equiv 0 \pmod p$ implies $y=py_1$ and $f/p^{t_1+1}\equiv\alpha x'^2+\beta\Omega' p y_1^2+\gamma\Omega'\Delta z^2/p \pmod{p^{t-t_1-1}}$. So continuing we have

If t_1 is even

$(-\alpha\beta\Omega'\mid p)$	Progressions (1) involving p	$(F\mid p)$
1	$p^{2r}(pn+\alpha_{-1})$, $p^{2s+1}a$, $s=0, 1, \cdots, t_1/2-1$	$(-\Omega'\mid p)$
-1	$p^{2r}(pn+\alpha_{-1})$, $p^{2s+1}a$, and if t_2 is odd, $p^{2k+1}(pn+\alpha_1)$	$-(-\Omega'\mid p)$

where in the second row $s=0, 1, \cdots, [(t-t_1-2)/2]$ and $(\alpha_1\mid p)=-(\gamma\Omega'\Delta')=(-\Delta'\mid p)$. In both cases $r=0, 1, \cdots, t_1/2-1$. It should be noted that the progressions in the first and second rows are not the same even if t_2 is even for $t_1/2-1\neq(t-t_1-2)/2$.

If t_1 is odd

Conditions	Progressions (1) involving p	$(F\mid p)$
t_2 even, $(-\alpha\Delta'\mid p)=1$	$p^{2r}(pn+\alpha_{-1})$, $p^{2s+1}a$, $p^{2s_1+1}(pn+\alpha_1)$	$-(-\alpha_1\Delta'\Omega'\mid p)$
t_2 even, $(-\alpha\Delta'\mid p)=-1$	$p^{2k}(pn+\alpha_{-1})$, $p^{2s+1}a$, $p^{2s_1+1}(pn+\alpha_1)$	$(-\alpha_1\Delta'\Omega'\mid p)$
t_2 odd, $(-\alpha_1\Delta'\mid p)=1$	$p^{2r}(pn+\alpha_{-1})$, $p^{2s+1}a$, $p^{2k+1}(pn+\alpha_1)$	$-(-\alpha\Delta'\Omega'\mid p)$
t_2 odd, $(-\alpha_1\Delta'\mid p)=-1$	$p^{2r}(pn+\alpha_{-1})$, $p^{2s+1}a$, $p^{2s_1+1}(pn+\alpha_1)$	$(-\alpha\Delta'\Omega'\mid p)$

where $r=0, 1, \cdots, [(t-t_1-1)/2]$, $s_1=0, 1, \cdots, [(t-t_1-2)/2]$ and $s=0, 1, \cdots, (t_1-3)/2$, the progressions in s being excluded unless $t_1 \geqq 3$, $-(\alpha_1|p)=(\beta\Omega'|p)=(\alpha\gamma\Omega'|p)=(\alpha F\Omega'|p)$, $(F|p)$ is determined from the progressions by the above tables.

III. We find completely the progressions (1) involving 2 and show that H, Δ, Ω, together with the progressions (1), determine the complete generic character of f.

If f and F are properly primitive we use Lemma 12 and have

$$\Delta' f \equiv \alpha\Delta' x^2 + \beta\Delta'\Omega y^2 + \gamma\Omega\Delta'' z^2 \qquad (\text{mod } 8\Omega''\Delta''),$$

$$\Omega' F \equiv \beta\gamma\Delta\Omega'' x^2 + \alpha\gamma\Delta\Omega' y^2 + \alpha\beta\Omega' z^2 \qquad (\text{mod } 8\Delta''),$$

where $\alpha\beta\gamma \equiv 1 \pmod{8\Delta''}$, $\Omega=\Omega'\Omega''$, $\Delta=\Delta'\Delta''$, Δ'' and Ω'' being the greatest powers of 2 dividing Δ and Ω respectively. Let $\alpha'=\alpha\Delta'$, $\beta'=\beta\Delta'\Omega'$, and $\gamma'=\gamma\Omega'$ and then replace α', β', γ' by α, β, γ respectively to get

$$\Delta' f \equiv \alpha x^2 + \beta\Omega'' y^2 + \gamma\Omega''\Delta'' z^2 \qquad (\text{mod } 8\Omega''\Delta''),$$

$$\Omega' F \equiv \alpha\Delta''\Omega'' x^2 + \beta\Delta'' y^2 + \gamma z^2, \quad \alpha\beta\gamma \equiv 1 \qquad (\text{mod } 8).$$

A. If $\Omega \not\equiv 0 \not\equiv \Delta \pmod 4$, and both f and F are properly primitive, the generic character involves a symbol Ψ defined in Smith's article. In his paper the generic character is given for the four cases in terms of this Ψ.

If H is odd we use Lemma 9 and referring to Smith's discussion, pages 465, 466, we have the following table:

Progressions (1) involving 2	Ψ
None	$+1$
$4^k(8n+7\Delta')$	-1

If $\Omega''=2$, $\Delta''=1$, we abbreviate Smith's notation $(-1)^{(f^2-1)/8}$ to read $(2|f)$, etc., and see from his case ii, page 466, that $f \not\equiv 3\Delta' \pmod 8$ implies $(2|f)\Psi=(2|\Delta)$ and $f \not\equiv 7\Delta' \pmod 8$ implies $(2|f)\Psi=-(2|\Delta)$. Thus, using the corollaries to Lemmas 10 and 11, we have the following table:

$\beta+\gamma \equiv \pmod 8$	Progressions (1) involving 2	Character
6 or 0	$8n+3\Delta'$	$(2\|f)\Psi=(2\|\Delta)$
4 or 2	$4^k(8n+7\Delta')$	$(2\|f)\Psi=-(2\|\Delta)$

If $\Omega'=1$, $\Delta''=2$, we have similarly

$\alpha+\beta\equiv$ (mod 8)	Progressions (1) involving 2	Character
6 or 0	None	$(2\mid F)\Psi=(2\mid\Omega)$
4 or 2	$4^k(16n+14\Delta')$	$(2\mid F)\Psi=-(2\mid\Omega)$

for if $\alpha+\beta\equiv 0$ or 6 (mod 8), ΩF represents no $8n+3$ and if $\alpha+\beta\equiv 2$ or 4 (mod 8), $\Omega F\not\equiv 7$ (mod 8).

If $\Delta''=\Omega''=2$ we note case iv in Smith's article on pages 465 and 467 and that $\Delta' f\equiv 0$ (mod 2) implies $x=2x_1$. We have the following:

$\alpha+\gamma\equiv$ (mod 8)	Progressions (1) involving 2	$(2\mid f)(2\mid F)(2\mid\Delta')(2\mid\Omega')\Psi$
0 or 6	$16n+6\Delta'$	$+1$
2 or 4	$4^k(16n+14\Delta')$	-1

This may be deduced as follows: $(2\mid f)(2\mid F)(2\mid\Delta')(2\mid\Omega')\Psi=(2\mid f\Delta')(2\mid F\Omega')\Psi=(-1)^\nu$, referring to page 465 of Smith's article, where $\nu=(\Delta'^2m^2+\Omega'^2M^2+2\Omega'M+2\Delta'm+2\Delta'm\Omega'M)/8=\{(\Delta'm+\Omega'M+1)^2-1\}/8$. Then, taking for $\Delta'm$ and $\Omega'M$ the first pair of values given on page 465: α, γ, we have $\nu\equiv\{(\alpha+\gamma+1)^2-1\}/8$ (mod 2).

B. If $\Omega''=1$ and $\Delta\equiv 0$ (mod 4) with f properly primitive, using the lemmas and the forms given at the beginning of section III for $\Delta' f$ and $\Omega' F$, we obtain the following table:

Δ''	$\alpha+\beta\equiv$ (mod 8)	Progressions (1) involving 2	$\Omega' F\equiv$ (mod 8) only
4	0 or 4	$4n+2$	3, 7
4	2 or 6	$4n+3\alpha\Delta'$ and, if $\alpha\equiv 1$ (mod 4), $4^k(8n+7\Delta')$	1, 5
8	0	$4n+2$	7
8	2	$4n+3\alpha\Delta'$, $8n+6\Delta'$, $4^k(16n+14\Delta')$	$2\alpha-1$
8	4	$4n+2$, $4^k(16n+14\Delta')$	3
8	6	$4n+3\alpha\Delta'$, $8n+2\Delta'$	$6\alpha-1$
$8\cdot 4^\tau$, $\tau>0$	0	$4n+2$	7
$8\cdot 4^\tau$, $\tau>0$	2, 4, or 6	4^r times the values given above for $\Delta''=8$ where $r=0, 1, \cdots, \tau$	see $\Delta''=8$

If $\Delta''=4\cdot 4^{\tau}$, $\tau>0$, we have the following table:

$\alpha\beta\equiv$ (mod 8)	Progressions (1) involving 2	$\Omega' F\equiv$ (mod 8) only
7	$4n+2$	7
5	$4^r(4n+3\alpha\Delta')$, $4^s(8n+2\alpha\Delta')$ and, if $\alpha\equiv 1 \pmod 4$, $4^k(8n+7\Delta')$	5
3	$4^r(4n+2)$	3
1	$4^r(4n+3\alpha\Delta')$, $4^s(8n+6\alpha\Delta')$ and, if $\alpha\equiv 1 \pmod 4$, $4^k(8n+7\Delta')$	1

where $r=0, 1, \cdots, \tau$ and $s=0, 1, \cdots, \tau-1$.

C. If $\Omega''=2$ and $\Delta\equiv 0 \pmod 4$, f and F are properly primitive. In the first line of the table note that $\alpha\equiv\beta \pmod 4$ and $\alpha\equiv 1$ or $3 \pmod 8$ implies that $\beta+\gamma\equiv 2$ or $4 \pmod 8$ to find the progressions (1). A similar situation exists for the rest of the table. If $\Delta''=4$, we have the table following:

Conditions on α and β	Progressions (1) involving 2	$\Omega' F\equiv$ (mod 8) only
$\alpha\equiv\beta \pmod 4$, $\alpha\equiv 1$ or $3 \pmod 8$	$8n+5\Delta'$, $4^k(8n+7\Delta')$	1, 5
$\alpha\equiv\beta \pmod 4$, $\alpha\equiv 5$ or $7 \pmod 8$	$8n+\Delta'$, $8n+3\Delta'$, $4(8n+3\Delta')$	1, 5
$\alpha\equiv 3\beta \pmod 4$, $\alpha\equiv 3$ or $5 \pmod 8$	$8n+\Delta'$, $4^k(8n+7\Delta')$	3, 7
$\alpha\equiv 3\beta \pmod 4$, $\alpha\equiv 1$ or $7 \pmod 8$	$8n+3\Delta'$, $8n+5\Delta'$, $4(8n+3\Delta')$	3, 7

Thus if $f\not\equiv\nu$ or $3\nu \pmod 8$ for some odd ν, only $\Omega' F\equiv 1$ or $5 \pmod 8$ is solvable, while $f\not\equiv\nu$ or $7\nu \pmod 8$ implies that only $\Omega' F\equiv 3$ or $7 \pmod 8$ is solvable.

If $\Delta''=8$, the multiples of 4 in progressions (1) are 4 multiplied by the progressions (1) given under the heading $\Omega''=2=\Delta''$. The remainder of the progressions are given below:

$\alpha\beta\equiv$ (mod 8)	Progressions (1) $\not\equiv 0 \pmod 4$ but involving 2	$\Omega' F\equiv$ (mod 8) only
1	$2^r(8n+5\alpha\Delta')$, $2^r(8n+7\alpha\Delta')$, $r=0, 1$	1
3	$8n+5\alpha\Delta'$, $8n+3\alpha\Delta'$, $16n+2\alpha\Delta'$, $16n+14\alpha\Delta'$	3
5	$8n+5\alpha\Delta'$, $8n+7\alpha\Delta'$, $16n+2\alpha\Delta'$, $16n+6\alpha\Delta'$	5
7	$2^r(8n+3\alpha\Delta')$, $2^r(8n+5\alpha\Delta')$, $r=0, 1$	7

Thus the third column is related to the second.

If

$$\Delta'' \equiv 0 \pmod{16}$$

the progressions (1) which are $\not\equiv 0 \pmod 4$ and the corresponding character of $\Omega' F$ are given by the table above for $\Delta''=8$. If

$$\Delta = 4 \cdot 4^\tau, \quad \tau \geqq 1,$$

the progressions (1) involving 2 and divisible by 4 are 4^s multipled by the odd progressions given in the table above for $\Delta''=8$, 4^{s_1} multiplied by the progressions $\equiv 2 \pmod 4$ for $\Delta''=8$, and 4^τ multiplied by the progressions for $\Delta''=4$ where $s=1, 2, \cdots, \tau$, and if $\tau>1$, $s_1=1, 2, \cdots, \tau-1$. If

$$\Delta'' = 8 \cdot 4^\tau,$$

the progressions (1) divisible by 4 are those given above for $\Delta''=8$ multiplied by 4^s, $s=1, 2, \cdots, \tau$, together with those for $\Omega''=\Delta''=2$ multiplied by $4^{\tau+1}$.

D. $\Delta''=1$, $\Omega\equiv 0 \pmod 4$ and F properly primitive. If $\Omega''=4$, we have

$\alpha \equiv \pmod 4$	Progressions (1) involving 2	$\Omega' F \equiv \pmod 8$ only
1	$4n+2, 4n+3\Delta'$ and, if $\beta\equiv\gamma\equiv 1 \pmod 4$, $4^k(8n+7\Delta')$	$\beta, 5\beta$
3	$4n+2, 4n+\Delta'$	1, 3, 5 or 7

If $\Omega''=8$ we have the following table:

$\beta+\gamma \equiv \pmod 8$	Progressions (1) involving 2	$\Omega' F \equiv \pmod 8$ only
6	$4n+2, 4n+3\Delta', 8n+5\alpha\Delta', 4(8n+3\Delta')$	$-(2\,\vert\alpha), -5(2\,\vert\alpha)$
2	$4n+2, 4n+3\Delta', 8n+5\alpha\Delta', 4^k(8n+7\Delta')$	$(2\,\vert\alpha), 5(2\,\vert\alpha)$
4	$4n+2, 4n+\Delta', 4^k(8n+7\Delta')$	1, 3, 5 or 7
0	$4n+2, 4n+\Delta', 4^r(8n+3\Delta'), r=0, 1$	1, 3, 5 or 7

If $\Omega''=4\cdot 4^\tau$, $\tau\geqq 1$, the progressions (1) involving 2 are the above for $\Omega''=4$ multiplied by 4^r, where $r=0, 1, \cdots, \tau$, and $4^s(8n+5\alpha\Delta')$, where $s=0, 1, \cdots, \tau-1$. The character of $\Omega' F$ is determined as for $\Omega''=4$.

If $\Omega''=8\cdot 4^\tau$ the progressions (1) are the above for $\Omega''=8$ multiplied by 4^r where $r=0, 1, \cdots, \tau$, and the character of $\Omega' F$ is determined as for $\Omega''=8$.

E. $\Delta''=2$ and $\Omega\equiv 0 \pmod 4$. We have the following table:

Ω''	$\alpha\equiv$ (mod 8)	$\alpha\equiv$ (mod 4)	Progressions (1) involving 2	$\Omega'F\equiv$ (mod 8) only
4	$6-\beta$ or $-\beta$	1	$4n+3\Delta', 4n+2$	5, 7
4	$6-\beta$ or $-\beta$	3	$4n+\Delta', 4n+2$	1, 7
4	$2-\beta$ or $4-\beta$	1	$4n+3\Delta', 4n+2, 4^k(16n+14\Delta')$	1, 3
4	$2-\beta$ or $4-\beta$	3	$4n+\Delta', 4n+2, 4^k(16n+14\Delta')$	3, 5
8	$6-\gamma$ or $-\gamma$	1	$4n+3\Delta', 4n+2, 8n+5\alpha\Delta', 4(16n+6\Delta')$	5α, 7α
8	$6-\gamma$ or $-\gamma$	3	$4n+\Delta', 4n+2, 8n+5\alpha\Delta', 4(16n+6\Delta')$	α, 7α
8	$2-\gamma$ or $4-\gamma$	1	$4n+3\Delta', 4n+2, 8n+5\alpha\Delta', 4^k(16n+14\Delta')$	α, 3α
8	$2-\gamma$ or $4-\gamma$	3	$4n+\Delta', 4n+2, 8n+5\alpha\Delta', 4^k(16n+14\Delta')$	3α, 5α

If $\Omega\equiv 0 \pmod{16}$ the same discussion applies as for case D.

F. If $\Delta''=2^{t_2}$, $\Omega''=2^{t_1}$, $t_1\geqq 2\leqq t_2$, the progressions (1) involving 2 are $4^r(4n+2)$, $4^r(4n+3\alpha\Delta')$, $4^s(8n+5\alpha\Delta')$ and 4^τ multiplied by the progressions given for $\Delta''=2^{t_2}\equiv 0 \pmod 4$ and $\Omega''=1$ or 2 according as t_1 is odd or even, where $r=0, 1, \cdots, [(t_1-2)/2]$, $\tau=[t_1/2]$, and $s=0, 1, \cdots, [(t_1-3)/2]$, the progressions in s being omitted if $t_1=2$.

The character of F is determined as for $\Delta''=2^{t_2}$ and $\Omega''=1$ or 2 according as t_1 is odd or even.

G. Since F has no character with respect to 2 if improperly primitive, it remains to consider f improperly primitive but F properly primitive. Now f is improperly primitive if and only if $2n+1$ occurs in progressions (1). Then $\Omega\equiv 1\not\equiv\Delta \pmod 2$. We use Lemma 13 to obtain

$$\Delta' f_2 \equiv \alpha\Delta' x^2 + \beta\Delta'\Omega y^2 + 4\gamma\Omega\Delta_1'' z^2 \pmod{8\Delta''},$$

$$\Omega F_2 \equiv \beta\gamma\Delta_1 x^2 + \alpha\gamma\Delta_1\Omega y^2 + \alpha\beta\Omega z^2 \pmod 8$$

where $\alpha\beta\Omega\equiv 3 \pmod 4$ and $x\equiv y \pmod 2$. ($\Delta_1''=\Delta''/2$.)

Let $\alpha'=\alpha\Delta'$, $\beta'=\beta\Delta'\Omega$, $\gamma'=\gamma\Omega$ and replace α', β', γ' by α, β, γ respectively, and

$$\Delta' f_2 \equiv \alpha x^2 + \beta y^2 + 4\Delta_1''\gamma z^2 \pmod{8\Delta''},$$

$$\Omega F_2 \equiv \alpha\Delta_1'' x^2 + \beta\Delta_1'' y^2 + \gamma z^2 \pmod 8,$$

with $x\equiv y \pmod 2$, $\alpha\beta\equiv 3 \pmod 4$ and $\alpha\beta\gamma\equiv 1 \pmod 8$. From Lemma 13 the integers represented by $\Delta' f$ are the halves of the evens represented by

$\Delta' f_2$ with $x \equiv y \pmod 2$ and the integers represented by ΩF are those represented by ΩF_2 with $x \equiv y \pmod 2$.

If $t_2 = 1$, i.e. $\Delta_1'' = 1$, no progressions (1) involving 2 occur except $2n+1$ and $\Omega F \equiv a \pmod 8$ is solvable if and only if $a \equiv 3 \pmod 4$.

If $t_2 = 2$, the progressions (1) involving 2 are $2n+1$ and, if $\alpha\beta \equiv 3 \pmod 8$, $4^k(8n+7\Delta')$. The congruence $\Omega F \equiv a \pmod 8$ is solvable if and only if $a \equiv \alpha\beta \pmod 8$.

If $t_2 > 2$ and $\alpha\beta \equiv 3 \pmod 8$ progressions (1) involving 2 are $4^r(2n+1)$ and, if t_2 is even, $4^k(8n+7\Delta')$, $r = 0, 1, \cdots, [(t_2-1)/2]$. Only $\Omega F \equiv 3 \pmod 8$ is solvable.

If $t_2 > 2$ and $\alpha\beta \equiv 7 \pmod 8$, the progression in (1) is $2n+1$ and only $\Omega F \equiv 7 \pmod 8$ is solvable.

IV. We have thus found associated with every prime factor of $2H$ progressions (1) such that for every integer a not contained in a progression involving an odd prime p it is true that $f \equiv a \pmod{p\mu_{ap}}$ is solvable and for every integer a not contained in a progression involving 2 it is true that $f \equiv a \pmod{8\mu_{a2}}$ is solvable. Thus if a is included in none of (1) it is true that $f \equiv a \pmod{8h\mu}$ is solvable. Also we have shown that these progressions determine the invariants Δ and Ω, the order and the genus of the form.

Conversely the invariants H, Δ and Ω together with order and generic character determine the progressions (1). This may be proved by inspection of the results listed above or from the theorem proved by Smith (pp. 480 ff.) that two forms of the same genus and order and having the same H, Ω and Δ may be transformed one into the other by a rational transformation of determinant 1, the denominators of the coefficients being prime to $2H$.

5. Examples. We apply this new definition of genus to the set of reduced properly primitive forms of Hessian 18.*

Consider the form $f = x^2 + 3y^2 + 6z^2$. Using the corollary of Lemma 4 we see the progression (1) involving 3 is $3n+2$. From Lemma 11, the progression (1) involving 2 is $4^k(16n+14)$.

Consider the form $g = 2x^2 + 3y^2 + 4z^2 - 2yz - 2xy$. We have $10g = 5(2x-y)^2 + (5y-2z)^2 + 36z^2$ and the integers represented by $5g$ are the halves of the multiples of 10 represented by $g' = 5X^2 + Y^2 + 36z^2$, for $g' \equiv 0 \pmod 5$ implies $Y^2 \equiv 4z^2 \pmod 5$ and the sign of Y may be so chosen that $5y - 2z = Y$ is solvable for y while $g' \equiv 0 \pmod 2$ implies $X \equiv Y \pmod 2$ and thus $2x - y = X$ is solvable for x. Now, by Lemma 4, no progressions involving 3 occur in (1)

* See Eisenstein's table, Journal für Mathematatik, vol. 41 (1851), p. 170.

for g' and therefore for g. By Lemma 6, no $4n+2$ occurs in progressions (1) for g' and therefore no $2n+1$ in progressions (1) for g. The condition $g'\equiv 0 \pmod 4$ implies $X=2X_1$, $Y=2Y_1$, and $g'/4=5X_1^2+Y_1^2+9z^2$, and thus, using Lemma 9, $g'/4$ represents no $4^k(16n+6)/2$ and the progressions (1) for g are $4^k(16n+14)$.

The other forms are similarly dealt with. We have the following table:

Form	Progressions (1)
(1, 1, 18)	$9n\pm 3$, $4^k(16n+14)$
(2, 2, 5, 0, −2, 0)	$9n\pm 3$, $4^k(16n+14)$
(1, 2, 9)	$4^k(16n+14)$
(2, 3, 4, −2, 0, −2)	$4^k(16n+14)$
(1, 3, 6)	$3n+2$, $4^k(16n+14)$
(2, 3, 3)	$9^k(3n+1)$

The first and second forms are of the same genus, also the third and fourth. Each of the last two forms represents the only class in its genus.

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THE REGULARITY OF A GENUS OF POSITIVE TERNARY QUADRATIC FORMS*

BY

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1. **Introduction.** L. E. Dickson‡ has defined a positive ternary quadratic form f to be regular when the integers not represented by the form f coincide with certain arithmetic progressions.

The methods previously used for proving a form regular have been chiefly modifications§ of a method used by Dirichlet‖ for the form $x^2+y^2+z^2$ together with certain elementary methods by which results for one form have been derived as corollaries from those of another. Now Dirichlet's method or modifications of it may be applied only when there is but one form in the genus or when it can otherwise be proved that all the integers represented by the forms of a genus are represented by one form. Heretofore it has been necessary to carry out a Dirichlet type of proof for each separate form or very restricted set of forms unless the proof could be referred back by elementary means to previous results for other forms, that is, with the exception of two cases when elliptic functions have been used.

In this paper it is proved that the integers represented by all of the forms of a genus¶ coincide with the positive integers in certain arithmetic progressions, i.e., a form f of genus G is regular if and only if f represents all the integers represented by every form of G. Using this result in the case of any form f, in order to prove f regular, it is necessary merely to show that there is only one class in the genus of f or that every integer represented by a form of the genus is represented by f. Similarly this establishes a new criterion for irregularity. Furthermore the form of these progressions in terms of the generic invariants is given by the author in his previous paper.

Examples are given at the end of this paper. Throughout this paper f

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* Presented to the Society, December 30, 1930; received by the editors in July, 1930.

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‡ Annals of Mathematics, (2), vol. 28 (1927), pp. 333–341.

§ See Annals of Mathematics, ibid.; Bulletin of the American Mathematical Society, vol. 33 (1927), p. 63.

‖ Journal für Mathematik, vol. 40 (1850), pp. 228–32.

¶ For the definition of genus see H. J. S. Smith, Collected Papers, vol. 1, pp. 455–509, and *A new definition of genus for ternary quadratic forms* by the author, in the present number of these Transactions, referred to in this article as "the previous paper." L. E. Dickson in his definition of genus (*Studies in the Theory of Numbers*, p. 52) omits certain of Smith's generic characters because of their redundancy. It is immaterial in this paper whether Dickson's or Smith's definition is used.

denotes a primitive positive ternary quadratic form. In the opinion of the author, this discussion could be easily modified to include indefinite forms, also, but he has refrained from doing this in order to avoid possible duplication with a paper on indefinite forms by Mr. Arnold Ross soon to appear in the Proceedings of the National Academy and Sciences.

2. **Lemmas.** We shall first prove some elementary lemmas.

LEMMA 1. *If* a_1 *is any integer primitively represented by a primitive form* $f=(a, b, c, 2r, 2s, 2t)$, *i.e.,* $f=ax^2+by^2+cz^2+2ryz+2sxz+2txy$, f *is equivalent to a form with leading coefficient* a_1.

This has been proved elsewhere but the proof is so simple that it is given here for completeness.

Suppose a primitive solution of $f=a_1$ is (x_1, y_1, z_1). Let $(x_1, y_1)=g$, the g.c.d. of x_1 and y_1, and choose integers u and v such that $x_1v-y_1u=g$. Then g is prime to z_1 since the solution is primitive and there thus exist a k and an n such that $gk-z_1n=1$. Then the transformation

$$x = x_1X + uY + x_1nZ/g, \quad y = y_1X + vY + y_1nZ/g, \quad z = z_1X + kZ$$

is of determinant 1, has integral coefficients, and takes f into a form with leading coefficient a_1.

LEMMA 2. *For any prime* q *and any positive integer* k, *the replacement of* z *by* $z+\tau_1y$ *and then* y *by* $y+\tau_2z$ *in* $f=(a_1, b, c, 2r, 2s, 2t)$ *yields, for a proper choice of* τ_1 *and* $\tau_2 \pmod{q^k}$, *a form*

$$f' = (a_1, b', c', 2r', 2s', 2t')$$

with $s'\equiv t' \pmod{q^k}$.

If we subject f to such a transformation, we see the coefficient of x^2 remains unaltered, $t'=t+s\tau_1$ and $s'=t\tau_2+s(1+\tau_1\tau_2)=\tau_2(t+s\tau_1)+s$. If s is prime to q, take $\tau_2\equiv 0 \pmod{q^k}$ and have $s'\equiv s \pmod{q^k}$. Then τ_1 may be chosen $\pmod{q^k}$ so that $t'=t+s\tau_1\equiv s \pmod{q^k}$. If t but not s is prime to q, proceed similarly. If $s\equiv 0\equiv t \pmod{q}$ and $k>1$, replace s, t, s' and t' above by s/q, t/q, s'/q, t'/q respectively, and prove in precisely the same manner that τ_1 and τ_2 may be so chosen that $s'/q\equiv t'/q \pmod{q^{k-1}}$ unless $s\equiv t\equiv 0 \pmod{q^2}$ and $k>2$. This process may be continued until the proof is complete.

Consider a form $f=(a_1, b, c, 2r, 2s, 2t)$. Transform it first by

$$x = x' + \tau y + \sigma z, \quad y = y, \quad z = z, \tag{1}$$

and have

$$\begin{aligned} f' &= a_1x'^2 + (a_1\tau^2 + 2t\tau + b)y^2 + (a_1\sigma^2 + 2s\sigma + c)z^2 + 2(r + a_1\tau\sigma + t\sigma + s\tau)yz \\ &\qquad + 2(a_1\sigma + s)xz + 2(a_1\tau + t)xy \\ &= a_1x'^2 + b'y^2 + c'z^2 + 2r'yz + 2s'xz + 2t'xy. \end{aligned}$$

Let $\lambda=(s', t')$ and transform f' by

(2) $$x' = x', \quad y = \alpha y' + \beta z', \quad z = \gamma y' + \delta z';$$

where

(3) $$\alpha = - s'/\lambda, \quad \gamma = t'/\lambda,$$

and β and δ are chosen so that

(4) $$\alpha\delta - \beta\gamma = 1.$$

This yields

$$f'' = a_1 x'^2 + b''y'^2 + c''z'^2 + 2r''y'z' + 2s''x'z' \text{ where } b'' = \alpha^2 b' + \gamma^2 c' + 2r'\alpha\gamma.$$

Let Ω denote the g.c.d. of the literal coefficients of the form adjoint to f, i.e. the g.c.d. of $bc-r^2$, a_1c-s^2, a_1b-t^2, $st-a_1r$, $rt-bs$, $rs-ct$. Let q be 2 or any odd prime factor of the Hessian H of a primitive form $f=(a_1, b, c, 2r, 2s, 2t)$ and ϵ the highest power of q occurring as a factor of H. We prove

LEMMA 3. *If* $a_1\not\equiv 0 \pmod q$ *and* $\Omega\equiv 0 \pmod{q^\omega}$ *but* $\Omega\not\equiv 0 \pmod{q^{\omega+1}}$, *where* ω *is a positive integer or zero, it is true that* τ_1 *and* τ_2 *in Lemma* 2 *and* τ *and* σ *in* (1) *may be so chosen modulo* q *or a power of* q *that* (2) *yields a form* f'' *with* $b''\equiv r''\equiv 0 \pmod{q^\omega}$ *but* $b''\not\equiv 0 \pmod{\pi q^{\omega+1}}$, *where* $\pi=2$ *or* 1 *according as* q *is even or odd.*

First note that Ω, being an invariant under transformations (1) and (2), is a divisor of a_1b'' and a_1r'', thus showing $b''\equiv r''\equiv 0 \pmod{q^\omega}$. From Lemma 2 we may consider the following congruence to hold: $t\equiv s \pmod{q^{\epsilon+2}}$. Note that $2\omega\leqq\epsilon$.

Now if $a_1b\not\equiv t^2 \pmod{\pi q^{\omega+1}}$, choose τ so that $a_1\tau+t=t'\equiv 0 \pmod{\pi q^{\omega+1}}$ and σ so that $a_1\sigma+s=s'\not\equiv 0 \pmod q$. Then $\gamma\equiv 0 \pmod{\pi q^{\omega+1}}$ and $b''\equiv\alpha^2(a_1\tau^2+2t\tau+b)\equiv\alpha^2(t\tau+b)\not\equiv 0 \pmod{\pi q^{\omega+1}}$ since $a_1(t\tau+b)\equiv -t^2+a_1b\not\equiv 0 \pmod{\pi q^{\omega+1}}$. If $a_1c\not\equiv t^2 \pmod{\pi q^{\omega+1}}$ the proof is similar.

If $a_1b\equiv a_1c\equiv t^2 \pmod{\pi q^{\omega+1}}$ and $a_1r\not\equiv t^2 \pmod{q^{\omega+1}}$, choose τ and σ so $t'\equiv s'\equiv 0 \pmod{\pi q^{\omega+1}}$ and $\alpha\gamma\not\equiv 0 \pmod q$. Then $b''\equiv\alpha^2(t\tau+b)+\gamma^2(t\sigma+c)-2(r+s\tau)\alpha\gamma\equiv -2(r+t\tau)\alpha\gamma\not\equiv 0 \pmod{\pi q^{\omega+1}}$.

Finally $a_1b\equiv a_1c\equiv a_1r\equiv t^2 \pmod{q^{\omega+1}}$ implies $\Omega\equiv 0 \pmod{q^{\omega+1}}$ which is contrary to hypothesis.

LEMMA 4. *If* $a_1\equiv 0 \pmod{q^{\omega_1}}$ *where* $\omega_1\geqq 1$, $\Omega\not\equiv 0 \pmod q$ *and* q *is* 2 *or an odd prime factor of* H, *then, for the proper choice in* (1) *of* τ *and* σ *and* τ_1 *and* τ_2 *in Lemma* 2 *modulo* q *or a power of* q, (2) *yields a form* f'' *with* $b''\not\equiv 0 \pmod{\pi q^{\omega_1+1}}$ *where* $\pi=2$ *or* 1 *according as* q *is even or odd and* $a_1\not\equiv 0 \pmod{q^{\omega_1+1}}$.

From Lemma 2 we need consider only $s\equiv t \pmod{q^{2\omega_1+3}}$. Then suppose

$s \equiv t \equiv 0 \pmod{q^{\omega_2}}$ where ω_2 is the greatest integer for which the congruence holds.

First: $\omega_2 \geqq \omega_1 > 0$. If $b \not\equiv 0 \pmod q$ choose τ and σ so that $\alpha \not\equiv 0 \equiv \gamma \pmod q$. Then $b'' \equiv \alpha^2 b \not\equiv 0 \pmod q$. Proceed similarly if $c \not\equiv 0 \pmod q$. If $b \equiv c \equiv 0 \pmod q$ we know $r \not\equiv 0 \pmod q$, since f is primitive, and, choosing τ and σ that so $\alpha\gamma \not\equiv 0 \pmod q$ we have $b'' \equiv 2r\alpha\gamma \not\equiv 0 \pmod q$ unless $q = 2$. If $b + c \equiv 0 \pmod 4$, choose σ and τ divisible by 4, and so that $\alpha\gamma$ is odd, and have $b'' \equiv b + c + 2r \equiv 2 \pmod 4$. If $b + c \equiv 2 \pmod 4$ and b and c are even, we complete the proof for $\omega_2 \geqq \omega_1 > 0$ as follows.

(1) If $t \equiv 0 \equiv b + 2 \pmod 4$ take τ so that $\gamma \equiv a_1\tau + t \equiv 0 \pmod 4$ and σ so that α is odd, and see $b'' \equiv t\tau + b \equiv 2 \pmod 4$. Make γ odd and $\alpha \equiv 0 \pmod 4$ if $c \equiv 2 \pmod 4$.

(2) If $t \equiv 2 \equiv a_1 \pmod 4$ and $b \equiv 0 \pmod 4$, take τ odd and σ even, have α odd, γ even and $b'' \equiv a_1 \equiv 2 \pmod 4$. Take τ even and σ odd if $c \equiv 0 \pmod 4$.

Second: $\omega_2 = 0$. Note $\alpha\gamma \not\equiv 0 \pmod q$ and let ω_3 be the greatest integer for which $b + c - 2r \equiv 0 \pmod{q^{\omega_3}}$. If $\omega_3 \leqq \omega_1$, take $\tau \equiv \sigma \equiv 0 \pmod{q^{\omega_3+1}}$ so that $\alpha \equiv -\gamma \pmod{q^{\omega_3+1}}$, and have $b'' \equiv \alpha^2(b + c - 2r) \not\equiv 0 \pmod{q^{\omega_3+1}}$. If $\omega_1 < \omega_3$, take $\sigma \equiv 0 \pmod{q^{\omega_1+1}}$, and have $b'' \equiv \alpha^2(a_1\tau^2 + 2t\tau + b) + \gamma^2 c + 2(r + t\tau)\alpha\gamma \pmod{q^{\omega_1+1}}$. Noting $\gamma\lambda = t + a_1\tau$ and $\alpha\lambda \equiv -t \pmod{q^{\omega_1+1}}$, where λ is prime to q, we have $b''\lambda^2 \equiv t\tau a_1\{-t\tau + 2(c - r)\} \pmod{q^{\omega_1+1}}$ and τ may be chosen prime to q so that $b'' \not\equiv 0 \pmod{q^{\omega_1+1}}$.

Third: $\omega_1 > \omega_2 > 0$. Note $\alpha\gamma \not\equiv 0 \pmod q$. If $\omega_3 \leqq \omega_1 - \omega_2 + \pi$ take $\tau \equiv \sigma \equiv 0 \pmod{q^{\omega_1-\omega_2+3}}$, so that $\alpha \equiv -\gamma \pmod{q^{\omega_1-\omega_2+3}}$, and have $b'' \equiv \alpha^2(b + c - 2r) \not\equiv 0 \pmod{q^{\omega_1-\omega_2+\pi+1}}$, i.e. $b'' \not\equiv 0 \pmod{\pi q^{\omega_1+1}}$. If $\omega_3 > \omega_1 - \omega_2 + \pi$, note that $b \not\equiv r \not\equiv c \pmod q$, since $b + c - 2r \equiv 0 \pmod q$ and $b \equiv r \pmod q$ would imply $c - r \equiv bc - r^2 \equiv \Omega \equiv 0 \pmod q$ contrary to hypothesis. Now take $\sigma \equiv 0 \pmod{q^{\omega_1+\omega_2+3}}$ and have, as in the paragraph above, $b''\lambda^2/q^{2\omega_2} \equiv ta_1\tau(-t\tau + 2\{c - r\})/q^{2\omega_2} + a_1^2\tau^2 c/q^{2\omega_2} \pmod{q^{\omega_1-\omega_2+\pi+1}}$.

Now if $q = p$ an odd prime, the first member on the right of the last congruence is $\equiv 2\tau t a_1(c - r)/p^{2\omega_2} \pmod{p^{\omega_1-\omega_2+1}}$, and the second $\equiv 0 \pmod{p^{\omega_1-\omega_2+1}}$ since $2\omega_1 - 2\omega_2 \geqq \omega_1 - \omega_2 + 1$. Thus, taking $\tau \not\equiv 0 \pmod p$ we have $b'' \not\equiv 0 \pmod{p^{\omega_1-\omega_2+1}}$, i.e., $b'' \not\equiv 0 \pmod{p^{\omega_1+1}}$.

If $q = 2$ we have $c - r \equiv 1 \pmod 2$, and thus $b'' \equiv A \cdot 2^{\omega_1}\tau^2 + B\tau \cdot 2^{\omega_1-\omega_2+1} + C\tau^2 \cdot 2^{2\omega_1-2\omega_2} \pmod{2^{\omega_1+3-\omega_2}}$, where AB is odd and $C \equiv c \pmod 4$. Call $C \equiv 2^{\omega_4} \pmod{2^{\omega_4+1}}$, $\omega_4 \geqq 0$. Then τ may be chosen so that $b'' \not\equiv 0 \pmod{2^{\omega_1+3-\omega_2}}$ as follows.

(1) If one of $\omega_1 - \omega_2 + 1$, ω_1, $2\omega_1 - 2\omega_2 + \omega_4$ is less than the other two, $\tau \equiv 1 \pmod{2^{\omega_1+3-\omega_2}}$.

(2) If $\omega_1-\omega_2+1=\omega_1\leqq 2\omega_1-2\omega_2+\omega_4$, τ may be chosen odd so that $\tau A+B \equiv 2 \pmod 4$ or $\equiv 0 \pmod 4$ according as $2\omega_1-2\omega_2+\omega_4 \geqq \omega_1+2$ or not.

(3) If $\omega_1-\omega_2+1=2\omega_1-2\omega_2+\omega_4<\omega_1$, we have $\omega_4=0$, $\omega_1=1+\omega_2$, and choose τ odd so that $\tau(B+C\tau)\equiv 2$ or $0 \pmod 4$ according as $\omega_1\geqq\omega_1-\omega_2+3$ or not. Now $2\omega_1-2\omega_2+\omega_4<\omega_1-\omega_2+1$ is impossible since $\omega_1-\omega_2+\omega_4\geqq 1$. This completes the proof if we note $\omega_1+3-\omega_2\leqq\omega_1+2$.

THEOREM 1. *Any form f of Hessian H representing primitively an integer a_1 prime to Ω is equivalent to a form $f=a_1x^2+by^2+cz^2+2ryz+2sxz$ with $b=\Omega m\beta$, where $m=m_1m_2$, m_1 being prime to Ω and a divisor of a_1 and therefore of H, $m_2=1$ or 2, β prime to $2a_1H$, and $c\equiv s\equiv r\equiv 0$* $\pmod{\Omega}$. (Ω is defined just preceding Lemma 3.)

Since the choice of τ and σ in Lemmas 3 and 4 and τ_1 and τ_2 in Lemma 2 is purely congruential mod q or powers of q for each q, a choice of τ and σ, τ_1 and τ_2 simultaneously fulfilling the requirements for every q (2 or a prime factor of H) is possible in Lemmas 2, 3, and 4, thus proving that we can make $b=\Omega m\beta$ as in the theorem. By Lemma 3 we have $r\equiv 0 \pmod{\Omega}$. To make $s\equiv 0 \pmod{\Omega}$ after the other conditions of the theorem, except for that on c, are satisfied, replace x by $x+\tau z$, note that the coefficients of x^2 and of the terms in y remain unaltered and that the coefficient of $2xz$ becomes $a_1\tau+s$. τ may be chosen so that $a_1\tau+s\equiv 0 \pmod{\Omega}$. Thus we may consider s in f to be $\equiv 0 \pmod{\Omega}$. Then $a_1c-s^2\equiv 0 \pmod{\Omega}$ implies $c\equiv 0 \pmod{\Omega}$ and the proof is complete.

3. Consider the forms $f=(a, b, c, 2r, 2s, 0)$ and $f'=(a', b', c', 2r', 2s', 0)$ of the same Hessian H. Let the adjoints of the two forms be $\mathfrak{F}$ and $\mathfrak{F}'$ and the reciprocal forms $F=\mathfrak{F}/\Omega$ and $F'=\mathfrak{F}'/\Omega'$ respectively. We prove

THEOREM 2. *Two forms f and f' of Hessian H and having the same properties as f in Theorem 1 are of the same genus if $a\equiv a'$* $\pmod{8h\mu}$, *$b\equiv b'$, $c\equiv c'$, $r\equiv r'$, $s\equiv s'$* $\pmod{8h\Omega}$, *where h is the product of the first powers of the odd prime factors of H and μ is the smallest integer for which a/μ is an integer prime to $2h$.*

It has been proved by H. J. S. Smith in his article previously referred to, that the genus of a form depends solely on the quadratic character of the integers represented by f and F with respect to the odd prime factors of the Hessian, the congruences mod 8 satisfied by the odds represented by f and F and, in certain cases, certain so-called "simultaneous characters." Two forms f and f' of the same Hessian are obviously of the same character with respect to $8h$ if the respective literal coefficients are $\equiv \pmod{8h}$. Now $ac-s^2\equiv a'c'-s'^2 \pmod{8h\Omega}$ since $ac/\Omega\equiv a'c'/\Omega \pmod{8h}$. The same is true of the other literal coefficients of $\mathfrak{F}$ and $\mathfrak{F}'$. Thus $\Omega'=\Omega$, since any factor of $2h$ dividing

all the literal coefficients of $\mathfrak{F}/\Omega$ divides all the literal coefficients of $\mathfrak{F}'/\Omega$ and conversely. Thus the literal coefficients of F and F' are respectively congruent mod $8h$. It is easily seen that this implies that the orders and the simultaneous characters of f and f' are the same and the proof is complete.

4. We shall prove the following theorem:

THEOREM 3. *For every form f representing primitively an integer a prime to Ω and any particular integer $a' \equiv a \pmod{16Hh\mu}$, where h and μ are defined in Theorem 2, there exists a form of the same Hessian and genus representing primitively a'.*

The method of proof here is a kind of generalization of that used by Dirichlet, as mentioned in the introduction to this paper.

We may consider f to be in the form of f in Theorem 1 with a_1 replaced by a. Then $H = a\delta - bs^2$ where $\delta = bc - r^2 = \Omega^2 dt$, $b = \Omega m\beta$ (see Theorem 1), d is a power of 2 and t is odd. Take $t = t_1 t_2$, where t_2 is the greatest factor of t prime to H. For a' then we seek integers b', c' and r' satisfying the conditions of Theorem 2 and such that $H = a'\delta' - b's^2$ where $\delta' = b'c' - r'^2 = \Omega^2 dt'$, $b' = \Omega m b_1$, b_1 prime to $2ha'$, and t' is odd. Take $t' = \tau + 16hHmt_1\Omega s^2 k$, where τ is chosen so that $a'\delta' \equiv a\delta \pmod{16Hh\Omega^2 t_1 d\mu m s^2}$. This is possible since this congruence is implied by $a't'/\mu \equiv at/\mu \pmod{16Hht_1 ms^2}$ and a'/μ as well as a/μ is prime to the modulus. (Any factor common to s and a divides H.) Note that this congruence implies that t'/t_1 is an integer prime to h, that $t'/t_1 \equiv t/t_1 \pmod{16Hh}$, and thus $\delta \equiv \delta' \pmod{16Hh\Omega^2 dt_1}$.

Then $b_1 = 16hHdt_1\Omega^2 a'k + (a'\Omega^2 d\tau - H)/(\Omega ms^2)$, since $H = a'\delta' - b's^2$. In view of the choice of τ, b_1 is an integer and $b_1 \equiv \beta \pmod{16h\Omega}$. Now the second member on the right of the last equation involving b_1 is an integer prime to $2H$ since β is, and is prime to a' since any factor common to a' and b_1 divides H. Thus the second member on the right and the coefficient of k are relatively prime, and by a classical theorem in the theory of numbers, we may choose k so that $b_1 = p$, a prime not dividing $2H$. Also β is prime to δ since any factor common to β and δ would divide H. For a similar reason p is prime to δ'. Since $bc - r^2 = \delta$, $(-\delta \mid \beta) = 1$.

Now $(-\delta' \mid p) = (-dt' \mid p) = (-d \mid p)(p \mid t')(-1)^\alpha$, where $\alpha = (t'-1)(p-1)/4 \equiv (t-1)(\beta-1)/4 \pmod 2$. Take $H = H'H''$, $m = m'm''$, $\Omega = \Omega'\Omega''$, the first factor in each equality being the greatest odd factor in the left member. Take $t_3' = (t', H)$, $t_3 = (t, H)$, $t'/t_3' = t_4'$, $t/t_3 = t_4$. Then $t_1 \equiv 0 \pmod{t_3'} \equiv 0 \pmod{t_3}$ and $t'/t_3 \equiv t/t_3 \pmod{16Hh}$. Thus $t_3 = t_3'$ and $t_4' \equiv t_4 \pmod{8Hh}$. Let $t_3 = t_{13}t_{23}$, where t_{23} is the greatest odd factor of t_3. Now, noting that $\Omega ms^2/t_3'$ is prime to t_4 and t_4' since H/t_3' is, and $H = a'\Omega^2 dt' - \Omega mps^2$, we have

$$\begin{aligned}(-\delta' \mid p) &= (-d \mid p)(\{-H''\Omega''m''/t_{13}^2\} \mid t_4')(\{H'm'\Omega'/t_{23}^2\} \mid t_4') \\ &\qquad \cdot (p \mid t_3')(-1)^\alpha \\ &= (-d \mid p)(\{-H''\Omega''m''/t_{13}^2\} \mid t_4')(p \mid t_3')(t_4' \mid \{H'm'\Omega'/t_{23}^2\})(-1)^\gamma;\end{aligned}$$

where

$$\gamma = \alpha + (t_4' - 1)(H'm'\Omega' - 1)/4 \equiv \alpha + (t_4 - 1)(H'm'\Omega' - 1)/4 \pmod 2.$$

Therefore

$$\begin{aligned}(-\delta' \mid p) &= (-d \mid p)(\{-H''\Omega''m''/t_{13}^2\} \mid t_4)(p \mid t_3)(t_4 \mid \{H'm'\Omega'/t_{23}^2\}), \\ (-1)^\gamma &= (-d \mid p)(\{-H''\Omega''m''/t_{13}^2\} \mid t_4)(p \mid t_3)(\{H'm'\Omega'/t_{23}^2\} \mid t_4), \\ (-1)^\alpha &= (-d \mid \beta)(\beta \mid t)(-1)^\alpha,\end{aligned}$$

since $(p \mid t_3) = (\beta \mid t_3)$, t_3 being a factor of H. Thus $(-\delta' \mid p) = (-d \mid \beta)(t \mid \beta) = (-\delta \mid \beta) = 1$.

Therefore there exists a ρ such that $-\delta' \equiv \rho^2 \pmod p$. Choose $r' \equiv \rho \pmod p$ and $\equiv r \pmod{16\Omega^2 dhH}$. Then $\delta + r^2 \equiv \delta' + r'^2 \pmod{16H\Omega}$, showing that $\delta' + r'^2 \equiv 0 \pmod{\Omega m}$, and we have shown the existence of a c' such that $\delta' = \Omega mpc' - r'^2$. Noting that $\delta = \Omega m\beta c - r^2$, we have $\Omega mpc' \equiv \Omega m\beta c \pmod{16Hh\Omega^2 d}$ and thus $pc' \equiv \beta c \pmod{16Hh\Omega d/m}$. Noting that $2H \equiv 0 \pmod m$ we have $c' \equiv c \pmod{8h\Omega}$ and the conditions of Theorem 2 that the forms f and f' be of the same genus are satisfied. Since the leading coefficient of f' is a', the representation of a' by f' is primitive.

THEOREM 3a. *For every form f representing an integer a prime to Ω and any particular $a' \equiv a \pmod{8h\mu}$, there exists a form f' of the same genus and Hessian representing a'.*

$16Hh\mu$ may be replaced by $8h\mu$ in the statement of Theorem 3 since $f \equiv a \pmod{16Hh\mu}$ solvable implies $f \equiv a \pmod{8h\mu}$ solvable and, by Lemma 17 of the previous paper* $f \equiv a \pmod{8h\mu}$ solvable implies $f \equiv a \pmod{16Hh\mu}$ solvable.

Suppose $a = k^2 a_1$, where f represents a_1 primitively and $k = k_1 k_2$, k_2 being the largest factor of k prime to $2h$. Then $a_1 k_1^2/\mu$ is prime to $2h$ and $f \equiv a_1 k_2^2 \pmod{8h\mu/k_1^2}$ is solvable.

Now since the quadratic characters of $a_1 k_1^2/\mu$ and of $a_1 k_1^2 k_2^2/\mu$ with respect to the factors of h are the same and since they are congruent mod 8, we have, by the lemmas of the previous paper, that $f \equiv a_1 \pmod{8h\mu/k_1^2}$ is solvable. Then $f \equiv a_1 k_2^2 \pmod{8h\mu/k_1^2}$ is solvable primitively, for, since f represents a_1 primitively, we may transform f to a form f_1 having its leading coefficient a_1 and a primitive solution of $f_1 \equiv a_1 k_2^2 \pmod{8h\mu/k_1^2}$ is $x = k_2$,

* See the sixth footnote of the Introduction.

$y=z=8h\mu/k_1^2$. Thus, from Theorem 3, there exists a form f' of the same genus representing $a'/k_1^2\equiv a_1k_2^2 \pmod{8h\mu/k_1^2}$. Thus f' represents a'.

5. We need two further lemmas.

LEMMA 5. *Any form f with $\Omega=p\Omega'\not\equiv 0 \pmod{p^2}$ and $H\not\equiv 0 \pmod{p^3}$, i.e., $H/\Omega^2=\Delta\not\equiv 0 \pmod{p}$, is equivalent to a form $f_1\equiv\alpha x^2+b'py^2+c'pz^2+2r'pyz \pmod{p^3}$ where $(b'c'|p)=-1$ and $\alpha\not\equiv 0 \pmod{p}$ (p is an odd prime).*

In $f=(\alpha, b_1, c_1, 2r_1, 2s_1, 2t_1)$ we may consider α to be prime to p. Replace x by $x+\tau y+\sigma z$, choosing τ and σ so the coefficients of $2xy$ and $2xz$ are $\equiv 0 \pmod{p^3}$, and have $f\equiv\alpha x^2+by^2+cz^2+2rzy \pmod{p^3}$. Now $\Omega\equiv 0 \pmod{p}$ implies $b=pb'$, $c=pc'$, $r=pr'$ and $f\equiv\alpha x^2+b'py^2+c'pz^2+2r'pyz \pmod{p^3}$. If $(b'c'|p)=-1$ the lemma is proved.

If $b'\equiv c'\equiv 0 \pmod{p}$ we know $r'\not\equiv 0 \pmod{p}$, since $\Omega\not\equiv 0 \pmod{p^2}$ and replacement of z by $y+z$ in $b'y^2+c'z^2+2r'yz$ yields a form with the coefficient of y^2 prime to p. This therefore reduces to the case below.

If b' or c' is prime to p, interchange y and z if necessary to have b', the coefficient of y^2, prime to p. In f replace y by $y+\tau z$ and see that the coefficients of y^2, xy, xz remain unaltered mod p^3 and that of pz^2 becomes $b'\tau^2+2r'\tau+c'$. To show that we may choose τ so that $b'(b'\tau^2+2r'\tau+c')=(b'\tau+r')^2+b'c'-r'^2$ is a non-residue of p, it is only necessary to show that for any a prime to p there exists an x such that x^2+a is a non-residue of p. ($b'c'-r'^2$ is prime to p since $H\equiv p^2\alpha(b'c'-r'^2)\not\equiv 0 \pmod{p^3}$.) This is obvious if $(a|p)=-1$. If $(a|p)=1$ the values $0, 1, \cdots, (p-1)/2$ of x give $(p+1)/2$ incongruent values of $x^2+a \pmod{p}$, one of which must be a non-residue, unless, for one of these values of x, $x^2+a\equiv 0 \pmod{p}$, and for the other values, x^2+a ranges in value over all the residues of p. This can happen only if $(-a|p)=1$, i.e. if $p\equiv 1 \pmod{4}$, and if, for every residue R of p, $x^2+a\equiv -Ra \pmod{p}$ is solvable for x, $-Ra$ being a residue of p. This is true only if $x^2\equiv -a(R+1) \pmod{p}$ is solvable for every R, i.e. if, for every $R\not\equiv -1 \pmod{p}$, $R+1$ is a residue of p. If this were true, since 1 is a residue of p, every positive integer less than p would be a residue of p which is false.

LEMMA 6. *If for a form $f_1=(\alpha', b, c, 2r, 2s, 2t)$, α' odd, $\Omega_1\equiv 2 \pmod{4}$ and $\Delta_1\equiv 2, 4$ or $6 \pmod{8}$, it is true that $f_1\sim f_1'\equiv\alpha'x^2+2\beta'y^2+2\Delta_1\gamma'z^2 \pmod{128\,\Delta_1\Omega_1}$ and $f_1\sim f_1''\equiv\alpha'x^2+2\beta''y^2+2\Delta_1\gamma''z^2 \pmod{128\,\Delta_1\Omega_1}$ where $\beta'\beta''\equiv 1+\Delta_1\beta'\gamma' \pmod{8}$ and α' prime to $2H$.*

Since $\Omega_1\equiv\Delta_1\equiv 0 \pmod{2}$ implies that f_1 and F_1 are properly primitive, $f_1\sim f_1'$ above from Lemma 12 in the previous paper, and $\beta'=\Omega_1\beta/2$, $\gamma'=\gamma\Omega_1/2$, where $\alpha'\beta\gamma$ is prime to $2\Omega_1\Delta_1$. Thus $\beta'\gamma'$ is odd. The substitution $y=y'$, $z=y'+z'$ transforms $\beta'y^2+\Delta_1\gamma'z^2$ into $(\beta'+\Delta_1\gamma')y'^2+cz'^2+2ry'z'$. Let $\beta''=\beta'$

$+\Delta_1\gamma'\equiv 1 \pmod 2$ and replace y' by $y'+\tau z'$, choosing τ so that the coefficient of $2y'z'$ is $\equiv 0 \pmod{64\Delta_1\Omega_1}$. This does not alter the coefficient of y'^2 and gives $f_1'\sim f_1''\equiv\alpha'x^2+2\beta''y^2+2c'z^2 \pmod{128\Delta_1\Omega_1}$, where $\beta'\beta''\equiv 1+\Delta_1\beta'\gamma' \pmod 8$. Now $H\equiv 4\alpha'\beta''c' \pmod{128\Delta_1\Omega_1}$, i.e., $H/(4\Omega')\equiv\alpha'\beta''c'/\Omega' \pmod{64\Delta_1}$ where $\Omega'=\Omega_1/2$. We have $\alpha'\beta''/\Omega'\not\equiv 0 \pmod{\Delta_1}$ and thus $c'\equiv H/(4\Omega')\equiv 0 \pmod{\Delta_1}$ and the lemma is proved.

6. We use the following abbreviation throughout the remainder of the paper: $g=f/n$ means "the multiples of n represented by f are n multiplied by the integers represented by g."

THEOREM 4. *For any genus G of primitive forms f and Hessian H and any prime factor q of Ω there exists a genus G_1 (or G_2) of primitive forms f_1 (or f_2) of Hessian H/q (or H/q^4) such that the multiples of q (or q^2) represented by a form f of G are q (or q^2) multiplied by the integers represented by a form f_1 (or f_2) of G_1 (or G_2).*

Use the part of the theorem in parentheses if $\Omega\equiv 0 \pmod{q^2}$. Furthermore if $\Omega\not\equiv 0 \pmod{q^2}$, Ω_1, the Ω-factor of f_1, is Ω or Ω/q according as $\Delta\equiv 0 \pmod q$ or not, i.e. according as $H\equiv 0 \pmod{q^3}$ or not. If $\Omega\equiv 0 \pmod{q^2}$, $\Omega_2=\Omega/q^2$. Note that by the notation explained above $f_1=f/q$ and $f_2=f/q^2$.

I. $q=p$ an odd prime. From Lemma 12 of the previous paper we may consider $f\equiv\alpha x^2+\beta\Omega y^2+\gamma\Omega\Delta z^2 \pmod{p^{t+1}}$ where t is the highest power of p in H and $\alpha\beta\gamma\equiv 1 \pmod{p^{t+1}}$. Then $f\equiv 0 \pmod p$ implies $x=px_1$ and, making the substitution in f and dividing by p, we have $f_1\equiv p\alpha x_1^2+\beta\Omega y^2/p+\gamma\Omega\Delta z^2/p \pmod{p^t}$. $f_1=f/p$.

First if $\Omega\not\equiv 0 \pmod{p^2}$, f_1 is primitive since the coefficients are altered only by multiples of p or $1/p$ and $\Omega/p\not\equiv 0 \pmod p$. The Hessian of f_1 is H/p. Take the genus G_1 to be the genus of f_1. Now the progressions (1) of the previous paper associated with f_1 are those obtained by dividing by p all the multiples of p in the progressions (1) associated with f. By the theorem of the previous paper all forms f of genus G have the same progressions (1) associated with them. Thus all forms f_1 obtained by the above process from forms f of G have the same progressions (1) associated with them, and, by the same theorem, are thus of the same genus G_1. Conversely consider any form f_1' of genus G_1 and Hessian H/p. Below we prove the existence of a form f' for which $f'/p=f_1'$ and the quadratic character of the integers prime to p represented by f' is the same as that of the integers prime to p represented by f. This proves that the progressions (1) associated with f' and f are the same and therefore that f and f' are of the same genus.

Take $f\equiv\alpha x^2+\beta\Omega y^2+\gamma\Omega\Delta z^2 \pmod{p^{t+1}}$ to be a form of genus G.

A. $\Omega_1=\Omega=p\Omega'$ and $\Delta_1=\Delta/p\equiv 0 \pmod p$. Then by Lemma 12 of the

previous paper, we take $f_1' \equiv \alpha' x^2 + \beta'\Omega y^2 + \gamma'\Omega\Delta_1 z^2 \pmod{p^t}$. Multiply f_1' by p, replace py by y and have $f' \equiv \alpha' p x^2 + \beta'\Omega' y^2 + \gamma'\Omega\Delta z^2 \pmod{p^{t+1}}$ and $f'/p = f_1'$. Furthermore $(\beta'\Omega' \mid p) = (\alpha \mid p)$, for the multiples of p represented by f_1' and by f_1 must be of the same character since f_1' and f_1 are of the same genus and $f_1'/p \equiv \alpha' p x_1^2 + \beta'\Omega' y^2 + \gamma\Omega'\Delta_1 z^2 \pmod{p^{t-1}}$, while $f_1/p \equiv \alpha x_1^2 + \beta\Omega y_1^2 + \gamma\Omega\Delta z^2/p^2 \pmod{p^{t-1}}$. Thus the conditions on f' required above are satisfied.

B. $\Omega_1 = \Omega = p\Omega'$ and $\Delta_1 = \Delta/p \not\equiv 0 \pmod{p}$. Then, by Lemma 5, we may consider $f_1' \equiv \alpha' x^2 + b' p y^2 + c' p z^2 + 2r' p yz \pmod{p^3}$ with $(b'c' \mid p) = -1$. Thus we can interchange y and z if necessary to make $(b' \mid p) = (\alpha \mid p)$. Then multiply f_1' by p and replace py by y, having $f' \equiv \alpha' p x^2 + b' y^2 + c' p^2 z^2 + 2r' p^2 yz \pmod{p^3}$, where $f'/p = f_1'$ and, as above, is of genus G.

C. $\Omega_1 = \Omega/p$. Then $\Delta_1 = \Delta p \not\equiv 0 \pmod{p^2}$ for $\Omega_1 F_1 \equiv \beta\gamma\Omega^2\Delta x^2/p^2 + \alpha\gamma\Omega\Delta y^2 + \alpha\beta\Omega z^2 \pmod{p}$ and $\Delta \equiv 0 \pmod{p}$ contradicts $\Omega_1 \not\equiv 0 \pmod{p}$. Thus we take $f_1' \equiv \alpha' x^2 + \beta'\Omega_1 y^2 + \gamma'\Omega\Delta z^2 \pmod{p^3}$. Multiply through by p, replace pz by z, and have $f' \equiv p\alpha' x^2 + \beta'\Omega y^2 + \gamma'\Delta\Omega_1 z^2 \pmod{p^3}$. Now $f'/p = f_1'$ and $(\gamma'\Delta\Omega_1 \mid p) = (\alpha \mid p)$ for f_1' and f_1, being of the same genus, have the same progressions (1) associated with them, and, by Lemmas 4 and 5 of the previous paper, $(-\beta\gamma\Delta \mid p) = (-\alpha'\beta'\Omega_1 \mid p)$, i.e. $(\alpha\Delta \mid p) = (\gamma'\Omega_1 \mid p)$.

Second, if $\Omega \equiv 0 \pmod{p^2}$, $f_1/p = f_2 \equiv \alpha x_1^2 + \beta\Omega y^2/p^2 + \gamma\Delta\Omega z^2/p^2 \pmod{p^{t-1}}$, where the Hessian of f_2 is H/p^4 and $\Omega_2 = \Omega/p^2$. Now $f/p^2 = f_2$ and, as above, we define G_2 to be the genus of f_2. All forms f_2 so obtained from forms of genus G are of the same genus G_2. Conversely any f_2' of genus G_2 may be taken $f_2' \equiv \alpha' x^2 + \beta'\Omega y^2/p^2 + \gamma'\Omega\Delta z^2/p^2 \pmod{p^{t-1}}$ where $(\alpha' \mid p) = (\alpha \mid p)$, for f_2' represents some integer α' prime to $2H$ for which $(\alpha' \mid p) = (\alpha \mid p)$, thus represents some such integer primitively, and, by Lemma 1 of this paper, is equivalent to a form with leading coefficient α'. Smith's process (ibid., pp. 460–462) by which f' is reduced to the form above leaves such a leading coefficient unaltered as is pointed out in the corollary to Lemma 12 of the previous paper. Now multiply the above form of f_2' by p and replace px by x having $f' \equiv \alpha' x^2 + \beta'\Omega y^2 + \gamma'\Omega z^2 \pmod{p^{t+1}}$, where $f'/p^2 = f_2'$. f' represents no multiple of $p \not\equiv 0 \pmod{p^2}$, $(\alpha' \mid p) = (\alpha \mid p)$, and therefore the progressions (1) of the previous paper associated with f' and f are the same and f' and f are of the same genus.

II. $q = 2$. Since $\Omega \equiv 0 \pmod{2}$ we know f is properly primitive. If further F is properly primitive we apply Lemma 12 of the previous paper as for $q = p$ and have $f \equiv \alpha x^2 + \beta\Omega y^2 + \gamma\Omega\Delta z^2 \pmod{2^{3+t}}$, $f_1 \equiv 2\alpha x_1^2 + \Omega\beta y^2/2 + \gamma\Delta z^2\Omega/2 \pmod{2^{2+t}}$ and $f/2 = f_1$. F is properly primitive if $\Delta \equiv 0 \pmod{2}$.

First, if $\Omega \equiv 2 \pmod{4}$ and F is properly primitive, 2 may be substituted for p in the corresponding discussion for $q = p$ if "the quadratic character of

an odd integer mod 2" we interpret to mean the congruence satisfied mod 8, i.e. a_1 and a_2 are of the "same quadratic character with respect to 2" if $a_1 \equiv a_2 \pmod 8$.

A. $\Omega_1 = \Omega = 2\Omega'$ and $\Delta_1 = \Delta/2 \equiv 0 \pmod 8$. Using Lemma 12 of the previous paper we may take $f_1' \equiv \alpha' x^2 + 2\beta'\Omega' y^2 + 2\gamma'\Omega'\Delta_1 z^2 \pmod{8\Delta_1}$. Multiply by 2, replace $2y$ by y, and have $f' \equiv 2\alpha' x^2 + \beta'\Omega' y^2 + 4\gamma'\Omega'\Delta_1 z^2 \pmod{16\Delta_1}$ and $f'/2 = f_1'$. Now the evens represented by f_1' and by f_1 satisfy the same congruences mod 16. Thus $f_1'/2 \equiv 2\alpha' x^2 + \beta'\Omega' y^2 + \gamma'\Omega'\Delta_1 z^2 \pmod{4\Delta_1}$ and $f_1/2 \equiv \alpha x_1^2 + 2\beta\Omega' y^2 + \gamma\Omega'\Delta_1 z^2 \pmod{4\Delta_1}$ implies that $\beta'\Omega'$ and $\beta'\Omega' + 2\alpha'$ are congruent $\pmod 8$ in some order to α and $\alpha + 2\beta\Omega'$, which proves that the odd integers represented by f and f' satisfy the same congruences mod 8.

B. $\Omega_1 = \Omega = 2\Omega'$ and $\Delta_1 = \Delta/2 \equiv 4 \pmod 8$. Using Lemma 6 we may take $f_1' \equiv \alpha' x^2 + 2\beta' y^2 + 2\Delta_1\gamma' z^2 \pmod{128\Delta_1}$ and also $f_1' \sim f_1'' \equiv \alpha' x^2 + 2\beta'' y^2 + 2\Delta_1 \gamma'' z^2 \pmod{128\Delta_1}$ where $\beta'\beta'' \equiv 5 \pmod 8$. Now f_1' represents primitively an integer $\equiv \beta\Omega' \pmod 8$, since f_1 does. Then we may take $\alpha' \equiv \beta\Omega' \pmod 8$. Noting that $f_1' \equiv \alpha' x^2 + 2\beta' y^2 \pmod 8$ and $f_1 \equiv 2\alpha x_1^2 + \beta\Omega' y^2 \pmod 8$ we see that $\alpha \equiv \beta' \pmod 4$. Replacing f' by f'' if necessary, we have $f_1' \equiv \alpha' x^2 + 2by^2 + 2\Delta_1\gamma' z^2 \pmod{128\Delta_1}$ with $b \equiv \alpha \pmod 8$ and $\alpha' \equiv \beta\Omega' \pmod 8$. Multiplying f_1' by 2 and replacing $2y$ by y we have $f' \equiv 2\alpha' x^2 + by^2 + 4\Delta_1\gamma' z^2 \pmod{128\Delta}$, $f'/2 = f_1'$, and f' is of genus G.

C. $\Omega_1 = \Omega = 2\Omega'$ and $\Delta_1 = \Delta/2 \equiv 2 \pmod 4$. Let $\Delta/4 = \Delta_1/2 = \Delta'$. Here

$$\Delta' f_1' \equiv \alpha' x^2 + 2\beta' y^2 + 4\gamma' z^2 \pmod{64},$$
$$\Delta' f \equiv \alpha x^2 + 2\beta y^2 + 8\gamma z^2 \pmod{128},$$

$\alpha'\beta'\gamma' \equiv 1 \pmod 8$ since $8\alpha'\beta'\gamma' \equiv \Delta' H_1 \equiv 8 \pmod{64}$. Similarly $\alpha\beta\gamma \equiv 1 \pmod 8$. Now, since $f_1' \equiv$ all odds $\pmod 8$ is solvable, we may consider $\alpha' \equiv \beta \pmod 4$. Since, by Lemma 6, β' may be replaced by $\beta'' \equiv 3\beta' \pmod 4$ and $\alpha'\beta''\gamma'' \equiv 1 \pmod 8$, we have $\gamma'' \equiv 3\gamma' \pmod 4$. We therefore can make $\gamma' \equiv \alpha'$ or $3\alpha' \pmod 4$ according as $\beta \equiv \gamma$ or $3\gamma \pmod 4$. Now the multiples of 4 represented by $\Delta' f_1'$ and by $\Delta' f_1$ satisfy the same congruences (mod 32), i.e. the evens represented by $\Delta' f_1'/2 \equiv 2\alpha' x^2 + \beta' y^2 + 2\gamma' z^2 \pmod{16}$ and by $\Delta' f_1/2 \equiv \alpha x^2 + 2\beta y^2 + 2\gamma z^2 \pmod{16}$ satisfy the same congruences mod 16. Thus, by the corollaries to Lemmas 10 and 11 of the previous paper, $\alpha' + \gamma' \equiv 2$ or $4 \pmod 8$ if and only if $\beta + \gamma \equiv 2$ or $4 \pmod 8$ in some order. If $\beta + \gamma \equiv 2 \pmod 8$, we see that $\beta \equiv \gamma \pmod 4$, and, by the above choice of γ', $\alpha' + \gamma' \equiv 2 \pmod 4$ and therefore $\alpha' + \gamma' \equiv 2 \pmod 8$. Similarly for all cases, the above choice of γ' is seen to make $\alpha' + \gamma' \equiv \beta + \gamma \pmod 8$.

Now if $\beta + \gamma \equiv 2a \pmod 8$, $\alpha \equiv 2a\beta - 1 \pmod 8$ and the odds represented by $\Delta' f$ are exclusively $\equiv 2a\beta - 1, 2\beta(a+1) - 1 \pmod 8$. Multiply $\Delta' f_1'$ by 2, replace $2y$ by y, and have $\Delta' f' \equiv 2\alpha' x^2 + \beta' y^2 + 8\gamma' z^2 \pmod{128}$. $\Delta' f'/2$

$=\Delta'/f_1'$, and the odds represented by $\Delta'f'$ are exclusively $\equiv\beta'$ and $\beta'+2\alpha'$ (mod 8). $\beta'\equiv 2a\alpha'-1\equiv 2a\beta-1$ (mod 8) and $\beta'+2\alpha'\equiv 2\beta(a+1)-1$ (mod 8) show that the odds represented by $\Delta'f'$ and by $\Delta'f$ satisfy the same congruences (mod 8) and thus f' and f are of the same genus.

D. $\Omega_1=\Omega=2\Omega'$ and $\Delta_1=\Delta/2=\Delta'\equiv 1$ (mod 2). Now f_1 represents $\beta\Omega'$ and $\beta\Omega'+2\alpha\equiv 3\beta\Omega'$ (mod 4) and thus from the form of f_1 in Lemma 14 of the previous paper we see this can happen only if F_1 is properly primitive. Furthermore for any form f_1' of genus G_1, F_1' is properly primitive since f_1', too, must represent integers $\equiv 1$ and 3 (mod 4). Thus only $f_1'\equiv\alpha'x^2+2\Omega'\beta'y^2+2\Omega'\Delta'\gamma'z^2$ (mod 8) need be considered. Multiply by 2, replace $2y$ by y and $f'\equiv 2\alpha'x^2+\Omega'\beta'y^2+4\Omega'\Delta'z^2$ (mod 16). f' and $f\equiv 1$, 3, 5 or 7 (mod 8) are solvable, $f'/2=f_1'$, and thus f' and f are of the same genus.

E. $\Omega_1=\Omega/2\equiv 1$ (mod 2) and F properly primitive. Then $\Delta_1=2\Delta\equiv 2$ (mod 4) as in the case IC. Here f_1 represents an odd and thus the genus is composed of properly primitive forms. We have

$$\Delta f=\Delta'f\equiv\alpha x^2+2\beta y^2+2\gamma z^2 \pmod{16};$$

$$\Delta f_1\equiv 2\alpha x_1^2+\beta y^2+\gamma z^2,\quad \alpha\beta\gamma\equiv 1 \pmod 8;$$

$$\Delta f_1'\equiv\alpha'x^2+\beta'y^2+2\gamma'z^2,\quad \alpha'\beta'\gamma'\equiv 1 \pmod 8.$$

Since the odds represented by Δf_1 and $\Delta f_1'$ satisfy the same congruences mod 8, we have, from Lemmas 10 and 11 of the previous paper, $\beta+\gamma\equiv 2$ or 4 (mod 8) if and only if $\alpha'+\beta'\equiv 2$ or 4 (mod 8) in some order. Multiply $\Delta f_1'$ by 2, replace $2z$ by z, and have $\Delta f'\equiv 2\alpha'x^2+2\beta'y^2+\gamma'z^2$ (mod 16), and, by virtue of the corollaries to Lemmas 10 and 11 of the previous paper, the odds represented by $\Delta f'$ and Δf satisfy the same congruences (mod 8). $\Delta f'/2=\Delta f_1'$ and thus f' and f are of the genus G.

F. $\Omega_1=\Omega/2\equiv 1$ (mod 2) and F improperly primitive. Then, as above, $\Delta_1=2\Delta\equiv 2$ (mod 4). Using Lemma 14 of the previous paper we take a form f of genus G as follows: $f\equiv ax^2+4by^2+4cz^2+4ryz$ (mod 16) with $br\equiv 1$ (mod 2). $f/2=f_1\equiv 2ax_1^2+2by^2+2cz^2+2ryz$ (mod 8) and the genus G_1 of f_1 is composed of improperly primitive forms. Now from the proof of Lemma 13 of the previous paper we may take $f_1'\equiv 2\alpha x^2+2b''xy+2\alpha'y^2+2\beta z^2$ (mod 16), where $\alpha b''$ is odd and $\beta=\gamma\Omega_1\Delta_1/2\equiv 1$ (mod 2) since γ is an odd integer. Multiply f_1' by 2, replace $2z$ by z and have $f'\equiv\beta z^2+4\alpha x^2+4b''xy+4\alpha'y^2$ (mod 16), $f'/2=f_1'$. Since f' represents only integers $\equiv\beta$ and $\beta+4\alpha$ (mod 8) and multiples of 4, it remains to prove $\beta\equiv a$ (mod 4). This is done as follows: From the form of f', $H\equiv 4\beta(4\alpha\alpha'-b''^2)\equiv 12\beta$ (mod 16), and likewise for f, $H\equiv 4a(4bc-r^2)\equiv 12a$ (mod 16). Thus $a\equiv\beta$ (mod 4).

Second: $\Omega\equiv 0$ (mod 4). Then by Lemma 14 of the previous paper we may consider any form f of genus G to be of the form

$$f \equiv ax^2 + 2^{t_1}by^2 + 2^{t_1}cz^2 + 2^{t_1+1}ryz \pmod{2^{t_1+3}}$$

where $\Omega \equiv 2^{t_1} \pmod{2^{t_1+1}}$, b is odd if F is properly primitive and $b \equiv c \equiv r+1 \equiv 0 \pmod 2$ if F is improperly primitive. Now $t_1 \geqq 2$ and $f \equiv 0 \pmod 2$ implies $x = 2x_1$ and $f_2 = f/4 \equiv ax_1^2 + 2^{t_1-2}by^2 + 2^{t_1-2}cz^2 + 2^{t_1-1}ryz \pmod{2^{t_1+1}}$. $H_2 = H/16$, $\Omega_2 = \Omega/4$ and $\Delta_2 = \Delta$. Now F_2 is improperly primitive if and only if $b \equiv c \equiv r+1 \equiv 0 \pmod 2$, i.e., if and only if F is improperly primitive. Define the genus G_2 to be the genus of f_2. Take any form f_2' of genus G_2 and see by Lemma 14 of the previous paper that we may take

$$f_2 \equiv a'x^2 + 2^{t_1-2}b'y^2 + 2^{t_1-2}c'z^2 + 2^{t_1-1}r'yz \pmod{2^{t_1+1}},$$

where $a' \equiv a \pmod 8$, b' is odd if F_2, and therefore F, is properly primitive, and $b' \equiv c' \equiv r'+1 \equiv 0 \pmod 2$ if F_2, and therefore F, is improperly primitive. Multiply by 4, replace $2x$ by x, and have

$$f' \equiv a'x^2 + 2^{t_1}b'y^2 + 2^{t_1}c'z^2 + 2^{t_1+1}r'yz \pmod{2^{t_1+3}}.$$

$f'/4 = f_2'$, f' represents no $4n+2$, and is of the same genus as f.

7. We prove the following theorem:

THEOREM 5. *The integers represented by no primitive form f of a given genus G are exclusively those occurring in progressions* (1), *of the previous paper, associated with f.*

Consider any integer a included in none of the progressions (1) associated with f. It follows from the theorem of the previous paper that $f \equiv a \pmod{8h\mu}$ is solvable. If a is prime to Ω, Theorem 3a applies to prove the theorem above. Otherwise we proceed as follows.

First let q be a prime factor of a which, when squared, divides Ω. Then $a \equiv 0 \pmod{q^2}$ since reference to the proofs of the previous paper shows $\Omega \equiv 0 \pmod{q^2}$ implies that qa occurs in progressions (1). Then there exists by Theorem 4 a form f_1 with $\Omega_1 = \Omega/q^2$ and $H_1 = H/q^4$ such that $f_1 = f/q^2$ and thus $f_1 \equiv a/q^2 \pmod{8h\mu/q^2}$. If $a/q^2 \equiv 0 \pmod q$ and $\Omega_1 \equiv 0 \pmod{q^2}$, we have $a/q^2 \equiv 0 \pmod{q^2}$ and the process may be repeated until, after r times, $f_r \equiv a/q^{2r} \pmod{8h\mu/q^{2r}}$ where $\Omega_r = \Omega/q^{2r}$, $f_r = f/q^{2r}$, $H_r = H/q^{4r}$ and either $a/q^{2r} \not\equiv 0 \pmod q$ or $\Omega_r \not\equiv 0 \pmod{q^2}$. If there is another prime factor, q_1, of a which, when squared, divides Ω, it is true that $\Omega_r \equiv 0 \pmod{q_1^2}$ and the above process may be applied to f_r. So this may be continued until we have the case below.

$g \equiv a/\mu_1^2 = a' \pmod{8h\mu/\mu_1^2}$ where $\Omega_g = \Omega/\mu_1^2$, $H_g = H/\mu_1^4$, $g = f/\mu_1^2$ and no prime factor of a' is, when squared, a factor of Ω_g. Let q' be a prime factor of a' dividing Ω_g. Since $\Omega_g \not\equiv 0 \pmod{q'^2}$, there exists a form $g_1 = g/q'$ of $\Omega_{g1} = \Omega_g$ or Ω_g/q' according as $H_g \equiv 0 \pmod{q'^3}$ or not. Then $g_1 \equiv a'/q'$

(mod $8h\mu/\{\mu_1^2 q'\}$) is solvable. If $H_g \equiv 0 \pmod{q'^3}$ and $a'/q' \equiv \Omega_{g1} \equiv 0$ (mod q'), we repeat the process until after t times we have $g_t = g/q'^t$ where $\Omega_{gt} = \Omega_g/q'^t$ or $\Omega_g/(q')^{t-1}$, $g_t \equiv a'/q'^t$ (mod $8h\mu/\{\mu_1^2 q'^t\}$) is solvable and either a'/q'^t or Ω_{gt} is prime to q'. Then any other factor q_1' dividing a' and Ω_g divides Ω_{gt} and the above process may be applied to g_t. This may be carried through for every factor q' dividing a' and Ω_g until we have a form $g' = f/\{\mu_1^2\mu_2\}$ where $g' \equiv a/\mu_1^2\mu_2$ (mod $8h\mu/\{\mu_1^2\mu_2\}$), $\Omega_g' = \Omega/(\mu_1^2\mu_2')$, where $\mu_2 \equiv 0$ (mod μ_2'), $H_{g'} = H/(\mu_1^4\mu_2)$ and $a/(\mu_1^2\mu_2)$ is prime to $\Omega_{g'}$.

Then by Theorem 3a there exists a form g'' of the same genus and Hessian as g' which represents $a/(\mu_1^2\mu_2)$. Following through the above process in the reverse order, applying the theorem of the previous paper at every step, we have the existence of a form f' of the same genus and Hessian as f such that $f'/(\mu_1^2\mu_2) = g''$, which proves that f' represents a.

COROLLARY. *A form f of genus G is regular if and only if f represents all the integers represented by every form of G.*

For suppose some integer k is not represented by f but is represented by some form f' of G. Then k occurs in no progression (1) and thus $f \equiv k$ (mod N) is solvable for N arbitrary, every arithmetic progression containing k contains an integer represented by f, and therefore, from the definition of regularity, f is irregular.

Note. Whenever f is the only reduced form of a genus it is regular, but, though most regular forms represent the only class in the genus, this is not always the case. See the following.

8. This section gives three examples.

I. $f = x^2 + 8y^2 + 24z^2$ and $g = x^2 + 2(2y+x)^2 + 6(2z+x)^2 = 8y^2 + 9x^2 + 24z^2 + 8xy + 24xz$ are of the same genus. They are not of the same class since g does not represent 1. Every odd integer n represented by g is represented by f, for if (x_1, y_1, z_1) is a solution of $g = n$, the $\pm$ sign may be so chosen in $2y_1 + x_1 \pm (2z_1 + x_1) = 4Y$ that the equation is solvable for Y and $n = x_1^2 + 8(2z + x_1 \mp Y)^2 + 24Y^2$. The evens represented by f and g are obviously the same. f is regular since f and g represent the only classes in the genus, but g is irregular.

II. The forms $x^2 + 3y^2 + 6z^2$ and $2x^2 + 3y^2 + 3z^2$ are regular, for each represents the only class in the genus. (See the examples of the previous paper.)

III. The forms $f = (1, 1, 18)$ and $g = (2, 2, 5, 0, -2, 0)$ are of the same genus and both are irregular, for f but not g represents 1 and g but not f represents 7.

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ON PARALLEL DISPLACEMENT IN A NON-FINSLER SPACE*†

BY

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Introduction

Preliminary definitions and concepts.‡ In modern geometry any set of objects in a one-to-one reciprocal correspondence with the totality of sets of ordered numbers $x \equiv (x^1, x^2, \cdots, x^n)$ is called an n-dimensional space; and any particular one-to-one correspondence between the objects and a set x is called a coördinate system. If in addition to a space there is given a curvilinear integral, the "arc length," the space is said to be metric. Thus any space can be regarded as the bearer of an indefinite number of metric spaces.

A set of n equations

$$y^i = y^i(x^1, x^2, \cdots, x^n)$$

such that the functions y^i are single-valued for all points x of our n-dimensional space, and which can be solved so as to yield a set of n equations

$$x^\alpha = x^\alpha(y^1, y^2, \cdots, y^n)$$

in which the functions x^α are single-valued, determines a one-to-one correspondence between the sets of numbers x and y. Such a set of equations is said to define a transformation of coördinates.

An object of any sort which is not changed by transformations of coördinates is called an invariant. For example any point is an invariant, and likewise any point function. In this paper we require that the functions which define the transformations involved be analytic and have a non-vanishing Jacobian. Invariants of this restricted group of transformations are known as differential invariants. An invariant may consist of a single function or of a number of functions. In the latter case the individual functions are called the components of the invariant. An invariant is said to be a tensor if its components transform according to equations of the type

* Presented to the Society, June 20, 1929; received by the editors May 8, 1930.

† This paper is a part of a thesis written at the University of Wisconsin under the direction of Professor J. H. Taylor.

‡ This material is taken from Oswald Veblen, *The Invariants of Quadratic Differential Forms*, Cambridge, 1927.

$$\overline{T}^{i_1 i_2 \cdots i_q}_{j_1 j_2 \cdots j_m} = \left|\frac{\partial x}{\partial y}\right|^n \sum_{\alpha,\beta}^{1,n} T^{\alpha_1\alpha_2\cdots\alpha_q}_{\beta_1\beta_2\cdots\beta_m} \frac{\partial y^{i_1}}{\partial x^{\alpha_1}} \cdots \frac{\partial y^{i_q}}{\partial x^{\alpha_q}} \frac{\partial x^{\beta_1}}{\partial y^{j_1}} \cdots \frac{\partial x^{\beta_m}}{\partial y^{j_m}} \cdot$$

The number n is called the weight of the tensor, m its order of covariance, and q its order of contravariance.

Notation. Throughout this paper we shall employ letters without indices to represent sets; and shall use primes to indicate differentiation with respect to the parameter which is used to define the curve in question. Thus x, x', x'', x''' shall designate respectively the sets

$$x^i, \frac{dx^i}{dt}, \frac{d^2x^i}{dt^2}, \frac{d^3x^i}{dt^3} \qquad (i = 1, 2, \cdots, n).$$

Partial derivatives will be denoted by means of subscripts; thus

$$F_{x'^\alpha}(x, x', x'') \equiv \frac{\partial F(x, x', x'')}{\partial x'^\alpha} \cdot$$

In the case of the functions F, f, and H to be introduced below, this notation will be modified by omitting the x from the subscript when the differentiation is with respect to the highest derivatives present, i.e.

$$F_\alpha(x, x', x'') = \frac{\partial F(x, x', x'')}{\partial x''^\alpha} \cdot$$

(We shall prove later that the quantities $F_{\alpha\beta\cdots\lambda}$ are the components of a covariant tensor.) Finally, summations are to be understood, as in tensor analysis, when repeated indices occur. Thus, for example, $x'^\alpha F_\alpha$ shall designate the sum

$$x'^1F_1 + x'^2F_2 + \cdots + x'^nF_n.$$

Riemannian geometry. If the metric of the space is given by the integral of the square root of a quadratic differential form,

$$g_{\alpha\beta}x'^\alpha x'^\beta,$$

the space is said to be Riemannian; and the theory of such forms is known as Riemannian geometry. The coefficients $g_{\alpha\beta}$ are the components of a covariant tensor, while the quantities $g^{\alpha\beta}$ (the matrix $(g^{\alpha\beta})$ is the inverse of the matrix $(g_{\alpha\beta})$) are the components of a contravariant tensor. These two tensors are called, respectively, the fundamental covariant tensor, and the fundamental contravariant tensor. An invariant of great importance $[\alpha\beta, \gamma]$ is derived from $g_{\alpha\beta}$ in accordance with the formula

$$[\alpha\beta, \gamma] \equiv \frac{1}{2}\left[\frac{\partial g_{\alpha\gamma}}{\partial x^\beta} + \frac{\partial g_{\beta\gamma}}{\partial x^\alpha} - \frac{\partial g_{\alpha\beta}}{\partial x^\gamma}\right].$$

The law of transformation of $[\alpha\beta, \gamma]$ is expressed by the equation

$$\overline{[ij, k]} = [\alpha\beta, \gamma]\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}\frac{\partial x^\gamma}{\partial y^k} + g_{\alpha\beta}\frac{\partial x^\alpha}{\partial y^k}\frac{\partial^2 x^\beta}{\partial y^i \partial y^j}.$$

On multiplying the two members of this relation by the corresponding members of the equality

$$\bar{g}^{lk}\frac{\partial x^\delta}{\partial y^l} = g^{\delta\gamma}\frac{\partial y^k}{\partial x^\gamma}$$

which expresses the contravariance of $g^{\delta\gamma}$, we obtain

$$\bar{g}^{lk}\overline{[ij, k]}\frac{\partial x^\delta}{\partial y^l} = g^{\delta\gamma}[\alpha\beta, \gamma]\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j} + \frac{\partial^2 x^\delta}{\partial y^i \partial y^j}.$$

This last relationship is known as the fundamental affine connection, while the quantities $[\alpha\beta, \gamma]$ and

$$g^{\delta\gamma}[\alpha\beta, \gamma] \equiv \left\{ {}_\alpha{}^\delta{}_\beta \right\}$$

are called, respectively, the Christoffel symbols of the first and second kind.

The fundamental affine connection enables one to derive tensors from tensors by differentiation. Consider, for example, a vector V defined along a given curve by means of the parameter t. Let us differentiate the equation for the transformation of V, i.e., $V^\delta = \overline{V}^i \partial x^\delta / \partial y^i$. We obtain thereby

$$V'^\delta = \overline{V}'^i\frac{\partial x^\delta}{\partial y^i} + \overline{V}^i\frac{\partial^2 x^\delta}{\partial y^i \partial y^j}y'^j,$$

and in this, the second derivatives may be eliminated by means of the fundamental affine connection; thus

$$V'^\delta = \overline{V}'^i\frac{\partial x^\delta}{\partial y^i} + \overline{V}^i y'^j\left[\left\{ \begin{matrix} l \\ i\ j \end{matrix} \right\}\frac{\partial x^\delta}{\partial y^l} - \left\{ \begin{matrix} \delta \\ \alpha\beta \end{matrix} \right\}\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}\right].$$

Transposing the final term, which may be written in the form $V^\alpha x'^\beta \left\{ {}_\alpha{}^\delta{}_\beta \right\}$, we note that

$$V'^\delta + V^\alpha x'^\beta \left\{ \begin{matrix} \delta \\ \alpha\beta \end{matrix} \right\}$$

is a contravariant vector. If the vector derived from V by this differentiation process is zero, then it is said that V "undergoes parallel displacement" or that V "remains parallel." If the tangent vector of a curve satisfies this definition of parallel displacement, then the curve is said to be autoparallel.

It is a theorem of Riemannian geometry that the magnitude of a vector, and the cosine of the angle between vectors, remain unaltered when the vectors undergo parallel displacement. The invariants magnitude of a vector and cosine of the angle between vectors are defined, respectively by the following formulas:

$$(g_{\alpha\beta}\xi^\alpha\xi^\beta)^{1/2}; \quad \frac{g_{\alpha\beta}\xi^\alpha\eta^\beta}{(g_{\alpha\beta}\xi^\alpha\xi^\beta)^{1/2}(g_{\alpha\beta}\eta^\alpha\eta^\beta)^{1/2}}.$$

The geometry of paths.* One generalization of Riemannian geometry has been obtained by replacing the quantities $\{{}_\alpha{}^\gamma{}_\beta\}$ in the affine connection by a set of point functions $\Gamma^\gamma_{\alpha\beta}$. The problem of finding a Riemannian metric such that

$$\begin{Bmatrix} \gamma \\ \alpha\beta \end{Bmatrix} = \Gamma^\gamma_{\alpha\beta}$$

does not always admit of a solution, and so Riemannian geometry is regarded as a special case of the geometry of paths. The system of curves defined by the set of differential equations

$$x''^i + \Gamma^i_{jk}x'^jx'^k = 0 \qquad (i = 1, 2, \cdots, n)$$

are regarded as autoparallel and are called the paths.

Finsler geometry. A second generalization of Riemannian geometry due to Emmy Noether,† Finsler,‡ and Berwald,§ may be obtained by replacing $(g_{\alpha\beta}x'^\alpha x'^\beta)^{1/2}$ by a more general function $F(x, x')$ of x and x'. The restrictions on F are for the most part those needed to insure the regularity of the problem of minimizing the integral $\int F(x, x')dt$. A fundamental tensor is derived by differentiating $\frac{1}{2}F^2$ with respect to x'^α and x'^β, and the parameter is so selected that F maintains the value one along the path of integration. We point out that the $g_{\alpha\beta}$ of Riemannian geometry may be obtained in this way from $(g_{\alpha\beta}x'^\alpha x'^\beta)^{1/2}$ and that as a consequence of the choice of parameter $\int F^2dt = \int Fdt$. The covariant tensor $F_{\alpha\beta}$ cannot be used as a fundamental tensor since its determinant vanishes (due to the conditions imposed on F). The invariants

$$(f_{\alpha\beta}\xi^\alpha\xi^\beta)^{1/2}; \quad \frac{f_{\alpha\beta}\xi^\alpha\eta^\beta}{(f_{\alpha\beta}\xi^\alpha\xi^\beta)^{1/2}(f_{\alpha\beta}\eta^\alpha\eta^\beta)^{1/2}} \qquad \left(f = \frac{1}{2}F^2\right)$$

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* See Oswald Veblen and L. P. Eisenhart, *The Riemannian geometry and its generalization*, Proceedings of the National Academy of Sciences, vol. 8 (1922), p. 19.

† Emmy Noether, *Invarianten beliebiger Differentialausdrucke*, Göttinger Nachrichten, Mathematisch-Physikalische Klasse, 1918, pp. 37–44.

‡ P. Finsler, *Über Kurven und Flächen in allgemeinen Räumen*, Dissertation, Göttingen, 1918.

§ L. Berwald, *Über Parallelübertragung in Räumen mit allgemeiner Massbestimmung*, Jahrsbericht der Deutschen Mathematiker Vereinigung, vol. 34 (1925).

are called, respectively, the magnitude of ξ, and the cosine of the angle between ξ and η, because of their analogy to the like named Riemannian invariants. J. H. Taylor* and independently J. L. Synge have constructed a differentiation process analogous to the one described in the section on Riemannian geometry. Their process gives rise to a theory of parallelism which has the cardinal properties that the magnitude, and the "direction" of vectors undergoing parallel displacement are invariant.

Purpose of this paper. The principle object of this paper is to develop a differentiation process and a theory of parallelism for a space whose metric is given by a function $F(x, x', x'')$ involving x'' as well as x' and x.

The metric. We shall require that (a), $F(x, x', x'')$ be of class† three, (b), the classical "F one" function‡ associated with the problem of minimizing the integral

$$J = \int_{t_1}^{t_2} F(x, x', x'')dt$$

be different from zero, (c), J be independent of the choice of parameter. Specifically, (a) and (b) are to hold not merely along a certain curve but throughout the region under consideration and for all values of the sets x', x'' excepting $x'^1=0; x'^2=0; \cdots; x'^n=0$; which values shall be excluded; (c) is to hold along all regular curves lying in the region for all choices of t_1 and t_2.

It is worthy of note that this invariance of J is equivalent to the condition that

$$F(x, x', x'') \equiv I(x, x', x'')H(x, x'),$$

where I is a function that is invariant in functional form under a change of parameter and H is homogeneous of degree plus one in x'. Since the invariance is supposed to hold for all values of t_1 and t_2 within certain limits, it is equivalent to the invariance of $F(x, x', x'')dt$,§ that is,

$$F(x, x', x'')dt = F\left(X(\tau), \frac{dX(\tau)}{d\tau}, \frac{d^2X(\tau)}{d\tau^2}\right) d\tau$$

where

$$t = t(\tau), \tau = \tau(t); \qquad dt = \frac{dt(\tau)}{d\tau}d\tau$$

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* J. H. Taylor, *A generalization of Levi-Civita's parallelism and the Frenet formulas*, these Transactions, vol. 27 (1925).

† Oscar Bolza, *Vorlesungen über Variationsrechnung*, 1909, p. 13.

‡ Oscar Bolza, loc. cit., p. 196.

§ See Oscar Bolza, loc. cit., p. 193.

is the parameter transformation, $t(\tau)$ being an increasing function of class C'', and $X(\tau)=x(t(\tau))$. Evidently F will have the desired invariance property if it is of the above form, and conversely, if $F(x, x', x'')dt$ is invariant, then

$$F(x, x', x'')/H(x, x'), \quad H \neq 0, \quad \text{and homogeneous,}$$

is invariant in functional form under a parameter transformation and $F=(F/H)H$. Because of this fact we shall adopt the view that our space was generated from a Finsler space $H(x, x')$ by a "warping" such that the length of each curve element Hdt of the original space is changed to $IHdt$. Accordingly, we shall consider that the metric properties of F depend upon the Finsler space from which F originated.

Zermelo has shown that the independence of J upon the parameter implies the following identities in x, x', x'':

$$x'^{\alpha}F_{\alpha} \equiv 0, \tag{1}$$

$$x'^{\alpha}F_{x'^{\alpha}} + 2x''^{\alpha}F_{\alpha} \equiv F \qquad \text{(the range of } \alpha \text{ is 1 to } n). \tag{2}$$

Evidently, (1) implies

$$x'^{\alpha}F_{\alpha\beta} \equiv 0, \tag{3}$$

and it follows that the determinant $|F_{\alpha\beta}|$ vanishes. We shall make it a part of our hypothesis on F that the rank of this determinant be $n-1$.

For use in examining a certain determinant which will appear presently, we insert here a few miscellaneous observations. As a consequence of equation (3) and the rank of $|F_{\alpha\beta}|$ the cofactors of the latter satisfy the following relations:

$$\frac{x'^1}{F^{\beta 1}} = \frac{x'^2}{F^{\beta 2}} = \cdots = \frac{x'^n}{F^{\beta n}}.$$

Noting that the quantities $F^{\alpha\beta}$ are symmetric in their indices, these equalities are seen to be expressible in the form

$$\frac{x'^{\alpha}x'^{\beta}}{F^{\alpha\beta}} = \frac{x'^{\gamma}x'^{\delta}}{F^{\gamma\delta}}, \tag{4}$$

where $\alpha, \beta, \gamma, \delta$ may each be any number of the set $1, 2, \cdots, n$ and no summation is to be understood. The reciprocal of the common value of the members of (4) is the F one function of our problem.

The fundamental tensor. It will be recalled that the fundamental covariant tensor of Finsler geometry was constructed by differentiating $\frac{1}{2}F^2$

with respect to x'. This may be looked upon as a procedure of differentiating, with respect to the highest derivatives present, a new function f having specific properties which make it equivalent, in some respects, to the original, F. The most essential of these properties are the following: (a), the determinant $|f_{\alpha\beta}|$ must be different from zero (this implies that f be dependent on the parameter); (b), for a certain choice of parameter

$$\int_{t_1}^{t_2} f dt = \int_{t_1}^{t_2} F dt.$$

We adopt the method outlined above and define our new function $f(x, x', x'')$ as follows:

$$f(x, x', x'') = \{H(x, x')\}^3 F(x, x', x'') + \tfrac{1}{2}\{H'(x, x')\}^2.*$$

The function H is to be of class three, homogeneous of degree plus one in x' and non-vanishing along all regular curves of our n-space. With these restrictions we may so select the parameter that H will maintain the value unity along the curve in question. For such a parameter it is evident that

$$\int_{t_1}^{t_2} f(x, x', x'')dt = \int_{t_1}^{t_2} F(x, x', x'')dt.$$

Furthermore the determinant of the components of the fundamental tensor

$$f_{\alpha\beta} \equiv H^3 F_{\alpha\beta} + H_\alpha H_\beta$$

is different from zero along all regular curves, since

$$\begin{aligned} |f_{\alpha\beta}| &= \begin{vmatrix} H_1H_1 + F_{11} \cdots H_1H_n + F_{1n} \\ H_nH_1 + F_{n1} \cdots H_nH_n + F_{nn} \end{vmatrix} = \begin{vmatrix} H_1H_1 & F_{12} \cdots F_{1n} \\ H_nH_1 & F_{n2} \cdots F_{nn} \end{vmatrix} \\ &+ \begin{vmatrix} F_{11} & H_1H_2 \cdots F_{1n} \\ F_{n1} & H_nH_2 \cdots F_{nn} \end{vmatrix} + \cdots + \begin{vmatrix} F_{11} & F_{12} \cdots F_{1n} \\ F_{n1} & F_{n2} \cdots F_{nn} \end{vmatrix} \\ &= H_\alpha H_\beta F^{\alpha\beta} = F_1' x'^\alpha x'^\beta H_\alpha H_\beta = F_1' \neq 0. \end{aligned}$$

(Here F_1' represents the F one function, t is the Finsler arc length and $x'^\alpha H_\alpha = H$ since H is homogeneous of degree $+1$ in x'.)

Invariants of $F(x, x', x'')$ and $H(x, x')$. We shall consider $F(x, x', x'')$ as an absolute scalar invariant (that is a tensor of weight and order zero)

* Although we find it convenient to select the parameter in a special way it is desirable to have the quantities $f_{\alpha\beta}$ independent of the parameter and it is for this reason that we introduce the factor H^3. One readily verifies that $H^4 F_{\alpha\beta}$ satisfies the set of necessary and sufficient conditions (1), (2) and so we conclude that $H^3 F_{\alpha\beta}$ has the desired property. Also, we note that H_α is homogeneous of degree zero in x'.

and accordingly transform it as below by means of the twice extended point transformation

$$x^\gamma = x^\gamma(y^1, y^2, \cdots, y^n); \qquad y^k = y^k(x^1, x^2, \cdots, x^n);$$

$$x'^\gamma = \frac{\partial x^\gamma}{\partial y^k} y'^k; \qquad x''^\gamma = \frac{\partial^2 x^\gamma}{\partial y^j \partial y^k} y'^j y'^k + \frac{\partial x^\gamma}{\partial y^k} y''^k.$$

(We note in passing that the x'^γ are the components of a tensor while the x''^γ are not.) Thus, the law of transformation of F is expressed by the equation

$$\overline{F}(y, y', y'') = F\left(x(y), \frac{\partial x}{\partial y^a} y'^a, \frac{\partial x}{\partial y^a} y''^a + \frac{\partial^2 x}{\partial y^a \partial y^b} y'^a y'^b\right).$$

Differentiating this equality with respect to y''^i, we have

$$\overline{F}_i = F_\alpha \frac{\partial x^\alpha}{\partial y^i} \cdot$$

Hence we see that the qth order partial derivatives $F_{\alpha_1 \cdots \alpha_q}$ of a scalar $F(x, x', x'')$ are the components of a tensor covariant of order q, and it is because of this that we have represented these derivatives in a special way. A moment's consideration of the extended point transformation will show that the above statement is true of scalars containing any order derivatives. Differentiating the equation of transformation of F, we have

$$\overline{F}_{y'^i} = F_{x'^\alpha} \frac{\partial x^\alpha}{\partial y^i} + 2F_\alpha \frac{\partial^2 x^\alpha}{\partial y^i \partial y^j} y'^j,$$

$$\overline{F}_i' = F_\alpha' \frac{\partial x^\alpha}{\partial y^i} + F_\alpha \frac{\partial^2 x^\alpha}{\partial y^i \partial y^j} y'^j,$$

and we see that

$$\overline{F}_{y'^i} - 2\overline{F}_i' = (F_{x'^\alpha} - 2F_\alpha') \frac{\partial x^\alpha}{\partial y^i},$$

and hence the quantities $(F_{x'^\alpha} - 2F_\alpha')$ are the components of a covariant tensor. In some respects this tensor is the analogue of H_α. For if the integral $\int H(x, x')dt$ is to be independent of the parameter, the following identity must hold:

$$x'^\alpha H_\alpha = H.^*$$

* Oscar Bolza, loc. cit., p. 193.

The corresponding conditions for $F(x, x', x'')$ are

$$x'^\alpha F_\alpha = 0; \quad x'^\alpha F_{x'^\alpha} + 2x''^\alpha F_\alpha = F,$$

and since, by the first of these relations, $x''^\alpha F_\alpha = -x'^\alpha F'_\alpha$, the last relation may be written in the form

$$x'^\alpha\{F_{x'^\alpha} - 2F'_\alpha\} = F.$$

THEOREM. *The left members of the Euler equations associated with the integral*

$$\int_{t_1}^{t_2} F(x, x', x'')dt$$

are the components of a covariant tensor of order one.

To prove this we differentiate the equation of transformation of F with respect to y^i and the equations of transformation of $\overline{F}'_i$ and $\overline{F}_{y'^i} - 2\overline{F}'_i$, with respect to t, and combine; thus

$$\overline{F}_{y^i} = F_{x^\alpha}\frac{\partial x^\alpha}{\partial y^i} + F_{x'^\alpha}\frac{\partial^2 x^\alpha}{\partial y^i \partial y^j} y'^j + F_\alpha\cdot\left(\frac{\partial^2 x^\alpha}{\partial y^i \partial y^j} y''^j + \frac{\partial^3 x^\alpha}{\partial y^i \partial y^j \partial y^k} y'^j y'^k\right),$$

$$\overline{F}''_i = F''_\alpha\frac{\partial x^\alpha}{\partial y^i} + 2F'_\alpha\frac{\partial^2 x^\alpha}{\partial y^i \partial y^j} y'^j + F_\alpha\cdot\left(\frac{\partial^2 x^\alpha}{\partial y^i \partial y^j} y''^j + \frac{\partial^3 x^\alpha}{\partial y^i \partial y^j \partial y^k} y'^j y'^k\right)$$

$$-(\overline{F}'_{y'^i} - 2\overline{F}''_i) = -(F'_{x'^\alpha} - 2F''_\alpha)\frac{\partial x^\alpha}{\partial y^i} - (F_{x'^\alpha} - 2F'_\alpha)\frac{\partial^2 x^\alpha}{\partial y^i \partial y^j} y'^j,$$

$$\overline{F}_{y^i} - \overline{F}'_{y'^i} + \overline{F}''_i = (F_{x^\alpha} - F'_{x'^\alpha} + F''_\alpha)\frac{\partial x^\alpha}{\partial y^i}\cdot$$

It is a theorem of the calculus of variations that this tensor contracted with x' is zero.

If ξ and η are two vectors defined along a regular curve C, we shall call the invariants

$$(\xi^\alpha \xi^\beta f_{\alpha\beta})^{1/2}, \quad \frac{\xi^\alpha \eta^\beta f_{\alpha\beta}}{(\xi^\alpha \xi^\beta f_{\alpha\beta})^{1/2}(\eta^\alpha \eta^\beta f_{\alpha\beta})^{1/2}}$$

(where the arguments of $f_{\alpha\beta}$ are taken along C) the θ magnitude of ξ and the θ cosine of the angle between ξ and η, with respect to the given curve. Likewise,

$$\xi^\alpha \eta^\beta f_{\alpha\beta}$$

will be referred to as the θ scalar product of ξ and η with respect to C. The similar quantities obtained by replacing $f_{\alpha\beta}$ with $I^2 f_{\alpha\beta}(I \equiv F/H)$ will be

designated τ invariants. In all of the subsequent work the arguments of f will be taken along a regular curve and we shall suppose the parameter t to be so chosen that $H(x, x')$ is unity along the curve in question. As a consequence of this choice of parameter and the homogeneity of H, it follows that the tangent vector x' of the curve is of θ magnitude unity, for

$$x'^{\alpha}x'^{\beta}f_{\alpha\beta} = 1,$$

since

$$x'^{\alpha}F_{\alpha\beta} = 0.$$

Likewise the τ magnitude of $dx/ds = x'/I$ $(s = \int_{t_1}^{t} F dt)$ is unity.

The differentiation process. The basis of our differentiation process is a relationship analogous to the fundamental affine connection of Riemannian geometry. We form this relationship from the equation of transformation of the first "Christoffel symbol" of our space by replacing irregular terms with quantities having a tensor character.

The law of transformation of the fundamental covariant tensor $f_{\alpha\beta} = H^3F_{\alpha\beta} + H_\alpha H_\beta$ is expressed by the equation

$$\bar{f}_{ij}(y, y', y'') = f_{\alpha\beta}\left(x(y), \frac{\partial x}{\partial y^a}y'^a, \frac{\partial x}{\partial y^a}y''^a + \frac{\partial^2 x}{\partial y^a \partial y^b}y'^a y'^b\right)\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}. \tag{5}$$

Differentiating with respect to y^k we obtain

$$\begin{aligned}\bar{f}_{ijy^k} = f_{\alpha\beta}\cdot&\left(\frac{\partial^2 x^\alpha}{\partial y^i \partial y^k}\frac{\partial x^\beta}{\partial y^j} + \frac{\partial x^\alpha}{\partial y^i}\frac{\partial^2 x^\beta}{\partial y^j \partial y^k}\right)\\ &+\left(f_{\alpha\beta x^\gamma}\cdot\frac{\partial x^\gamma}{\partial y^k} + f_{\alpha\beta x'^\gamma}\cdot\frac{\partial x'^\gamma}{\partial y^k} + f_{\alpha\beta\gamma}\cdot\frac{\partial x''^\gamma}{\partial y^k}\right)\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^k},\end{aligned}$$

from which the Christoffel symbol $[\alpha\beta,\gamma]$ is formed in accordance with the defining equation

$$[\alpha\beta, \gamma] \equiv \tfrac{1}{2}(f_{\alpha\gamma x^\beta} + f_{\gamma\beta x^\alpha} - f_{\alpha\beta x^\gamma}).$$

The law of transformation of these symbols is evidently

$$\begin{aligned}[\overline{ij, k}] = f_{\alpha\beta}\cdot&\frac{\partial^2 x^\alpha}{\partial y^i \partial y^j}\frac{\partial x^\beta}{\partial y^k} + [\alpha\beta, \gamma]\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}\frac{\partial x^\gamma}{\partial y^k}\\ &+\frac{1}{2}\left[\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^k}\frac{\partial x'^\gamma}{\partial y^j} + \frac{\partial x^\beta}{\partial y^k}\frac{\partial x^\alpha}{\partial y^j}\frac{\partial x'^\gamma}{\partial y^i} - \frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}\frac{\partial x'^\gamma}{\partial y^k}\right]f_{\alpha\beta x'^\gamma}\\ &+\frac{1}{2}\left[\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^k}\frac{\partial x''^\gamma}{\partial y^j} + \frac{\partial x^\beta}{\partial y^k}\frac{\partial x^\alpha}{\partial y^j}\frac{\partial x''^\gamma}{\partial y^i} - \frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^j}\frac{\partial x''^\gamma}{\partial y^k}\right]f_{\alpha\beta\gamma}.\end{aligned} \tag{6}$$

Multiply both members of (6) by y'^j and sum. The resulting equation with the lines somewhat rearranged is

$$
\begin{aligned}
[\overline{ij,k}]y'^j &= f_{\alpha\beta}\cdot\frac{\partial^2 x^\alpha}{\partial y^i\partial y^j}y'^j\frac{\partial x^\beta}{\partial y^k} + [\alpha\beta,\gamma]x'^\beta\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\gamma}{\partial y^k}\\
&+\frac{1}{2}\left[f_{\alpha\beta\gamma}\cdot\frac{\partial x''^\gamma}{\partial y^j}y'^j + f_{\alpha\beta x'^\gamma}\cdot\frac{\partial x'^\gamma}{\partial y^j}y'^j\right]\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^k}\\
&+\frac{1}{2}\left[\frac{\partial x^\beta}{\partial y^k}x'^\alpha f_{\alpha\beta x'^\gamma}\cdot\frac{\partial x'^\gamma}{\partial y^i} - \frac{\partial x^\alpha}{\partial y^i}x'^\beta f_{\alpha\beta x'^\gamma}\cdot\frac{\partial x'^\gamma}{\partial y^k}\right]\\
&+\frac{1}{2}x'^\alpha f_{\alpha\beta\gamma}\cdot\frac{\partial x^\beta}{\partial y^k}\frac{\partial x''^\gamma}{\partial y^i} - \frac{1}{2}x'^\beta f_{\alpha\beta\gamma}\cdot\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x''^\gamma}{\partial y^k}.
\end{aligned}\tag{7}
$$

Certain of these terms will be replaced, presently, by equivalent expressions. The last two of the eight terms, however, drop out. In fact we have assumed that F satisfies the identity $x'^\alpha F_\alpha = 0$. Differentiating this with respect to x''^β and x''^γ, we obtain $x'^\alpha F_{\alpha\beta\gamma}=0$. But $H(x, x')$ does not contain x'' and so $f_{\alpha\beta\gamma}=H^3F_{\alpha\beta\gamma}$. Hence (7) reads

$$
\begin{aligned}
[\overline{ij,k}]y'^j &= f_{\alpha\beta}\cdot\frac{\partial^2 x^\alpha}{\partial y^i\partial y^j}y'^j\frac{\partial x^\beta}{\partial y^k} + [\alpha\beta,\gamma]x'^\beta\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\gamma}{\partial y^k}\\
&+\frac{1}{2}\left[f_{\alpha\beta\gamma}\cdot\frac{\partial x''^\gamma}{\partial y^j}y'^j + f_{\alpha\beta x'^\gamma}\cdot\frac{\partial x'^\gamma}{\partial y^j}y'^j\right]\frac{\partial x^\alpha}{\partial y^i}\frac{\partial x^\beta}{\partial y^k}\\
&+\frac{1}{2}\left[\frac{\partial x^\beta}{\partial y^k}x'^\alpha f_{\alpha\beta x'^\gamma}\cdot\frac{\partial x'^\gamma}{\partial y^i} - \frac{\partial x^\alpha}{\partial y^i}x'^\beta f_{\alpha\beta x'^\gamma}\cdot\frac{\partial x'^\gamma}{\partial y^k}\right].
\end{aligned}\tag{8}
$$

We shall now proceed to develop formulas which are to be used, as just indicated, to modify the form of the right member of (8). The first six formulas listed below are obtained without difficulty from the extended point transformation and consequently require no explanation:

$$\frac{\partial x'^\gamma}{\partial y^j} = \frac{\partial^2 x^\gamma}{\partial y^i\partial y^j}y'^i;\tag{9}$$

$$x''^\gamma = \frac{\partial^2 x^\gamma}{\partial y^i\partial y^j}y'^iy'^j + \frac{\partial x^\gamma}{\partial y^i}y''^i;\tag{10}$$

$$x'''^\gamma = \frac{\partial^3 x^\gamma}{\partial y^i\partial y^j\partial y^k}y'^iy'^jy'^k + 3\frac{\partial^2 x^\gamma}{\partial y^i\partial y^j}y''^iy'^j + \frac{\partial x^\gamma}{\partial y^i}y'''^i;\tag{11}$$

$$y''^j = \frac{\partial^2 y^j}{\partial x^\alpha\partial x^\beta}x'^\alpha x'^\beta + \frac{\partial y^j}{\partial x^\beta}x''^\beta;\tag{12}$$

$$(13)\qquad \frac{\partial x''^{\gamma}}{\partial y^{j}} = \left[\frac{\partial^3 x^{\gamma}}{\partial y^a \partial y^b \partial y^j} y'^a y'^b + \frac{\partial^2 x^{\gamma}}{\partial y^a \partial y^j} y''^a\right] y'^i.$$

Multiplying (10) by $\partial y^k/\partial x^\gamma$, summing and changing indices yields

$$(14)\qquad y''^j = -\frac{\partial^2 x^\alpha}{\partial y^i \partial y^k} y'^i y'^k \frac{\partial y^j}{\partial x^\alpha} + \frac{\partial y^j}{\partial x^\alpha} x''^\alpha.$$

Changing the index j to k, in equation (5) differentiating with respect to y'^j, and multiplying by y''^j, we obtain the relation

$$(15)\qquad \bar{f}_{iky'^j} \cdot y''^j = \left[f_{\alpha\beta x'^\gamma} \cdot \frac{\partial x^\gamma}{\partial y^j} \cdot y''^j + 2 f_{\alpha\beta\gamma} \cdot \frac{\partial^2 x^\gamma}{\partial y^a \partial y^j} y'^a y''^j\right] \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^j}.$$

Multiplying (14) by $f_{\alpha\beta x'\gamma} \cdot \partial x^\gamma/\partial y^j$ and summing j, we have

$$f_{\alpha\beta x'^\gamma} \cdot \frac{\partial x^\gamma}{\partial y^j} y''^j = \left[-\frac{\partial^2 x^\gamma}{\partial y^i \partial y^k} y'^i y'^k + x''^\gamma\right] f_{\alpha\beta x'^\gamma}.$$

Substituting this in (15) and transposing the second term of the right member of (15) we get the relation

$$(16)\qquad \begin{aligned} -2 f_{\alpha\beta\gamma} \cdot \frac{\partial^2 x^\gamma}{\partial y^a \partial y^j} \cdot y'^a y''^j \cdot \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^k} = \left[-\frac{\partial^2 x^\gamma}{\partial y^a \partial y^j} y'^a y'^j + x''^\gamma\right] f_{\alpha\beta x'^\gamma} \\ \cdot \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^k} - \bar{f}_{iky'^j} \cdot y''^j. \end{aligned}$$

The addition of the identity

$$2\frac{\partial^2 x^\gamma}{\partial y^a \partial y^j} y''^a y'^j - 2\frac{\partial^2 x^\gamma}{\partial y^a \partial y^j} y''^a y'^j = 0$$

to (13) and the multiplication of the result by $\frac{1}{2} f_{\alpha\beta\gamma}(\partial x^\alpha/\partial y^i)(\partial x^\beta/\partial y^k)$ yields the equality

$$(17)\qquad \begin{aligned} \frac{1}{2}\left\{ f_{\alpha\beta\gamma} \cdot \frac{\partial x''^\gamma}{\partial y^j} y'^j = f_{\alpha\beta\gamma}\left[\frac{\partial^3 x^\gamma}{\partial y^a \partial y^b \partial y^j} y'^a y'^b y'^j + 3\frac{\partial^2 x^\gamma}{\partial y^a \partial y^j} y''^a y'^j\right]\right. \\ \left. - 2 f_{\alpha\beta\gamma} \frac{\partial^2 x^\gamma}{\partial y^a \partial y^j} y''^a y'^j \right\} \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^k}. \end{aligned}$$

By adding

$$\left\{\frac{1}{2} f_{\alpha\beta x'^\gamma} \frac{\partial x'^\gamma}{\partial y^j} y'^j = \frac{1}{2} f_{\alpha\beta x'^\gamma} \frac{\partial x'^\gamma}{\partial y^j} y'^j\right\} \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^k},$$

and substituting by means of (11) and (16), the relation (17) may be put into the symmetric form

$$
\begin{aligned}
(18)\quad & \left[\frac{1}{2} f_{\alpha\beta\gamma} \frac{\partial x''^{\gamma}}{\partial y^j} y'^j + \frac{1}{2} f_{\alpha\beta x'^\gamma} \frac{\partial x'^{\gamma}}{\partial y^j} y'^j \right] \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^k} \\
& = \frac{1}{2} f_{\alpha\beta\gamma} \left[x'''^{\gamma} - y'''^{a} \frac{\partial x^\gamma}{\partial y^a} \right] \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^k} + \frac{1}{2} \left[- \frac{\partial^2 x^\gamma}{\partial y^a \partial y^j} y'^a y'^j + x''^{\gamma} \right] f_{\alpha\beta x'^\gamma} \\
& \qquad \cdot \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^k} - \frac{1}{2} \bar{f}_{iky'^j} \cdot y''^j + \frac{1}{2} f_{\alpha\beta x'^\gamma} \frac{\partial x'^\gamma}{\partial y^j} y'^j \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\beta}{\partial y^k} .
\end{aligned}
$$

Since

$$
\frac{\partial x'^\gamma}{\partial y^j} y'^j = \frac{\partial^2 x^\gamma}{\partial y^a \partial y^j} y'^a y'^j
$$

the third and sixth terms of the right member of (18) cancel. Differentiating the identity $x'^\beta F_\beta = 0$ we see that $x'^\beta F_{\alpha\beta x'^\gamma} = -F_{\alpha\gamma}$ and so $x'^\beta f_{\alpha\beta x'^\gamma} = x'^\beta [H^3 F_{\alpha\beta x'^\gamma} + 3H^2 H_\gamma F_{\alpha\beta} + H_{\alpha\gamma} H_\beta + H_\alpha H_{\beta\gamma}] = -H^3 F_{\alpha\gamma} + H H_{\alpha\gamma}$, that is, $x'^\beta f_{\alpha\beta x'^\gamma}$ is a symmetric covariant tensor.

We now express $\partial x'^\gamma / \partial y^i$ by means of Taylor's differentiation scheme,† designating the expression $\frac{1}{2} H^2$ by means of h; thus

$$
\frac{\partial x'^\gamma}{\partial y^i} = \bar{T}_i{}^m \frac{\partial x^\gamma}{\partial y^m} - T_\alpha{}^\gamma \frac{\partial x^\alpha}{\partial y^i},
$$

where

$$
\bar{T}_i{}^m = \overset{*}{\Gamma}{}^m_{ij} y'^j + \tfrac{1}{2} y''^k \bar{h}_{ijk} \bar{h}^{jm},
$$

$$
T_\alpha{}^\gamma = \overset{*}{\Gamma}{}^\gamma_{\alpha\beta} x'^\beta + \tfrac{1}{2} x''^\delta h_{\alpha\beta\delta} h^{\beta\gamma}
$$

(the Γ^* being the Christoffel symbols of the second kind associated with the Finsler space $H(x, x')$).

Multiplying the above equation for $\partial x'^\gamma / \partial y^i$ by $(\partial x^\alpha / \partial y^k)(x'^\beta f_{\alpha\beta x'^\gamma})$ we get

$$
\begin{aligned}
(19)\quad & \frac{\partial x^\alpha}{\partial y^k} x'^\beta f_{\alpha\beta x'^\gamma} \frac{\partial x'^\gamma}{\partial y^i} = \frac{\partial x^\alpha}{\partial y^k} x'^\beta f_{\alpha\beta x'^\gamma} \left[\bar{T}_i{}^m \frac{\partial x^\gamma}{\partial y^m} - T_\alpha{}^\gamma \frac{\partial x^\alpha}{\partial y^i} \right] \\
& = y'^j \bar{f}_{kjy'^m} \bar{T}_i^m - x'^\beta f_{\gamma\beta x'^\lambda} T_\alpha{}^\lambda \frac{\partial x^\alpha}{\partial y^i} \frac{\partial x^\gamma}{\partial y^k} .
\end{aligned}
$$

† J. H. Taylor, *A generalization of Levi-Civita's parallelism and the Frenet fomulas*, loc. cit., p. 255, formula 21.

Equations (18) and (19) will now be used to modify the form of equation (8). In fact we modify the second line of (8) by means of (18) and the third line by means of (19), thus

$$\begin{aligned}\overline{[ij, k]}y'^{j} + \frac{1}{2}\bar{f}_{iky'^{j}}\, y''^{j} + \frac{1}{2}\bar{f}_{ikj}\, y'''^{j} - \frac{1}{2}y'^{j}\bar{f}_{kjy'^{m}}\overline{T}_{i}^{m} \\ + \frac{1}{2}y'^{j}\bar{f}_{ijy'^{m}}\overline{T}_{k}^{m} = f_{\alpha\beta}\frac{\partial x^{\beta}}{\partial y^{k}}\frac{\partial^{2}x^{\alpha}}{\partial y^{i}\partial y^{j}}y'^{j} + \Big[[\alpha\beta, \gamma]x'^{\beta} \\ + \frac{1}{2}f_{\alpha\gamma x'^{\lambda}}x''^{\lambda} + \frac{1}{2}f_{\alpha\gamma\lambda}x'''^{\lambda} - \frac{1}{2}x'^{\beta}f_{\gamma\beta x'^{\lambda}}T_{\alpha}^{\lambda} \\ + \frac{1}{2}x'^{\beta}f_{\alpha\beta x'^{\lambda}}T_{\gamma}^{\lambda}\Big]\frac{\partial x^{\alpha}}{\partial y^{i}}\frac{\partial x^{\gamma}}{\partial y^{k}}\,.\end{aligned}$$

We shall write this last equation symbolically as

$$\overline{[i, k]} = f_{\alpha\beta}\frac{\partial x^{\beta}}{\partial y^{k}}\frac{\partial^{2}x^{\alpha}}{\partial y^{i}\partial y^{j}}y'^{j} + [\alpha, \gamma]\frac{\partial x^{\alpha}}{\partial y^{i}}\frac{\partial x^{\gamma}}{\partial y^{k}}\,. \tag{20}$$

Multiplying both sides of this equation by $\partial y^{k}/\partial x^{\rho}$, we obtain

$$\overline{[i, k]}\frac{\partial y^{k}}{\partial x^{\rho}} = f_{\alpha\rho}\frac{\partial^{2}x^{\alpha}}{\partial y^{i}\partial y^{j}}y'^{j} + [\alpha, \rho]\frac{\partial x^{\alpha}}{\partial y^{i}}\,.$$

Multiply by $f^{\sigma\rho}$ and note

$$f^{\sigma\rho}\frac{\partial y^{k}}{\partial x^{\rho}} = \bar{f}^{lk}\frac{\partial x^{\sigma}}{\partial v^{l}}\,.$$

The result is

$$\bar{f}^{lk}\frac{\partial x^{\sigma}}{\partial y^{l}}\overline{[i, k]} = \frac{\partial^{2}x^{\sigma}}{\partial y^{i}\partial y^{j}}y'^{j} + [\alpha, \rho]f^{\sigma\rho}\frac{\partial x^{\alpha}}{\partial y^{i}}\,. \tag{20'}$$

Application to covariant vectors. Differentiating the equation of transformation of a covariant vector V,

$$\overline{V}_{i} = V_{\sigma}\frac{\partial x^{\sigma}}{\partial y^{i}}$$

with respect to t, multiplying (20′) by $-V^{\sigma}$, and adding, we obtain

$$\overline{V}_{i}' - \overline{V}_{l}\bar{f}^{lk}\overline{[i, k]} = \{V_{\sigma}' - V_{\delta}f^{\delta\rho}[\sigma, \rho]\}\frac{\partial x^{\sigma}}{\partial y^{i}}\,.$$

Indicating this operation on V_{i} by θV_{i} we have as a conclusion

If V_σ is a covariant tensor of rank one, then θV_σ is also a covariant tensor of rank one.

A similar process may be applied to contravariant vectors. For, if V^σ is a contravariant vector, then

$$V^\sigma = \overline{V}^i \frac{\partial x^\sigma}{\partial y^i}$$

and

$$V'^\sigma = \overline{V}'^i \frac{\partial x^\sigma}{\partial y^i} + \overline{V}^i \frac{\partial^2 x^\sigma}{\partial y^i \partial y^j} y'^j.$$

Eliminating the second derivative as before gives the result

$$\begin{aligned} V'^\sigma + V^\alpha[\alpha, \rho] f^{\sigma\rho} &= \overline{V}'^i \frac{\partial x^\sigma}{\partial y^i} + \overline{V}^i [\overline{i, k}] \bar{f}^{lk} \frac{\partial x^\sigma}{\partial y^l} \\ &= [\overline{V}'^i + \overline{V}^j [\overline{j, k}] \bar{f}^{ik}] \frac{\partial x^\sigma}{\partial y^i} . \end{aligned}$$

Designating this process as before by θ we have

If V^σ is a contravariant vector, then θV^σ is a contravariant vector.

In general the θ process, with respect to a given curve, will consist of forming the derivative of the tensor $X:::$, with respect to t, and adding

$$X^\alpha :::[\alpha, \rho] f^{\sigma\rho}$$

for each contravariant index, and subtracting

$$X_\delta ::: f^{\delta\rho}[\sigma, \rho]$$

for each covariant index, all arguments being taken along the curve in question.

We next point out that the rule of ordinary calculus for forming the derivative of a product is conserved. It will suffice for this to examine a special case, say that of the product $\overline{V}_i\overline{V}_k$. By the law of composition of tensors $\overline{V}_i\overline{V}_k$ is a second-order covariant tensor and so

$$\overline{V}_i\overline{V}_k = V_\sigma V_\tau \frac{\partial x^\sigma}{\partial y^i} \frac{\partial x^\tau}{\partial y^k} .$$

Then

$$\begin{aligned} \overline{V}_i' \overline{V}_k + \overline{V}_i \overline{V}_k' = {} & (V_\sigma' V_\tau + V_\sigma V_\tau') \frac{\partial x^\sigma}{\partial y^i} \frac{\partial x^\tau}{\partial y^k} \\ & + V_\sigma V_\tau \left[\frac{\partial x^\sigma}{\partial y^i} \frac{\partial^2 x^\tau}{\partial y^k \partial y^j} y'^j + \frac{\partial^2 x^\sigma}{\partial y^i \partial y^j} y'^j \frac{\partial x^\tau}{\partial y^k} \right]. \end{aligned}$$

Hence

$$\bar{V}_i'\bar{V}_k+\bar{V}_i\bar{V}_k' = (V_\sigma' V_\tau + V_\sigma V_\tau')\frac{\partial x^\sigma}{\partial y^i}\frac{\partial x^\tau}{\partial y^k}$$
$$+ V_\sigma V_\tau \frac{\partial x^\sigma}{\partial y^i}\left[\bar{f}^{lm}\frac{\partial x^\tau}{\partial y^l}\overline{[k, m]} - [\alpha, \rho]f^{\tau\rho}\frac{\partial x^\alpha}{\partial y^k}\right]$$
$$+ V_\sigma V_\tau \frac{\partial x^\tau}{\partial y^k}\left[\bar{f}^{lm}\frac{\partial x^\sigma}{\partial y^l}\overline{[i, m]} - [\alpha, \rho]f^{\sigma\rho}\frac{\partial x^\alpha}{\partial y^i}\right].$$

Transposing, we get

$$\bar{V}_i'\bar{V}_k+\bar{V}_i\bar{V}_k' - \bar{V}_i\bar{V}_l\bar{f}^{lm}\overline{[k, m]} - \bar{V}_l\bar{V}_k\bar{f}^{lm}\overline{[i, m]}$$
$$= \left[V_\sigma' V_\tau + V_\sigma V_\tau' - V_\sigma V_\alpha f^{\alpha\rho}[\tau, \rho] - V_\alpha V_\tau f^{\alpha\rho}[\sigma, \rho]\right]\frac{\partial x^\sigma}{\partial y^i}\frac{\partial x^\tau}{\partial y^k}.$$

Rewriting the above we have, for the left member,

$$[\bar{V}_i' - \bar{V}_l\bar{f}^{lm}\overline{[i, m]}]\bar{V}_k + \bar{V}_i[\bar{V}_k' - \bar{V}_l\bar{f}^{lm}\overline{[k, m]}];$$

that is,

$$\theta(\bar{V}_i\bar{V}_k) = (\theta\bar{V}_i)\bar{V}_k + \bar{V}_i(\theta\bar{V}_k).$$

The θ process applied to a scalar expressed as the contraction of tensors is equivalent to the ordinary derivative.

Let us consider, for example, the scalar $V_\sigma V^\sigma$. Forming the θ derivative of this by the rule just discussed, we have

$$\theta(V_\sigma V^\sigma) = (\theta V_\sigma)\, V^\sigma + V_\sigma\,(\theta V^\sigma);$$

that is,

$$\theta(V_\sigma V^\sigma) = [V_\sigma' - V_\delta f^{\delta\rho}[\sigma, \rho]]V^\sigma + V_\sigma[V'^\sigma + V^\alpha[\alpha, \rho]f^{\sigma\rho}]$$
$$= V'^\sigma V_\sigma + V_\sigma' V^\sigma.$$

A most important property of our θ process is expressed by the equation

$$\theta f_{\alpha\gamma} \equiv 0.$$

We have

$$\theta f_{\alpha\gamma} = f_{\alpha\gamma}' - f_{\sigma\gamma}f^{\sigma\rho}[\alpha, \rho] - f_{\alpha\sigma}f^{\sigma\rho}[\gamma, \rho]$$
$$= f_{\alpha\gamma}' - [\alpha, \gamma] - [\gamma, \alpha].$$

One readily verifies that

$$[\alpha, \gamma] + [\gamma, \alpha] = [[\alpha\beta, \gamma] + [\gamma\beta, \alpha]]x'^\beta + f_{\alpha\gamma x'^\lambda}\cdot x''^\lambda + f_{\alpha\gamma\lambda}\cdot x'''^\lambda.$$

Then, since

$$[\alpha\beta, \gamma] + [\gamma\beta, \alpha] = \frac{\partial f_{\alpha\gamma}}{\partial x^\beta},$$

it follows that

$$\theta f_{\alpha\gamma} \equiv 0.$$

If a contravariant vector ξ^α, defined at each point of a curve, satisfies the differential equations

$$\theta\xi^\alpha = 0 \qquad (\alpha = 1, 2, \cdots, n)$$

along the curve, we shall say the vector remains parallel θ. This parallelism reduces to Taylor's if $F_{\alpha\beta}$ and F are functions of x and x' alone and H is chosen as F.

Theorem. *If two vectors, ξ and η, defined along a curve, each remain parallel θ along the curve, then the θ magnitude of the vectors and the θ cosine of the angle between them are constant along the curve.*

Each of the three expressions

$$f_{\alpha\beta}\xi^\alpha\xi^\beta, \quad f_{\alpha\beta}\eta^\alpha\eta^\beta, \quad f_{\alpha\beta}\xi^\alpha\eta^\beta$$

is a scalar formed by the contraction of tensors, and so the θ process applied to them is equivalent to differentiating them in the ordinary way with respect to t. Thus, we have

$$\frac{d}{dt}[f_{\alpha\beta}\xi^\alpha\xi^\beta] = \theta[f_{\alpha\beta}\xi^\alpha\xi^\beta] = [\theta f_{\alpha\beta}]\xi^\alpha\xi^\beta + 2f_{\alpha\beta}(\theta\xi^\alpha)\xi^\beta = 0,$$

and

$$\frac{d}{dt}[f_{\alpha\beta}\xi^\alpha\eta^\beta] = [\theta f_{\alpha\beta}]\xi^\alpha\eta^\beta + f_{\alpha\beta}(\theta\xi^\alpha)\eta^\beta + f_{\alpha\beta}\xi^\alpha(\theta\eta^\beta) = 0,$$

since $\theta f_{\alpha\beta} = 0$, and $\theta\xi^\alpha = \theta\eta^\beta = 0$ by hypothesis. And so the two cardinal properties of parallelism are preserved.

Of special interest, perhaps, are the curves along which the fundamental tensors $f_{\alpha\beta}$ and $h_{\alpha\beta}(h_{\alpha\beta} = HH_{\alpha\beta} + H_\alpha H_\beta)$ coincide. Let C be one of these curves and let us apply the θ process to a vector V^σ defined along C; thus

$$\begin{aligned}\theta V^\sigma &= V'^\sigma + V^\alpha[\alpha, \rho]f^{\sigma\rho} \\ &= V'^\sigma + V^\alpha\{[\alpha\beta, \rho]x'^\beta + \tfrac{1}{2}f_{\alpha\rho x'^\lambda}\cdot x''^\lambda + \tfrac{1}{2}f_{\alpha\rho\lambda}\cdot x''^\lambda \\ &\quad - \tfrac{1}{2}x'^\beta f_{\rho\beta x'^\lambda}\cdot T_\alpha{}^\lambda + \tfrac{1}{2}x'^\beta f_{\alpha\beta x'^\lambda}T_\rho{}^\lambda\}f^{\sigma\rho}.\end{aligned}$$

We have noted that $x'^\beta f_{\rho\beta x'^\lambda} = -H^3F_{\rho\lambda} + HH_{\rho\lambda}$, and so the last two terms of θV reduce to zero. A moment's consideration will be sufficient to verify that

$$x'^{\beta}[\alpha\beta,\rho] = x'^{\beta}[\alpha\beta,\rho]' - \frac{1}{2}\frac{\partial}{\partial x^{\beta}}(HH_{\alpha\rho})\cdot x'^{\beta} + \frac{1}{2}\frac{\partial}{\partial x^{\beta}}(H^{3}F_{\alpha\rho})\cdot x'^{\beta}$$
$$= x'^{\beta}[\alpha\beta,\rho]' - \frac{1}{2}h_{\alpha\rho x^{\beta}}\cdot x'^{\beta} + f_{\alpha\rho x^{\beta}}\cdot x'^{\beta}.$$

(The symbol $[\alpha\beta,\rho]'$ appearing above is used to designate the first Christoffel symbol of the Finsler space $H(x, x')$.) Moreover

$$f'_{\alpha\rho} - h_{\alpha\rho x^{\beta}}\cdot x'^{\beta} = h_{\alpha\rho\beta}\cdot x''^{\beta}$$

since $h_{\alpha\beta}=f_{\alpha\beta}$. Hence we conclude that, with respect to C, the Taylor-Synge derivatives* of V^{σ} and θV^{σ} are identical. This property is somewhat analogous to the defining property of Levi-Civita's parallelism, namely: A vector W defined along a curve K of the surface S is said to remain parallel provided it remains parallel when K is considered as belonging to the developable surface tangent to S along K.

A second differentiation process, which we shall call the τ process, may be obtained by multiplying (20) by I^2 and adding II' to both members. Evidently $\tau(I^2 f_{\alpha\beta})=0$ and consequently the "τ magnitude" and the "τ direction" of vectors undergoing a displacement parallel (τ) are unaltered.

An example

$$I = \left[1 + \frac{1}{2}\frac{(x'y'' - y'x'')^2}{(x'^2 + y'^2)^3}\right]; \qquad H = (x'^2 + y'^2)^{1/2};$$

$$F = IH; \quad \frac{\partial F}{\partial x''} = -\frac{(x'y'' - y'x'')y'}{(x'^2 + y'^2)^{5/2}}; \quad \frac{\partial F}{\partial y''} = \frac{(x'y'' - y'x'')x'}{(x'^2 + y'^2)^{5/2}};$$

$$\frac{\partial^2 F}{\partial x''\partial x''} = \frac{y'^2}{(x'^2 + y'^2)^{5/2}}; \quad \frac{\partial^2 F}{\partial x''\partial y''} = \frac{-x'y'}{(x'^2 + y'^2)^{5/2}}; \quad \frac{\partial^2 F}{\partial y''\partial y''} = \frac{x'^2}{(x'^2 + y'^2)^{5/2}};$$

$$\frac{\partial^2 H}{\partial x'\partial x'} = \frac{y'^2}{(x'^2 + y'^2)^{3/2}}; \quad \frac{\partial^2 H}{\partial x'\partial y'} = \frac{-x'y'}{(x'^2 + y'^2)^{3/2}}; \quad \frac{\partial^2 H}{\partial y'\partial y'} = \frac{x'^2}{(x'^2 + y'^2)^{3/2}}.$$

We note that I is independent of the parameter, the determinant $F_{\alpha\beta}$ is of rank $n-1$, the F one function is not zero, and $f_{\alpha\beta}=h_{\alpha\beta}(f_{\alpha\beta}=H^3F_{\alpha\beta}+H_{\alpha}H_{\beta}$, $h_{\alpha\beta}=HH_{\alpha\beta}+H_{\alpha}H_{\beta})$. The geometry of this space is very similar to euclidean geometry. The measurement of angles, magnitude of vectors and parallel displacement of vectors are the same as in the euclidean plane. In particular the autoparallel curves of both spaces are straight lines and the two metrics F and H coincide along straight lines.

* See J. H. Taylor, loc. cit., p. 255.

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ON THE MAXIMUM ABSOLUTE VALUE OF THE DERIVATIVE OF $e^{-x^2}P_n(x)$*

BY

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A remarkable theorem due to S. Bernstein† asserts that if L is the maximum absolute value of an arbitrary polynomial $P_n(x)$ of degree n in the interval (a, b) then the maximum absolute value of the derivative $P_n'(x)$ does not exceed $nL[(b-x)(x-a)]^{-1/2}$ on (a, b). A related theorem for trigonometric sums states that if L is the maximum of the absolute value of a trigonometric sum of order n, then the maximum absolute value of its derivative does not exceed nL.‡

A similar theorem is here given for the function $e^{-x^2}P_n(x)$, where $P_n(x)$ is an arbitrary polynomial of degree n.

THEOREM. *If L is the maximum absolute value of $e^{-x^2}P_n(x)$ in the interval $-\infty<x<\infty$, then the maximum absolute value of the derivative is less than $n^{1/2}L[1.0951+O(n^{-1})]$ in the infinite interval.*

It is convenient to establish the corresponding result for functions of the form $f_n(x)=e^{-x^2/4}P_n(x)$ and then to obtain the stated theorem by the change of variable $x=2x'$. The proof follows the line of attack adopted by de la Vallée Poussin, and is accomplished with the aid of the following propositions.

I. *If $f_n'(x)$ attains its maximum absolute value at x_0, then*

$$x_0^2 < 2k(n+1),$$

where k is a constant which may be taken as 3.69264.

II. *There exists an analytic function $\psi_m(x)$, where $4m+2>2k(n+1)$, such that*

(a) *$\psi_m'(x)$ has an extremum equal to $f_n'(x_0)$ at $x=x_0$;*

(b) *$\psi_m(x)$ becomes infinite at $-\infty$, at $+\infty$, and has $m+1$ extrema, with alternating signs at these $m+3$ points (counting $\pm\infty$);*

(c) *the least extremum of $\psi_m(x)$ is greater in absolute value than*

$$|f_n'(x_0)|\,[m+\tfrac{1}{2}-x_0^2/4]^{1/4}[m+\tfrac{1}{2}]^{-3/4}[1+O(m^{-1})].$$

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* Presented to the Society, June 20, 1930; received by the editors March 24, 1930.

† S. Bernstein, *Sur l'ordre de la meilleure approximation des fonctions continues par des polynomes de degré donné*, Mémoire Couronné, Brussels, 1912.

‡ de la Vallée Poussin, *Sur le maximum du module de la dérivée d'une expression trigonométrique d'ordre et de module bornés*, Comptes Rendus, vol. 166 (1918), pp. 843–846.

III. *If*

$$R(x) = \psi_m(x) - f_n(x),$$

then

(a) $R'(x)$ *has a double root at* $x = x_0$;

(b) $R'(x)$ *has not more than* $m+1$ *roots in the interval* $-\infty < x < \infty$.

Before demonstrating these propositions let us show how they establish the theorem. Let the maximum of $|f_n(x)|$ and of $|f_n'(x)|$ be L and M respectively, and suppose if possible that the least maximum of $|\psi_m(x)|$ is greater than L. Then $R(x)$ has the sign of $\psi_m(x)$ at $-\infty$, at $+\infty$ and at $m+1$ intermediate points. Because of the alternation of signs at these $m+3$ points, $R(x)$ has at least $m+2$ distinct roots and $R'(x)$ has at least $m+1$ distinct roots. Therefore, by III(a), $R'(x)$ has at least $m+2$ roots. But this contradicts III(b), and hence L cannot be less than the least maximum of $|\psi_m(x)|$. Consequently, by II(c),

$$L > M[m + \tfrac{1}{2} - x_0^2/4]^{1/4}[m + \tfrac{1}{2}]^{-3/4}[1 + O(m^{-1})].$$

Now, $x_0^2 < 2k(n+1)$, and we shall choose m as an integer in such a manner that

$$m = \tfrac{3}{4}kn[1 + O(n^{-1})].$$

With this value of m and the value of k given in I the inequality

$$M < n^{1/2}L[2.19018 + O(n^{-1})]$$

follows, from which the inequality of the theorem is derived by the change of variable $x = 2x'$.

We turn now to the proof of I. If L denotes the maximum of $|e^{-x^2/4}P_n(x)|$, then in the interval where $x^2 \leqq 2n$

$$|P_n(x)| \leqq Le^{n/2}.$$

Hence by a theorem* due to Tchebycheff we have

$$|P_n(x)| \leqq L|2ex^2/n|^{n/2},$$

and consequently

$$|e^{-x^2/4}P_n(x)| \leqq L(2e/n)^{n/2}|x|^n e^{-x^2/4},$$

for $x^2 > 2n$. When $x > (2n)^{1/2}$ the function $e^{-x^2/4}x^n$ is decreasing, so that when $x^2 > 2kn$ we merely strengthen the inequality in replacing x^2 on the right

* For statement and proof see S. Bernstein, *Leçons sur les Propriétés Extrémales et la Meilleure Approximation des Fonctions Analytiques d'une Variable Réelle*, Paris, 1926, pp. 7–8.

hand side by $2kn$. We then find upon calculating the value of the right hand expression with the given value of k that

$$|e^{-x^2/4}P_n(x)| < L$$

when $x^2 > 2kn$. Since the derivative of $e^{-x^2/4}P_n(x)$ is $e^{-x^2/4}$ times a polynomial of degree $n+1$, the proof of I is completed.

To establish II we let w_1 and w_2 be two solutions of

$$d^2w/dx^2 + [m + \tfrac{1}{2} - x^2/4]w = 0, \tag{1}$$

with the initial values

$$w_1 = (m + \tfrac{1}{2})^{-1/4}, \qquad w_2 = 0,$$
$$w_1' = 0, \qquad w_2' = (m + \tfrac{1}{2})^{1/4},$$

at $x=0$, and express the solution of (1) in the form

$$w_m(x) = [w_1^2 + w_2^2]^{1/2} \cos[\psi(x) - \theta], \tag{2}$$

where

$$\psi(x) = \int_{-\infty}^{x} [w_1^2 + w_2^2]^{-1} dx.$$

As θ increases the extrema of $w_m(x)$ move continuously to the right* in the interval $-(4m+2)^{1/2} < x < (4m+2)^{1/2}$, so that if we select m as an integer such that

$$4m + 2 > 2k(n + 1) \tag{3}$$

it is clearly possible to choose θ so that an extremum of $w_m(x)$ occurs at $x = x_0$, since, by I, $|x_0| < (2kn)^{1/2}$.

Corresponding to a given integral value of m there is a single critical value of θ, $0 \leqq \theta < \pi$, for which $w_m(x)$ vanishes at $\pm\infty$, while for all other values of θ, $w_m(x)$ becomes infinite at $\pm\infty$.† We desire to construct a function that will always become infinite at $\pm\infty$ and therefore, if the θ chosen above should prove to be critical, we shall take a new m equal to the original m increased by unity. Since the critical function is

$$Ce^{-x^2/4}H_m(x),$$

where $H_m(x)$ is an Hermitian polynomial, we see from the known properties of $H_m(x)$ and $H_{m+1}(x)$ that if θ is critical for m it will not be so for $m+1$. By this arrangement we are sure that $w_m(x)$ will always become infinite at $\pm\infty$.

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* The proof is similar to that for the behavior of the roots. See W. E. Milne, these Transactions, vol. 30 (1928), pp. 797–802, especially p. 800, formula (16).

† Cf. W. E. Milne, loc. cit., pp. 799–800.

The function $\psi_m(x)$ is now defined as follows:

$$\psi_m(x) = [f_n'(x_0)/w_m(x_0)]\int_a^x w_m(s)ds, \tag{4}$$

in which a denotes the abscissa of the extremum of $w_m(x)$ nearest the origin (or one of the two nearest). It is clear from (4) that II(a) and III(a) are verified.

Next consider the roots of $w_m(x)$. When θ is critical $w_m(x)$ becomes $Ce^{-x^2/4}H_m(x)$ and is known to have exactly m real distinct roots. As θ increases each root moves continuously to the right, no root is gained or lost in the finite interval, but a new root appears at $-\infty$. Hence, for non-critical values of θ, $w_m(x)$ has exactly $m+1$ real distinct roots. Therefore $\psi_m(x)$ has $m+1$ distinct extrema, and obviously becomes infinite at $\pm\infty$.

Finally it is known that the amplitudes of the oscillations and the intervals between the roots of $w_m(x)$ increase as x recedes from the origin, so that the areas bounded by the successive arches increase. This assures us of the alternation in sign of $\psi_m(x)$ at the extrema and at $\pm\infty$, and completes the proof of II(b).

The proof of II(c) is easily effected with the aid of (2) and (4) and certain inequalities previously established.*

Finally, to prove III(b) we note that

$$e^{x^2/4}R'(x) = v_m(x) + P_n'(x) - xP_n(x)/2,$$

where $v_m(x)$ is a solution of the differential equation

$$v'' - xv' + mv = 0.$$

Differentiating this equation m times we get

$$v^{(m+2)} - xv^{(m+1)} = 0,$$

whence

$$v^{(m+1)} = Ce^{x^2/2}.$$

The value $C=0$ gives the critical solution, hence $C\neq 0$. Therefore in view of the fact that $m>n+1$ because of (3)

$$(d^{m+1}/dx^{m+1})(e^{x^2/4}R'(x)) = Ce^{x^2/2} \neq 0,$$

which shows that $R'(x)$ has not more than $m+1$ roots.

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* W. E. Milne, these Transactions, vol. 31, pp. 907–918. See pp. 909–910, formulas (8) to (15).

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THE TRANSFORMATION E OF NETS*

BY

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1. Introduction

Let there be given a surface S in euclidean space of three dimensions. Suppose that on S there are two one-parameter families of curves such that through each point on S there passes one curve of each family, the two tangents being distinct. We have called such a set of curves a net. Suppose that through each point of S there passes a line g of a congruence G, such that the developables of the congruence G intersect S in the curves of the given net N. Let $\overline{S}$ be another surface in the same space and in one-to-one point correspondence with S, corresponding points lying on the lines of G. The developables of G intersect $\overline{S}$ in a net $\overline{N}$. If neither S nor $\overline{S}$ is a focal surface of G, N and $\overline{N}$ are said to be *in relation* C, or to be C *transforms*.† If N and $\overline{N}$ are conjugate nets in relation C, they are in the relation of a transformation‡ F.

In this paper we extend the notion of the transformation of Ribaucour to nets not necessarily the lines of curvature on the sustaining surfaces. Two nets in relation C will be said to be in *relation* E or to be E *transforms* if and only if every point on the line of intersection of corresponding tangent planes to the sustaining surfaces is equally distant from the corresponding points P and $\overline{P}$. If N and $\overline{N}$ are the lines of curvature of S and $\overline{S}$, the transformation E is a transformation of Ribaucour.

We also extend the notion of semi-parallel nets in relation F to nets in relation C. We shall say that two nets N and $\overline{N}$ are *semi-parallel* if the tangents to one and only one of the families of curves of N are parallel to the tangents to the corresponding curves of $\overline{N}$ at corresponding points. If the tangents to both families of curves of N are parallel to the corresponding tangents of the curves of $\overline{N}$, the nets N and $\overline{N}$ are parallel nets. If N and $\overline{N}$ are parallel nets they are conjugate nets.

Let the Cartesian coördinates of the point P of S be (x_1, x_2, x_3), the direction cosines of the normal to S at P be (X_1, X_2, X_3), and the direction cosines of g be $(\lambda_1, \lambda_2, \lambda_3)$. The corresponding quantities for $\overline{S}$ will be denoted by

* Presented to the Society, December 30, 1930; received by the editors July 10, 1930.

† V. G. Grove, *Transformations of nets*, these Transactions, vol. 30 (1928), p. 483. Hereafter referred to as Grove, *Transformations*.

‡ L. P. Eisenhart, *Transformations of Surfaces*, Princeton University Press, 1923, p. 34.

barred letters. Let the parametric equations of S be $x_i = x_i(u, v)$, $i = 1, 2, 3$. The three pairs of functions (x, λ) satisfy a system of differential equations of the form

$$
\begin{aligned}
x_{uu} &= \alpha x_u + \beta x_v + L\lambda, \\
x_{uv} &= a x_u + b x_v + M\lambda, \\
x_{vv} &= \gamma x_u + \delta x_v + N\lambda, \\
\lambda_u &= m x_u + s x_v + A\lambda, \\
\lambda_v &= t x_u + n x_v + B\lambda,
\end{aligned} \tag{1}
$$

wherein

$$L = D \sec \phi, \qquad M = D' \sec \phi, \qquad N = D'' \sec \phi, \tag{2a}$$

D, D', D'' being the second fundamental coefficients of S, and ϕ the angle between the line g and the normal to S at P. The remaining coefficients are obtained by solving the equations

$$
\begin{aligned}
\alpha E + \beta F &= \tfrac{1}{2}E_u - L \cos \theta^{(u)}, \\
\alpha F + \beta G &= F_u - \tfrac{1}{2}E_v - L \cos \theta^{(v)};
\end{aligned} \tag{2b}
$$

$$
\begin{aligned}
aE + bF &= \tfrac{1}{2}E_v - M \cos \theta^{(u)}, \\
aF + bG &= \tfrac{1}{2}G_u - M \cos \theta^{(v)};
\end{aligned} \tag{2c}
$$

$$
\begin{aligned}
\gamma F + \delta G &= \tfrac{1}{2}G_v - N \cos \theta^{(v)}, \\
\gamma E + \delta F &= F_v - \tfrac{1}{2}G_u - N \cos \theta^{(u)};
\end{aligned} \tag{2d}
$$

$$
\begin{aligned}
mE^{1/2} \cos \theta^{(u)} + sG^{1/2} \cos \theta^{(v)} + A &= 0, \\
mE + sF + AE^{1/2} \cos \theta^{(u)} &= e, \\
mF + sG + AG^{1/2} \cos \theta^{(v)} &= f;
\end{aligned} \tag{2e}
$$

$$
\begin{aligned}
tE^{1/2} \cos \theta^{(u)} + nG^{1/2} \cos \theta^{(v)} + B &= 0, \\
tF + nG + BG^{1/2} \cos \theta^{(v)} &= g, \\
tE + nF + BE^{1/2} \cos \theta^{(u)} &= f',
\end{aligned} \tag{2f}
$$

wherein E, F, G are the first fundamental coefficients of S; $\theta^{(u)}, \theta^{(v)}$ the angles between g and the tangents to $v=\text{const.}$, and $u=\text{const.}$ respectively, and where

$$e = \sum x_u \lambda_u, \quad f = \sum x_v \lambda_u, \quad f' = \sum x_u \lambda_v, \quad g = \sum x_v \lambda_v.$$

The coördinates of the point $\overline{P}$ of $\overline{S}$ corresponding to P of S are of the form

$$\bar{x} = x + \lambda d,$$

where d is a scalar function of u, v. We may readily verify that the three pairs of functions $(x, \bar{x})$ are solutions of the following system of differential equations:

$$
\begin{aligned}
x_{uu} &= \alpha x_u + \beta x_v - Lx/d + L\bar{x}/d, \\
x_{uv} &= ax_u + bx_v - Mx/d + M\bar{x}/d, \\
x_{vv} &= \gamma x_u + \delta x_v - Nx/d + N\bar{x}/d, \\
\bar{x}_u &= (1 + md)x_u + sdx_v - (Ad + d_u)x/d + (Ad + d_u)\bar{x}/d, \\
\bar{x}_v &= tdx_u + (1 + nd)x_v - (Bd + d_v)x/d + (Bd + d_v)\bar{x}/d.
\end{aligned} \tag{3}
$$

The parametric nets on S and $\bar{S}$ are therefore in relation* C if and only if $s = t = 0$. We shall hereafter assume that the parametric nets are in relation C.

The first fundamental coefficients of $\bar{S}$ may be written in the form

$$
\begin{aligned}
\bar{E} &= (1 + md)^2E + (Ad + d_u)[2(1 + md)E^{1/2}\cos\theta^{(u)} + Ad + d_u], \\
\bar{F} &= (1 + md)(1 + nd)F + (1 + md)(Bd + d_v)E^{1/2}\cos\theta^{(u)} \\
&\quad + (Ad + d_u)(Bd + d_v) + (1 + nd)(Ad + d_u)G^{1/2}\cos\theta^{(v)}, \\
\bar{G} &= (1 + nd)^2G + (Bd + d_v)[2(1 + nd)G^{1/2}\cos\theta^{(v)} + Bd + d_v].
\end{aligned} \tag{4}
$$

From (2e) and (2f) we find readily that

$$
f - f' = (EG)^{1/2}(m - n)[\cos\omega - \cos\theta^{(u)}\cos\theta^{(v)}] \tag{5}
$$

where ω is the angle between the parametric tangents on S.

The focal points of g are defined by the expressions

$$
y = x - \lambda/m, \quad z = x - \lambda/n. \tag{6}
$$

We shall call the surfaces generated by these points the first and second focal surfaces respectively. The parametric curves on these surfaces are conjugate. They will be also orthogonal if the functions F_1, F_2 defined below are respectively zero:

$$
\begin{aligned}
F_1 &= (m_u - Am)[mG^{1/2}(m - n)\cos\theta^{(v)} + m_v - Bm]/m^4, \\
F_2 &= (n_v - Bn)[nE^{1/2}(n - m)\cos\theta^{(u)} + n_u - An]/n^4.
\end{aligned} \tag{7}
$$

By means of the integrability conditions† of system (1), we may write the expressions for F_1 and F_2 as follows:

$$
\begin{aligned}
F_1 &= (m - n)(m_u - Am)(nG^{1/2}\cos\theta^{(v)} - a)/m^4, \\
F_2 &= (n - m)(n_v - Bn)(nE^{1/2}\cos\theta^{(u)} - b)/n^4.
\end{aligned} \tag{8}
$$

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* Grove, *Transformations*, p. 484.

† G. M. Green, *Memoir on the general theory of surfaces and rectilinear congruences*, these Transactions, vol. 20 (1919), p. 150. Hereafter referred to as Green, *Surfaces*.

From (8) we derive the fact that a normal congruence G is a congruence of Guichard if, and only, if the congruence in Green's relation R to G with respect to the net in which the developables of G intersect any surface to which G is normal, consists of the rulings of the ruled plane at infinity. This theorem may readily be proved from geometrical considerations.

2. The transformation E

We shall say that the nets N and $\overline{N}$ in relation C are *E transforms* or *in relation E* if and only if every point on the line h of intersection of corresponding tangent planes is equidistant from P and $\overline{P}$. We may readily show that these conditions may be written in the following form:

$$\begin{aligned} 2(1+md)E^{1/2}\cos\theta^{(u)} + Ad + d_u &= 0, \\ 2(1+nd)G^{1/2}\cos\theta^{(v)} + Bd + d_v &= 0. \end{aligned} \tag{9}$$

Under conditions (9) formulas (4) may be reduced to the following form:

$$\overline{E} = (1+md)^2E,\quad \overline{F} = (1+md)(1+nd)F,\quad \overline{G} = (1+nd)^2G. \tag{10}$$

Hence *if a net N is orthogonal, any E transform of N is orthogonal.* From (10) we find that the cross ratio* of the corresponding points P and $\overline{P}$ and the focal points on g in the proper order is defined by

$$R^2 = \overline{E}G/(\overline{G}E). \tag{11}$$

Hence *if one of two nets in relation E is isothermally orthogonal, the other will be isothermally orthogonal if and only if the nets are K_R transforms.*†

From (10) we observe that *S is mapped conformally on $\overline{S}$ by a transformation E of non-radial nets N and $\overline{N}$ if and only if N and $\overline{N}$ are orthogonal nets in relation K_{-1}.*

We may readily prove that the nets which are the spherical representations of the normal congruences of the sustaining surfaces of nets N and $\overline{N}$ in relation E are in relation C if and only if N and $\overline{N}$ are in the relation of a transformation of Ribaucour.

We have said that nets N and $\overline{N}$ are L transforms‡ if they are in relation C, and if the developables of the congruence of lines of intersection of corresponding tangent planes correspond to the curves of the nets. In our present notation N and $\overline{N}$ are L transforms if and only if

$$L\overline{M}(1+md) - M\overline{L}(1+nd) = N\overline{M}(1+nd) - M\overline{N}(1+md) = 0. \tag{12}$$

---

* Grove, *Transformations*, p. 493.

† Ibid., p. 493.

‡ Ibid., p. 487.

If we use formulas (10), (11), and (12), we may prove the following theorems:

(a) *If the asymptotic nets N and $\bar{N}$ are E transforms, the lines of curvature on S and $\bar{S}$ correspond if N and $\bar{N}$ are radial transforms or K_{-1} transforms.*

(b) *If the non-conjugate, non-asymptotic nets N and $\bar{N}$ are E transforms in the relation of a transformation L, the lines of curvature on S and $\bar{S}$ correspond if and only if N and $\bar{N}$ are radial.*

(c) *Let N and $\bar{N}$ be two non-conjugate, non-asymptotic nets in relation E and also L. If one of the sustaining surfaces is minimal (developable) the other is also minimal (developable).*

(d) *A necessary condition that two orthogonal nets N and $\bar{N}$ in relation E be isothermally orthogonal is that they shall also be in relation K_R.*

(e) *Let N and $\bar{N}$ be orthogonal non-radial nets in relation E. A necessary and sufficient condition that $\bar{N}$ be an isometrical map of N on $\bar{S}$ is that E be also K_{-1}.*

3. Semi-parallel nets in relation C

Suppose that N and $\bar{N}$ are semi-parallel nets in relation C. Let the parallel tangents be the tangents to the curves $v=$const. of N and $\bar{N}$. It follows from (3) that

$$Ad + d_u = 0, \quad Bd + d_v \neq 0. \tag{13}$$

If N is orthogonal, it follows from (4) that $\bar{N}$ will be orthogonal if and only if $\cos \theta^{(u)} = 0$. From (2e), (2f) and (13), we observe that

$$f = f' = A = d_u$$

if N and $\bar{N}$ are orthogonal semi-parallel nets in relation C. Hence *if one of two semi-parallel nets in relation C by means of a congruence G is orthogonal, the other will be orthogonal if and only if the lines of G are orthogonal to the parallel tangents at corresponding points. The congruence G is moreover a normal congruence. The curves of the nets with parallel tangents are parallel curves.* If through the origin are drawn lines parallel to the lines of G, there is obtained on the unit sphere a net N', the spherical indicatrix of the congruence G. If $f=f'=0$, the tangents to the curves $v=$const. ($u=$const.) of N' are perpendicular to the curves $u=$const. ($v=$const.) of N and $\bar{N}$; *N' is also an orthogonal net.* Suppose that the curves $v=$const. on S and $\bar{S}$ are parallel curves. In that case we may show that

$$\bar{D}' = \cos \bar{\phi}[M(1 + md) - bR(Bd + d_v)].$$

It follows also from (2e) that $\cos \theta^{(u)} = 0$, and from (8) that $F_2 = 0$. We may state our results in the following way: *If N and $\bar{N}$ are in relation C and the*

curves of one pair of the two families composing the nets are parallel curves, the nets are also in relation F if and only if the line in Green's relation R to N (and consequently $\overline{N}$) is parallel to the parallel tangents. Moreover *the developables of the congruence G intersect the focal surface corresponding to the parallel curves in its lines of curvature. The congruence G will be a normal congruence if and only if N ($\overline{N}$) is orthogonal.*

We see readily that the two semi-parallel nets (with the tangents to v=const. parallel) are in relation E if and only if

$$\cos\theta^{(u)} = Ad + d_u = 0,$$
$$2(1 + md)G^{1/2}\cos\theta^{(v)} + Bd + d_v = 0.$$

If use be made of equations (2), these conditions may be written

$$\cos\theta^{(u)} = A = d_u = 0, \qquad B(2 + nd) - nd_v = 0. \tag{14}$$

If we differentiate the second of (14) with respect to u and use the integrability* conditions of system (1), we obtain

$$Mn(2 + nd) - 2bB = 0.$$

Hence *if the nets N and $\overline{N}$ are semi-parallel nets in relation E, and if one of the nets is conjugate, the other is conjugate, and the rays of the points P and $\overline{P}$ with respect to N and $\overline{N}$ respectively are parallel to the parallel tangents of the curves of the nets.*

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* Green, *Surfaces*, p. 150.

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THE CHARACTERISTIC NUMBER OF A SEQUENCE OF INTEGERS SATISFYING A LINEAR RECURSION RELATION*

BY

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1. Introduction. Let

$$(W)_n: \qquad W_0, W_1, \cdots, W_n, \cdots$$

denote a sequence of integers satisfying the linear difference equation of order $r=3$

$$(1.1) \qquad \Omega_{n+3} = P\Omega_{n+2} - Q\Omega_{n+1} + R\Omega_n, \; R \neq 0,$$

where P, Q, R, W_0, W_1, W_2 are fixed integers,† and let $m>1$ be any positive integer. If

$$W_n \equiv A_n \qquad (\text{mod } m; 0 \leqq A_n \leqq m-1; n = 0, 1, \cdots)$$

we shall call $(A)_n$ the *reduced sequence* corresponding to $(W)_n$ modulo m.

If after s terms in the reduced sequence, a cycle of t terms keeps repeating itself indefinitely, $(W)_n$ will be said to *admit the period t, modulo m*. The least period that $(W)_n$ admits (modulo m) is called its *characteristic number*.‡

In this paper, I give a number of new results on the form of the characteristic number of a sequence. The principal result is the following:

If $m=p_1^{a_1}\cdots p_k^{a_k}$ is the resolution of m into its prime factors, then the characteristic number of any sequence modulo m is the least common multiple of its characteristic numbers modulis $p_1^{a_1}, \cdots, p_k^{a_k}$.

The restriction to the case of a difference equation of order 3 is mainly for convenience of notation and ease of illustration. The theorems in the first seven sections of the paper, which include my main result, may be immediately extended to the general case of a difference equation of order r.

* Presented to the Society, November 29, 1929; received by the editors in January, 1930.

† The arithmetical properties of such sequences do not seem to have been extensively investigated. Besides the references in Dickson's *History*, there is an important paper by Carmichael on the linear recursion relation (1.1) for general r (Quarterly Journal of Mathematics, vol. 48 (1920), pp. 343–372). We shall refer to this paper as Carmichael I, giving page reference. Many of Carmichael's results are summarized in a more recent paper (American Mathematical Monthly, vol. 36 (1929), pp. 132–143). Draeger (*Ueber rekurrente Reihen von höherer, inbesondere von der dritten Ordnung*, Dissertation, Jena, 1919) has discussed (1.1) in detail and given some arithmetical results for the cases $m=2, 3$ and $P\equiv 0$ (mod m).

‡ Carmichael I, p. 345. We shall omit the phrase "modulo m" when no confusion can arise.

2. **Periodicity of sequences.** We shall employ the notation

$$A_0, A_1, \cdots, A_{\lambda-1}; \dot{A}_\lambda, A_{\lambda+1}, \cdots, \dot{A}_{\lambda+\mu-1}$$

for a reduced sequence $(A)_n$ having λ non-repeating terms $A_0, A_1, \cdots, A_{\lambda-1}$ and μ repeating terms* $A_\lambda, A_{\lambda+1}, \cdots, A_{\lambda+\mu-1}$. If $\lambda=0$, $(W)_n$ is said to be *purely periodic* modulo m.

If μ is the characteristic number of $(W)_n$, then a necessary and sufficient condition that $(W)_n$ admit the period r is that† $\mu \mid r$.

THEOREM 2.1. *Every sequence $(W)_n$ becomes periodic,‡ modulo m. Moreover, if μ is the characteristic number of $(W)_n$ and λ the maximum number of non-repeating terms in the reduced sequence $(A)_n$ corresponding to $(W)_n$, then*

$$\lambda \leqq m^3 - 1; \quad 1 \leqq \mu \leqq m^3 - \lambda.$$

Call an ordered set of three consecutive elements of $(A)_n$ a triad. Then the first m^3+3 terms of $(A)_n$ contain the m^3+1 triads

$$A_0, A_1, A_2; A_1, A_2, A_3; \cdots; A_{m^3}, A_{m^3+1}, A_{m^3+2} \tag{2.1}$$

of which at most m^3 are distinct, since $0 \leqq A_n \leqq m-1$. Hence if λ, μ are the least values of s, t such that

$$A_s = A_{s+t}, A_{s+1} = A_{s+t+1}, A_{s+2} = A_{s+t+2}$$

in (2.1), the first part of the theorem follows from the linearity of (1.1). The remainder of the theorem follows from the inequalities

$$s \leqq m^3 - 1; \quad s + t + 2 \leqq m^3 + 2.$$

3. **Reduction to prime powers.** We shall now show that there is no loss of generality in supposing that m is a power of a prime.

THEOREM 3.1. *Let $(W)_n$ be any particular solution of the difference equation (1.1), and assume that $m=a\cdot b$ where $(a, b)=1$; $a, b>1$. Then the characteristic number of $(W)_n$ modulo m is the L.C.M. of its characteristic numbers modulis a and b.*

Let $\mu(x) \equiv \mu_x$ denote the characteristic number of $(W)_n$ modulo x, and let κ denote the L.C.M. of μ_a and μ_b where, by hypothesis, $a\cdot b=m$; $(a, b)=1$.

$(W)_n$ admits the period μ_m modulis a and b; therefore $\mu_a \mid \mu_m$, $\mu_b \mid \mu_m$, so that $\kappa \mid \mu_m$.

* It is understood that λ is the greatest number of non-repeating terms, and μ the smallest number of repeating terms in the reduced sequence.

† We use the customary abbreviations (a, b) for the greatest common divisor of the integers a and b, $a \mid b$ for a divides b, and L.C.M. of a and b for the least common multiple of a and b.

‡ For another proof, see Carmichael I, p. 344.

$(W)_n$ also admits the period κ modulis a and b; therefore

(3.1) $W_{\lambda+\kappa+n} - W_{\lambda+n} \equiv 0 \pmod{a}$, $W_{\lambda+\kappa+n} - W_{\lambda+n} \equiv 0 \pmod{b}$ $(n = 0, 1, \cdots)$

where λ is the number of non-repeating terms in the reduced sequence $(A)_n$ corresponding to $(W)_n$ modulo $m = a \cdot b$.

Since $(a, b) = 1$, (3.1) implies that

$$W_{\lambda+\kappa+n} - W_{\lambda+n} \equiv 0 \qquad (\text{mod } m;\ n = 0, 1, \cdots).$$

Hence $(W)_n$ admits the period κ modulo m and $\mu_m | \kappa$. Since $\kappa | \mu_m$, $\kappa = \mu_m$. The following fundamental result is a direct corollary of this theorem.

THEOREM 3.11. *Let $(W)_n$ be any particular solution of the difference equation* (1.1) *and let*

$$m = p_1^{a_1} \cdots p_k^{a_k}$$

be the resolution of m into its prime factors. Then the characteristic number of $(W)_n$ modulo m is the L.C.M. of its characteristic numbers modulis $p_1^{a_1}, \cdots, p_k^{a_k}$.

To illustrate this theorem, consider the difference equation

$$\Omega_{n+3} = \Omega_{n+2} + \Omega_{n+1} + \Omega_n$$

with the particular solution $(U)_n$ whose first few terms are

$$1, 0, 0, 1, 1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, 927, \cdots.$$

Let $\mu(m)$ denote the characteristic number of $(U)_n$ modulo m. Taking $(U)_n$ modulis 2, 3, 4, 5, 6, 7, 9, 42, we find that $\mu(2) = 4$, $\mu(3) = 13$, $\mu(4) = 8$, $\mu(6) = 52$, $\mu(7) = 48$, $\mu(9) = 39$, and $\mu(42) = 624$. Thus $\mu(42)$, for example, equals $3 \cdot 13 \cdot 16$ which is the L.C.M. of $\mu(2)$, $\mu(3)$ and $\mu(7)$; and $\mu(6)$ is the L.C.M. of $\mu(2)$ and $\mu(3)$.

4. Purely periodic sequences. We shall now give some conditions that a sequence $(W)_n$ be purely periodic, modulo m. It is easily shown that a sufficient condition* that the sequence $(W)_n$ be purely periodic is that $(R, m) = 1$. This condition is not, however, a necessary one. On the other hand, we shall prove

THEOREM 4.1. *A necessary condition that the sequence $(W)_n$ be purely periodic modulo m is that*

$$W_2 \equiv PW_1 - QW_0 \pmod{d}, \tag{4.1}$$

where d is the greatest common divisor of R and m.

* Carmichael, I, p. 344, §2.

Assume that $(W)_n$ is purely periodic modulo m, and has the characteristic number μ. Then if $(m, R)=d$, $d\,|\,m$ and $d\,|\,R$, so that

$$W_{\mu+2} \equiv PW_{\mu+1} - QW_\mu + RW_{\mu-1} \equiv PW_{\mu+1} - QW_\mu \pmod{d},$$

giving (4.1) immediately.

Unfortunately, this condition is not sufficient for pure periodicity. Consider for example the difference equation

$$\Omega_{n+3} = 2\Omega_{n+2} + \Omega_{n+1} + 3\Omega_n, \text{ with } m = 9.$$

Here $d=3$, and if we take $W_0=0$, $W_1=0$, $W_2=3$, then $W_2 \equiv 2W_1+W_0 \pmod 3$. Nevertheless, in this case $(A)_n$ is $0; \dot{0}, 3, 6, 6, 0, 6, 3, \dot{3}$.

We can, however, prove as in Theorem 3.1 that if $(W)_n$ is purely periodic modulis a and b, where $(a, b)=1$, then $(W)_n$ is purely periodic modulo $a\cdot b$. Consequently, we have the following criterion for pure periodicity:

THEOREM 4.2. *If $m=p_1^{a_1}\cdots p_k^{a_k}$ is the decomposition of m into its prime factors, then a necessary and sufficient condition that $(W)_n$ be purely periodic modulo m is that it be purely periodic modulis $p_1^{a_1}, \cdots, p_k^{a_k}$.*

We shall consider henceforth only purely periodic solutions of (1.1).

5. Singular and non-singular sequences. Let $(W)_n$ stand as usual for a particular solution of (1.1), and let $D=D(W)$ denote the determinant

$$\begin{vmatrix} W_0, & W_1, & W_2 \\ W_1, & W_2, & W_3 \\ W_2, & W_3, & W_4 \end{vmatrix}.$$

The solution $(W)_n$ is said to be non-singular (modulo m) if $(D, m)=1$ and singular if $(D, m)=d>1$.

THEOREM 5.1. *All purely periodic non-singular sequences satisfying* (1.1) *have the same characteristic number, τ, modulo m. Moreover, the characteristic number modulo m of any singular sequence is a divisor of τ.*

Let $(W)_n$ be any solution of (1.1), and $(T)_n$ any non-singular solution, and let the characteristic numbers of $(W)_n$ and $(T)_n$ modulo m be μ and τ respectively. Then we can determine integers K_0, K_1, K_2 such that

$$W_n \equiv K_0T_n + K_1T_{n+1} + K_2T_{n+2} \quad (\text{mod } m;\ n = 0, 1, \cdots) \tag{5.1}$$

where

$$0 \leqq K_0, K_1, K_2 \leqq m - 1. \tag{5.2}$$

For on account of the linearity of (1.2), (5.1) will be true provided that it is true for $n=0$, 1, and 2.

But a sufficient condition that the congruences

$$W_i \equiv K_0T_i + K_1T_{i+1} + K_2T_{i+2} \qquad (\text{mod } m;\ i = 0, 1, 2)$$

have a solution satisfying the conditions (5.2) is that $(D(T), m)=1$.

From (5.1), we see that $(W)_n$ admits all the periods of $(T)_n$, so that $\mu|\tau$. If $(W)_n$ is also non-singular, a repetition of the argument shows that $\tau|\mu$, so that $\tau=\mu$.

The characteristic number τ is called the *principal period of* (1.1) modulo m.

It is easily shown that if $(m, c)=1$, the sequences $(W)_n$ and cW_0, cW_1, $\cdots$ or for short $c(W)_n$, have the same characteristic number* modulo m. If $(m, c)>1$, this is not usually the case.

For instance, consider the difference equation and particular solution given to illustrate Theorem 3.11. The characteristic number of $(U)_n$ modulo 6 is 52. Nevertheless, the characteristic number of $3(U)_n$ modulo 6 is only 4.† Now 4 is the characteristic number of $(U)_n$ modulo $2=6/3$. We have here an illustration of the following theorem:

THEOREM 5.2. *If $(W)_n$ is any particular solution of* (1.1) *and c is any integer, the characteristic number of $c(W)_n$ modulo m equals the characteristic number of $(W)_n$ modulo m/d, where d is the greatest common divisor of m and c.*

Let $c=c'\cdot d$, $m=m'\cdot d$; $(m', c')=1$. From the congruences

$$cW_{n+3} \equiv cPW_{n+2} - cQW_{n+1} + cRW_n \qquad (\text{mod } m),$$

we obtain

$$(5.3) \qquad c'W_{n+3} \equiv c'PW_{n+2} - c'QW_{n+1} + c'RW_n \qquad (\text{mod } m';\ n = 0, 1, \cdots).$$

Since $(c', m')=1$, the characteristic number of $c'(W)_n$ modulo m' is the same as the characteristic number, κ, of $(W)_n$ modulo m'. Let μ denote the characteristic number of $c(W)_n$ modulo m. From (4.3), $(W)_n$ admits the period μ modulo m', so that $\kappa|\mu$.

But we also have

$$W_{k+\kappa} - W_k \equiv 0 \qquad (\text{mod } m';\ k = 0, 1, \cdots).$$

Hence $c'W_{k+\kappa}-c'W_k\equiv 0$ (mod m'), $cW_{k+\kappa}-cW_k\equiv 0$ (mod m), so that $c(W)_n$ admits the period κ, modulo m. Thus $\mu|\kappa$, so that $\mu=\kappa$.

* If the periodic parts of $(A)_n$ and $c(A)_n$ are merely cyclic permutations of each other, c is called a multiplier of $(W)_n$. The theory of the multipliers of a sequence is considered in §9, for m a prime p.

† Note that $D(3U)=27$, which is not prime to the modulus 6.

We can derive the following important consequence from Theorem 5.2.

THEOREM 5.3. *Let* $(S)_n$ *be any singular solution of* (1.1) *and let* d *be the greatest common divisor of* $D(S)$ *and* m. *Then the characteristic number of* $(S)_n$ *modulo* m *is a multiple of the principal period of* (1.1) *modulo* m/d.*

If $(T)_n$ is any non-singular solution and if $(D(S), m)=d$, then it is easily shown that we can determine constants K_0, K_1, K_2 such that

$$dT_n \equiv K_0S_n + K_1S_{n+1} + K_2S_{n+2} \qquad (\text{mod } m;\ n = 0, 1, \cdots),$$

where $0 \leqq K_0, K_1, K_2 \leqq m-1$. Hence $d(T)_n$ admits the periods of $(S)_n$. The theorem now follows immediately from Theorem 5.2, since $(T)_n$ is also a non-singular solution of (1.1) modulo m/d.

6. **The binomial congruence.** Consider the binomial congruence

$$x^n \equiv 1 \qquad (\text{modd } m, F(x)) \tag{6.1}$$

where it should be noted that

$$F(x) = x^3 - Px^2 + Qx - R$$

is the characteristic function of the difference equation (1.1).

The problem which immediately suggests itself is to find those values of n for which (6.1) is an identity in x. We shall see that they are the periods of the non-singular sequences of (1.1), modulo m.

If

$$(U)_n: \qquad U_0, U_1, U_2, \cdots, U_n, \cdots$$

denotes that particular solution of

$$\Omega_{n+3} = P\Omega_{n+2} - Q\Omega_{n+1} + R\Omega_n, \quad R \neq 0, \tag{1.1}$$

with the initial values $U_0=1/R$, $U_1=0$, $U_2=0$, then it may be shown by induction that

$$x^n = U_{n+1}x^2 + (U_{n+2} - PU_{n+1})x + RU_n + Q_n(x)F(x), \tag{6.2}$$

where

$$Q_0(x) = 0; \qquad Q_n(x) = \sum_{r=1}^{n} U_r x^{n-r} \qquad (n = 1, 2, \cdots).$$

Suppose that

$$U_n \equiv H_n \qquad (\text{mod } m;\ 0 \leqq H_n \leqq m-1;\ n = 0, 1, \cdots).$$

* One might conjecture from Theorem 4.3 that all singular solutions $(S)_n$ for which the greatest common divisor of $D(S)$ and m has the same value would have the same characteristic number, but it is easy to construct examples showing that this is not the case.

(6.2) then gives us the fundamental formula

$$x^n \equiv H_{n+1}x^2 + (H_{n+2} - PH_{n+1})x + RH_n \qquad (\text{modd } m, F(x)).$$

Hence $x^n \equiv 1$ (modd m, $F(x)$) identically in x when and only when $U_{n+1} \equiv U_{n+2} \equiv 0$ (mod m) and $RU_n \equiv 1$ (mod m). Thus we have the following theorem:

THEOREM 6.1. *Necessary and sufficient conditions that* (6.1) *hold identically in x for $n=\mu$ are that R be prime to m, and that the sequence $(U)_n$ admit the period μ modulo m.*

We shall assume henceforth that $(R, m) = 1$. Since

$$D(U) = \begin{vmatrix} R^{-1}, & 0, & 0 \\ 0, & 0, & 1 \\ 0, & 1, & P \end{vmatrix} = (-R)^{-1},$$

the characteristic number of $(U)_n$ is the principal period τ of (1.2) modulo m. It then follows from Theorem 6.1 that the least value of n for which (6.1) is an identity in x is τ.*

If we put $x = \alpha$ in the identity (6.2), where α is a root of $F(x) = 0$, we have the congruence

$$\alpha^n \equiv 1 \qquad (\text{mod } m).$$

Thus τ is divisible by the exponent to which α belongs modulo m, which gives us the following theorem:

THEOREM 6.2. *The principal period of* (1.1) *modulo m is divisible by the L.C.M. of the exponents to which the roots of $F(x) = 0$ belong, modulo m.*

7. Characteristic number for powers of a prime. Assume now that

$m = p^t$ $(t \geqq 1)$ is a power of a prime, p.

THEOREM 7.1. *If $(W)_n$ is any solution of* (1.1) *and $\mu^{(t)} = \mu(p^t)$, $\mu = \mu(p)$ the characteristic numbers of $(W)_n$ modulo $m = p^t$ and modulo p respectively, then*

$$\mu^{(t)} = p^b\mu \tag{7.1}$$

where† $0 \leqq b \leqq t-1$.

By Theorem 6.1, $x^{\mu^{(t)}} \equiv 1$ (modd p^t, $F(x)$), so that $x^{\mu^{(t)}} \equiv 1$ (modd p, $F(x)$) and $\mu \mid \mu^{(t)}$. Also,

* Compare the relationship between the binomial congruence $x^n \equiv 1$ (mod m) and the difference equation $\Omega_{n+1} \equiv R\Omega_n$ (mod m); $(R, m) = 1$. The characteristic number of any solution of the difference equation is an admissible value of n for the congruence.

† Carmichael (I, p. 352) gives the limits $0 \leqq b \leqq t$ for b.

$$(7.2) \qquad x^{\mu} = 1 + pP(x) + F(x)Q(x)$$

where $P(x)$ and $Q(x)$ are polynomials in x with integral coefficients. On raising both sides of (7.2) to the p^{t-1} power, we see that $x^{\mu p^{t-1}} \equiv 1$ (modd p^t, $F(x)$). By Theorem 6.1, $\mu^{(t)} \mid \mu p^{t-1}$; since $\mu \mid \mu^{(t)}$, (7.1) follows.

For illustrations of this theorem, see the examples following Theorem 3.11.*

By the same method of proof used in Theorem 7.1, we can establish the following result:

THEOREM 7.2. *If* $\sigma \geqq 1$ *and* $x^{\mu(p)} \equiv 1$ (modd p^σ, $F(x)$) *but* $x^{\mu(p)} \not\equiv 1$ (modd $p^{\sigma+1}$, $F(x)$), *then*

$$\mu(p^t) = \mu(p) \qquad \textit{if} \qquad \sigma \geqq t \geqq 1,$$
$$\mu(p^t) = p^{t-\sigma}\mu(p) \qquad \textit{if} \qquad t \geqq \sigma.$$

The problem of determining the exponent b in (7.1) is thus a generalization of Abel's famous problem† of finding the highest power of p which will divide $a^{p-1}-1$.

8. **Characteristic number for prime modulus.** Assume now that m is a prime, p. The factorization of $F(x)$ modulo p may be described by a partition of three; for example, if $F(x)$ is irreducible, we shall say it is of "type [3]", if it can be factored into an irreducible quadratic factor and a linear factor, we shall say that it is of "type [2, 1]" and so on. In any case, the factorization is unique; denote the roots which correspond to linear factors by small italic letters a, b, c and the roots which correspond to irreducible quadratic or cubic factors by small greek letters α, β, γ.

Let $L(\alpha)$; $L(a)$ denote the exponents to which the roots α; a belong modulo p and $L(\alpha, b)$; $L(\alpha, \beta, c)$, etc., the L.C.M. of the exponents to which α, b; α, β, c etc. belong modulo p.

Finally, let Δ denote the discriminant of $F(x)$, and W the matrix

$$\begin{pmatrix} W_0, & W_1, & W_2 \\ W_1, & W_2, & W_3 \\ W_2, & W_3, & W_4 \end{pmatrix}.$$

Then it is easily shown from the known algebraic theory of (1.1) that the characteristic number of $(W)_n$ is given by the following table:

* b in (7.2) may be zero; for example, take $F(x)=x^3-2x^2+x-1$ and $p=2$. The first few terms of $(U)_n$ are 1, 0, 0, 1, 2, 3, 5, 9, 16, 28, $\cdots$. Taking the sequence modulo 2 and modulo 4, we obtain 1, 0, 0, 1, 0, 1, 1, and 1, 0, 0, 1, 2, 3, 1 so that $\mu(2^2)=\mu(2)=7$.

† Crelle's Journal, vol. 3 (1828), p. 212. See also Dickson's *History*, Chapter IV.

CHARACTERISTIC NUMBERS MODULO p

Case	Type of $F(x)$	Quadratic character* of Δ modulo p	Algebraic form of W_n in terms of roots of $F(x)\equiv 0 (\mathrm{mod}\ p)$	Rank of W	Characteristic number
I	[3]	$+1$	$A\alpha^n+B\beta^n+C\gamma^n$	3	$L(\alpha)=L(\beta)=L(\gamma)$.
II	[2, 1]	-1	$A\alpha^n+B\beta^n+C\gamma^n$	3	$L(\alpha, c)=L(\beta, c)$,
			$A\alpha^n+B\beta^n$	2	$L(\alpha)=L(\beta)$,
			Cc^n	1	$L(W_1/W_0)$.
III	[1, 1, 1]	$+1$	$Aa^n+Bb^n+Cc^n$	3	$L(a, b, c)$,
			$\left.\begin{matrix}Aa^n+Bb^n\\ Bb^n+Cc^n\\ Cc^n+Aa^n\end{matrix}\right\}$	2	$\left\{\begin{matrix}L(a, b),\\ L(b, c),\\ L(c, a),\end{matrix}\right.$
			Aa^n; Bb^n; Cc^n	1	$L(W_1/W_0)$.
IV	[1², 1]	0 $PQ-9R\not\equiv 0$	$\left.\begin{matrix}(A+Bn)a^n+Cc^n\\ Bna^n+Cc^n\end{matrix}\right\}$	3	$pL(a, c)$,
			$\left.\begin{matrix}Bna^n\\ (A+Bn)a^n\end{matrix}\right\}$	2	$pL(a)$,
			Aa^n+Cc^n	2	$L(a, c)$,
			Aa^n; Cc^n	1	$L(W_1/W_0)$.
V	[1³]	0 $PQ-9R\equiv 0$	$\left.\begin{matrix}(A+Bn+Cn^2)a^n\\ (Bn+Cn^2)a^n\\ (A+Cn^2)a^n\\ Cn^2a^n\end{matrix}\right\}$	3	$pL(a)$,
			$(A+Bn)a^n$	2	$pL(a)$,
			Aa^n	1	$L(W_1/W_0)$.

The problem of determining the characteristic number for a prime modulus is thus equivalent to the problem of determining the exponent to which a given element in a Galois field of order p^3, p^2 or p belongs.†

* There exists no convenient criterion for distinguishing the cases when $F(x)$ is of type [3], and of type [1, 1, 1]. See Dickson's *History*, vol. I, pp. 252–256.

† If we call a difference equation *primitive* (modulo p) when there is only one sequence belonging to it, then, just as in the allied theory of primitive marks in a Galois field, or primitive roots of p^n, we can show that for every prime p, there exist primitive difference equations of any order r.

We shall devote the concluding two sections of the paper to studying case I. The characteristic number modulo p of all sequences satisfying (1.1) is then the same, and equals the exponent to which any root of $F(x)=0$ belongs in the Galois field $[p^3]$ associated with $F(x)$. We shall call this number the *period* of $F(x)$, and denote it by τ. It is of course a divisor of p^3-1.

If α, β, γ are the roots of $F(x)=0$, and $S_n = \alpha^n+\beta^n+\gamma^n$, then*

$$\beta \equiv \alpha^p, \ \gamma \equiv \alpha^{p^2}; \ R \equiv \alpha^{1+p+p^2}; \ S_n \equiv \alpha^n + \alpha^{pn} + \alpha^{p^2 n}. \tag{8.1}$$

9. **Multipliers of cycles.** If $(A)_n$ is any reduced sequence of residues, so that

$$A_{n+3} \equiv PA_{n+2} - QA_{n+1} + RA_n \qquad (0 \leqq A_n \leqq p-1, n = 0, \pm 1, \cdots)$$

the τ residues $A_0, \cdots, A_{\tau-1}$ are said to form a cycle (A) belonging to $F(x)$. Two such cycles are said to be equal if either can be obtained from the other by a cyclic permutation of its elements.

Let L be any residue. If the cycle $LA_0, \cdots, LA_{\tau-1}$ equals the cycle (A), then L is called a multiplier of (A); we have

$$LA_n \equiv A_{n+l} \qquad (n = 0, \cdots, \tau - 1). \tag{9.1}$$

Since any other cycle (B) of $F(x)$ may be expressed in the form

$$B_n \equiv K_0 A_n + K_1 A_{n+1} + K_2 A_{n+2} \qquad (n = 0, \cdots, \tau - 1),$$

where K_0, K_1, K_2 are residues, the following theorem is apparent:

THEOREM 9.1. *If L is a multiplier of one cycle of $F(x)$, it is a multiplier of all the cycles of $F(x)$, and the integer l in equation* (9.1) *does not depend on the particular cycle (A) used in defining L.*

We shall call l the span of L.

The following three theorems are easily established:

THEOREM 9.2. *The multipliers of the cycles of $F(x)$ form a group with respect to multiplication modulo p.*

THEOREM 9.3. *Two multipliers with the same span are identical modulo p.*

THEOREM 9.4. *The group of the multipliers of the cycles of $F(x)$ is cyclic, and a generator is the unique multiplier of least span.*

Let M denote this unique multiplier. From Theorem 9.4, there follows:

* It is understood that all congruences in which the modulus is not indicated are to be taken to the modulus p over the field of the p residues $0, 1, \cdots, p-1$. For the properties of Galois fields which are assumed, see Dickson, work cited.

THEOREM 9.41. *The span of M divides the span of every other multiplier.*

THEOREM 9.5. *If $p^2+p+1\equiv\pi$ (mod τ), then R is a multiplier of span π.*

Since $p^3\equiv 1$ (mod τ),

$$p^2\pi \equiv p\pi \equiv \pi \qquad (\text{mod } \tau).$$

By (8.1),

$$RS_n \equiv \alpha^\pi(\alpha^n + \alpha^{pn} + \alpha^{p^2n}) \equiv \alpha^{n+\pi} + \alpha^{p(n+\pi)} + \alpha^{p^2(n+\pi)} \equiv S_{n+\pi}.$$

Hence by Theorem 9.1, R is a multiplier of span π.

As an immediate consequence of these theorems, we see that R is congruent to a power of M, modulo p, and that the span of M divides π.

THEOREM 9.6. *If $\epsilon(M)$ is the exponent to which the multiplier M belongs modulo p, and if μ is its span, then*

$$\tau = \epsilon(M)\mu \tag{9.2}$$

*where τ is the period of $F(x)$.**

$\mu\,|\,\tau$; for if $\tau=s\mu+t$ $(0\leqq t\leqq \mu-1)$, then by (9.1)

$$M^{\tau-s}A_n \equiv A_{n+(\tau-s)\mu} \equiv A_{n+t} \qquad (n=0,\cdots,\tau-1)$$

so that $\mu\,|\,t$; $t=0$. Similarly, $\epsilon(M)\,|\,\tau$ so that $\epsilon(M)\cdot\mu\,|\,\gamma\tau$ where $\gamma=(\epsilon(M),\mu)=3$ or 1. But

$$A_n \equiv M^{\epsilon(M)}A_n \equiv A_{n+\epsilon(M)\mu} \qquad (n = 0,\cdots,\tau-1).$$

Hence $\tau\,|\,\epsilon(M)\mu$, so that either $\tau=\epsilon(M)\mu$ or $3\tau=\epsilon(M)\mu$.

The latter case can occur only when $(\epsilon(M),\mu)=3$; but then

$$A_{n+\tau} \equiv A_n \equiv M^{\epsilon(M)/3}\cdot A_n$$

so that $M^{\epsilon(M)/3}\equiv 1$, contradicting the definition of $\epsilon(M)$.

We shall call μ the *restricted period* of $F(x)$; since it is a divisor of p^2+p+1, we may write

$$p^2+p+1 = \kappa\cdot\mu. \tag{9.3}$$

We easily find that

$$M^\kappa \equiv R,\ \ M^3 \equiv R^\mu, \tag{9.31}$$

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* (9.2) is a special case of a theorem given in Carmichael I, p. 355. Carmichael calls μ the restricted period of the sequence whose characteristic number is τ.

so that

$$\epsilon(R) \mid \epsilon(M) \mid 3\epsilon(R), \tag{9.32}$$

where $\epsilon(R)$ is the exponent to which R belongs, modulo p.

If $p \equiv 2 \pmod 3$, then $p^2+p+1 \equiv 1 \pmod 3$ and it follows from Theorem 9.6 and equation (9.32) that $\tau = \epsilon(R)\mu$ where $\mu \mid p^2+p+1$ and $(\mu, 3) = 1$. The concluding section of the paper is devoted to the more interesting case when $p \equiv 1 \pmod 3$.

10. Period for primes of form $3m+1$. We shall assume throughout this section that

$$p = 3^k n + 1, \ (n, 3) = 1; \ k \geqq 1. \tag{10.1}$$

Then $p^2+p+1 \equiv 0 \pmod 3$, $\not\equiv 0 \pmod 9$, and from (9.3),

$$\kappa\mu/3 \equiv 1 \pmod{\epsilon(M)} \equiv 1 \pmod{\epsilon(R)}. \tag{10.2}$$

THEOREM 10.1. *If p is of the form $3^k n+1$, and μ denotes the restricted period of $F(x)$, then $\mu \equiv 0 \pmod 3$ when and only when $\epsilon(R) \equiv 0 \pmod{3^k}$.*

If this last condition holds, then

$$\tau = \epsilon(R)\mu, \ M \equiv R^{\mu/3} \pmod p. \tag{10.3}$$

Let $\mu = 3\mu'$. Then $(\mu', 3) = 1$, $(\kappa, 3) = 1$, and from (10.2) $\kappa\mu' \equiv 1 \pmod{\epsilon(M)}$. From (9.31), $R \equiv M^{\kappa} \pmod p$ and

$$M \equiv R^{\mu'} \pmod p. \tag{10.31}$$

Hence by (9.32), $\epsilon(M) = \epsilon(R)$. Assume that

$$\epsilon(M) = 3^s \cdot \sigma, \ (\sigma, 3) = 1; \ s \leqq k;$$

then by (9.2), $\tau = 3^{s+1}\tau'$;

$$\tau'; \quad (\tau', 3) = 1.$$

Now it is easily seen that $\alpha^{\tau'}$ is a primitive 3^{s+1} root of unity, modulo p, and hence a residue of p if and only if $s < k$. Assume that $s < k$. Then if $\alpha^{\tau'} \equiv Q$, $\alpha^{p\tau'} \equiv \alpha^{p^2\tau'} \equiv Q$, so that by (8.1),

$$QS_n \equiv S_{n+\tau'},$$

and by Theorem 9.1, Q is a multiplier. By Theorem 9.4, $3^{s+1} = \epsilon(Q)$; but $\epsilon(Q) \mid \epsilon(M)$. Hence $s = k$ and

$$\epsilon(R) = \epsilon(M) \equiv 0 \pmod{3^k}. \tag{10.32}$$

Conversely, if $\epsilon(R)\equiv 0 \pmod{3^k}$, then (10.32) follows from (9.32). If $\mu\not\equiv 0 \pmod 3$, then $\kappa\equiv 0 \pmod 3$, and (9.32) gives

$$M^{\kappa\epsilon(M)/3} \equiv 1 \equiv R^{\epsilon(M)/3} \equiv R^{\epsilon(R)/3} \pmod p,$$

contrary to the definition of $\epsilon(R)$. Equation (10.3) now follows from Theorem 9.6, (10.32) and (10.31).

THEOREM 10.2. *If* $\epsilon(R)\equiv 0 \pmod 3$, $\not\equiv 0 \pmod{3^k}$, *then* $\tau = 3\epsilon(R)\mu$, *where* $\mu \mid (p^2+p+1)/3$.

The last part of the theorem follows immediately from Theorem 10.1, so that it is sufficient, in view of Theorem 9.6, to prove that $\epsilon(M)=3\epsilon(R)$. But this equality follows from (9.31), (9.32), since $(\mu, \epsilon(R))=1$.

THEOREM 10.21. *If R is not a cubic residue of* p, τ *is of the form* $3\epsilon(R)\sigma$, *where* $\sigma \mid (p^2+p+1)/3$.

If R is not a cubic residue of p, $\epsilon(R)\equiv 0 \pmod 3$, and the theorem follows from Theorems 10.1 and 10.2.

If $\epsilon(R)\not\equiv 0 \pmod 3$, then $(\mu, 3)=1$, but I have not found a criterion to distinguish whether $\tau = 3\epsilon(R)\mu$ or $\tau=\epsilon(R)\mu$. The discovery of such a criterion would fill a serious lacuna in the theory. To illustrate the two cases possible, take $p=7$. Then $p^2+p+1=57=3\cdot 19$, so that $\mu=19$. For the irreducible polynomial modulo 7, $F(x)=x^3+x-1$, $\epsilon(R)=1$ and we find by direct computation that the period τ is $57=3\epsilon(R)\mu$. However, for x^3-3x^2+4x-1, the period is only $19=\epsilon(R)\mu$.

Finally, the case $p=3$ may be easily treated by a direct enumeration of the possible cases.*

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* Draeger's Thesis contains such an enumeration for certain forms of $F(x)$.

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THE DISTRIBUTION OF RESIDUES IN A SEQUENCE SATISFYING A LINEAR RECURSION RELATION*

BY

MORGAN WARD

I. Introduction

1. Statement of problem. Let

$(W)_n$: $$W_0, W_1, \cdots, W_n, \cdots$$

denote a sequence of integers satisfying the linear difference equation of order $r=3$,

$$\Omega_{n+3} = P\Omega_{n+2} - Q\Omega_{n+1} + R\Omega_n, \quad R \neq 0, \tag{1.1}$$

where P, Q, R, W_0, W_1, W_2 are fixed integers.†

If m is a positive integer, and if

$$W_n \equiv A_n \mod m, \quad 0 \leqq A_n \leqq m-1,$$

we shall call

$(A)_n$: $$A_0, A_1, \cdots, A_n, \cdots$$

the *reduced sequence corresponding to* $(W)_n$, *modulo* m.

It is easily shown the $(A)_n$ is periodic; following Carmichael,‡ we shall call its smallest period, τ, the *characteristic number* of $(W)_n$ modulo m.

The object of this memoir is to attack the following fundamental distribution problem:§

Given the numerical values of the integers $P, Q, R, W_0, W_1, W_2, m$ *and* τ, *to determine the distribution of the residues* $0, 1, 2, \cdots, m-1$ *among any* τ *terms of the reduced sequence* $(A)_n$.

There are really two distinct problems involved here: the determination of the particular place a given residue occurs in $(A)_n$ and the determination of the number of times a given residue occurs in any τ terms of $(A)_n$. Both

* Presented to the Society, November 29, 1929; received by the editors in January, 1930.

† For references to investigations of (1.1) see Dickson's *History*, vol. 1, chapter 17. For a general discussion of the problems in number theory connected with (1.1), see Carmichael, American Mathematical Monthly, vol. 36 (1929), pp. 132–143.

‡ Carmichael, Quarterly Journal of Mathematics, vol. 48 (1920), pp. 344–345.

§ As far as I am aware, this problem has not been explicitly considered for difference equations of order greater than two. In a paper which has already appeared in these Transactions I have considered the problem of determining τ, given P, Q, R, W_0, W_1, W_2 and m.

problems may be readily solved in particular cases. Consider for example the difference equation $\Omega_{n+3}=\Omega_{n+2}+\Omega_{n+1}-\Omega_n$ with $W_0=0$, $W_1=1$, $W_2=2$. But the general solution of either problem presents considerable difficulties.*

I shall confine myself here almost entirely to the second, simpler, distribution problem for the special case when m is a prime p and the characteristic function of (1.1),

$$(1.2) \qquad F(x) = x^3 - Px^2 + Qx - R,$$

is irreducible modulo p. A discussion of this case is a necessary preliminary to the more complicated cases when m is composite or when the characteristic function (1.2) is reducible modulo p.

2. **Plan of paper and principal results.** Let $k(i)=k_i$ denote the number of times the least positive residue i (mod p) occurs in the first τ terms

$$(A): \qquad A_0, A_1, \cdots, A_{\tau-1}$$

of any reduced sequence $(A)_n$ (mod p).† Regarding k_i as a function of i, we shall speak of it as the *distribution function for the cycle* (A) *of* $F(x)$ *associated with* $(A)_n$ *and* $(W)_n$.

If we know the distribution function for the cycle (A), then we will know it for the cycle (B) if the three initial values B_0, B_1, B_2 of (B) happen to be three consecutive elements of (A). It is thus important to be able to tell from the initial values of two sequences whether or not their cycles are distinct. This problem is dealt with in §§6 and 7, where it is reduced to the problem of determining whether or not any given three consecutive residues appear in a *fixed* cycle (K) of $F(x)$. The preliminary definitions and results needed there and in the body of the paper are developed in §§3, 4, and 5. In §8 I digress slightly to give some results connected with the first distribution problem.

In §9 I prove that the number of zeros that can occur in each cycle of $F(x)$ are not independent of one another, but must satisfy two simple diophantine equations. In §10, I apply this result to determine completely the number of zeros which can occur in any cycle of $F(x)$ when $\tau=(p^2+p+1)/3$. In §11 I prove that if $\tau=p^2+p+1$, then every residue occurs in every cycle at least once.

In §12 it is proved that the distribution problem is essentially the same for all difference equations (1.1) with the same characteristic number τ

* In connection with the first problem, probably the best known result is that if $W_0=3$, $W_1=P$, $W_2=P^2-2Q$ and p is a prime, then $W_n\equiv W_{np}\equiv W_{np^2}$ mod p. In §8 of this paper I shall give several new results of a similar character.

† We shall omit the words "modulo p" when no confusion can arise.

modulo p, and that it can be reduced to the case when τ divides p^2+p+1 and is prime to 3.

In §13, I show that the distribution function $k(n)$ for any cycle (A) is known as soon as we know the least positive residues of k_i modulis p and 3. In particular, k_0 is known as soon as its residue modulo p is known.

In §15, I give an explicit formula which determines k_i modulo p as the residue of a summation taken over the solutions of a certain diophantine system. This system is discussed fully in §14, and a general method of solving it is given. I have been unable to determine the residue of k_i modulo 3 save in special cases.

In §16, I apply my results to various special cases, obtaining theorems like the following:

If $p=3N+1$, $3\tau=p^2+p+1$, *and* k_0 *is the number of terms divisible by* p *in the first* τ *terms of* $(S)_n$, *where* $S_0=3$, $S_1=P$, $S_2=P^2-2Q$, *then* k_0 *is the least positive residue modulo* p *of* $(2N+1)(1+3N!/(N!)^3)$.

Finally in §17, I give a method for obtaining an upper limit to the size of k_i for any (A) and τ.

3. **Preliminary definitions. Triads.** Let the roots of $F(x)=0$ in the Galois field of order p^3 associated with $F(x)$ be denoted by* α, α^p, α^{p^2}, and suppose that

$$\alpha^n + \alpha^{pn} + \alpha^{p^2n} \equiv S_n \mod p \quad \left(\begin{matrix} 0 \leqq S_n \leqq p-1, \\ n = 0, \pm 1, \pm 2, \cdots \end{matrix}\right).$$

Then

$$S_1 \equiv P,\ RS_{-1} \equiv Q,\ \alpha^{1+p+p^2} \equiv R \qquad \mod p.$$

We shall refer to the p numbers

$$0, 1, 2, \cdots, p-1$$

which form a sub-field in the Galois field as *residues*. The characteristic number τ is simply the exponent to which α belongs in the Galois field; we shall also refer to it as the *period* of $F(x)$ (modulo p).

The τ residues $A_0, A_1, \cdots, A_{\tau-1}$ of any reduced sequence $(A)_n$ will be said to form a *cycle belonging to* $F(x)$. The cycle (S), where S_n is defined above, will be called the *principal cycle* of $F(x)$.

An ordered set of three residues (or more generally, of three rational integers) A', B', C' will be called a *triad*, and denoted by $[A', B', C']$. The τ triads

$$[A_0, A_1, A_2],\ [A_1, A_2, A_3], \cdots, [A_{\tau-2}, A_{\tau-1}, A_0],\ [A_{\tau-1}, A_0, A_1]$$

will be called the *triads belonging to the cycle* (A).

* For the properties of Galois fields which are assumed here, see Dickson, *Linear Groups*, Leipzig, 1901.

Two triads are equal when and only when they are identical modulo p; two cycles are equal if one may be derived from the other by a cyclic permutation of its elements. It is clear that any cycle is completely specified by any one of its triads; furthermore, two given cycles have either all or none of their triads in common.

The cycles whose initial triads are $[1, 0, 0]$, $[0, 1, 0]$, and $[0, 0, 1]$ will be denoted by (X), (Y), and (Z) respectively.*

4. **Multipliers and blocks.** If L is any residue such that the cycles

$$LA_0, LA_1, \cdots, LA_{\tau-1} \text{ and } A_0, A_1, \cdots, A_{\tau-1}$$

are equal (modulo p), L is called a *multiplier* of (A).

In the paper previously referred to, I have shown that every cycle of $F(x)$ has the same multipliers, and that there exists a unique "basic multiplier" M such that every other multiplier is congruent to some power of M.

It follows that if $e=\epsilon(M)$ denotes the exponent to which M belongs modulo p, then there are exactly e distinct multipliers. e moreover divides τ and the quotient divides p^2+p+1. If we write $\tau=\epsilon(M)\mu$, $\mu \mid p^2+p+1$, then μ is called the restricted period† of $F(x)$.

Let

$$et = p - 1, \quad \mu\kappa = p^2 + p + 1.$$

Then there are exactly t distinct cycles modulo p among the $p-1$ cycles

$$XA_0, XA_1, \cdots, XA_{\tau-1} \qquad (X = 1, 2, \cdots, p - 1).$$

These t cycles will be said to form a *block* of cycles. There are in all exactly κ distinct blocks of cycles; we shall denote them by the capital German letters $\mathfrak{B}_1, \mathfrak{B}_2, \cdots, \mathfrak{B}_\kappa$.

In particular, it will be understood that $\mathfrak{B}_1$ is the block containing the principal cycle (S).

The number of times a given residue appears in a given block is given by the following easily established theorem.

THEOREM 4.1. *If b_0 denotes the number of times the residue* 0 *appears in the first μ terms of any cycle of a given block $\mathfrak{B}$, then every residue other than zero appears in $\mathfrak{B}$ exactly $\mu-b_0$ times, while the residue zero appears $(p-1)b_0$ times.*

5. **Illustration.** In order to clarify the definitions of the preceding two sections, we give all the cycles of $F(x)=x^3+3x^2+4x+1$ for $p=7$, and a list of the notations introduced.

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* For a number of algebraic properties of the associated sequences, see Bell, Tôhoku Mathematical Journal, vol. 24 (1924), pp. 169–184. The terms of the principal cycle (S) and (X), (Y), (Z) are connected by the simple relation $S_n \equiv X_n+Y_{n+1}+Z_{n+2} \bmod p$.

† This term is due to Carmichael, who uses it in a slightly more general sense. See Quarterly Journal paper, p. 354.

Complete cycles, grouped by blocks

$\mathfrak{B}_1$

$$\begin{cases} 3,\ 4,\ 1,\ 6,\ 2,\ 4,\ 2,\ 4,\ 4,\ 5,\ 0,\ 4,\ 4,\ 0,\ 1,\ 0,\ 3,\ 4,\ 4, \\ 4,\ 3,\ 6,\ 1,\ 5,\ 3,\ 5,\ 3,\ 3,\ 2,\ 0,\ 3,\ 3,\ 0,\ 6,\ 0,\ 4,\ 3,\ 3. \end{cases}$$

$$\begin{cases} 6,\ 1,\ 2,\ 5,\ 4,\ 1,\ 4,\ 1,\ 1,\ 3,\ 0,\ 1,\ 1,\ 0,\ 2,\ 0,\ 6,\ 1,\ 1, \\ 1,\ 6,\ 5,\ 2,\ 3,\ 6,\ 3,\ 6,\ 6,\ 4,\ 0,\ 6,\ 6,\ 0,\ 5,\ 0,\ 1,\ 6,\ 6. \end{cases}$$

$$\begin{cases} 2,\ 5,\ 3,\ 4,\ 6,\ 5,\ 6,\ 5,\ 5,\ 1,\ 0,\ 5,\ 5,\ 0,\ 3,\ 0,\ 2,\ 5,\ 5, \\ 5,\ 2,\ 4,\ 3,\ 1,\ 2,\ 1,\ 2,\ 2,\ 6,\ 0,\ 2,\ 2,\ 0,\ 4,\ 0,\ 5,\ 2,\ 2. \end{cases}$$

$\mathfrak{B}_2$

$$\begin{cases} 0,\ 0,\ 1,\ 4,\ 5,\ 3,\ 2,\ 5,\ 2,\ 0,\ 1,\ 2,\ 4,\ 0,\ 3,\ 1,\ 6,\ 3,\ 1, \\ 0,\ 0,\ 6,\ 3,\ 2,\ 4,\ 5,\ 2,\ 5,\ 0,\ 6,\ 5,\ 3,\ 0,\ 4,\ 6,\ 1,\ 4,\ 6. \end{cases}$$

$$\begin{cases} 0,\ 0,\ 2,\ 1,\ 3,\ 6,\ 4,\ 3,\ 4,\ 0,\ 2,\ 4,\ 1,\ 0,\ 6,\ 2,\ 5,\ 6,\ 2, \\ 0,\ 0,\ 5,\ 6,\ 4,\ 1,\ 3,\ 4,\ 3,\ 0,\ 5,\ 3,\ 6,\ 0,\ 1,\ 5,\ 2,\ 1,\ 5. \end{cases}$$

$$\begin{cases} 0,\ 0,\ 3,\ 5,\ 1,\ 2,\ 6,\ 1,\ 6,\ 0,\ 3,\ 6,\ 5,\ 0,\ 2,\ 3,\ 4,\ 2,\ 3, \\ 0,\ 0,\ 4,\ 2,\ 6,\ 5,\ 1,\ 6,\ 1,\ 0,\ 4,\ 1,\ 2,\ 0,\ 5,\ 4,\ 3,\ 5,\ 4. \end{cases}$$

$\mathfrak{B}_3$

$$\begin{cases} 0,\ 1,\ 3,\ 1,\ 5,\ 6,\ 3,\ 4,\ 5,\ 1,\ 1,\ 2,\ 3,\ 3,\ 5,\ 5,\ 4,\ 5,\ 6, \\ 0,\ 6,\ 4,\ 6,\ 2,\ 1,\ 4,\ 3,\ 2,\ 6,\ 6,\ 5,\ 4,\ 4,\ 2,\ 2,\ 3,\ 2,\ 1. \end{cases}$$

$$\begin{cases} 0,\ 2,\ 6,\ 2,\ 3,\ 5,\ 6,\ 1,\ 3,\ 2,\ 2,\ 4,\ 6,\ 6,\ 3,\ 3,\ 1,\ 3,\ 5, \\ 0,\ 5,\ 1,\ 5,\ 4,\ 2,\ 1,\ 6,\ 4,\ 5,\ 5,\ 5,\ 3,\ 1,\ 4,\ 4,\ 6,\ 4,\ 2. \end{cases}$$

$$\begin{cases} 0,\ 3,\ 2,\ 3,\ 1,\ 4,\ 2,\ 5,\ 1,\ 3,\ 3,\ 6,\ 2,\ 2,\ 1,\ 1,\ 5,\ 1,\ 4, \\ 0,\ 4,\ 5,\ 4,\ 6,\ 3,\ 5,\ 2,\ 6,\ 4,\ 4,\ 1,\ 5,\ 5,\ 6,\ 6,\ 2,\ 6,\ 3. \end{cases}$$

For this case, $p-1=6$, $p^2+p+1=57$, $M=6$, $e=2$, $t=3$, $\mu=19$, $\kappa=3$, $\tau=38$ and in $\mathfrak{B}_1$, $b_0=3$.

List of notation

$\left.\begin{array}{l}\text{Characteristic number} \\ \quad\text{period of } F(x) \\ \text{Number of elements or} \\ \quad\text{triads in a cycle}\end{array}\right\}\tau$

Restricted period of $F(x)$: μ

Basic multiplier: M

Exponent to which M belongs modulo p: $e=\epsilon(M)$.

Number of cycles in a block: t

Number of blocks: κ

Connecting relations:

$\mu\kappa=p^2+p+1$,

$et=p-1$,

$\tau=e\mu$.

Triad: $[U, V, W]$

Cycle: (A)

Block: $\mathfrak{B}$

II. The distribution of triads

6. **Invariant of a cycle.** Let us assume that we know the distribution functions of the cycles (U), (V), $\cdots$, (W) and that we are given the initial values $A_0=K$, $A_1=L$, $A_2=M$ of some cycle (A). Then if the triad $[K, L, M]$ occurs in one of the cycles (U), (V), $\cdots$, (W), the distribution function of (A) is also known. We are thus led to consider the problem of determining to what cycle any given triad belongs. In this section we shall restrict ourselves to the simplest case, when we know beforehand that the triad belongs to a certain block.

First, if α, β, γ are the roots of

$$F(x) = x^3 - Px^2 + Qx - R = 0 \tag{6.1}$$

and

$$W_n = \sum_{(\alpha)}(K_0 + K_1\alpha + K_2\alpha^2)\alpha^n \qquad (n = 0, 1, \cdots)$$

is the general term of any sequence $(W)_n$ satisfying (1.1), then it is easily shown that the determinant

$$D_n(W) = \begin{vmatrix} W_n, & W_{n+1}, & W_{n+2} \\ W_{n+1}, & W_{n+2}, & W_{n+3} \\ W_{n+2}, & W_{n+3}, & W_{n+4} \end{vmatrix}$$

has the value

$$D_n(W) = R^n\Delta N(w), \tag{6.11}$$

where R is the constant term of the characteristic equation (6.1), Δ is the discriminant of $F(x)$, and $N(w)$ the norm of the algebraic number $w=K_0+K_1\alpha+K_2\alpha^2$.

Consequently, if $\epsilon(R)$ denotes the exponent to which R belongs modulo p, and if (A) is the cycle corresponding to $(W)_n$ modulo p, then the value of

$$J(A) \equiv [\Delta_n(A)]^{\epsilon(R)} \qquad \mod p$$

is independent of n. We shall call this residue the *invariant* of the cycle (A).

By means of (1.1), we can express the determinant $\Delta_n(W)$ as a polynomial in W_n, W_{n+1}, W_{n+2}. If we define $\Lambda(K, L, M)$ for all values of its arguments to be the polynomial

$$\begin{aligned} \Lambda(K, L, M) = &- R^2K^3 + 2QRK^2L - PRK^2M - (PR + Q^2)KL^2 \\ &+ (PQ + 3R)KLM - QKM^2 + (PQ - R)L^3 - (P^2 + Q)L^2M \\ &+ 2PLM^2 - M^3, \end{aligned}$$

then

$$\Delta_n(W) = \Delta(W_n, W_{n+1}, W_{n+2}). \tag{6.12}$$

Thus if A_0, A_1, A_2 are the initial values of any cycle (A), the invariant $J(A)$ is determined by the congruence

$$J(A) \equiv [\Delta(A_0, A_1, A_2)]^{\epsilon(R)} \qquad \text{mod } p.$$

Now if L is any constant residue, $\Delta_n(L\cdot W) = L^3\Delta_n(W)$. Hence if $(L\cdot A)$ denotes the cycle $LA_0, LA_1, \cdots, LA_{\tau-1}$,

$$J(L\cdot A) \equiv L^{3\epsilon(R)}J(A) \qquad \text{mod } p. \tag{6.2}$$

In the work previously referred to, I have shown that either $\epsilon(M) = \epsilon(R)$ or $\epsilon(M) = 3\epsilon(R)$, where it will be recalled that $\epsilon(M)$ is the exponent to which the basic multiplier M belongs modulo p. If $p\equiv 2$ mod 3, then $\epsilon(M)$ necessarily equals $\epsilon(R)$. Moreover, $L^{3\epsilon(M)}\equiv 1$ mod p when and only when $L^{\epsilon(M)}\equiv 1$ mod p; hence, from (6.2) $J(L\cdot A)\equiv J(A)$ mod p when and only when L is a multiplier of (A). A precisely similar result holds if $p\equiv 1$ mod 3 and $\epsilon(M) = 3\epsilon(R)$.*

It follows that in these two cases if $(A^{(1)}), \cdots, (A^{(t)})$ are the t distinct cycles of a given block $\mathfrak{B}$, the invariants $J(A^{(1)}), \cdots, J(A^{(t)})$ are all incongruent to one another modulo p. We thus obtain the following theorem.

THEOREM 6.1. *If $p\equiv 2$ mod 3 or $p\equiv 1$ mod 3, and $\epsilon(M) = 3\epsilon(R)$, and if $[K, L, M]$ is any triad of the block $\mathfrak{B}$, then a necessary and sufficient condition that $[K, L, M]$ belong to the cycle (A) of $\mathfrak{B}$ is that*

$$\{\Delta(K, L, M)\}^{\epsilon(R)} \equiv J(A) \qquad \text{mod } p. \tag{6.3}$$

If $p\equiv 1$ mod 3 and $\epsilon(M) = \epsilon(R)$ (which implies that $\epsilon(M)\not\equiv 0$ mod 3), we cannot go quite so far. For if ω is a primitive cube root of unity modulo p, then

$$\Delta_n(\omega\cdot A) = \omega^3\Delta_n(A) \equiv \Delta_n(A) \qquad \text{mod } p.$$

Consequently, since ω is not a multiplier, the three cycles (A), $(\omega\cdot A)$, $(\omega^2\cdot A)$ are distinct, and will have the same invariant. (6.3) must then be replaced by

$$\{\Delta(K, L, M)\}^{\epsilon(R)} \equiv J(A) = J(\omega\cdot A) = J(\omega^2\cdot A),$$

and for any given triad $[K, L, M]$ of the block $\mathfrak{B}$, we can ascertain merely that it must be in one of three cycles of $\mathfrak{B}$.

* If $p = 3^lN+1$ sufficient conditions for $\epsilon(M) = 3\epsilon(R)$ are $\epsilon(R)\equiv 0$ mod 3, $\not\equiv 0$ mod 3^k; or R not a cubic residue of p.

7. Distribution of triads in cycles. Let

$$(K):\qquad K_0, K_1, \cdots, K_{\tau-1}$$

denote a fixed cycle of $F(x)$. We shall show that we can determine whether or not two triads $[A, B, C]$ and $[A', B', C']$ belong to the same cycle if we know all the triads which belong to (K).

If L_0, L_1, L_2 are determined by the congruences

$$\begin{aligned} A &\equiv L_0K_0 + L_1K_1 + L_2K_2, \\ B &\equiv L_0K_1 + L_1K_2 + L_2K_3, \\ C &\equiv L_0K_2 + L_1K_3 + L_2K_4, \end{aligned} \tag{7.1}$$

then a necessary and sufficient condition that $[A', B', C']$ should belong to the same cycle as $[A, B, C]$ is that for some value of m there should exist congruences of the form

$$\begin{aligned} A' &\equiv L_0K_m + L_1K_{m+1} + L_2K_{m+2}, \\ B' &\equiv L_0K_{m+1} + L_1K_{m+2} + L_2K_{m+3}, \\ C' &\equiv L_0K_{m+2} + L_1K_{m+3} + L_2K_{m+4}. \end{aligned} \tag{7.2}$$

Now, by means of the difference equation (1.1), we can express K_{m+3} and K_{m+4} in (7.2) linearly in terms of K_m, K_{m+1}, K_{m+2}. Write A'', B'', C'' for K_m, K_{m+1}, K_{m+2}. Then if we introduce the abbreviations $[LU]_i$ ($U=X, Y, Z$; $i=1, 2, 3$) for the sums $L_0U_i+L_1U_{i+1}+L_2U_{i+2}$, the equations (7.2) give the following values for A'', B'', C'':

$$A'' \equiv \frac{|A', [LY]_1, [LZ]_2|}{|[LX]_0, [LY]_1, [LZ]_2|}, \text{ etc.}, \tag{7.3}$$

where $|A', [LY]_1, [LZ]_2|$ stands for the determinant

$$\begin{vmatrix} A', & [LY]_0, & [LZ]_0 \\ B', & [LY]_1, & [LZ]_1 \\ C', & [LY]_2, & [LZ]_2 \end{vmatrix},$$

and so on.

If we treat (7.1) in a similar manner, letting $\{KX\}_i$ stand for the sum $K_0X_i+K_1Y_i+K_2Z_i$ $(i=0, 1, 2, 3, 4)$ we find that

$$L_0 \equiv \frac{|A, \{KX\}_2, \{KX\}_4|}{|\{KX\}_0, \{KX\}_2, \{KX\}_4|}, \text{ etc.}, \tag{7.4}$$

where $|A, \{KX\}_2, \{KX\}_4|$ stands for the determinant

$$\begin{vmatrix} A, & \{KX\}_1, & \{KX\}_2 \\ B, & \{KX\}_2, & \{KX\}_3 \\ C, & \{KX\}_3, & \{KX\}_4 \end{vmatrix}$$

and so on.

We thus obtain the following theorem.

THEOREM 7.1. *A necessary and sufficient condition that the triads* $[A, B, C]$ *and* $[A', B', C']$ *should belong to the same cycle is that the triad* $[A'', B'', C'']$ *determined by* (7.3) *and* (7.4) *should belong to the cycle* (K).

We have thus reduced the problem of determining to what cycle any triad belongs to the problem of determining whether or not a triad belongs to some fixed cycle, say the principal cycle of $F(x)$.

8. **The distribution of zeros in a cycle.** We shall assume in this section that the cycles of $F(x)$ have no multiplier other than the trivial multiplier unity which implies that τ divides p^2+p+1. The distribution of zeros in an arbitrary cycle (U) of $F(x)$ then depends in a remarkable manner upon the distribution of residues in the principal cycle (S), as is shown by the following theorem:

THEOREM 8.1. *Let* (U) *denote a definite cycle of* $F(x)$ *in which it is known that*

$$U_a \equiv U_b \equiv 0 \mod p \qquad (a \neq b). \tag{8.1}$$

Then a necessary and sufficient condition that $U_c \equiv 0 \bmod p$ *is that**

$$S_{a+bp+cp^2} \equiv S_{a+bp^2+cp} \mod p. \tag{8.2}$$

Let

$$U_n \equiv K_0 S_n + K_1 S_{n+1} + K_2 S_{n+2} \mod p. \tag{8.3}$$

Then (8.1) gives

$$K_0 : K_1 : K_2 = \begin{vmatrix} S_{a+1}, & S_{a+2} \\ S_{b+1}, & S_{b+2} \end{vmatrix} : \begin{vmatrix} S_{a+2}, & S_a \\ S_{b+2}, & S_b \end{vmatrix} : \begin{vmatrix} S_a, & S_{a+1} \\ S_b, & S_{b+1} \end{vmatrix}.$$

Hence by (8.3)

$$U_n \equiv LD(a, b, n) \mod p,$$

where $D(a, b, n)$ denotes the determinant

$$\begin{vmatrix} S_a, & S_{a+1}, & S_{a+2} \\ S_b, & S_{b+1}, & S_{b+2} \\ S_n, & S_{n+1}, & S_{n+2} \end{vmatrix}$$

and L is a constant residue.

* In numerical cases the subscripts of the S are reduced modulo τ.

On expanding this determinant and substituting for S_a, S_b, etc.,

$$\alpha^a + \alpha^{pa} + \alpha^{p^2a}, \quad \alpha^b + \alpha^{pb} + \alpha^{p^2b} \text{ etc.},$$

we find that

$$D(a, b, n) \equiv S_{a+bp+np^2} - S_{a+bp^2+np} \qquad \text{mod } p,$$

so that

$$(8.31) \qquad U_n \equiv L(S_{a+bp+np^2} - S_{a+bp^2+np}) \qquad (n = 0, 1, \cdots, \tau - 1).$$

This proof would fail if

$$\begin{vmatrix} S_{a+1}, & S_{a+2} \\ S_{b+1}, & S_{b+2} \end{vmatrix}, \quad \begin{vmatrix} S_{a+2}, & S_a \\ S_{b+2}, & S_b \end{vmatrix}, \quad \begin{vmatrix} S_a, & S_{a+1} \\ S_b, & S_{b+1} \end{vmatrix}$$

should all be congruent to zero modulo p. But in this case

$$S_a \equiv MS_b, \quad S_{a+1} \equiv MS_{b+1}, \quad S_{a+2} \equiv MS_{b+2}$$

where M is a constant residue. Since the only multiplier of (S) is unity, $M=1$ and $a=b$ contrary to hypothesis.

The following two theorems are direct corollaries of Theorem 8.1.

THEOREM 8.11. *If* (Z) *denotes the cycle* $0, 0, 1, \cdots$, *then a necessary and sufficient condition that* $Z_n \equiv 0 \text{ mod } p$ *is that* $S_{n+p} \equiv S_{n+p^2} \text{ mod } p$.

THEOREM 8.12. *If* (Y) *denotes the cycle* $0, 1, 0, \cdots$, *then a necessary and sufficient condition that* $Y_n \equiv 0 \text{ mod } p$ *is that* $S_{n+2p} \equiv S_{n+2p^2} \text{ mod } p$.

We have several times used the fact that if (S) is the principal cycle, $S_n \equiv S_{np} \equiv S_{np^2} \text{ mod } p$. The following limited converse of this result is a direct consequence of Theorem 8.1.

THEOREM 8.2. *Let* (U) *be any cycle of* $F(x)$. *If for any* $m \neq 0$ *it is known that* $U_m \equiv U_{pm} \equiv U_{p^2m} \equiv 0 \text{ mod } p$, *then* $U_n \equiv KS_n$ $(n=0, 1, \cdots, \tau-1)$.

The following congruences, which are all special cases of the easily established general formula

$$U_{n+m} + U_{n+pm} + U_{n+p^2m} \equiv U_n S_m \qquad \text{mod } p,$$

give some curious arithmetical properties of cycles:

$$(8.4) \qquad \begin{aligned} U_n + U_{pn} + U_{p^2n} &\equiv U_0 S_n, \\ U_0 + U_{(p-1)n} + U_{(p^2-1)n} &\equiv U_{-n} S_n, \\ S_{n+m} + S_{n+pm} + S_{n+p^2m} &\equiv S_n S_m, \\ X_n + X_{pn} + X_{p^2n} &\equiv S_n, \\ Y_n + Y_{pn} + Y_{p^2n} &\equiv 0, \\ Z_n + Z_{pn} + Z_{p^2n} &\equiv 0 \qquad \text{mod } p. \end{aligned}$$

From Theorem 8.2, we see that it is impossible for Y_n, Y_{pn}, Y_{p^2n} or Z_n, Z_{pn}, Z_{p^2n} to be all congruent to zero modulo p simultaneously. From the last two formulas of (8.4), we see that it is also impossible for Y_n and Y_{pn} or Z_n and Z_{pn} to be congruent to zero simultaneously. In the succeeding section we shall prove a much more precise result of this character, which will enable us to obtain valuable information about the number of zeros in any cycle.

9. **Diophantine relations for the number of zeros in a cycle.** If the cycles of $F(x)$ have no multiplier save unity, each block of cycles contains $p-1$ distinct cycles. Let $\mathfrak{B}_1, \mathfrak{B}_2, \cdots, \mathfrak{B}_\kappa$ be the separate blocks, and let $(p-1)b_i$ be the number of zeros in $\mathfrak{B}_i$, so that each cycle of $\mathfrak{B}_i$ contains b_i zeros. Clearly, $\sum_{i=1}^{\kappa}(p-1)b_i = p^2-1$; hence

$$(9.1) \qquad b_1 + b_2 + \cdots + b_\kappa = p + 1.$$

In this section I shall establish the additional formula

$$(9.2) \qquad b_1^2 + b_2^2 + \cdots + b_\kappa^2 = \tau + p.$$

THEOREM 9.1. *Let a, b be any two distinct numbers $\geqq 0$, $<\tau$. Then there exists a cycle $U_0, U_1, \cdots, U_{\tau-1}$ such that*

$$U_a \equiv U_b \equiv 0 \qquad \text{mod } p.$$

The τ residues

$$U_n \equiv \begin{vmatrix} S_n, & S_a, & S_b \\ S_{n+1}, & S_{a+1}, & S_{b+1} \\ S_{n+2}, & S_{a+2}, & S_{b+2} \end{vmatrix} \qquad \text{mod } p$$

clearly form a cycle satisfying the conditions of the theorem.

THEOREM 9.2. *Let (U), (V) be any two cycles of $F(x)$, in which it is known that*

$$U_a \equiv U_b \equiv 0, \qquad V_c \equiv V_d \equiv 0 \qquad \text{mod } p.$$

Then a sufficient condition that (U) and (V) should belong to the same block is that $a-b \equiv c-d$ mod τ.

Let $V_n = W_{n+a-c}$ $(n=0, 1, \cdots, \tau-1)$. Then $W_a \equiv W_b \equiv 0$ mod p. Hence if

$$U_n \equiv K_0S_n + K_1S_{n+1} + K_2S_{n+2}, \quad W_n \equiv L_0S_n + L_1S_{n+1} + L_2S_{n+2},$$

then

$$K_0S_a + K_1S_{a+1} + K_2S_{a+2} \equiv 0, \qquad L_0S_a + L_1S_{a+1} + L_2S_{a+2} \equiv 0,$$
$$K_0S_b + K_1S_{b+1} + K_2S_{b+2} \equiv 0, \qquad L_0S_b + L_1S_{b+1} + L_2S_{b+2} \equiv 0.$$

Hence* $L_0:L_1:L_2=K_0:K_1:K_2$, so that

$$V_{n+c-a} \equiv W_n \equiv MU_n \mod p \quad (n = 0, 1, \cdots, \tau - 1),$$

and (V) and (W) belong to the same block.

THEOREM 9.3. *If the cycle (U) has no multiplier save unity, and if U_{a_1}, $U_{a_2}, \cdots, U_{a_b}$ are the b residues of (U) congruent to zero modulo p, then the $b(b-1)$ differences $a_i-a_j (i, j=1, \cdots, b; i\neq j)$ are all incongruent modulo τ.*

If $a_i-a_j\equiv a_k-a_l \mod \tau$, then, by the previous theorem,

$$U_{a_i-a_k+n} \equiv MU_n \qquad (n = 0, 1, \cdots, \tau - 1).$$

Hence $M=1$ and $a_i-a_k\equiv 0 \mod \tau$; $i=k, j=l$.

THEOREM 9.4. *Let $(U^{(1)}), \cdots, (U^{(\kappa)})$ be κ cycles belonging to the blocks $\mathfrak{B}_1, \cdots, \mathfrak{B}_\kappa$, respectively, so that $(U^{(i)})$ contains exactly b_i zeros. Then*

$$\sum_{i=1}^{k} b_i(b_i - 1) = \tau - 1. \tag{9.21}$$

If $b_i<2$, $b_i(b_i-1)=0$. If $b_i\geqq 2$, then as in Theorem 9.3 the cycle $(U^{(i)})$ furnishes $b_i(b_i-1)$ differences a_i-a_j which are all incongruent modulo τ. But by Theorems 9.1 and 9.2, to each of the $\tau-1$ distinct differences a_k-a_l modulo τ there corresponds exactly one block such that for every cycle (U) of this block $U_{a_k}\equiv U_{a_l}\equiv 0 \mod p$. Hence (9.21) follows.

Formula (9.2) now follows from (9.21) and (9.1) by addition.

10. Application to the case $\tau=(p^2+p+1)/3$. We shall now apply the formulas of §9 to the case when $p=3N+1$ and when the characteristic number τ equals $(p^2+p+1)/3$. There are then only three blocks of cycles and no multipliers, so that (9.1) and (9.2) become

$$\begin{aligned} b_1 + b_2 + b_3 &= p + 1, \\ b_1^2 + b_2^2 + b_3^2 &= (p^2 + 4p + 1)/3. \end{aligned} \tag{10}$$

Moreover, since $\mathfrak{B}_1$ contains the principal cycle (S), $b_1\equiv 0 \mod 3$; for $S_n\equiv 0 \mod p$ implies that $S_{np}\equiv S_{np^2}\equiv 0 \mod p$. Thus we have the additional restrictions

$$b_1 \equiv 0 \mod 3, \quad 0 \leqq b_1, \; b_2, \; b_3 \leqq p + 1. \tag{10.1}$$

The theory of the diophantine system (10) and (10.1) is a special case of a theory of simultaneous quadratic and linear representation given by Dr. Gordon Pall in a forthcoming paper. I am indebted to Dr. Pall for the following result:

* It is impossible for the ratios to be indeterminate; see the proof of Theorem 8.1.

The system (10), (10.1) *always has a unique solution in positive integers. If* (ξ, η) *is that solution of* $\xi^2+3\eta^2=p$ *satisfying the condition* $\xi\equiv 2 \bmod 3$, *then the solution of* (10) *is given by*

$$b_1 = N + \tfrac{2}{3}(\xi + 1), \quad b_2 = N + \eta - \tfrac{1}{3}(\xi - 2), \quad b_3 = N - \eta - \tfrac{1}{3}(\xi - 2).$$

It should be noted that b_2 and b_3 are both congruent to 1 modulo 3, and distinct.

In §17, we shall return to this case and obtain the values of b_1, b_2, b_3 by quite a different method.

The next simplest case is when $\tau=(p^2+p+1)/7$. Pall's theory allows us to reduce the solution of (9.1) and (9.2) to a single equation in six variables, but unfortunately this new equation has a large number of possible solutions.

11. **Minimum number of residues of a cycle.** I shall conclude this part of the paper by proving the following theorem:

THEOREM. *If the characteristic number* τ *equals* p^2+p+1, *then every residue appears in every cycle of* $F(x)$ *at least once.*

Under the hypothesis of the theorem, there are $p-1$ cycles grouped in a single block $\mathfrak{B}$; hence it is sufficient to prove that every residue appears in the cycle (S) at least once.

Consider the $(p-1)^3-1$ triads which do not contain a particular residue K. If U, V, W stand for distinct residues, these triads may be grouped into five classes; namely,

$$\begin{array}{lll}
m_1 = (p-1)(p-2)(p-3) & \text{triads of type} & [U, V, W]; \\
m_2 = (p-1)(p-2) & \quad\text{“} & [U, V, U]; \\
m_3 = (p-1)(p-2) & \quad\text{“} & [U, U, V]; \\
m_4 = (p-1)(p-2) & \quad\text{“} & [U, V, V]; \\
m_5 = (p-2) & \quad\text{“} & [U, U, U].
\end{array}$$

Let μ_i be the number of triads of type i which appear in (S). To each triad of type 1 there correspond $p-4$ distinct triads of type 1 in the block of cycles $L(S)$; namely, those for which

$$LU \not\equiv K, \quad LV \not\equiv K, \quad LW \not\equiv K \quad \bmod p \quad (L = 1, 2, \cdots, p-1).$$

Therefore,

$$(p-4)\mu_1 \leqq m_1, \quad \text{or} \quad \mu_1 < (p-1)(p-2).$$

Similarly,

$$\mu_2 < p-1; \quad \mu_3 < p-1; \quad \mu_4 < p-1; \quad \mu_5 < 1.$$

Accordingly, if the residue K does not appear in (S),

$$\tau = \mu_1 + \mu_2 + \mu_3 + \mu_4 + \mu_5 < (p-1)(p-2) + 3(p-1) + 1 = p^2,$$

giving a contradiction.

III. Determination of distribution function

12. **Reduction to the case when τ is prime to 3 and divides p^2+p+1.** We shall now show that it is sufficient to determine the distribution functions for difference equations whose characteristic number is prime to 3 and divides p^2+p+1.

Let $F(x)$ and $F'(x)$ be two irreducible cubics modulo p with the periods τ and τ', where $F'(x)$ is so chosen that τ' divides τ. Write

$$\tau = \tau' k'. \tag{12.1}$$

Let the roots of $F(x)=0$ and $F'(x)=0$ be denoted by $\alpha, \beta, \gamma; \alpha', \beta', \gamma'$ respectively. We then have in the Galois field associated with $F(x)$ a congruence of the form

$$\alpha' \equiv \alpha^{k's} \qquad \text{mod } p. \tag{12.2}$$

s here is a fixed integer prime to τ depending on our choice of $F'(x)$.

Now let (U') be any cycle of $F'(x)$. Then

$$U'_n \equiv \sum_{(\alpha')} (K_0 + K_1\alpha' + K_2\alpha'^2)\alpha'^n \qquad (n = 0, 1, \cdots, \tau'-1).$$

Hence by (12.2)

$$U'_n \equiv \sum_{(\alpha)} (L_0 + L_1\alpha + L_2\alpha^2)\alpha^{nk's}$$

where

$$\begin{aligned} L_0 &\equiv X_0K_0 + X_{k's}K_1 + X_{2k's}K_2; \quad L_1 \equiv Y_0K_0 + Y_{k's}K_1 + Y_{2k's}K_2; \\ L_2 &\equiv Z_0K_0 + Z_{k's}K_1 + Z_{2k's}K_2. \end{aligned} \tag{12.3}$$

L_0, L_1, L_2 cannot moreover all be congruent to zero modulo p, for if Δ and Δ' are the discriminants of $F(x)$ and $F'(x)$,

$$\begin{vmatrix} X_0, & X_{k's}, & X_{2k's} \\ Y_0, & Y_{k's}, & Y_{2k's} \\ Z_0, & Z_{k's}, & Z_{2k's} \end{vmatrix} \equiv \Delta'/\Delta \not\equiv 0 \qquad \text{mod } p.$$

Write

$$\begin{aligned} A &\equiv L_0S_0 + L_1S_1 + L_2S_2, \\ B &\equiv L_0S_1 + L_1S_2 + L_2S_3, \\ C &\equiv L_0S_2 + L_1S_3 + L_2S_4. \end{aligned} \tag{12.31}$$

Then A, B, C are the initial values of the sequence (U) where

$$U_n \equiv \sum_{(\alpha)} (L_0 + L_1\alpha + L_2\alpha^2)\alpha^n \qquad (n = 0, 1, \cdots, \tau - 1)$$

and we have the formula

$$U'_n \equiv U_{nk's} \mod p \qquad (n = 0, 1, \cdots, \tau' - 1). \tag{12.4}$$

The process by which we obtained (U) from (U') fixed the initial elements of (U); if we remove this restriction, then if we are given (U), τ' and s, by letting $[A, B, C]$ in (12.31) equal successively the triads $[U_0, U_1, U_2]$, $[U_1, U_2, U_3]$, $\cdots$, $[U_{k'-1}, U_{k'}, U_{k'+1}]$, K_0, K_1, K_2 are obtained from (12.3), so that (U') is known, and (12.4) is replaced by the more general relation

$$U'_n \equiv U_{nk's+r} \qquad (n = 0, 1, \cdots, \tau' - 1; 0 \leqq r < k'). \tag{12.41}$$

Since ns in (12.4) or (12.41) runs through a complete residue system modulo τ', the result we have obtained may be formulated as follows:

THEOREM 12.1. *Any cycle (U') of $F'(x)$ can be obtained from some cycle (U) of $F(x)$ by selecting from (U) terms whose subscripts lie in an arithmetical progression of constant difference k', and arranging them in a suitable order.*

A cycle (U') obtained in this manner from some cycle (U) of $F(x)$ will be called a *derived* cycle of (U). On taking $\tau' = \tau$, we obtain the following:

THEOREM 12.2. *If $F(x)$ and $F'(x)$ are two irreducible cubics modulo p with the same characteristic number τ, and if (U') is any cycle of $F'(x)$, then there exists a cycle (U) of $F(x)$ with the same distribution function as (U').*

Thus in a certain sense the distribution function of a cycle is independent of the characteristic function to which the cycle belongs. For example, the distribution functions of the derived cycles (U') of (U) defined in (12.41) are independent of the value of s, provided only that s is prime to τ.

Now let us take* $\tau = e\mu$ where e denotes as usual the period modulo p of the basic multiplier, and take $\tau' = \mu$. We shall show that the distribution function of any cycle (U) of $F(x)$ is determined if we know the distribution function for a single one of its derived cycles (U') of period μ.

Let

$$\begin{array}{llll} a_0 M, & a_0 M^2, & \cdots, & a_0 M^e; \\ a_1 M, & a_1 M^2, & \cdots, & a_1 M^e; \\ \cdot\ \cdot\ \cdot & & & \\ \cdot\ \cdot\ \cdot & & & \\ a_{t-1} M, & a_{t-1} M^2, & \cdots, & a_{t-1} M^e \end{array}$$

* It can be shown that $(e, \mu) = 1$ save when $p = 3^k N + 1$, $\epsilon(R) \equiv 0 \mod 3^k$. We shall exclude this case from consideration here.

represent the separation of the multiplicative group $\{1, 2, \cdots, p-1\}$ of the $p-1$ non-zero residues of p into co-sets with respect to its cyclic sub-group $\{M\}$. Then if $k(n)$ is the distribution function of (U), and $l(n)$ the distribution function of the first μ terms of (U),

$$k(a_iM^s) = [l(a_iM) + l(a_iM^2) + \cdots + l(a_iM^e)]$$
$$(i = 0, 1, \cdots, \tau; s = 1, 2, \cdots, e).$$

Now let $k'(n)$ be the distribution function of any derived cycle (U') of (U) with the period μ. From the properties of multipliers, it is evident that if $a \equiv b \bmod \mu$, then $U_a \equiv M^\sigma U_b \bmod p$, the exponent σ depending of course on our choice of U_a and U_b. Now the subscript $n\kappa' s + r(\kappa' = e)$ of (U) in (12.41) runs through a complete residue system modulo μ, for $(e, \mu) = 1$; hence to each term U_a' of (U') there corresponds a unique term U_b in the first μ terms of (U) such that

$$U_a' \equiv M^\sigma U_b \qquad \bmod p.$$

Consequently, U_a' and U_b always lie in the same co-set, and

$$k'(a_iM) + k'(a_iM^2) + \cdots + k'(a_iM^e) = l(a_iM) + l(a_iM^2) + \cdots + l(a_iM^e).$$

Thus we have the formula

$$k(a_iM^s) = [k'(a_iM) + k'(a_iM^2) + \cdots + k'(a_iM^e)]$$
$$(i = 0, 1, \cdots, \tau; s = 0, 1, \cdots, e).$$

In a similar manner, we find that

$$k(0) = ek'(0).$$

These two formulas express the distribution function of (U) in terms of the distribution function of any one of its derived cycles (U') of period μ. It can be readily shown that μ divides p^2+p+1 and is prime to 3.

If we assume that the period τ of $F(x)$ is p^2+p+1 while the period τ' of $F'(x)$ is a divisor of p^2+p+1, we can deduce the following important result from Theorem 12.1.

THEOREM 12.3. *If τ divides p^2+p+1, then no residue can appear in any cycle of period τ more times than it appears in any cycle of period p^2+p+1.*

13. **Reduction to determination of residues modulis P and 3.** Consider an irreducible cubic $F(x) = x^3 - Px^2 + Qx - R$ with the period $\tau = p^2+p+1$, modulo p, so that its cycles are grouped into a single block $\mathfrak{B}$, and let $k_n = k(n)$ be the distribution function of the principal cycle (S).

Then since

$$S_n \equiv S_{np} \equiv S_{np^2} \qquad \bmod p,$$

we see that if $p=3N+2$, then

$$k_i \equiv 0 \ (i \neq 3), \ k_3 \equiv 1 \qquad \mod 3.$$

If $p=3N+1$, it is easily verified that $p\tau/3 \equiv \tau/3 \mod \tau$. Furthermore, if ω denotes a primitive cube root of unity modulo p,

$$S_0 \equiv 3, \ S_{\tau/3} \equiv 3\omega, \ S_{2\tau/3} \equiv 3\omega^2 \qquad \mod p.$$

Hence

$$k_i \equiv 0 \mod 3 \qquad (i \neq 3, 3\omega, 3\omega^2),$$

$$k_3 \equiv k_{3\omega} \equiv k_{3\omega^2} \equiv 1 \qquad \mod 3.$$

Consequently,

$$\begin{aligned} k_i &= 3l_i \ (i \neq 3), \ k_3 = 3l_3 + 1, \ p = 3N + 2, \\ k_i &= 3l_i \ (i \neq 3, 3\omega, 3\omega^2), \\ k_{3\omega^a} &= 3l_{3\omega^a} + 1 \ (a = 0, 1, 2), \qquad p = 3N + 1. \end{aligned} \tag{13.1}$$

Now all of the cubics

$$x^3 - S_n x^2 + S_{-n} x - 1, \ 0 < n < \tau \qquad \left(n \neq \frac{\tau}{3}, \ \frac{2\tau}{3} \text{ if } p = 3N + 1\right)$$

are irreducible modulo p and have periods which divide τ; thus l_i is the number of irreducible cubics modulo p among the p cubics

$$x^3 - ix^2 + ux - 1 \qquad (u = 0, 1, \cdots, p - 1).$$

Hence

$$0 \leqq l_i \leqq p - 2, \tag{13.2}$$

for the cubics $x^3 - ix^2 + ix - 1$, $x^3 - ix^2 - (i+2)x - 1$ are obviously reducible for any p.

Consequently, for $p > 3$, k_i is completely determined if we know its residues modulis p and 3.

Since the other cycles of $F(x)$ are obtained simply by multiplying the cycle (S) by some constant residue, their distribution functions are merely permutations of the distribution function of (S), so that (13.2) holds for all the cycles of $F(x)$.

Now let $F'(x)$ be any other irreducible cubic with the period τ', a divisor of p^2+p+1, and let $k'(n)$ be the distribution function of some cycle (U') of $F'(x)$. If $k(n)$ is the distribution function of the cycle (U) of $F(x)$ from which (U') is derived in accordance with Theorem 12.1, then by Theorem 12.2,

$$k'(n) \leqq k(n) \qquad (n = 0, 1, \cdots, p - 1).$$

We thus have the following important result.

THEOREM 13.1. *If $k(n)$ is the distribution function of any cycle (U) whose characteristic number divides p^2+p+1, then $k(n)$ is completely determined if we know its residues modulo p and modulo* 3.

Since by (9.1), $k(0) \leqq p+1$, we merely need to know the residue of $k(0)$ modulo p.

14. **Digression on diophantine systems.** The diophantine system

$$(\mathrm{D}) \qquad r_1 + r_2 + r_3 = m, \quad r_1 + pr_2 + p^2 r_3 = s\tau,$$

where the integers m, p, τ are given, plays such an important part in the developments which are to follow, that it is necessary to discuss its solution rather fully.

The parameters p and τ are defined as follows. Let κ be any fixed integer of the sequences 1, 3, 7, 13, 19, 21, 31, 39, $\cdots$, of all possible divisors of the form x^2+x+1, and let p be a prime such that p^2+p+1 is exactly divisible by κ.

If $p = k\kappa + \rho$, $1 < \rho < \kappa$, then ρ is zero if $\kappa = 1$ and unity if $\kappa = 3$. In all other cases, ρ is a primitive cube root of unity modulo κ. p is thus restricted to certain linear forms $n\kappa + \rho$.

τ is defined to be the quotient obtained by dividing p^2+p+1 by κ. If $\rho^2+\rho+1 = \sigma\kappa$, then

$$(14.1) \qquad \tau = k(k\kappa + \rho) + (\rho + 1)k + \sigma, \quad 0 < (\rho + 1)k + \sigma < 2p.$$

Finally m is restricted to be less than p, and the solutions r_1, r_2, r_3 must all be $\geqq 0$. There are no restrictions on s other than that it be an integer.

Since $p^3 \equiv 1 \bmod \tau$, if $r_1 = u$, $r_2 = v$, $r_3 = w$, or for short (u, v, w) is a solution of (D), (v, w, u) and (w, u, v) are also solutions. We can accordingly restrict ourselves to finding those solutions of (D) for which $r_3 \leqq r_1, r_2$.

Now

$$r_1 + pr_2 + p^2 r_3 - r_3 \kappa\tau = (k\kappa + \rho)(r_2 - r_3) + (r_1 - r_3) \equiv 0 \qquad \mathrm{mod}\ \tau.$$

Thus if we let $r_2 - r_3 = s_1$, $r_1 - r_3 = s_2$, we obtain

$$(14.2) \qquad (k\kappa + \rho)s_1 + s_2 = u\tau \quad (s_1, s_2, u \geqq 0; \quad 0 \leqq s_1, s_2 < p).$$

Moreover, it is easily shown that $0 \leqq u < \kappa$, $s_1 + s_2 < p$, and

$$(14.21) \qquad m \geqq s_1 + s_2; \; m \equiv s_1 + s_2 \qquad \mathrm{mod}\ 3.$$

The method for solving (D) is then as follows: For a given value of κ, ρ and σ are determined, so that τ is known as a function of k from (14.1). For

each value of u between 0 and $\kappa-1$ we can determine from (14.2) a pair of values for s_1 and s_2 in terms of k. We reject all solutions of (14.2) for which $s_1+s_2\geqq p$. (14.21) then gives the restrictions upon m, and the corresponding solution of (D) is

$$\left(\frac{m - s_1 + 2s_2}{3},\quad \frac{m + 2s_1 - s_2}{3},\quad \frac{m - s_1 - s_2}{3}\right).$$

The following theorems for the cases $\kappa=1$ and $\kappa=3$ will serve to illustrate the method.

THEOREM 14.1. *If* $\kappa=1$, *there is no solution of* (D) *unless* $m\equiv 0 \bmod 3$. *If* $m=3M$, *there is the single solution* (M, M, M).

We have $\rho=0$, $p=k$, so that (14.5) becomes

$$ps_1 + s_2 = u(p^2 + p + 1);\ u = 0 \text{ giving } s_1=s_2=0,\ m\equiv 0 \bmod 3;$$

$$(r_1, r_2, r_3) = (M, M, M).$$

THEOREM 14.2. *If* $\kappa=3$, *there is no solution of* (D) *unless* $p\equiv 1$, $m\equiv 0 \bmod 3$. *If* $m=3M$, *there is the single solution* (M, M, M).

Since $p^2+p+1\equiv 0 \bmod 3$, $p\equiv 1 \bmod 3$. Let $p=3k+1$. Then $\tau=k(3k+1)+2k+1$, and (14.2) becomes

$$(3k + 1)s_1 + s_2 = uk(3k + 1) + u(2k + 1) \qquad (u = 0, 1, 2).$$

Case (i) $u=0$. We have $s_1=s_2=0$, so that from (14.2), $m=3M$, giving the solution (M, M, M).

Case (ii) $u=1$. Then $s_1=k$, $s_2=2k+1$ so that $s_1+s_2=p$ and there is no solution.

Case (iii) $u=2$. From (14.5) $s_2\equiv 4k+2\equiv k+1 \bmod p$. Since $s_2<p$, $s_2=k+1$ and consequently $s_1=2k+1$. Hence $s_1+s_2>p$ and there is no solution.

In general, if $p\equiv 1 \bmod 3$, we see from (14.2) and (14.21) that $m\equiv\tau\equiv 0 \bmod 3$.

When $\kappa=7$, which requires that p be of the form $7n+2$ or $7n+4$, we find by the same method the following solutions of (D).

SOLUTIONS FOR $p\equiv 2, 4 \bmod 7$, $7\tau=p^2+p+1$

Form of p	Form of m	Restriction on m	Solution
	$3M$	$\geqq 0$	(M, M, M)
$21L+2$	$3M+1$	$\geqq 12L+1$	$(M+5L+1, M-L, M-4L)$
	$3M+2$	$\geqq 15L+2$	$(M+L+1, M+4L+1, M-5L)$
	$3M$	$\geqq 18L+3$	$(M-3L, M+9L+1, M-6L-1)$

SOLUTIONS FOR $p\equiv 2, 4 \bmod 7,\ 7\tau=p^2+p+1$ (continued)

Form of p	Form of m	Restriction on m	Solution
$21L+16$	$3M$	$\geqq 0$	(M, M, M)
	$3M$	$\geqq 12L+9$	$(M+5L+4,\ M-L-1,\ M-4L-3)$
	$3M$	$\geqq 15L+12$	$(M+L+1,\ M+4L+3,\ M-5L-4)$
	$3M$	$\geqq 18L+15$	$(M-3L-2,\ M+9L+7,\ M-6L-5)$
$21L+4$	$3M$	$\geqq 0$	(M, M, M)
	$3M$	$\geqq 12L+3$	$(M-L,\ M+5L+1,\ M-4L-1)$
	$3M$	$\geqq 15L+3$	$(M+4L+1,\ M+L,\ M-5L-1)$
	$3M$	$\geqq 18L+3$	$(M+9L+2,\ M-3L-1,\ M-6L-1)$
$21L+11$	$3M$	$\geqq 0$	(M, M, M)
	$3M+1$	$\geqq 12L+5$	$(M-L,\ M+5L+3,\ M-4L-2)$
	$3M+2$	$\geqq 15L+8$	$(M+4L+3,\ M+L+1,\ M-5L-2)$
	$3M$	$\geqq 18L+9$	$(M+9L+5,\ M-3L-2,\ M-6L-3)$

For other small values of k the explicit solution of (D) may be obtained in a similar manner without undue labor.

15. **Determination of distribution function modulo p.** Let $k(n)=k_n$ be the distribution function modulo p of any cycle (U) whose characteristic number τ divides p^2+p+1. We shall determine the residue of k_n modulo p.

Let i be any residue of p. Then by Fermat's theorem

$$\begin{aligned}(U_n - i)^{p-1} &\equiv 1 \quad \bmod p, \quad U_n \not\equiv i;\\ &\equiv 0 \quad \bmod p, \quad U_n = i.\end{aligned}$$

Hence

$$\sum_{n=0}^{\tau-1}(U_n - i)^{p-1} \equiv \tau - k_i \quad \bmod p \quad (i = 0, 1, \cdots, p-1).$$

On expanding $(U_n-i)^{p-1}$ by the binomial theorem, we obtain after a few easy reductions

$$\tau - k_i \equiv \sum_{n=0}^{\tau-1}\sum_{m=0}^{p-1}(i)^{-m}U_n{}^m\,\tau - k_0 \equiv \sum_{n=0}^{\tau-1}U_n{}^{p-1} \qquad \bmod p. \tag{15.1}$$

Suppose that

$$U_n \equiv A\alpha^n + B\beta^n + C\gamma^n,$$

where $A=K_0+K_1\alpha+K_2\alpha^2$, etc., so that

$$N(u) = ABC \equiv N(K_0 + K_1\alpha + K_2\alpha^2) \qquad \bmod p.$$

Then by the multinomial theorem

$$U_n{}^m \equiv \sum_{(r)} \frac{m!}{r_1!r_2!r_3!} A^{r_1}B^{r_2}C^{r_3}\alpha^{n(r_1+pr_2+p^2r_3)} \qquad \text{mod } p.$$

Since

$$\sum_{n=0}^{\tau-1} \alpha^{nR} \equiv 0 \quad \text{mod } p \text{ if } \tau \text{ does not divide } R,$$

$$\equiv \tau \quad \text{mod } p \text{ if } \tau \text{ divides } R,$$

we obtain, on substituting in (15.1), the fundamental formulas

$$(15.2) \qquad \begin{aligned} k_i &\equiv -\tau \sum_{m=1}^{p-1} \sum_{(r)} \frac{m!}{r_1!r_2!r_3!} \frac{A^{r_1}B^{r_2}C^{r_3}}{i^m} && \text{mod } p, \\ k_0 &\equiv \tau\left(1 + \sum_{(r)} \frac{A^{r_1}B^{r_2}C^{r_3}}{r_1!r_2!r_3!}\right) && \text{mod } p, \end{aligned}$$

where in the expression for k_i the summation variables satisfy the conditions

$$(\mathrm{D}) \qquad r_1 + r_2 + r_3 = m, \quad r_1 + pr_2 + p^2r_3 \equiv 0 \qquad \text{mod } \tau,$$

while in the expression for k_0,

$$r_1 + r_2 + r_3 = p - 1, \quad r_1 + pr_2 + p^2r_3 \equiv 0 \qquad \text{mod } \tau.$$

For the principal cycle (S), $A = B = C = 1$ and the formulas (15.2) assume the simpler form

$$(15.3) \qquad \begin{aligned} k_i &\equiv \tau \sum_{m=1}^{n-1} \sum_{(r)} \frac{m!}{r_1!r_2!r_3!i^m} && \text{mod } p, \\ k_0 &\equiv \tau\left(1 + \sum_{(r)} \frac{1}{r_1!r_2!r_3!}\right) && \text{mod } p. \end{aligned}$$

The problem of determining the residue of k_i modulo 3 offers very serious difficulties, principally because $U_0, U_1, \cdots, U_{\tau-1}$ do not satisfy a difference equation when taken modulo 3. The only cases in which I have succeeded in determining the residue are given in the formulas (13.1) for the distribution function of the principal cycle (S), and their obvious extension to the remaining cycles of the block $\mathfrak{B}_1$ to which (S) belongs.

16. **Applications.** By applying the results of §14 on the solutions of the diophantine equations (D) to formulas (15.2) and (15.3), we obtain a number of interesting special cases. Throughout this section, (U) denotes a fixed cycle of $F(x)$ whose general term is $U_n \equiv A\alpha^n + B\beta^n + C\gamma^n$ mod p, $A = K_0 + K_1\alpha + K_2\alpha^2$ etc., and whose distribution function is $k(n)$.

From formulas (6.11), (6.13),

$$(16.1) \qquad ABC \equiv \Lambda(U_0, U_1, U_2)/\Delta \qquad \text{mod } p,$$

where Λ is the polynomial defined in formula (6.12), and Δ is the discriminant of $F(x)$.

If $p\equiv 2 \bmod 3$ and $\tau=p^2+p+1$, then by Theorem 14.1 formulas (15.2) become

$$k_i \equiv \sum_{n=1}^{(p-2)/3} \frac{(3n)!}{(n!)^3}\left(\frac{\Lambda(U_0, U_1, U_2)}{\Delta i^3}\right)^n \quad \bmod p,$$

$$k_0 \equiv 1 \quad \bmod p.$$

Since in this case the residues of k_i modulo 3 are known, these formulas determine the distribution function $k(n)$ completely.

If $p\equiv 1 \bmod 3$ and $\tau=(p^2+p+1)/3$, then on writing $p=3N+1$, $\tau\equiv 2N+1 \bmod p$, and by Theorem 14.2, formulas (15.2) become

$$(16.2)\qquad \begin{aligned} k_i &\equiv N \sum_{n=1}^{N} \frac{(3n)!}{(n!)^3}\left(\frac{\Lambda(U_0, U_1, U_2)}{\Delta i^3}\right)^n && \bmod p,\\ k_0 &\equiv (2N+1)\left(1+\frac{(\Lambda(U_0, U_1, U_2))^N}{(N!)^3}\right) && \bmod p. \end{aligned}$$

These formulas will determine the distribution function $k(n)$ for any cycle (U) belonging to the block $\mathfrak{B}$, since the residues of k_i modulo 3 are known. For the other blocks, the residues of k_i modulo 3 are unknown.

Formula (16.2) has some important consequences. We have seen in §10 that if $k_0^{(1)}$, $k_0^{(2)}$, $k_0^{(3)}$ are the number of zeros in three cycles $(U^{(1)})$, $(U^{(2)})$, $(U^{(3)})$ belonging to the blocks $\mathfrak{B}_1$, $\mathfrak{B}_2$, $\mathfrak{B}_3$ respectively, then $h_0^{(1)}$, $h_0^{(2)}$, $h_0^{(3)}$ are all distinct from one another. Hence if (U) is allowed to range over all the cycles of $F(x)$, we see from (16.1), (16.2) that $(ABC)^N$ must take three distinct values modulo p.

Since $(ABC)^{3N}\equiv 1 \bmod p$, $(ABC)^N\equiv\omega^a \bmod p$ where ω is a primitive cube root of unity modulo p, and the exponent a of ω depends on the block to which (U) belongs. In particular, for (S), $ABC=1$ so that $a=0$. We thus obtain from (16.1) and formulas (6.11), (6.12) the following simple criterion to decide whether or not two triads belong to the same block.

THEOREM 16.1. *If $[A', B', C']$ and $[A'', B'', C'']$ are any two triads of $F(x)$ and if $F(x)$ has the period $(p^2+p+1)/3$, then a necessary and sufficient condition that $[A', B', C']$ and $[A'', B'', C'']$ belong to the same block is that $\Lambda(A', B', C')$ and $\Lambda(A'', B'', C'')$ have the same cubic character modulo p.*

For the cycle (Z): $0, 0, 1, \cdots, ABC\equiv(-1/\Delta) \bmod p$. Hence (Z) lies in $\mathfrak{B}_1$ when and only when $\Delta^N\equiv 1 \bmod p$. But obviously (Z) lies in $\mathfrak{B}_1$ when and

only when the cycle (S) contains two consecutive zeros; we thus obtain the following interesting theorem:

THEOREM 16.2. *If* $p\equiv 1 \bmod 3$ *and* $F(x)$ *is any irreducible cubic with the period* $\tau=(p^2+p+1)/3 \bmod p$, *then the sequence* $(S)_n$ *of* $F(x)$ *will contain pairs of consecutive elements divisible by* p *when and only when the discriminant of* $F(x)$ *is a cubic residue of* p.

Formulas (15.1) serve to determine the distribution function for all the cycles of $\mathfrak{B}_1$, regardless of the value of τ, but if the period τ is less than $(p^2+p+1)/3$, they become increasingly complicated as τ is taken smaller. The table at the close of §14 allows us to give explicit formulas for the residues of $k(n)$ modulo p when $\tau=(p^2+p+1)/7$. The simplest of these results is contained in the following theorem.

THEOREM 16.3. *If* $\tau=(p^2+p+1)/7$, $p\equiv 2 \bmod 3$ *and if* b_0 *denotes the number of zeros in the principal cycle* (S), *then*

$$b_0 \equiv (9L+1)\left(1+\frac{3}{(12L+1)!6L!3L!}\right) \qquad \bmod p$$

or

$$b_0 \equiv (15L+8)\left(1+\frac{3}{(6L+3)!(12L+6)!(3L+1)!}\right) \qquad \bmod p$$

according as p *is of the form* $21L+2$ *or* $21L+11$.

17. **Determination of upper limit to distribution function.** On account of the great increase in complexity in the formulas (15.2) as τ is taken smaller, it is desirable to have an upper limit to the number of times a given residue can appear in a given cycle. The results I have obtained in this connection are incomplete in the same sense as those I have obtained to determine $k(n)$; it is necessary to know this upper limit modulis p and 3, whereas I have determined it only modulo p. They suffice nevertheless to give an upper limit to the number of times the residue zero can appear in any cycle, and the number of times any residue can appear in the principal cycle.

We have seen, in §13, that if (U') is any sequence of $F'(x)$ of period τ', where $\tau'\kappa'=\tau$, the period of $F(x)$, then the terms of (U') consist of the rth, $(\kappa'+r)$th, $(2\kappa'+r)$th, $\cdots$, $((\tau'-1)\kappa'+r)$th terms of some definite sequence (U) of $F(x)$ written usually in a different order. We shall now regard (U), τ, and r as given, but τ' as unknown, and endeavor to obtain an upper limit to $k'(n)$, the distribution function of (U'). Let us take U_r, U_{r+1}, U_{r+2} as the initial values of (U), which amounts to replacing (U) by the cycle (W), where

$$W_n = U_{n+r} \qquad (n=0,1,\cdots,\tau-1).$$

This change does not affect the distribution function $k(n)$ of (U). Then if m_i denotes the number of times the residue i appears in those terms of (U) whose indices are prime to τ, it is apparent that

$$k_i' \leqq k_i - m_i \qquad (i = 0, 1, \cdots, p-1). \tag{17.1}$$

m_i is determined if we know its residues modulis p and 3, in particular, m_0 is determined if we know its residue modulo p. Moreover, it is easily shown that if (W) belongs to the block of the principal cycle, $m_i \equiv 0 \bmod 3$. We shall now determine m_i modulo p.

By Fermat's theorem, we have the fundamental formula

$$\phi(\tau) - m_i \equiv \sum_{(n,\tau)=1} (W_n - i)^{p-1} \qquad \bmod p,$$

where the summation extends over all the terms of (W) whose subscripts are prime to τ, and $\phi(\tau)$ denotes as usual the totient of τ.

On proceeding as in §17, we find that, if $i \neq 0$,

$$\sum_{(n,\tau)=1} (W_n - i)^{p-1} \equiv \sum_{(n,\tau)=1} \sum_{m=0}^{p-1} \sum_{(r)} \frac{(i)^{-m} m!}{r_1! r_2! r_3!} A^{r_1} B^{r_2} C^{r_3} \alpha^{(r_1 + pr_2 + p^2 r_3)n},$$

where $r_1 + r_2 + r_3 = m$ and $W_n \equiv A\alpha^n + B\beta^n + C\gamma^n \bmod p$.

Now if $\mu(n)$ denotes Möbius' function, it is easily shown that

$$\begin{aligned} \sum_{(n,\tau)=1} \alpha^{Rn} &\equiv \mu(\tau) \quad \bmod p, \quad \tau \text{ does not divide } R, \\ &\equiv \phi(\tau) \quad \bmod p, \quad \tau \text{ divides } R. \end{aligned}$$

Hence after a slight transformation, we find that

$$\begin{aligned} \phi(\tau) - m_i \equiv \mu(\tau) \sum_{m=0}^{p-1} \sum_{(r)} \frac{(i)^{-m} m!}{r_1! r_2! r_3!} A^{r_1} B^{r_2} C^{r_3} \\ + (\phi(\tau) - \mu(\tau)) \sum_{m=0}^{p-1} \sum_{(r)} \frac{(i)^{-m} m!}{r_1! r_2! r_3!} A^{r_1} B^{r_2} C^{r_3}, \end{aligned}$$

where in the first summation $r_1 + r_2 + r_3 = m$, but in the second summation

$$r_1 + r_2 + p^2 r_3 \equiv 0 \quad \bmod \tau, \quad r_1 + r_2 + r_3 = m. \tag{D}$$

By the multinomial theorem, the first sum is found to be congruent modulo p to $\mu(\tau)[W_0(W_0 - i)]^{p-1}$.

Referring back to the formulas (15.2), the second sum is congruent to $(\phi(\tau) - \mu(\tau))(1 - k_i/\tau)$. Thus using the fact that $\kappa\tau = p^2 + p + 1 \equiv 1 \bmod p$, we obtain

$$(17.2)\qquad m_i \equiv \kappa(\phi(\tau \;-\mu(\tau))k_i + \epsilon\mu(\tau) \qquad \text{mod } p,$$

where $\epsilon = -1$, if $W_0 = 0$ or i, $\epsilon = 0$, otherwise.

In a similar manner, we find that

$$(17.3)\qquad m_0 \equiv \kappa(\phi(\tau) - \mu(\tau))k_0 + \epsilon'\mu(\tau) \qquad \text{mod } p,$$

where $\epsilon' = 1$, $W_0 = 0$, $\epsilon' = 0$ otherwise.

Thus if τ has a square factor, we have the simple formula*

$$m_i \equiv \kappa\phi(\tau)h_i \quad \text{mod } p \qquad (i = 0, 1, \cdots, p-1).$$

For $\tau = p^2 + p + 1$ or $(p^2 + p + 1)/3$, these formulas give a practicable determination of m_i for any given p.

* It is perhaps worth noting that p never divides $\phi(\tau)$.

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ON LINEAR CONNECTIONS*

BY

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With the introduction of infinitesimal parallelism, by T. Levi-Civita† in 1917, and independently by J. A. Schouten‡ in 1918, tangent spaces began to play a leading rôle in differential geometry. The tangent space at a point, x, is the totality of all contravariant vectors, or differentials, associated with that point. By means of an affine connection§ the tangent spaces at any two points on a curve are related by an affine transformation, which will in general depend on the curve.

Linear connections of another kind were defined by R. König,|| who associated with each point of a given n-dimensional manifold a space of m dimensions. A linear connection arises in differential equations of the form¶

$$dZ^\alpha + Z^\beta L^\alpha_{\beta i} dx^i = 0, \tag{0.1}$$

by means of which the associated spaces at different points are related to each other, and which are said to define a linear displacement.

Even if $m=n$, a linear connection of the König type has nothing to do with an affine connection** unless we require explicitly that the associated space at each point is the tangent space of differentials at that point.

Schouten has proposed the use of linear connections in handling a scheme†† by which differential geometry is based on group theory, in the spirit of Klein's Erlanger Program. The associated spaces are to be the spaces, according to Klein, of some group, and are related through linear displacement

* Presented to the Society, December 30, 1930; received by the editors June 8, 1930.

† Rendiconti del Circolo Matematico di Palermo, vol. 42 (1917), pp. 173–205.

‡ Proceedings, Koninklijke Akademie van Wetenschappen, Amsterdam, vol. 21 (1918), pp. 607–613.

§ H. Weyl, Mathematische Zeitschrift, vol. 2 (1918), pp. 384–411. See also G. Hessenberg, Mathematische Annalen, vol. 78 (1918), p. 199.

¶ Jahresbericht der Deutschen Mathematiker Vereinigung, vol. 28 (1919), pp. 213–228.

|| Greek letters, used as indices, will take on the values $1, \cdots, m$, and italic letters the values $1, \cdots, n$ ($m > n$ or $< n$).

** I mean by an affine connection any invariant with the transformation law

$$\bar{\Gamma}^i_{jk} = \left(\Gamma^a_{bc}\frac{\partial x^b}{\partial \bar{x}^j}\frac{\partial x^c}{\partial \bar{x}^k} + \frac{\partial^2 x^a}{\partial \bar{x}^j \partial \bar{x}^k}\right)\frac{\partial \bar{x}^i}{\partial x^a}.$$

†† Rendiconti del Circolo Matematico di Palermo, vol. 50 (1926), pp. 142–169. In particular Schouten has applied linear connections to the non-holonomic projective ($m=n+1$) and conformal ($m=n+2$) geometries.

by transformations of this group.* The general problem of imposing conditions upon the associated spaces, in order that they may be suitably related to the underlying manifold, has been discussed by Weyl† who solved the problem for the projective group.

Without touching on the questions which arise out of this scheme, there is a definite field for research in studying invariant properties of the differential equations (0.1) under transformations of the form (4.1). L. Schlesinger‡ has gone some distance in this direction, and we adopt this point of view in the present paper. Though most of our results refer to linear connections of the König type, they can all be interpreted in terms of affine connections and arbitrary n-uples. Given any affine connection we take $m=n$, and the associated spaces as the tangent spaces of differentials. In order to have a theory in which transformations of the form (4.1) are allowable, where (4.1a) is independant of (4.1b), we take

$$L^{\alpha}_{\beta i} = \gamma^{\alpha}_{\beta\sigma} u^{\sigma}_{i} \tag{0.2}$$

where u^{σ}_{i} are the covariant vectors of any n-uple, and $\gamma^{\alpha}_{\beta\sigma}$ are the scalar functions§ analogous to Ricci's coefficients of rotation. The equations (4.1a) will define a change over from one n-uple to another.

In §1 we give a geometrical proof of a theorem established by B. V. Williams|| and the author, in which we showed how to obtain an integrable connection which osculates (see §1 of this paper) a given linear connection. In §2 we prove a theorem about affine connections, which bears a formal re-

* This idea is mainly due to E. Cartan and is formulated by him in a paper (Bulletin de la Société Physico-Mathématique de Kazan, (2), vol. 3 (1927)), where he discusses Schouten's plan.

† Bulletin of the American Mathematical Society, vol. 35 (1929), pp. 716–725. Immediately preceding this, O. Veblen (Journal of the London Mathematical Society, vol. 4 (1929), pp. 140–160) had dealt with projective displacement from a different point of view. He showed how the space of projective vectors at any point, which plays the part of the associated space, is related to the space of differentials.

‡ Mathematische Annalen, vol. 99 (1928), pp. 413–434.

§ L. P. Eisenhart, *Non-Riemannian Geometry*, p. 47. These scalars are given by

$$\gamma^{\alpha}_{\beta\sigma} = u^{\alpha}_{i;j} v^{i}_{\beta} v^{j}_{\sigma},$$

where the semi-colon denotes covariant differentiation with respect to the affine connection, and v^{i}_{β} are the contravariant vectors of the n-uple. In his treatment of non-holonomic affine spaces Cartan (Annales de l'Ecole Normale Supérieure, 1923) uses n^2+n Pfaffian forms, ω^{α} and ω^{α}_{β}. The former give the coördinates of a point in each tangent space, and the latter define the affine connection. According to (0.2) these forms are given by

$$\omega^{\alpha} = u^{\alpha}_{i} dx^{i}, \qquad \omega^{\alpha}_{\beta} = L^{\alpha}_{\beta i} dx^{i}.$$

|| Annals of Mathematics, vol. 31 (1930), pp. 151–157. This paper will be referred to as T. L. C.

semblance to, but which differs essentially from either of those proved in T. L. C. We define a family of coördinate systems, which, like normal coördinate systems, have the property that each of them is uniquely determined by an affine connection, a point, and a given coördinate system. In §3 we show how the theorems of §§1 and 2 can be applied simultaneously to the theory of a linear connection together with an affine connection. In §4 we pass on to the study of invariants under transformations of the form (4.1). We prove a theorem for linear connections, and show that a similar theorem is true for symmetric affine connections. In the case of the latter this amounts to expressing

$$T^i_{a[j;k_1;l_1;\cdots;k_p;l_p]}$$

in terms of T^i_a and the curvature tensor. In §5 we return to the study of a linear connection together with an affine connection, and show how a complete set of invariants may be obtained which are closely analogous to affine normal tensors. Dynamical systems with non-holonomic constraints provide a field of application for this theory, as we show briefly in §6. In §7 we apply the existing theory of Pfaffian forms to the equations for linear displacement, and show how the theorem in §1 is relevant to the study of integral subspaces.

As in T. L. C. we follow Schlesinger in his use of matrices. Instead of (0.1) we deal with the equations

$$dZ^\alpha_\beta + Z^\sigma_\beta L^\alpha_{\sigma i} dx^i = 0,$$

which we write as one equation

$$dZ + ZL_i dx^i = 0,$$

with a matrix for the unknown. This equation is completely integrable if, and only if, it is satisfied by a non-singular (i.e., with non-zero determinant) matrix $V(x)$. In this case we have

$$L_i = - V^{-1}V_{,i},$$

where we use the comma to denote partial differentiation.

1. **Osculating connections.** The necessary and sufficient conditions that the equation

$$dZ + ZL_i dx^i = 0$$

is completely integrable are that*

$$\tfrac{1}{2}R_{ij} = L_{[i,j]} + L_{[i}L_{j]} = 0.$$

---

* We follow J. A. Schouten in writing $p!A_{[i_1\ldots i_p]}$ for the alternating sum of the quantities $A_{i_1\ldots i_p}$.

In this case the connection L, and the displacement defined by this connection, are said to be integrable.

Williams and I showed that, given any linear connection L, and a coordinate system, x, there exists a unique integrable connection Γ such that

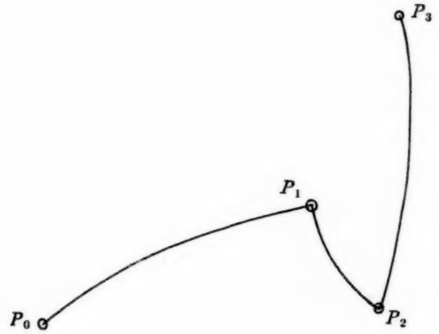
$$
\begin{aligned}
&\Gamma_n = L_n, \\
&\Gamma_{n-1} = L_{n-1} \quad \text{for} \quad x^n = 0, \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&\Gamma_p = L_p \quad \text{for} \quad x^{p+1} = \cdots = x^n = 0, \qquad (1.1)\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&\Gamma_1 = L_1 \quad \text{for} \quad x^2 = \cdots = x^n = 0.
\end{aligned}
$$

We shall refer to this as Theorem C. Since the connection Γ is completely determined by these conditions we shall say that it osculates the connection L, in the manner described by (1.1). Notice that Γ is determined not by L alone, but by L together with the series of subspaces, each contained in the next, on which the components $\Gamma_1, \Gamma_2, \cdots$ agree with $L_1, L_2, \cdots$.

We shall give another proof of this theorem. Let $P_x \equiv (x^1, \cdots, x^n)$ be any point in the neighborhood of the point $P_0 = (0, \cdots, 0)$, and let

$$
\begin{aligned}
&P_1 \text{ be the point } (x^1, 0, \cdots, 0), \\
&P_2 \quad `` \quad `` \quad `` \quad (x^1, x^2, 0, \cdots, 0), \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&P_r \quad `` \quad `` \quad `` \quad (x^1, \cdots, x^r, 0, \cdots, 0), \qquad (1.2)\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&P_n \quad `` \quad `` \quad `` \quad P_x \equiv (x^1, \cdots, x^n).
\end{aligned}
$$

There is a unique curve, $P_0P_1 \cdots P_x$, joining P_0 to each point P_x in the neighborhood of P_0. Each of these curves is analytic except at a finite number of points.



Let V_0 be any non-singular matrix. Then at each point, x, is defined a non-singular matrix, $V(x)$, by the linear displacement of V_0 from P_0 to P_x, along the curve $P_0P_1 \cdots P_x$. Along this curve we shall have

$$dV + VL_i dx^i = 0. \tag{1.3}$$

By repeated applications of the theorem that a solution of

$$\frac{dy^\alpha}{dt} = \phi^\alpha(t; a^1, \cdots, a^N),$$

where ϕ^α are analytic in all their arguments, is itself analytic in t and $a^1, \cdots, a^N$, we see that V is an analytic function of x. We can, therefore, define an integrable connection by the equations

$$\Gamma_i = - V^{-1}V_{,i}, \tag{1.4}$$

and, from (1.3) and (1.4), we have

$$(\Gamma_i - L_i)dx^i = 0 \tag{1.5}$$

along each curve $P_0P_1 \cdots P_x$. On the segment $P_{p-1}P_p$ we have

$$\Gamma_p = L_p \quad \text{for} \quad x^{p+1} = \cdots = x^n = 0, \tag{1.6}$$

and by giving p the values* 1, $\cdots$, n we have the relations (1.1), and the theorem is established.

2. **A theorem on affine connections.** In this section we shall prove the following

Theorem. *Let D be any affine connection (not necessarily symmetric). Then there exist coördinate systems in which the components, D^i_{jk}, of D satisfy the following conditions:*

$$\begin{aligned}
&D^i_{nn} = 0; \\
&D^i_{\rho n-1} = 0, \quad \rho \geqq n-1, \quad \textit{for} \quad y^n = 0; \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&D^i_{\rho p} = 0, \quad \rho \geqq p, \qquad \textit{for} \quad y^{p+1} = \cdots = y^n = 0; \\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
&D^i_{j1} = 0, \quad j = 1, \cdots, n, \quad \textit{for} \quad y^2 = \cdots = y^n = 0.
\end{aligned} \tag{2.1}$$

Any such coördinate system is uniquely determined by a point and n independent contravariant vectors associated with that point.

* When $p=n$ we have simply $\Gamma_n = L_n$.

By taking these as the unit vectors tangent to the coördinate lines, in any coördinate system, we can associate, with each coördinate system and each point, a unique coördinate system in which (2.1) are satisfied. We shall thus have a class of coördinate systems, the totality of which will be an invariant of the affine connection D.

An essential difference between this theorem and those proved in T. L. C. is that it is concerned with the affine connection itself, and not with the affine connection together with a given system of curves and surfaces.

Let P_0 be any point in the space bearing the affine connection D. Let p_a be n independent contravariant vectors associated with P_0. A coördinate system may be constructed by the following procedure. The coördinates of P_0 are to be $(0, \cdots, 0)$. Let C_1 be the path which passes through P_0 in the direction determined by the vector p_1. Move the matrix (p^i_a) by parallel displacement along C_1 from P_0 to a point P_1. The coördinates of P_1 are to be $(y^1, 0, \cdots, 0)$. Let $v_a(y^1, 0, \cdots, 0)$ be the components of the vectors thus obtained, and let C_2 be the path through P_1 in the direction v_2. Then move the matrix v by parallel displacement along C_2 from P_1 to a point P_2, whose coördinates are to be $(y^1, y^2, 0, \cdots, 0)$. Repeating* this process we shall eventually reach a point P_n whose coördinates are to be $(y^1, \cdots, y^n)$.

The proof that this process gives an allowable coördinate system (i.e., a coördinate system obtained by an analytic transformation from a given coördinate system) is of the same nature as that required in §1.

Let H^i_{jk} be the components of D in a coördinate system x, in which the equations to C_1 are $x^i = \phi^i(y^1)$. The components of the n-uple v^i_a, at P_1, are given by those sets of solutions to

$$\frac{dx^i}{dy^1} + X^j H^i_{jk} \frac{d\phi^k}{dy^1} = 0 \tag{2.2}$$

which reduce to p^i_a for $y^1 = 0$. The equations to the path C_2 are given by those solutions, $\phi^i_1(y^1, t)$, to the equations

$$\frac{d^2x^i}{dt^2} + H^i_{jk}\frac{dx^j}{dt}\frac{dx^k}{dt} = 0.$$

which satisfy the initial conditions

$$\begin{aligned} \phi^i_1(y^1, 0) &= \phi^i(y^1), \\ \left(\frac{d\phi^i_1}{dt}\right)_{t=0} &= v^i_2(y^1, 0, \cdots, 0). \end{aligned} \tag{2.3}$$

---

* At the rth step we shall move the matrix along the path C_r, which passes through P_{r-1} in the direction v_r, from P_{r-1} to a point P_r, whose coördinates are to be $(y^1, \cdots, y^r, 0, \cdots, 0)$.

The components v_a^i at P_2 are given by the sets of solutions to the equations

$$\frac{dX^i}{dt} + X^j H^i_{jk} \frac{d\phi_1^k}{dt} = 0$$

which reduce to $v_a^i(y^1, 0, \cdots, 0)$ for $t=0$. If we put $y^2=t$ the coördinates of P_2, and the components of v_a^i at this point, will be given by

$$(2.4) \qquad \begin{aligned} x^i &= \phi_1^i(y^1, y^2), \\ v_a^i &= v_a^i(y^1, y^2, 0, \cdots, 0). \end{aligned}$$

Since $\phi(y^1)$ and H^i_{jk} are analytic functions of y^1 and x respectively, ϕ_1^i and v_a^i, given by (2.4), will be analytic functions of y^1 and y^2. Proceeding in this way we shall eventually obtain, for the coördinates of P_n,

$$x^i = x^i(y^1, \cdots, y^n),$$

and for the components of v_a^i at this point,

$$v_a^i(y^1, \cdots, y^n),$$

where x^i and v_a^i are analytic functions of y. The process given above in descriptive terms does, therefore, give an allowable coördinate system, and an n-uple of contravariant vectors whose components are analytic functions of y.

From the construction for v_a^i it follows that

$$(2.5) \qquad \begin{aligned} v_n^i &= \delta_n^i, \\ v_{n-1}^i &= \delta_{n-1}^i \text{ for } \quad y^n = 0, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ v_p^i &= \delta_p^i \quad \text{for } y^{p+1} = \cdots = y^n = 0, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ v_1^i &= \delta_1^i \quad \text{for } \quad y^2 = \cdots = y^n = 0. \end{aligned}$$

These equations may be concentrated into

$$v_\rho^i = \delta_\rho^i,\ \rho \geqq p, \text{ for } y^{p+1} = \cdots = y^n = 0,\ p = 1, \cdots, n.$$

Let

$$\overset{a}{u}_i \overset{i}{v}_b = \delta^a_b.$$

Then from (2.5) we have

$$(2.6) \qquad \overset{a}{u}_\rho = \delta^a_\rho,\ \rho \geqq p \text{ for } y^{p+1} = \cdots = y^n = 0 \quad (p = 1, \cdots, n).$$

By an argument used in §1 we have

$$
\begin{aligned}
&D^i_{jn} = E^i_{jn},\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&D^i_{jp} = E^i_{jp} \text{ for } y^{p+1} = \cdots = y^n = 0,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&D^i_{j1} = E^i_{j1} \text{ for } y^2 = \cdots = y^n = 0,
\end{aligned}
\tag{2.7}
$$

where $E^i_{jk} = v^i_a u^a_{j,k}$, and D^i_{jk} are the components, in y, of the given connection. From (2.6) we have

$$
E^i_{\rho p} = 0,\ \rho \geqq p,\ y^{p+1} = \cdots = y^n = 0 \quad (p = 1, \cdots, n),
\tag{2.8}
$$

which, combined with (2.7), give (2.1). The coördinate system is uniquely determined by the point P_0 and the matrix (p^i_a). The theorem is therefore established.

We shall give another proof that (2.1) hold, which will bring out their geometrical significance. The curves of the congruence defined by $v^i_n(=\delta^i_n)$ are paths. Hence

$$D^i_{nn} = 0.$$

The curves of the congruence defined by v_{n-1}, which lie in the hypersurface $y^n=0$, are paths. Since $v^i_{n-1} = \delta^i_{n-1}$ for $y^n=0$, we have

$$D^i_{n-1,n-1} = 0 \text{ for } y^n = 0.$$

But the vectors v_n are parallel at different points of these curves. Hence

$$D^i_{n,n-1} = 0 \text{ for } y^n = 0.$$

The remaining conditions may be obtained by a repetition of this argument.

In case D is symmetric all its components figure in the equations

$$D^i_{\rho p} = 0,\ \rho \geqq p, \text{ for } y^{p+1} = \cdots = y^n = 0 \quad (p = 1, \cdots, n),$$

which may be written

$$
D^i_{pq} = 0 \text{ for } y^{s+1} = \cdots = y^n = 0,\ s = \min(p, q) \quad (p, q = 1, \cdots, n).
\tag{2.9}
$$

3. **Linear connections together with affine connections.** The theorem proved in §1 belongs to the combined theory of a linear connection and an affine connection, for it refers to the connection L and the sub-spaces given in the coördinate system x by $x^2 = \cdots = x^n = 0$, $x^3 = \cdots = x^n = 0$, and so on.

These loci are flat sub-spaces, defined by a flat affine connection for which x is a cartesian coördinate system. If we are concerned with the general theory of a linear connection L, and an affine connection D, we can construct a coördinate system y and an n-uple v_a^i by the process given in §2. Theorem C, referred to the coördinate system y, will belong to the combined theory of L and D. In place of (1.1) we can write the relations

$$(\Gamma_i - L_i)v_p^i = 0 \text{ for } y^{p+1} = \cdots = y^n = 0 \qquad (p = 1, \cdots, n). \tag{3.1}$$

The methods of §1 can be used to give other osculating integrable connections. The simplest of these is constructed by taking normal coördinates, y, for D at any point P_0, and considering the matrix function, V, given by the linear displacement of a non-singular matrix, V_0, from P_0 to any point y along the path joining these points. The equations giving V are

$$dV + VL_i dx^i = 0, \tag{3.2}$$

or

$$(\Delta_i - L_i)dy^i = 0, \tag{3.3}$$

where Δ is the integrable connection given by

$$V_{,i} + V\Delta_i = 0. \tag{3.4}$$

Since y^i are normal coördinates, and (3.3) refer to displacement along paths through the origin, we have

$$(\Delta_i - L_i)y^i = 0. \tag{3.5}$$

As in §1 the connection Δ is uniquely determined by this condition.

4. **Invariant theory.** In this section we take up the invariant theory of a linear connection under transformations of the form

$$(4.1)\quad \begin{array}{ll} \text{(a)} & \bar{Z}^\alpha = Z^\beta p_\beta^\alpha, \\ \text{(b)} & \bar{x}^i = \bar{x}^i(x), \end{array}$$

where $\|p_\beta^\alpha\|$ is a non-singular matrix depending on x only. A coördinate system for the underlying manifold, together with a frame of reference in each of the associated spaces, will be called a *representation*; and a transformation of the form (4.1) will be called a change of representation. On this basis an invariant may be defined in terms of its transformation law* under changes of representation. We shall deal only with linear connections, and with tensors having m^2n^p components which obey the transformation law

* Schlesinger, loc. cit., p. 423.

$$\bar{T}^{\lambda}_{\beta(i)} P^{\alpha}_{\lambda} = P^{\lambda}_{\beta} T^{\alpha}_{\lambda(a)} \frac{\partial x^{(a)}}{\partial \bar{x}^{(i)}},$$

where the symbol (i) stands for any number of italic indices, and $P^{\lambda}_{\beta} p^{\alpha}_{\lambda} = \delta^{\alpha}_{\beta}$. The transformation law for a linear connection is given by

$$\bar{L}^{\alpha}_{\beta j} = (P^{\lambda}_{\beta,i} + P^{\mu}_{\beta} L^{\lambda}_{\mu i}) p^{\alpha}_{\lambda} \frac{\partial x^i}{\partial \bar{x}^j},$$

and we shall write these formulas*

$$(4.2)\qquad \begin{aligned} &\text{(a)} \qquad \bar{T}_{(i)} P = P T_{(a)} \frac{\partial x^{(a)}}{\partial \bar{x}^{(i)}}, \\ &\text{(b)} \qquad \bar{L}_j P = (P_{,i} + P L_i) \frac{\partial x^i}{\partial \bar{x}^j} \cdot \end{aligned}$$

From (4.2b) we see that there exist representations in which all the components of an integrable connection vanish. For if Γ is an integrable connection, there will be a non-singular matrix-function V, such that

$$V_{,i} + V\Gamma_i = 0,$$

and the components of Γ will vanish in the representation given by

$$\bar{Z}^{\alpha} = Z^{\beta} U^{\alpha}_{\beta},$$
$$\bar{x}^i = x^i,$$

where $U = V^{-1}$. All representations in which the components of the connection vanish are related by equations of the form (4.1), where p is a constant matrix.

An operation analogous to covariant differentiation arises from the following considerations. Let V be a matrix which satisfies the equation

$$(4.3)\qquad dV + VL_i dx^i = 0$$

* Since the transformations (4.1a) and (4.1b) are independent, it might, for some purposes, be desirable to borrow from group-theory the notion of conjugacy. Two tensors K and H may be described as conjugate if there exists a non-singular matrix, V, such that

$$K_{(i)} V = V H_{(i)}.$$

The set of all tensors conjugate to a given tensor may be called the class of that tensor. Similarly two linear connections are in the same class if there exists a non-singular matrix, V, such that

$$M_i V = V_i + V L_i.$$

All tensors or linear connections belonging to the same class are seen to be equivalent under transformations of the form (4.1a).

along a curve C. It follows that

$$dV^{-1} - L_i V^{-1} dx^i = 0. \tag{4.4}$$

Let $T_{(i)}$ be a given tensor and let

$$A_{(i)} = VT_{(i)}V^{-1}.$$

Differentiating along C we have, from (4.3) and (4.4),

$$dA_{(i)} = VT_{(i)/k}V^{-1},$$

where*

$$T_{(i)/k} = T_{(i),k} + T_{(i)}L_k - L_k T_{(i)}. \tag{4.5}$$

Direct calculation shows that

$$2T_{(i)/[j/k]} = T_{(i)}R_{jk} - R_{jk}T_{(i)}. \tag{4.6}$$

Let

$$(TR)_s = R_{j_1k_1} \cdots R_{j_{s-1}k_{s-1}} T_{(i)} R_{j_sk_s} \cdots R_{j_pk_p}, \tag{4.7}$$

and let an operator, α, be defined by the equation

$$\alpha(TR)_s = (TR)_{s+1}.$$

We can describe α as an operator which moves T one place to the right in any expression such as (4.7), without respect to particular values of s and p $(p>s)$. We can write (4.6) as

$$T_{(i)/[j/k]} = \tfrac{1}{2}(1-\alpha)T_{(i)}R_{jk}.$$

In T. L. C. (p. 154) it was shown that

$$R_{[jk/l]} = 0.$$

Hence

$$T_{(i)/[j_1/k_1/j_2/k_2]} = \tfrac{1}{4}(1-\alpha)^2 T_{(i)} R_{[j_1k_1} R_{j_2k_2]},$$

and, in general,

$$T_{(i)/[j_1/k_1/\cdots/j_p/k_p]} = \frac{1}{2^p}(1-\alpha)^p T_{(i)} R_{[j_1k_1} \cdots R_{j_pk_p]}. \tag{4.8}$$

---

* We cannot derive tensors from a given tensor by repeated applications of this operation, as there is no way of eliminating the second derivatives

$$\frac{\partial^2 x^i}{\partial \bar{x}^j \partial \bar{x}^k}.$$

Similar identities will occur in the theory of a symmetric affine connection. For if $m=n$ and L is an affine connection such that

$$L^{\alpha}_{\beta i} = L^{\alpha}_{i\beta},$$

we have

$$T^{\alpha}_{\beta[i;j]} = T^{\alpha}_{\beta[i/j]},$$

where the semi-colon denotes ordinary covariant differentiation, and $T^{\alpha}_{\beta i/j}$ means the same as before. The relation (4.8) was obtained by purely formal methods, and we have, therefore,

$$(4.9) \qquad T_{[i;j_1;k_1;\cdots;j_p;k_p]} = \frac{1}{2^p}(1-\alpha)^p T_{[i}R_{j_1k_1}\cdots R_{j_pk_p]}.$$

5. **Normal representations.** In the theory of a linear connection together with an affine connection, the comma on the right hand side of (4.5) can be taken to define covariant differentiation with respect to the latter. If, for example, C^i_{jk} are the components of the affine connection, we shall have*

$$T_{i/j} = \frac{\partial T_i}{\partial x^j} - T_s C^s_{ij} + T_i L_j - L_j T_i.$$

It will then be possible to obtain successive tensor invariants from a given tensor.

Let y be the normal coördinate system at a point q for the affine connection and the coördinate system, x, in some given representation. There will be representations in which the components of the integrable connection Δ, defined by (3.4) and (3.5), are zero. There is just one of these representations, the normal coördinate system, y, being retained throughout, which determines in the associated space at q the same frame of reference as the given representation. This is obtained by imposing the initial conditions

$$(V^{\alpha}_{\beta})_{y=0} = \delta^{\alpha}_{\beta},$$

in the equations (3.4), and may be called the *normal representation* at q for the linear connection together with the affine connection, and for the given representation. In this representation we have

$$(5.1) \qquad L_i y^i = 0,$$

and

* This is a simple application of a scheme introduced by A. W. Tucker in a paper which will shortly appear in the Annals of Mathematics.

$$L_i = \sum_{p=1}^{\infty} \frac{1}{p!} H_{i\,k_1\cdots k_p} y^{k_1} \cdots y^{k_p}, \tag{5.2}$$

where

$$H_{i\,k_1\cdots k_p} = \left(\frac{\partial^p L_i}{\partial y^{k_1} \cdots \partial y^{k_p}}\right)_{y=0}. \tag{5.3}$$

From (5.1) and (5.2) it follows that

$$H_{i\,k_1+\cdots+k_p} + H_{k_1 i\cdots k_p} + \cdots + H_{k_1\cdots k_p i} = 0. \tag{5.4}$$

Let $\overline{H}_{ik}$, $\overline{H}_{ik_1k_2}$, $\cdots$, $\overline{H}_{i(k)_p}$, $\cdots$ be the quantities obtained in the same way as $H_{i(k)_p}$, at the same point, but starting with a different representation. Just as in the affine theory, it follows that $H_{i(k)_p}$ and $\overline{H}_{i(k)_p}$ are related by the transformation law for a tensor. Hence a sequence of tensors,

$$H_{i\,k}, \cdots, H_{i\,k_1\cdots k_p}, \cdots,$$

analogous to affine normal tensors, is defined by the condition that the components of $H_{i(k)_p}$, at each point q, shall be given by (5.3). These, together with the normal tensors for the affine connection, constitute a complete set of invariants for the linear connection together with the affine connection. This may be proved by methods similar to those used in proving the analogous theorem for affine connections.*

6. **Application to dynamics.** The mathematical machinery used by G. Vranceanu† in his treatment of dynamical systems with non-holonomic constraints, may be regarded as the combined theory of a linear connection and a Riemannian metric. The metric $g_{ij}dx^i dx^j$ represents the kinetic energy, and the constraints can be represented by m unit orthogonal vectors $\xi_1, \cdots, \xi_m$ $(m \leqq n)$. If $\overset{\alpha}{\gamma}_{\beta\sigma}$ are the rotation functions, given by

$$\overset{\alpha}{\gamma}_{\beta\sigma} = \xi_{\alpha i}\overset{i}{\xi}_{\beta;\sigma},$$

where $\xi_{\beta;\sigma}$ is the intrinsic derivative of ξ_β, a linear connection is defined by

$$\overset{\alpha}{L}_{\beta i} = \overset{\alpha}{\gamma}_{\beta\sigma}\xi_{\sigma i}.$$

We should limit (4.1a) to orthogonal transformations by imposing the condition

$$\overset{\sigma}{p}_\alpha \overset{\sigma}{p}_\beta = \delta_{\alpha\beta}.$$

---

* T. Y. Thomas, Mathematische Zeitschrift, vol. 25 (1926), pp. 723–733. Thomas was considering a special type of affine connection, but the method is general.

† Comptes Rendus, vol. 183 (1926), p. 852, also p. 1083.

The associated space at each point can be identified with the sub-space spanned in the tangent space by the vectors ξ_α, but the essential feature which distinguishes the theory of a linear connection from that of an affine connection is retained: namely that the frame of reference may be changed in each associated space independently of coördinate transformations.

7. **Integral sub-spaces.** In this section we shall show how some of the general ideas in the theory of Pfaffian forms* can be interpreted in terms of linear displacement, when considering the equations

$$dZ^\alpha + Z^\beta L^{\alpha}_{\beta i} dx^i = 0. \tag{7.1}$$

It will be convenient to say that a set of numbers $(x^1, \cdots, x^n; Z^1, \cdots, Z^m)$ determine, on the one hand a point x in the underlying manifold V_n, together with a point Z in the linear space associated with x, and on the other hand a point in a space of $m+n$ dimensions, which we shall denote by S_{n+m}. We shall discuss some of the simpler properties of the integral sub-spaces, in S_{n+m}, of the equations (7.1).

Let R_{ij} be the curvature tensor derived from the linear connection L. We shall say that any two vectors ξ and η which satisfy the condition

$$R_{ij}\xi^i\eta^j = 0 \tag{7.2}$$

are in involution† with respect to L. Any two vectors linearly dependent on ξ and η will also satisfy (5.2). Let a set of vectors $\xi_1, \cdots, \xi_p$, such that

$$\xi^i_{\lambda,j}\xi^j_\mu - \xi^i_{\mu,j}\xi^j_\lambda = c^{\nu}_{\lambda\mu}\xi^i_\nu \qquad (\nu = 1, \cdots, p), \tag{7.3}$$

be mutually in involution with respect to L. In virtue of (7.3) we can find a set of vectors $X_1, \cdots, X_p$, linearly dependent on $\xi_1, \cdots, \xi_p$, and such that the equations

$$\frac{\partial x^i}{\partial t^\lambda} = X^i_\lambda \qquad (\lambda = 1, \cdots, p) \tag{7.4}$$

are completely integrable. The vectors $X_1, \cdots, X_p$ will, therefore, define a congruence‡ of p-spaces given by

* All these ideas are to be found in Goursat's *Leçons sur le Problème de Pfaff*, especially in chapters VI and VIII. The latter chapter is mainly an exposition of Cartan's work.

† This is not the same as saying that ξ and η are in involution with respect to the equations (7.1), the conditions for which are

$$Z^\beta R^{\alpha}_{\beta ij}\xi^i\eta^j = 0.$$

We require that ξ^i and η^i shall not depend on Z, in which case these equations imply (7.2).

‡ By a congruence we mean a family of p-spaces such that one and only one passes through each point of some given n-cell in V_n.

$$x^i - x_0^i = x^i(t^1, \cdots, t^p; x_0), \tag{7.5}$$

where $x^i(t^1, \cdots, t^p; x_0)$ satisfy (7.4). Since the vectors $X_1, \cdots, X_p$ are mutually in involution with respect to L, we shall have

$$R_{ij}\frac{\partial x^i}{\partial t^\lambda}\frac{\partial x^j}{\partial t^\mu} = 0, \tag{7.6}$$

and so the equations

$$\frac{\partial Z^\alpha}{\partial t^\lambda} + \overset{\beta}{Z}\overset{\alpha}{L}_{\beta i}\frac{\partial x^i}{\partial t^\lambda} = 0 \qquad (\lambda = 1, \cdots, p) \tag{7.7}$$

will be completely integrable. On each p-space of the congruence given by (7.5) the connection L determines, therefore, an integrable displacement. Any solution to (7.7) is of the form

$$\overset{\alpha}{Z} = \overset{\beta}{Z_0}\overset{\alpha}{\phi_\beta}(t^1, \cdots, t^p; x_0),$$

where $\overset{\alpha}{Z_0}$ are arbitrary constants.

In terms of the space S_{n+m} we say that the equations

$$\begin{aligned} \overset{\alpha}{Z} &= \overset{\beta}{Z_0}\overset{\alpha}{\phi_\beta}(t; x_0), \\ x^i &= x_0^i + x^i(t; x_0) \end{aligned} \tag{7.8}$$

define a congruence of integral p-spaces (in S_{n+m}) with respect to the equations (7.1). Such a family of integrals is called *generic*, since (7.4) and (7.7) are completely integrable at a "typical point" of S_{n+m}.

It may happen that there are singular* integral sub-spaces in S_{n+m}. Singular integrals arise in any of the three following cases:

(1) The equations (7.6) are satisfied by a complete system of vectors $X_1, \cdots, X_p$, but only on some sub-space of V_n (i.e. subject to certain conditions, $\phi(x)=0, \psi(x)=0, \cdots$).

(2) The equations (7.4) admit solutions, but are not completely integrable.

(3) The equations (7.7) admit solutions, but are not completely integrable. In the third case let (7.7) admit a complete set of q independent solutions $U_1, \cdots, U_q$, $q<m$. Then a^sU_s, $s=1, \cdots, q$, where a^s are constants, will also be a solution, and so, for $x=$const., the totality of solutions to (7.7) will be the linear space spanned by $U_1, \cdots, U_q$. In terms of the linear connection L we have an integrable displacement of linear q-spaces in

* An integral sub-space is called singular if it does not belong to a congruence, but to a family which is entirely contained in some sub-space of higher dimensionality.

the associated spaces.* Since the matrix (U_s^α), $\alpha=1, \cdots, m$, $s=1, \cdots, q$, is of rank q we may assume that the determinant $|U_s^t|$, $s, t=1, \cdots, q$, does not vanish. Apply the change of representation given by

$$Z^\alpha = \bar{Z}^s U_s^\alpha + \bar{Z}^\rho \delta_\rho^\alpha \quad (s = 1, \cdots, q, \rho = q+1, \cdots, m),$$
$$\bar{x}^i = x^i.$$

Then the linear q-spaces in question are given, in the new representation, by $\bar{Z}^\rho=0$. Since the equations (7.1) are invariant in form under all changes of representation, it follows that

$$L_{si}^{\rho} dx^i = 0 \tag{7.9}$$

for values of dx tangent to any sub-space on which (7.7) admit the solutions $U_1, \cdots, U_q$.

In the remainder of this section we shall suppose that (7.7) either admit no solutions, or else are completely integrable, in which case the vectors $X_1, \cdots, X_p$ are mutually in involution with respect to L. Singular integrals will occur, therefore, only in the event of (1) or (2) arising, and we shall combine these into the case where V_n admits a family of sub-spaces, on each of which L defines an integrable displacement, and which, in a suitable coordinate system, are defined by equations of the form

$$x^{p+1} = c^{p+1}, \cdots, x^q = c^q, x^{q+1} = \cdots = x^n = 0, \qquad n > q > p, \tag{7.10}$$

where $c^{p+1}, \cdots, c^q$ are arbitrary constants. If $q=n$ this family is a congruence, and the corresponding integrals generic. We shall show how Theorem C is relevant to this simplified theory of integral sub-spaces, and to the study of the characteristics. A vector ξ will be described as a characteristic of the connection L if it is in involution, with respect to L, with every other vector. The necessary and sufficient conditions for this to be the case are that†

$$R_{ij}\xi^i = 0. \tag{7.11}$$

Let ξ_ρ, $\rho=p+1, \cdots, n$, be a complete set of solutions to these equations. Differentiating

$$R_{ij}\xi_\rho^j = 0,$$

---

* If L were an affine connection $U_1, \cdots, U_q$ would be parallel fields of contravariant vectors, defined on some sub-space in V_n.

† A vector ξ is a characteristic for the Pfaffian system (7.1) if

$$Z^\beta R_{\beta ij}^\alpha \xi^j = 0,$$

and we can only deduce (7.11) from these equations when ξ^i are independent of Z.

we can write the result in the forms

$$R_{ij/k}\xi^j_\rho + R_{ij}\xi^j_{\rho,k} = 0,$$
$$R_{ik/j}\xi^k_\sigma + R_{ik}\xi^k_{\sigma,j} = 0,$$
$$R_{jk/i}\xi^j_\rho + R_{jk}\xi^j_{\rho,i} = 0,$$
$$R_{jk/i}\xi^k_\sigma + R_{jk}\xi^k_{\sigma,i} = 0.$$

Multiplying these equations by ξ^k_σ, $-\xi^j_\rho$, $\frac{1}{2}\xi^k_\sigma$, and $\frac{1}{2}\xi^j_\rho$ respectively, adding them together, and taking into account the relations $R_{[ij/k]}=0$ and (7.11), we obtain

$$R_{ij}\xi^j_{[\rho,\sigma]} = 0,$$

where

$$\xi^i_{\rho,\sigma} = \xi^i_{\rho,j}\xi^j_\sigma.$$

Since ξ_ρ form a complete set of solutions to (7.11) we have,* from these equations,

$$\xi^i_{[\rho,\sigma]} = C^\tau_{\rho\sigma}\xi^i_\tau \qquad (\tau = p+1,\cdots,n). \tag{7.12}$$

There exists, therefore, a coördinate system x, in which δ^i_ρ form a complete set of solutions to (7.11), and in this coördinate system,

$$R_{i\rho} = 0. \tag{7.13}$$

The vectors δ^i_ρ define the congruence of $(n-p)$-spaces given by

$$x^1 = c^1, \cdots, x^p = c^p,$$

where $c^1, \cdots, c^p$ are arbitrary constants. This will be called the congruence of characteristic sub-spaces for the connection L. Applying Theorem C to the connection L, in the coördinate system x, and choosing a representation in which the components of the integrable connection Γ vanish, we have

$$\text{(7.14)}\qquad \begin{array}{llll} (a_n) & L_n = 0; & & \\ (a_{n-1}) & L_{n-1} = 0 & \text{for} & x^n = 0; \\ \cdot & \cdot & \cdot & \cdot \\ (a_p) & L_p = 0 & \text{for} & x^{p+1} = \cdots = x^n = 0; \\ \cdot & \cdot & \cdot & \cdot \\ (a_1) & L_1 = 0 & \text{for} & x^2 = \cdots = x^n = 0. \end{array}$$

* The linear elements, in S_{n+m}, given by $(\xi^i_\rho, -Z^\beta L^\alpha_{\beta i}\xi^i_\rho)$, do not necessarily define the characteristic manifolds (in S_{n+m}) for the equations (7.1). The relations (7.12) do not, therefore, follow from the general theory of Pfaffian systems.

From (7.13) and ($7.14a_n$) we have, if $p<n$,

$$L_{i \cdot n} = 0.$$

The condition $x_n=0$ may, therefore, be discarded in all the relations (7.14). We have, therefore, $L_{n-1}=0$, and from (7.13), if $p<n-1$,

$$L_{i \cdot n-1} = 0.$$

Hence the condition $x^{n-1}=0$ may also be discarded in (7.14). Repeating this argument we can replace (7.14) by the relations

$$
\begin{aligned}
& L_\rho = 0, \ \rho = p+1, \cdots, n; \\
& L_p = 0; \\
(7.15) \qquad & L_{p-1} = 0 \quad \text{for} \quad x^p = 0; \\
& \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\
& L_1 = 0 \quad \text{for} \quad x^2 = \cdots = x^p = 0,
\end{aligned}
$$

with the condition $L_{i,\rho}=0$, $\rho=p+1, \cdots, n$. The study of the connection L may, therefore, be confined to its behavior on the sub-space V_p, of V_n, given by $x^{p+1}= \cdots =x^n=0$. The number p may be called the class of the connection* L.

Let us suppose the class to be n, and consider the equations (7.14). The equations ($7.14a_n$) assert that the connection L defines an integrable displacement along the curves of parameter x^n. If L defines an integrable displacement on each surface of the congruence given by

$$x^a = c^a \qquad (a = 1, \cdots, n-2),$$

where c^a are arbitrary constants, we can discard the condition $x^n=0$ in ($7.14a_{n-1}$), and conversely. In this case the surfaces in S_{n+m}, given by

$$x^1 = c^1, \cdots, x^{n-2} = c^{n-2}, Z^1 = A^1, \cdots, Z^m = A^m,$$

where c and A are arbitrary constants, belong to a generic family of integrals with respect to the equations (7.1).

In order that the surfaces in S_{n+m} given by

$$x^1 = c^1, \cdots, x^{p-1} = c^{p-1}, Z^\alpha = A^\alpha$$

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* If p is the class of a linear connection L, the tensor (see §4) whose components are

$$R_{[i_1k_1} \cdots R_{i_qk_q]}$$

will vanish for $2q>p$. This follows from the existence of coördinate systems in which $R_{i\rho}=0$, for $\rho>p$.

shall be a generic family of integrals, it is necessary and sufficient that $L_p = \cdots = L_n = 0$.

Starting from the other end, the sub-space in S_{n+m}, given by

$$x^{p+1} = \cdots = x^n = 0, \quad Z^\alpha = A^\alpha,$$

will be a singular integral if, and only if, $L_1 = \cdots = L_p = 0$ for $x^{p+1} = \cdots = x_n = 0$. The connection L will then define an integrable displacement over the p-space in V_n, given by $x^{p+1} = \cdots = x^n = 0$.

In general let a_q^p, $n \geqq q > p$, stand for the condition $x^q = 0$ which is imposed in (7.14a_p). If the connection L is such that any given set of these conditions, $a_{q_1}^{p_1}, \cdots, a_{q_s}^{p_s}$, can be discarded, there will be a family of integrals, whose equations will be apparent from (7.14). If, for example, the conditions a_{n-2}^{n-1}, a_{n-2}^n are unnecessary, the surfaces in S_{n+m} given by

$$x^1 = c^1, \cdots, x^{n-3} = c^{n-3}, x^{n-1} = c^{n-1}, Z^\alpha = A^\alpha,$$

will be generic integrals, and those given by

$$x^1 = c^1, \cdots, x^{n-3} = c^{n-3}, x^n = 0, Z^\alpha = A^\alpha$$

singular integrals.

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ON SEQUENCES DEFINED BY LINEAR RECURRENCE RELATIONS*

BY

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I. Introduction

A sequence of rational integers

$$(1)\qquad u_0,\ u_1,\ u_2,\ \cdots$$

is defined in terms of an initial set $u_0, u_1, \cdots, u_{k-1}$ by the recurrence relation

$$(2)\qquad u_{n+k} + a_1 u_{n+k-1} + \cdots + a_k u_n = a, \quad n \geqq 0,$$

where $a, a_1, a_2, \cdots, a_k$ are given rational integers. The purpose of this paper is to investigate the periodicity of such sequences with respect to a rational integral modulus m. Carmichael‡ has studied the period for a modulus m whose prime divisors exceed k and are prime to a_k. In this paper, I give a solution to the problem without restriction on m. If m is prime to a_k the sequence (1) is periodic from the start; otherwise, it is periodic after a definite number of initial terms.

Definition 1. *We say that π is a general period of the recurrence* (2) *for the modulus m if every sequence of rational integers satisfying* (2) *has the period π* (mod m).

Theorem 1. *The minimum period μ* (mod m) *of a sequence* (1) *satisfying* (2) *is a divisor of any general period π* (mod m) *of* (2).

For, since (1) has the period π, $\pi \geqq \mu$. Suppose μ does not divide π, that is, $\pi = q\mu + \rho$, where $0 < \rho < \mu$. Then $u_{i+q\mu+\rho} \equiv u_i$ (mod m), that is, $u_{i+\rho} \equiv u_i$ (mod m) and (1) has the period ρ, which is contradictory.

The algebraic equation

$$(3)\qquad F(x) = x^k + a_1 x^{k-1} + \cdots + a_k = 0$$

is said to be associated with the recurrence (2). We obtain general periods (mod m) of (2) in terms of the decompositions

$$(4)\qquad F(x) \equiv \phi_1(x)^{e_1}\phi_2(x)^{e_2} \cdots \phi_r(x)^{e_r} \pmod{p}$$

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* Presented to the Society, September 11, 1930; received by the editors in August, 1930.

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‡ R. D. Carmichael, *On sequences of integers defined by recurrence relations*, Quarterly Journal of Mathematics, vol. 41 (1920), pp. 343–372.

for the prime divisors p of m, where the $\phi_i(x)$ are prime functions (mod p) whose degrees we denote by k_i.

For the case of periodicity (mod p) it is shown in Section II that we may choose a polynomial $f(x) \equiv F(x)$ (mod p) so that p is not a divisor of the index* of $f(x)$ and hence, by the theorem of Dedekind, (4) implies a corresponding prime ideal decomposition of p in the field generated by a root of $f(x)=0$. General periods of (2) (mod p) are obtained directly from the general solution of (2) by use of the theorem of Fermat in an algebraic field.

The results for the prime power modulus p^α are obtained directly from those (mod p) by the theorem of Section IV. In Section V the solution for a composite modulus m is expressed in terms of the solutions for the prime divisors of m.

The theorems obtained include those given by Carmichael for primes greater than k. The methods may be readily extended to the study of periodicity for an ideal modulus of algebraic sequences defined by linear recurrence relations.

II. Periodicity (mod p)

2.1. It is seen that any change of $F(x)$ (mod p) such that the new polynomial is of degree k with leading coefficient unity does not change the associated sequences (mod p). We prove the following lemma:

Lemma 1. *We may choose a polynomial* $f(x) \equiv F(x)$ (mod p) *with the following properties*:

(i) $f(x)$ *is irreducible of degree* k *with leading coefficient unity.*

(ii) p *does not divide the index of* $f(x)$.

(iii) If θ *is a root of* $f(x)=0$ *and* p *contains precisely the* α*th power of a prime ideal* $\mathfrak{p}$ *in* $K(\theta)$, *then* $f'(\theta)$ *contains precisely* $\mathfrak{p}^{\alpha-1+\rho}$ *where* $\rho=1$ *or* 0 *according as* α *is or is not divisible by* p.

(iv) $1-\theta \not\equiv 0$ (mod $\mathfrak{p}^2$) *for any prime ideal divisor* $\mathfrak{p}$ *of* p *in* $K(\theta)$.

If, in (4), $e_i>1$, we write

$$f_i(x) = \phi_i(x)^{e_i} + p(1 + \phi_i(x)). \tag{5}$$

If $e_i=1$ we write

$$f_i(x) = \phi_i(x) + p. \tag{6}$$

The discriminant of the product

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* If θ is any root of the irreducible equation $f(x)=0$, d the discriminant of the field $K(\theta)$, and D the discriminant of $f(x)$, then $D=\kappa^2 d$, where κ is a rational integer which is called the index of θ or of $f(x)$.

$$P(x) = \prod_{i=1}^{r} f_i(x)$$

is not zero. For the discriminant of each $f_i(x)$ is not zero since the $f_i(x)$ are algebraically irreducible* and the resultant of $f_i(x)$ and $f_j(x)$ is not zero for $i \neq j$. Suppose the discriminant of $P(x)$ contains precisely p^δ. We set

$$(7) \qquad f(x) = P(x) + p^{\delta+2}R(x),$$

where $R(x)$ is a polynomial of degree $k-1$ chosen so that $f(x)$ satisfies the Eisenstein irreducibility criterion for another prime q. Then $f(x)$ is irreducible and of degree k with leading coefficient unity.

Since

$$(8) \qquad f(x) = \phi_1(x)^{e_1}\phi_2(x)^{e_2} \cdots \phi_r(x)^{e_r} + pM(x),$$

where $M(x) \not\equiv 0$ (modd p, $\phi_i(x)$), $i = 1, 2, \cdots, r$, it follows by the criterion of Dedekind that p is not a divisor of the index of $f(x)$.

Hence, by the theorem of Dedekind, (4) implies the prime ideal decomposition

$$(9) \qquad p = \mathfrak{p}_1^{e_1}\mathfrak{p}_2^{e_2} \cdots \mathfrak{p}_r^{e_r}, \quad N(\mathfrak{p}_i) = p^{k_i}$$

in the field defined by a root θ of $f(x) = 0$. Furthermore,

$$(10) \qquad \mathfrak{p}_i = (p, \phi_i(\theta)) \qquad (i = 1, 2, \cdots, r).$$

From (10), $f_j(\theta)$ is prime to $\mathfrak{p}_i$ for $i \neq j$. Hence, from (7), since $f(\theta) = 0$, $f_i(\theta) \equiv 0 \pmod{\mathfrak{p}_i^{e_i(\delta+2)}}$, and

$$(11) \qquad f'(\theta) \equiv f_i'(\theta)Q(\theta) \qquad \pmod{\mathfrak{p}_i^{e_i(\delta+2)}},$$

where $Q(\theta)$ is prime to $\mathfrak{p}_i$. Hence $f'(\theta)$ contains the same power of $\mathfrak{p}_i$ as $f_i'(\theta)$. If $e_i = 1$, $f_i'(\theta) = \phi_i'(\theta)$ is prime to $\mathfrak{p}_i$. If $e_i > 1$,

$$f_i'(\theta) = [e_i\phi_i(\theta)^{e_i-1} + p]\phi_i'(\theta).$$

But since $\phi_i(x)$ is a prime function (mod p), $\phi_i'(\theta)$ is prime to $\mathfrak{p}_i$. Hence $f'(\theta)$ contains $\mathfrak{p}_i^{e_i}$ or $\mathfrak{p}_i^{e_i-1}$ according as e_i is or is not divisible by p, and (iii) is proved.

If $F(1) \equiv 0 \pmod{p}$ we choose $\phi_1(x) = x - 1$. Then, from (8), $f(1) \not\equiv 0 \pmod{p^2}$. Suppose $1 - \theta \equiv 0 \pmod{\mathfrak{p}_i^2}$ for some i. We have the contradiction $N(1-\theta) = f(1) \equiv 0 \pmod{p^2}$. Hence (iv) is proved. Furthermore, by (10), the only prime ideal divisor of p which divides $1 - \theta$ is the ideal $\mathfrak{p} = (p, \theta - 1)$.

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* Cf. Ö. Ore, *Zur Theorie der Irreduzibilitätskriterien*, Mathematische Zeitschrift, vol. 18 (1923), p. 287.

2.2. We shall consider the sequence

$$v_0, v_1, v_2, \cdots \tag{12}$$

associated with $f(x)$ such that $v_i = u_i$, $i = 0, 1, 2, \cdots, k-1$. Then $v_i \equiv u_i$ (mod p) for all i. If $\theta_1, \theta_2, \cdots, \theta_k$ are the roots of $f(x) = 0$, the general term of the sequence (12) is given by

$$v_n = \frac{a}{f(1)} + \beta_1\theta_1^n + \beta_2\theta_2^n + \cdots + \beta_k\theta_k^n, \tag{13}$$

where the β_i are algebraic constants, that is, independent of n. If we set $n = 0, 1, 2, \cdots, k-1$ and insert the initial values $v_0, v_1, \cdots, v_{k-1}$, on solving for the β_i we obtain

LEMMA 2. *The general term of the sequence* (12) *is given by* (13), *where*

$$\beta_j = \frac{\gamma_j}{f'(\theta_j)} + \frac{a\delta_j}{(1-\theta_j)f'(\theta_j)} \tag{14}$$

and γ_j, δ_j *are integers in* $K(\theta_j)$.

2.3. We shall develop some modifications of the theorem of Fermat. Let p have the decomposition (9) in $K(\theta)$. If ω is an integer of $K(\theta)$ prime to p, and $P_i = p^{k_i} - 1$, then by the theorem of Fermat

$$\omega^{P_i} \equiv 1 \pmod{\mathfrak{p}_i} \qquad (i = 1, 2, \cdots, r), \tag{15}$$

or

$$\omega^{P_i} = 1 + \pi,$$

where π is an integer in $K(\theta)$ divisible by $\mathfrak{p}_i$. Taking the pth power we have

$$\omega^{pP_i} \equiv 1 \qquad (\mathrm{mod}\ \mathfrak{p}_i^{e_i+1}) \text{ or } (\mathrm{mod}\ \mathfrak{p}_i^{p}) \tag{16}$$

according as $p > e_i$ or $p \leqq e_i$. Suppose $p^{\epsilon_i} \leqq e_i < p^{\epsilon_i+1}$. Taking successive pth powers of (15) and writing $E_i = p^{\epsilon_i}$, we obtain

$$\omega^{E_iP_i} \equiv 1 \qquad (\mathrm{mod}\ \mathfrak{p}^{E_i}), \tag{17}$$

and hence, if ω is prime to p,

$$\omega^{pE_iP_i} \equiv 1 \qquad (\mathrm{mod}\ \mathfrak{p}^{e_i+E_i}) \text{ or } (\mathrm{mod}\ \mathfrak{p}^{pE_i}) \tag{18}$$

according as $e_i + p^{\epsilon_i} < p^{\epsilon_i+1}$ or $\geqq p^{\epsilon_i+1}$.

2.4. From (4) and Lemma 1, p has the prime ideal decomposition

$$p = \mathfrak{p}_{1j}^{e_1}\mathfrak{p}_{2j}^{e_2} \cdots \mathfrak{p}_{rj}^{e_r}, \quad N\mathfrak{p}_{ij} = p^{k_i}$$

in the field $K(\theta_j)$. Let G denote the Galois field formed by composition of the fields $K(\theta_j)$, $j=1, 2, \cdots, k-1$. We have

LEMMA 3. *The ideals* $\prod_{i=1}^{r}\mathfrak{p}_{ij}$, $j=1, 2, \cdots, k$, *have a common ideal divisor* $\mathfrak{P}$ *in* G.

For if $\mathfrak{P}$ is a prime ideal divisor of p in G, then for each j there exists an i such that $\mathfrak{p}_{ij}$ is divisible by $\mathfrak{P}$.

2.5. Let e denote the maximum e_i in (4) and l the least common multiple of $p^{k_i}-1$, $i=1, 2, \cdots, r$. We consider first the case $(a_k, p)=1$, $a\equiv 0 \pmod{p}$. Then the roots θ_j are prime to p. If $e=1$, the denominators in (14) are prime to p by Lemma 1. Hence, by (15),

$$\theta_j^{\,l} \equiv 1 \left(\mathrm{mod}\ \prod_{i=1}^{r}\mathfrak{p}_{ij}\right) \qquad (j = 1, 2, \cdots, k),$$

and, by Lemma 3,

$$v_{n+l} \equiv v_n \pmod{\mathfrak{P}} \qquad (n = 0, 1, 2, \cdots).$$

Since the v_i are rational integers and congruent (mod p) to the u_i, we have

$$u_{n+l} \equiv u_n \pmod{p} \qquad (n = 0, 1, 2, \cdots),$$

and hence obtain

THEOREM 2. *If* $(a_k, p)=1$, $a\equiv 0 \pmod{p}$ *in* (2) *and* $e=1$ *in* (4), *then* (2) *has the general period* l, *where* $e=\max e_i$ *in* (4) *and* l *is the least common multiple of* $p^{k_i}-1$, $i=1, 2, \cdots, r$.

Consider the case $(a_k, p)=1$, $a\equiv 0 \pmod{p}$ and $e>1$. From (18) we have

$$\theta_j^{\,p^{\epsilon+1}l} \equiv 1 \pmod{\mathfrak{p}_{ij}^{\,e_i+1}}. \tag{19}$$

Since the denominators in (14) contain at most $\mathfrak{p}_{ij}^{\,e_i}$ we have the following theorem:

THEOREM 3. *If* $(a_k, p)=1$, $a\equiv 0 \pmod{p}$, $p^\epsilon \leqq e < p^{\epsilon+1}$, $\epsilon \geqq 0$, *then* (2) *has the general period* $p^{\epsilon+1}l$.

2.6. We shall now consider the periodicity in the non-homogeneous case $a\not\equiv 0 \pmod{p}$. If $F(1)\not\equiv 0 \pmod{p}$ as in Lemma 1, it is seen that $1-\theta_j$ is prime to p, $j=1, 2, \cdots, k$. Hence, as above, we have the following theorem:

THEOREM 4. *If* $(a_k, p)=1$, $a\not\equiv 0 \pmod{p}$ *and* $F(1)\not\equiv 0 \pmod{p}$, *then if* $e=1$, *the recurrence* (2) *has the period* l; *if* $e>1$, *the recurrence* (2) *has the period* $p^{\epsilon+1}l$.

Suppose $a\not\equiv 0 \pmod{p}$ and $F(1)\equiv 0 \pmod{p}$. Let $\phi_1(x)=x-1$ and hence $\mathfrak{p}_{ij}=(p, \theta_j-1)$, $j=1, 2, \cdots, k$. Suppose $p^\epsilon \leqq e < p^{\epsilon+1}$, $e=\max e_i$, $\epsilon\geqq 0$. If

$(e, p)=1$, the denominators in (14) contain at most $\mathfrak{p}_{ij}^{e_i}$, $i=1, 2, \cdots, r$. Hence it follows from (19) that (2) has the period $p^{\epsilon+1}l$. If p divides e_1, then $\epsilon \geqq 1$ and $p^{\epsilon+1} \geqq e_1+2$. Hence (18) gives

$$\theta_j^{p^{\epsilon+1}l} \equiv 1 \qquad (\text{mod } \mathfrak{p}_{ij}^{e_1+2})$$

while (19) holds for $i \neq 1$. But the denominators in (14) contain at most $\mathfrak{p}_{ij}^{e_i}$ for $i \neq 1$ and $\mathfrak{p}_{ij}^{e_1+1}$. Hence we have the following theorem:

THEOREM 5. *If* $(a_k, p)=1$, $a \not\equiv 0$ (mod p) *and* $p^\epsilon \leqq e < p^{\epsilon+1}$, *then* (2) *has the general period* $p^{\epsilon+1}l$ (mod p) *where* $e = \max e_i$ *in* (4) *and* l *is the least common multiple of* $p^{k_i}-1$, $i=1, 2, \cdots, r$.

We state a corollary of these theorems:

COROLLARY. *If* $(a_k, p)=1$ *and* $p > e$ *then* (2) *has the general period* l (mod p) *when* $e=1$ *and* $F(1) \not\equiv 0$ (mod p), *otherwise it has the general period* pl (mod p).

The results of Carmichael for $p > k$ are contained in Theorems 1 to 4.

2.7. We now consider the case where p divides a_k, that is, $f(x)$ contains the factor x (mod p). Suppose that $f(x)$ contains precisely x^{e_2} (mod p), that is, $a_{k-i} \equiv 0$ (mod p), $i=0, 1, \cdots, e_2-1$ and $a_{k-e_2} \not\equiv 0$ (mod p). We may write $\mathfrak{p}_{2j}=(p, \theta_j)$, $j=1, 2, \cdots, k$. Then $(\mathfrak{p}_{ij}, \theta_j)=1$ for $i \neq 2$. Hence $\theta_j^\beta \equiv \theta_j^{e_2}$ (mod $\mathfrak{p}_{2j}^{e_2}$) for all $\beta > e_2$ while the results of §2.5 hold for the ideals $\mathfrak{p}_{ij}$, $i \neq 2$. Furthermore $1-\theta_j$ is not divisible by $\mathfrak{p}_{2j}$ and the denominators in (14) contain at most $\mathfrak{p}_{2j}^{e_2}$. Hence we have the following theorem:

THEOREM 6. *If* p *divides* a_k *and the last* s *coefficients of* (2) *are divisible by* p, $a_{k-s} \not\equiv 0$ (mod p), *the sequence* (1) *is periodic* (mod p) *except for the initial terms* $u_0, u_1, \cdots, u_{s-1}$, *and* (2) *has the general period given by Theorems* 2 *to* 5 *inclusive.*

III. PERIODICITY (mod p). A SECOND METHOD

3.1. We consider again in this section the periodicity of (2) (mod p) for $(a_k, p)=1$ and $a=0$ and obtain an improved result for the case $e=p$. Instead of the $f(x)$ of II we consider the associate polynomial

$$\pi(x) = \prod_{i=1}^{r} \phi_i(x)^{e_i}. \tag{20}$$

Let ρ_{ij}, $j=1, 2, \cdots, k_i$, be the roots of $\phi_i(x)=0$. Then the general solution of a homogeneous recurrence associated with (20) is given by

$$v_n = \sum_{i,j} \left\{ c_{ij}^{(1)} + c_{ij}^{(2)}\binom{n}{1} + \cdots + c_{ij}^{(e_i)}\binom{n}{e_i - 1} \right\} \rho_{ij}^n. \tag{21}$$

If we set $n=0, 1, \cdots, k-1$ and insert the initial terms $v_0, v_1, \cdots, v_{k-1}$, it is seen that the determinant Δ of the coefficients of the c's is precisely a determinant of Bonolis* whose value is

$$(22) \qquad \Delta = \pm \left[\prod_{i,j} \rho_{ij}{}^{(1/2)e_i(e_i-1)} \right] \prod (\rho_{\alpha\beta} - \rho_{\gamma\delta})^{e_\alpha e_\gamma},$$

where the second product extends over all differences of distinct roots ρ_{ij}, only one permutation of a given pair being included. We prove

LEMMA 4. *If* $(a_k, p)=1$, *then* Δ *is prime to* p.

For since $(a_k, p)=1$, the roots ρ_{ij} are all prime to p. If $\alpha \neq \gamma$, since the resultant of $\phi_\alpha(x)$ and $\phi_\beta(x)$ is prime to p, the differences $\rho_{\alpha\beta}-\rho_{\gamma\delta}$ are prime to p. If $\alpha=\gamma$ the differences $\rho_{\alpha\beta}-\rho_{\alpha\delta}$ are prime to p since the discriminant of a prime function $\phi_\alpha(x)$ is prime to p.

By the theorem of Dedekind, p is a prime ideal of degree k_i in the fields $K(\rho_{ij}), j=1, 2, \cdots, k$. Hence by the theorem of Fermat

$$(23) \qquad \rho_{ij}{}^{P_i} \equiv 1 \pmod{p},$$

where $P_i=p^{k_i}-1$. We obtain the following theorem directly from (2) since the denominators of the c's are prime to p by Lemma 4.

THEOREM 7. *If* $p \geqq e$ *and* $(a_k, p)=1$, $a=0$, *then* (2) *has the general period* l *or* pl (mod p) *according as* $e=1$ *or* $e>1$.

The period given by Theorem 7 is less than that of II for the single case $p=e$. The denominators in (21) contain, in general, a higher power of p than those in (14). It is possible, however, that the results of II may be obtained from (21) by an analysis of the minors of Δ.

IV. PERIODICITY (mod p^α)

4.1. In this section we prove a theorem which gives a general period of (2) (mod p^α) directly from the results already obtained (mod p). Let us first consider the case $(a_k, p)=1$. We prove the following lemmas:

LEMMA 5. *If a non-homogeneous recurrence* (2) *has the general period* π (mod m), *then* π *is a period* (mod m) *of the corresponding homogeneous recurrence.*

For if $[u_i]$ is a sequence satisfying (2) for $a \neq 0$, then $[u_i]$ has the period π (mod m). If $[v_i]$ is any sequence satisfying (2) for $a=0$, then $[u_i-v_i]$ is a sequence satisfying (2) for $a \neq 0$. Hence $[u_i-v_i]=[w_i]$ has the period π (mod m). It follows that $[u_i-w_i]=[v_i]$ has the period π (mod m).

* A. Bonolis, *Sviluppi di alcuni determinanti*, Giornale di Matematiche, vol. 15 (1877), p. 133.

LEMMA 6. *If the recurrence* (2) *has the general period* π (mod p^β) *then it has the period* $p\pi$ (mod $p^{\beta+1}$), $\beta \geqq 1$.

For, replacing n by $n+\pi$ in (2) and subtracting (2), we obtain

$$(24) \qquad (u_{n+\pi+k} - u_{n+k}) + a_1(u_{n+\pi+k-1} - u_{n+k-1}) + \cdots + a_k(u_{n+\pi} - u_n) = 0.$$

Hence, since π is a period of (2) (mod p^β),

$$(25) \qquad U_i = (u_{i+\pi} - u_i)/p^\beta \qquad (i = 1, 2, \cdots)$$

is a sequence of integers satisfying (2) with $a=0$. By Lemma 5, (25) has the period π (mod p^β). Consider the subsequence $U_{i+j\pi}$ where i is fixed but arbitrary and j has the range 0, 1, 2, $\cdots$. The first differences of this subsequence and hence the $(p-1)$th differences are divisible by p^β. If $\Delta_j{}^\gamma$ denotes the γth difference for variable j, we have

$$(26) \qquad \Delta_j{}^{p-1}U_{i+j\pi} = \Delta_j{}^p u_{i+j\pi}/p^\beta \equiv 0 \qquad (\text{mod } p^\beta).$$

Hence

$$[\Delta_j{}^p u_{i+j\pi}]_{j=0} \equiv 0 \qquad (\text{mod } p^{2\beta}),$$

that is,

$$(27) \qquad \begin{aligned} &u_{i+p\pi} - \binom{p}{1} u_{i+(p-1)\pi} \\ &\quad + \binom{p}{2} u_{i+(p-2)\pi} + \cdots + (-1)^p u_i \equiv 0 \qquad (\text{mod } p^{2\beta}). \end{aligned}$$

If p is odd we may group the terms in (27) and obtain

$$(28) \qquad \begin{aligned} &(u_{i+p\pi} - u_i) + \binom{p}{1}(u_{i+(p-1)\pi} - u_{i+\pi}) + \cdots \\ &\quad + \binom{p}{(p+1)/2}(u_{i+(p+1)\pi/2} - u_{i+(p-1)\pi/2}) \equiv 0 \quad (\text{mod } p^{2\beta}). \end{aligned}$$

But the differences on the left are divisible by p^β, and the binomial coefficients are divisible by p. Hence

$$(29) \qquad u_{i-p\pi} - u_i \equiv 0 \qquad (\text{mod } p^{\beta+1}).$$

If $p=2$, (27) becomes

$$u_{i+2\pi} - 2u_{i+\pi} + u_i \equiv 0 \qquad (\text{mod } p^{2\beta}),$$

or

$$(u_{i+2\pi} - u_i) - 2(u_{i+\pi} - u_i) \equiv 0 \qquad (\text{mod } 2^{2\beta}).$$

Hence (29) follows for $p=2$ and the lemma is proved.

The following theorem is obtained directly from Lemma 6 and is sufficient to determine a period (mod p^α) of (2) from the results (mod p) of II.

THEOREM 8.* *If* $(a_k, p)=1$, *and the recurrence* (2) *has the general period* π (mod p), *then it has the general period* $p^{\alpha-1}\pi$ (mod p^α).

Let $e=\max e_i$ in (4) and l the least common multiple of $p^{k_i}-1$, $i=1, 2, \cdots, r$. We state an immediate corollary:

COROLLARY. *If* $p>e$, *then* (2) *has the general period* $p^{\alpha-1}l$ *or* $p^\alpha l$ (mod p^α) *according as* $e=1$ *or* $e>1$.

4.2. Suppose $(a_k, p)\neq 1$ and $F(x)\equiv x^s F_1(x)$ (mod p), where $F_1(x)$ does not contain x (mod p). We have shown in §2.7 that (1) is periodic (mod p) after s terms. We shall show by induction that (1) is periodic (mod p^α) after αs terms. For suppose (2) has the general period π (mod p^β) after βs terms, $\beta\geqq 1$. Then (25) defines a sequence of integers for $i\geqq\beta s$; namely, $U_{\beta s}, U_{\beta s+1}, \cdots$. This sequence has the period π (mod p) after s terms, that is, for $i\geqq(\beta+1)s$. Hence we obtain the congruence (27) for the modulus $p^{\beta+1}$ for $i\geqq(\beta+1)s$ and as above

$$u_{i+p\pi} - u_i \equiv 0 \pmod{p^{\beta+1}}, \quad i \geqq (\beta + 1)s.$$

By induction we obtain the following theorem:

THEOREM 9. *If the last s coefficients of* (2) *are divisible by* p, $a_{k-s}\not\equiv 0$ (mod p), *then* (1) *is periodic* (mod p^α) *after* αs *terms and a period* (mod p^α) *is determined by Theorem* 8.

V. PERIODICITY (mod m)

For the general rational integral modulus m the following theorem suffices for obtaining a general period of (2) (mod m) from the previous results.

THEOREM 10. *If* $m=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_t^{\alpha_t}$ *the least common multiple* L *of a set of general periods* λ_i *of* (2) (mod $p_i^{\alpha_i}$), $i=1, 2, \cdots, t$, *is a general period of* (2) (mod m).

For $u_{n+L}\equiv u_n$(mod $p_i^{\alpha_i}$), $i=1, 2, \cdots, t$, and hence $u_{n+L}\equiv u_n$ (mod m).

We have obtained in this paper general periods of the recurrence (2), that is, periods of (1) for arbitrary initial values. Whether or not there exists a set of initial values for which the sequence has the general period obtained has not been discussed. Furthermore, it is possible that improved results may be obtained for sequences with special initial values such as the fundamental sequences of Lucas.

* For $e=1$, $(a_k, p)=1$, $F(1)\not\equiv 0$ (mod p), this theorem gives the period $p^{\alpha-1}l$(mod p^α). The period obtained by Carmichael for the same case with $p>k$ is $p^\alpha l$.

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ON DIRECT PRODUCTS, CYCLIC DIVISION ALGEBRAS, AND PURE RIEMANN MATRICES*

BY

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1. **Introduction.** The present paper is the result of a consideration of several related topics in the theory of linear associative algebras and the application of the results obtained to the theory of Riemann matrices. We first consider a linear algebra problem of great importance in its application to Riemann matrix theory, the question as to when a normal division algebra of order n^2 over F is representable by an algebra of m-rowed square matrices with elements in F. It is shown that this is possible if and only if n^2 divides m and this is applied to prove that.

The multiplication index h of a pure Riemann matrix of genus p is a divisor† *of $2p$.*

The algebras called cyclic (Dickson) algebras are the simplest normal division algebras structurally. J. H. M. Wedderburn has given sufficient conditions that constructed cyclic algebras be division algebras but it seems to have been overlooked that these conditions have not been shown to be necessary. In the fundamental problem of the construction of all such division algebras necessary and sufficient conditions are of course needed. The question of the necessity of the Wedderburn conditions is considered here and the results applied to obtain what seems a remarkable restriction on the types of algebras which may be the multiplication algebras of pure Riemann matrices. Also certain general theorems on direct products of normal simple algebras are proved and the conclusions used to reduce the problem of the construction of cyclic division algebras to the case where the order of the algebra is a power of a single prime.

2. **Results presupposed and elementary theorems.** We shall assume that F is any non-modular field and shall use the definitions of direct product, division algebras, and other terms as in L. E. Dickson's *Algebren und ihre Zahlentheorie*. We shall assume

THEOREM 1. *The direct product A of two total matric algebras B and C of orders n^2 and m^2 respectively is a total matric algebra of order $(nm)^2$.*

* Presented to the Society, April 18, 1930; received by the editors in June, 1930.

† It was merely known until now that $h \leqq 2p$, a very much milder condition.

THEOREM 2. *Every simple algebra over F, which is not a zero algebra of order one, is expressible as a direct product of a division algebra B over F and a total matric algebra M over F in one and only one way in the sense of equivalence. Conversely every such direct product is simple.*

We shall henceforth restrict the algebras of this paper to be not zero algebras when they are simple. Moreover we shall study only associative algebras. If A is the direct product of B and C where B and C are simple, then, by the operation of passing to algebras equivalent to B and C, we may assume that the moduli of A, B, and C are all the same quantity and similarly for the zero quantities. We shall assume this operation performed in all cases and that $\times$ is the notation for direct product, while if M and P are linear sets that MP is their product.

Definition. An algebra A with a modulus e over a field F is called *normal* if the only quantities of A commutative with all quantities of A are multiples of e by scalars in F.

THEOREM 3. *Let A be a normal simple algebra. Then A is the direct product of a normal division algebra and a total matric algebra and conversely.*

For A is the direct product of a total matric algebra M and a division algebra B. If B were not normal then there would be a quantity x in B such that x is not a scalar multiple of the modulus e of A and yet $xb=bx$ for every b of B. But $xm=mx$ for every m of M since $A=B\times M$. The quantities of A are sums of multiples of quantities of B by quantities of M so that $xa=ax$ for every a of A, a contradiction of the hypothesis that A is normal.

Conversely if B is a normal division algebra and M is a total matric algebra, then $A=B\times M$ is simple. Let $M=(e_{ij})$, $e_{ij}e_{jk}=e_{ik}$, $e_{ij}e_{tk}=0$ $(j\neq t)$. Then if

$$x = \sum_{i,j} b_{ij}e_{ij} \qquad (b_{ij} \text{ in } B)$$

is commutative with all of the quantities of A we have

$$xe_{kt} = \sum_i b_{ik}e_{it} = e_{kt}x = \sum_j b_{kj}e_{kj}$$

for all k and t. It follows from the definition of direct product that $b_{ij}=0$ $(i\neq j)$, $b_{ii}=b_{11}$, and that x is in B. But the only quantities of B commutative with all quantities of B are scalar multiples of its modulus, the modulus of A, so that x is a multiple of the modulus by a scalar of F and A is normal.

THEOREM 4. *Let $A=B\times C$ where B and C are normal simple algebras over F. Then A is a normal simple algebra over F.*

For by adjoining a scalar ξ to F, the algebra B_1 with the same basal units

and constants of multiplication as B but over $F(\xi)=F'$ is a total matric algebra. Also ξ may be so chosen that C simultaneously reduces to a total matric algebra by the well known theorem on the adjunction of a finite number of scalars to a nonmodular field being equivalent to the adjunction of a single scalar. But A' over F' is a total matric algebra. By Theorem 3, A' is normal and if A were not normal obviously neither would be A', having the same basis and constants of multiplication as A. Hence A is normal. Moreover by a known* theorem A' is semi-simple if and only if A is simple. If A were reducible then so obviously would be A' so that, since A' is simple and not reducible, A is not reducible. A semi-simple algebra which is not reducible is simple and A is thus a normal simple algebra.

We shall assume a theorem† of J. H. M. Wedderburn.

THEOREM 5. *Let A be a linear associative algebra over a non-modular field F. Let e be the modulus of A and suppose that B, a normal simple sub-algebra of A, has the same modulus e as A. Then A is the direct product of B and another algebra C with the same modulus e.*

We also have for A an algebra with modulus the same as that of B, C_1, C

THEOREM 6. *Let $A=B\times C=B\times C_1$, where B is a normal simple algebra. Then $C_1=C$.*

For let c_1 be in C_1. Then c_1 is in A and is necessarily expressible in the form

$$c_1 = \sum_i b_i u_i,$$

where b_i are in B and the u_i are a basis of C, with $u_1=e$, the modulus of A. But if b is any quantity of B, then

$$bc_1 = c_1 b, \quad 0 = bc_1 - c_1 b = \sum_i (b_i b - bb_i)u_i.$$

By the definition of direct product this implies that $b_i b = bb_i$ for all b's of B, so that the b_i are in F and the quantities of C_1 are in C. Similarly the quantities of C are in C_1 and $C=C_1$.

THEOREM 7. *Let $A=B\times C$ where A is a normal simple algebra. Then both B and C are normal simple algebras.*

For it evidently suffices to prove the above true for C. Suppose that C were not simple so that C would contain an invariant proper sub-algebra N. Then $NC\leqq N$, $CN\leqq N$. But if $P=B\times N$, then $AP=(B\times C)\ (B\times N)=B^2 \times(CN)\leqq P$, while $PA=(B\times N)\ (B\times C)=B^2\times(NC)\leqq P$, so that A would

* Dickson (loc. cit.) p. 110, Corollary to Theorem 18.

† Proceedings of the Edinburgh Mathematical Society, vol. 25 (1906–1907), pp. 1–3.

have the invariant proper sub-algebra P contrary to the hypothesis that A is simple. If C were not normal, then there would exist an x in C such that $xc=cx$ for every c of C while x is not a multiple of the modulus of C by a quantity of F. But then evidently $xa=ax$ for every a of A contrary to the hypothesis that A is a normal algebra. It follows that C is a normal simple algebra.

THEOREM 8. *Let $A=B\times C$ where A and B are total matric algebras. Then C is a total matric algebra.*

For A is a normal simple algebra, so that, by Theorem 7, so is C. Hence $C=M\times D$ where M is a total matric algebra and D is a normal division algebra. It follows that $A=B\times C=(B\times M)\times D$, where, by Theorem 1, $B\times M$ is a total matric algebra. By the uniqueness in Theorem 2 and the fact that A is a total matric algebra, algebra D has order one and $C=M$ is a total matric algebra.

3. **On direct products of normal division algebras.** The theory of the complex multiplications of pure Riemann matrices has been studied in detail by various authors.* Of utmost importance in this theory is the question as to what are necessary and sufficient conditions that a division algebra B of order h over F be expressible as an algebra of m-rowed square matrices with elements in F. The author has reduced this question to the case where B is a normal division algebra over F.† Using this reduction, and by a consideration of certain theorems on direct products of normal division algebras, we shall completely answer the above question. We shall also obtain a theorem on the direct products of normal division algebras of relatively prime orders for use in the next section.

THEOREM 9. *Let B be a normal division algebra of order n^2 over F. Then there exists a normal simple algebra B_1 of order n^2 over F such that $A=B\times B_1$ is a total matric algebra over F.*

For it is well known that every linear associative algebra with a modulus and having order t over F is equivalent to an algebra of t-rowed square matrices. Hence B is equivalent to an algebra C of n^2-rowed square matrices with elements in F and, if M is the algebra of all n^2-rowed square matrices with elements in F, then by Theorem 5, $M=C\times C_1$ where C_1 is a normal simple algebra. Let B_1 be an abstract algebra defined so that it is equivalent

* For references see the Bulletin of the National Research Council, No. 63, *Selected Topics in Algebraic Geometry*, chapters 15, 16, 17, 1928.

† In a paper *On the structure of pure Riemann matrices with non-commutative multiplication algebras*, to be published in the Rendiconti del Circolo Matematico di Palermo, probably in January, 1931.

to C_1. Since M has order n^4 and C has order n^2, algebra C_1 has order n^2. Hence B_1 has order n^2 and is a normal simple algebra. Let A be the direct product of B and B_1. Evidently A is equivalent to M and is a total matric algebra over F.

THEOREM 10. *Let $M=B\times C$ be a total matric algebra over F, where B and C are normal division algebras. Then B and C have the same order.*

For if the order of B is n^2 and that of C is r^2, then, without loss of generality, we may take $n\leqq r$. Let B_1 be the algebra of Theorem 9 such that $A=B\times B_1$ is a total matric algebra. By the associative law and $A=B_1\times B$, the algebra $G=B_1\times M$ may also be written $G=A\times C$. Now $B_1=D\times T$ where D is a normal division algebra of order $t^2\leqq n^2$ and T is a total matric algebra. Hence $G=B_1\times M=D\times(T\times M)$, where $T\times M$ is a total matric algebra. But, by Theorem 2, G is simple, $G=D\times(T\times M)=C\times A$, and C is equivalent to D. Hence $t=r$. But $t\leqq n\leqq r$. It follows that $r=n$ as desired.

We are now in a position to prove the fundamental theorem on the representation of a normal division algebra over F as an algebra of matrices with elements in F.

THEOREM 11. *A normal division algebra B of order n^2 over F is expressible as a sub-algebra of the algebra of all m-rowed square matrices with elements in F if and only if m is divisible by n^2.*

For if $m=n^2t$ then B can be expressed first as a sub-algebra of H, the algebra of all n^2-rowed square matrices with elements in F. Then if T is a t-rowed total matric algebra, the algebra $M=H\times T$ is an m-rowed total matric algebra by Theorem 1. Evidently B is a sub-algebra of M and its representation in M provides an expression for B as a sub-algebra of the algebra of all m-rowed square matrices with elements in F which is equivalent to M.

Conversely let B be a sub-algebra of M, an m-rowed total matric algebra over F. Then $M=B\times(D\times T)$ by Theorem 5, where D is a normal division algebra and T is a total matric algebra, so that $m=ndt$ where d^2 is the order of D and t^2 the order of T. By Theorem 8 the algebra $B\times D$ is a total matric algebra, so that, by Theorem 10, $d=n$, and $m=n^2t$.

As a corollary we have

THEOREM 12. *The algebra B_1 of Theorem 9 is a normal division algebra.*

For let $B_1=D\times T$, where T is a total matric algebra and D a normal division algebra. Then $A=B\times B_1=(B\times D)\times T$, so that, by Theorems 8 and 10, D has order n^2. But B_1 has order n^2. It follows that B_1 is D and is a normal division algebra.

We shall finally prove

THEOREM 13. *Let $A = B\times C$ where B and C are normal division algebras whose orders m^2 and n^2 respectively are relatively prime. Then A is a normal division algebra.*

For by Theorem 4 algebra A is a normal simple algebra and $A = H\times D$ where H is a total matric algebra of order h^2 and D a normal division algebra of order d^2. Hence $mn = hd$. Let B_1 be the normal division algebra of Theorems 9 and 12 so that $B\times B_1$ is a total matric algebra. The direct product $B_1\times D$ is a normal simple algebra by Theorem 4, and $B_1\times D = Q\times P$ where Q is a total matric algebra and P a normal division algebra. But $G = B_1\times A = B_1\times (H\times D) = H\times(B_1\times D) = H\times(Q\times P) = (H\times Q)\times P = (B_1\times B)\times C$. It follows that $H\times Q$ is equivalent to $B_1\times B$ from the uniqueness in Theorem 2. Hence if Q has order q^2, then $hq = m^2$. By forming the direct product $C_1\times A$ we prove similarly that h divides n^2. Hence all prime factors of h are factors of both m and n. But m and n are relatively prime so that $h = 1$ and $A = D$ is a division algebra.

4. **Cyclic normal division algebras.** Let x satisfy an equation $\phi(\xi) = 0$ of degree n, with leading coefficient unity and further coefficients in F, and with the cyclic group with respect to F. Then there exists a polynomial $\theta(x)$ such that if we define its iteratives $\theta^r(x) = \theta[\theta^{r-1}(x)]$, $\theta^0(x) = x (r = 0, 1, \cdots)$, then $\theta^n(x) = x$ and $\phi[\theta^r(x)] = 0$. The algebra D with the basis

$$x^r y^s \qquad (r, s = 0, 1, \cdots, n-1)$$

and the multiplication table

$$\phi(x) = 0,\ y^n = \gamma,\ y^r f(x) = f[\theta^r(x)]y^r \quad (r = 0, 1, \cdots, n-1),$$

for every $f(x)$ of $F(x)$, where γ is in F, is an associative normal algebra over F. For every $f = f(x)$ in $F(x)$ we write

$$N(f) = f[\theta^{n-1}(x)] \cdots f[\theta(x)]\cdot f(x),$$

a quantity in F, and call $N(f)$ the norm of f. J. H. M. Wedderburn has proved*

THEOREM 14. *Algebra D is a division algebra if no power $\gamma^r (r < n)$ is the norm of any f in $F(x)$.*

The above condition is a sufficient condition that D be a division algebra but is not known to be necessary. For $n = 2$, 3 the above condition is also necessary but may be replaced by the simpler condition that $\gamma \neq N(f)$ for

* These Transactions, vol. 15 (1914), pp. 162–166.

any f of $F(x)$. We shall show that a like result holds *for n any prime* and shall reduce the problem of constructing all cyclic algebras to the *case n a power of a prime.*

THEOREM 15. *Let $F(x)$ be a cyclic field of order $r=nm$ where n and m are relatively prime integers. Then $F(x)$ is the direct product of two cyclic fields of orders m and n respectively, and conversely.*

For let G be the cyclic substitution group of order r which is the Galois group of the minimum equation of x. Then G will consist of the powers of a single substitution P such that $P^r=I$, the identity substitution. Since m and n are relatively prime there exist integers g and h such that $1=gm+hn$. Then for every $\alpha<r$ we have $g\alpha\equiv\beta(\text{mod } n)$ and $h\alpha\equiv\delta(\text{mod } m)$, where $0\leqq\beta<n$ and $0\leqq\delta<m$. But then

$$P^{\alpha} = (P^m)^{g\alpha}\cdot(P^n)^{h\beta} = P^{m\beta}\cdot P^{n\delta},$$

since $P^r=I$. Hence every substitution of G is expressible as a product of substitutions of the groups $G_1=(P^{m\beta})$ and $G_2=(P^{n\delta})$. It follows that G is the direct product of G_1 and G_2. It is known that there exists a quantity a in $F(x)$, which *belongs to the group* G_2 and hence has grade n with respect to F and G_1 as a representation of the Galois group of its minimum equation. Then this equation has the cyclic group with respect to F. Similarly there exists a quantity b in $F(x)$, *which belongs to* G_1 and has grade m with respect to F and G_2 as a representation of the Galois group of its minimum equation. The field $F(a, b)$ contains $F(a)$ and $F(b)$ as sub-fields and hence its order is divisible by both m and n. But m and n are relatively prime so that the order of $F(a, b)$ is divisible by mn. The order of $F(a, b)$, a sub-field of $F(x)$ of order mn, is at most mn, so that $F(a, b)$ has order mn and $F(a, b)=F(x)$. But when $F(a, b)$ has order mn, the product of the orders of $F(a)$ and $F(b)$, it is the direct product of $F(a)$ and $F(b)$ by the definition of direct product. It follows that $F(x)$ is the direct product of the two cyclic fields $F(a)$ and $F(b)$.

Conversely let X be the direct product of two cyclic fields $F(a)$ and $F(b)$ of relatively prime orders n and m respectively. Let ξ be a scalar root of the minimum equation of a and let η be a scalar root of the minimum equation of b. The field $F(\xi, \eta)$ has $F(\xi)$ and $F(\eta)$ as sub-fields and hence has order mn. It follows that $F(\xi, \eta)$ is the direct product of $F(\xi)$ and $F(\eta)$ and is equivalent to X. Thus $X=F(a, b)$ is a commutative division algebra or field. Let x generate X so that $X=F(x)$, $x=Q(a, b)$, a polynomial in a and b with coefficients in F. Let the roots of the minimum equation of a which are in $F(a)$ be denoted by $\lambda^{\beta}(a)$, $(\beta=0, 1, \cdots, n-1)$, and the roots of the minimum equation of b in $F(b)$ by $\mu^{\delta}(b)$, $(\delta=0, 1, \cdots, m-1)$. If $K=F(b)$ then any rational

function of the $\lambda^\beta(a)$ with coefficients in K which is symmetric in the $\lambda^\beta(a)$ is in K, since the minimum equation of a with respect to K is the same as its minimum equation with respect to F. The polynomial

$$\phi(\omega) \equiv \prod_{\beta,\gamma} \omega - Q[\lambda^\beta(a);\, \mu^\delta(b)]$$

has coefficients in K since they are symmetric in the $\lambda^\beta(a)$. But they are symmetric in the $\mu^\delta(b)$, so they are in F. The equation $\phi(\omega)=0$ has degree mn. It has leading coefficient unity, further coefficients in F and $\phi(x)=0$. Hence $\phi(\omega)=0$ is the minimum equation of x since x has grade mn. We may choose integers g and h such that

$$gm + hn = 1, \quad g\alpha \equiv \beta \pmod{n}, \quad h\alpha \equiv \delta \pmod{m},$$

so that every integer less than mn is expressible in the form

$$\alpha = \beta m + \delta n + \alpha_1 mn \qquad (0 \leqq \beta < n,\, 0 \leqq \delta < m).$$

Now if S is the substitution replacing a by $\lambda(a)$ and T is the substitution replacing b by $\mu(b)$ then the substitution P which is the substitution replacing x by

$$\theta(x) = Q[\lambda(x),\, \mu(x)]$$

is equivalent to the substitution product $S\cdot T$. Hence P^α replaces x by

$$\theta^\alpha(x) = Q[\lambda^\beta(x),\, \mu^\delta(x)] \qquad (\alpha = 0, 1, \cdots, r-1),$$

with β and δ determined as above. It follows that the Galois group of the minimum equation of x has a cyclic sub-group of order mn. But since the roots in $F(x)$ of the minimum equation of x are mn in number the group of this equation is actually the above cyclic group

$$(P^\alpha)\ (\alpha = 0, 1, \cdots, r-1),\ r = mn.$$

THEOREM 16. *Let A be a cyclic normal division algebra of order r^2 over F, and $r=mn$ where m and n are relatively prime integers. Then A is the direct product of a cyclic normal division algebra B of order n^2 and a cyclic normal division algebra C of order m^2, each having the same γ as A. Conversely the direct product A of two cyclic normal division algebras B and C of relatively prime orders n^2 and m^2 respectively is a cyclic normal division algebra whose γ may be taken to be the γ of B and C.*

Let A be a cyclic algebra of order r^2 over F so that A contains a quantity x whose minimum equation with respect to F has degree $r=mn$ and roots $\theta^\alpha(x)$ in $F(x)$. Moreover, A contains a quantity y such that

$$y^\alpha f(x) = f[\theta^\alpha(x)]y^\alpha, \quad y^r = \gamma \text{ in } F.$$

Let A be a normal division algebra. Let a and b be defined as in the proof of Theorem 15 so that a is unaltered by the substitutions of the group $G_2=(P^{n\delta})$ where P generates the Galois group of the minimum equation of x, and $G_1=(P^{m\beta})$ is a representation of the Galois group of the minimum equation of a. Then evidently $y^{n\delta}a=ay^{n\delta}$, while $(y^m)^n=\gamma$, $y^{m\beta}a=\lambda^\beta(a)y^{m\beta}$. Similarly $(y^n)^m=\gamma$, $y^{m\beta}b=by^{m\beta}$, $y^{n\delta}b=\mu^\delta(b)y^{n\delta}$. Let B be the algebra with the basis

$$a^g y^{m\beta} \qquad (g, \beta = 0, 1, \cdots, n-1),$$

and C be the algebra with the basis

$$b^h y^{n\delta} \qquad (h, \delta = 0, 1, \cdots, m-1).$$

The algebras B and C are cyclic algebras with the same γ as A. Moreover every quantity of B is commutative with every quantity of C. Every quantity of A is expressible in the form

$$\sum_{\alpha=0}^{r-1} f_\alpha(x)y^\alpha \qquad [f_\alpha(x) \text{ in } F(x)].$$

But any such quantity has the form

$$\sum g_{\beta\delta}y^{m\beta}y^{n\delta}$$

since we may find integers β, δ for which

$$y^\alpha = y^{m\beta}\cdot y^{n\delta}\cdot\gamma^{\alpha_1}$$

and express each $f_\alpha(x)$ in the form $g_{\beta\gamma}(a, b)\cdot\gamma^{-\alpha_1}$ since $F(x)$ is the direct product of $F(a)$ and $F(b)$. Hence the set A is a set whose quantities are sums of products of quantities of B and quantities of C and is contained in the set BC. But BC is a sub-set of A so that A is equal to BC. Since the quantities of B are commutative with the quantities of C, A is the direct product of B and C.

Conversely let B be a cyclic normal division algebra over F and have order n^2 so that B contains a quantity a such that the minimum equation of a with respect to F is cyclic with respect to F and with polynomial roots $\lambda^\beta(a)$ $(\beta=0, 1, \cdots, n-1)$ such that $\lambda^0(a)=\lambda^n(a)=a$. Then B contains also a quantity y_1 such that

$$y_1^n = \gamma_1 \text{ in } F, \quad y_1^\beta g(a) = g[\lambda^\beta(a)]y_1^\beta \qquad (\beta = 0, 1, \cdots, n-1),$$

for every $g(a)$ in $F(a)$. Similarly let C have order m^2 where m and n are relatively prime, contain the cyclic m-ic field $F(b)$, and contain the quantity y_2 such that

$$y_2^m = \gamma_2 \text{ in } F, \ y_2^\delta g(b) = g[\mu^\delta(b)]y_2^\delta \quad (\delta = 0, 1, \cdots, m-1).$$

If A is the direct product of B and C then A is a normal division algebra by Theorem 13. Moreover the direct product of $F(a)$ and $F(b)$ is a cyclic field $F(x)$ whose polynomial roots are

$$\theta^\alpha(x) = Q[\lambda^\beta(a), \mu^\delta(b)],$$

and if we let $y = y_1y_2$ then $y^\alpha x = \theta^\alpha(x)y^\alpha$ with β and δ properly chosen integers such that $\alpha = \beta m + \delta n + \alpha_1 mn$. We thus have

$$y^{nm} = y_1^{nm}y_2^{mn} = \gamma_1^m\gamma_2^n,$$

and A is a cyclic algebra whose γ is $\gamma_1^m\gamma_2^n$. If we let $y_{11} = \gamma_2 y_1^m$, $y_{22} = \gamma_1 y_2^n$, then

$$y_{11}^n = \gamma_2^n\gamma_1^m = \gamma, \quad y_{22}^m = \gamma_1^m\gamma_2^n = \gamma,$$

and we may replace y_1 in the basis of B by y_{11}, y_2 in the basis of C by y_{22}, and each of these algebras will have the same γ as A. In fact since m and n are relatively prime we may choose an integer s such that $ms \equiv 1 \pmod n$, so that y_{11}^s is a constant multiple of y_1 and we have a basis of B when we replace y_1 by y_{11}.

As an immediate corollary we have

THEOREM 17. *Let $r = p_1^{e_1} \cdot p_2^{e_2} \cdots p_t^{e_t}$ where the p_i are distinct primes. Every cyclic algebra of order r^2 over F which is a division algebra is a direct product of t cyclic division algebras of order $p_i^{2e_i}$, and conversely all such direct products are cyclic division algebras.*

We have thus reduced the problem of constructing all cyclic division algebras to the case where the order is a power $2e$ of a prime p. This latter problem is of great difficulty even in the case where $p = 2$, $e = 2$. When $e = 1$ we may make a simple discussion as follows.

THEOREM 18. *A cyclic algebra A of order p^2 over F, p a prime, is a division algebra if and only if γ is not the norm of any polynomial in x.*

For suppose that γ be not the norm of any polynomial in x but that A were not a division algebra. By the sufficient condition of Theorem 14 there must exist an integer $\alpha < p$, such that

$$\gamma^\alpha = N(g), \qquad g \text{ in } F(x).$$

Since p is a prime there exists an integer σ such that $\sigma\alpha \equiv 1 \pmod p$. Let then $\sigma\alpha = 1 + tp$, $\sigma > 0$, $t > 0$, so that

$$\gamma \cdot \gamma^{tp} = \gamma^{\sigma\alpha} = [N(g)]^\sigma = N(g^\sigma).$$

If $\gamma=0$ then $\gamma=N(0)$, a contradiction. Hence

$$\gamma = (\gamma^{-t})^p \cdot N(g^\sigma) = N(g^\sigma \gamma^{-t}),$$

a contradiction.

Conversely let $\gamma=N(f)$, where f is in $F(x)$. If $f=0$ then $y^p=0$ which is impossible in a division algebra when $y\neq 0$. But $1, y, \cdots, y^{p-1}$ are linearly independent with respect to F so that y is not zero and f is not zero. Then f has an inverse in $F(x)$, and if h is this inverse we have $(hy)^p=N(hf)=1$. But if $y_1=hy$ then $1, y_1, y_1^2, \cdots, y_1^{p-1}$ are linearly independent with respect to F, since the quantities $x^r y^s$ $(r, s=0, 1, \cdots, p-1)$ are linearly independent with respect to F. It follows that $y_1-1\neq 0$, and $y_1^{p-1}+y_1^{p-2}+\cdots+y_1+1\neq 0$. But y_1^p-1 is the product of the two aforesaid non-zero quantities and is zero, which is impossible in a division algebra.

We shall now obtain some simple but interesting limitations on the order of a cyclic division algebra A when the sufficient condition of Theorem 14 is not satisfied.

THEOREM 19. *Let A be a cyclic division algebra of order r^2 where $r=p_1^{e_1}\cdot p_2^{e_2}\cdots p_t^{e_t}$ and the p_i are distinct prime integers. Then if γ^s is the norm of a polynomial in x, the integer s is divisible by $p_1\cdot p_2\cdots p_t$.*

For suppose that s were not divisible by p, a factor of r. Then there would exist an integer α such that $\alpha s\equiv 1 \pmod{p^e}$. Let $\alpha s=1+tp^e$ and $r=p^e m$, so that we can express A as the direct product of a cyclic division algebra B of order p^{2e} and a cyclic division algebra C of order m^2 by Theorem 16. The norm of any polynomial in x may be written as the norm of a polynomial in a where a is the cyclic quantity of Theorem 16 for the algebra B. We have then that

$$\gamma^s = N_a(c)$$

where c is a polynomial in a so that

$$\gamma^{\alpha s} = \gamma\gamma^{tp^e} = N_a(c), \quad \gamma = N_a(c\gamma^{-t}).$$

If y_1 is the y of algebra B we have

$$(y_1)^{p^e} = N_a(c), \quad (c^{-1}y_1)^{p^e} - 1 = 0,$$

from which, as in the proof of Theorem 18, it quickly follows that B is not a division algebra. But this is impossible since B is a sub-algebra of the division algebra A.

As a corollary we have

THEOREM 20. *Let A be a cyclic division algebra of order r^2. Then if p is a prime and γ^p is the norm of a polynomial in x, the integer r is a power of p.*

For if r were not a power of p then it would contain a factor not dividing p.

For the case where B is a cyclic algebra of order p^2, p a prime, we may explicitly determine the structure of algebra B_1 of Theorem 9. We showed that B_1 was a normal division algebra of order p^2 in Theorem 12. We shall now prove

THEOREM 21. *A direct product $A = B\times C$ of a cyclic division algebra B of order p^2, p a prime, and a normal division algebra C of order p^2 is a total matric algebra if and only if C is equivalent to B.*

We use the form of the multiplication table of B which is reciprocal to our previous form, assuming that

$$y^s x^r \qquad (r, s = 0, 1, \cdots, p-1)$$

are a basis of B and that

$$y^p = \gamma, \quad xy^s = y^s\theta^s(x) \qquad (s = 0, 1, \cdots, p-1).$$

Assume first that C is equivalent to B and let X in C correspond to x in B and Y in C to y of B. Consider the quantities

$$e_{ii} = \frac{(x - X_1)(x - X_2)\cdots(x - X_{i-1})(x - X_{i+1})\cdots(x - X_p)}{(X_i - X_1)(X_i - X_2)\cdots(X_i - X_{i-1})(X_i - X_{i+1})\cdots(X_i - X_p)},$$

$$e_{1i} = y^{i-1}e_{ii}, \quad e_{i1} = \frac{1}{\gamma}y^{p+1-i}e_{11} \qquad (i = 2, \cdots, p),$$

$$e_{ij} = e_{i1}e_{1j} \qquad (i, j = 1, \cdots, p),$$

where $X_i = \theta^i(X)$. Then Wedderburn has shown* that, with the agreement that $e_{j+mp,\,j+mp} = e_{jj}$, we have

$$e_{ii}y^k = y^k e_{k+i,\,k+i}, \quad x = \sum_i X_i e_{ii},$$

$$y = e_{12} + e_{23} + \cdots + \gamma e_{p1},$$

for all integer values of k and i. In particular $e_{ii}y^{-1} = y^{-1}e_{i-1,\,i-1}$. Since $X_iY = YX_{i+1}$ we have obviously from the form of the e_{ii} that $e_{ii}Y = Ye_{i+1,\,i+1}$. Hence if $Z = Yy^{-1}$ then $e_{ii}Z = Ze_{ii}$. Also since $A = C\times B$ we have $Zy^k = y^kZ$ whence from the definitions of the e_{ij}, $Ze_{ij} = e_{ij}Z$. Wedderburn has also shown that the e_{ij} form a basis of an algebra M which is a total matric algebra of order p^2. The linear set $M_1 = (Z^sX^r)$ $(r, s = 0, 1, \cdots, p-1)$ is an algebra with

* These Transactions, vol. 15 (1914), pp. 162–166.

the multiplication table given by $XZ^k = Z^k\theta^k(X)$, $Z^p = Y^p y^{-p} = \gamma\gamma^{-1} = 1$. Since the quantities of M are commutative with those of M_1 and since both sets are algebras, the set MM_1 is an algebra. But MM_1 is contained in A. Also the quantities x, y, X, Y are all in MM_1 by the above displayed equations and hence all of the quantities of A are in MM_1. It follows that $A = MM_1$, and since the quantities of M are commutative with those of M_1, $A = M \times M_1$. Algebra $A = B \times C$ is a normal simple algebra by Theorem 4. Hence algebra M_1 is a normal simple algebra by Theorem 7. But algebra A has order p^4 and M has order p^2 so that M_1 has order p^2. Since p is a prime, M_1 is either a normal division algebra or a total matric algebra. But $Z^p = 1$ which is impossible in a normal division algebra when $1, Z, Z^2, \cdots, Z^{p-1}$ are linearly independent with respect to F. By our choice of the basis of M_1 and the fact that the order of M_1 is p^2 we have the independence of the above quantities so that M_1 is a total matric algebra. By Theorem 1, A is a total matric algebra.

Conversely let $A = B \times C$ be a total matric algebra, where B is a cyclic algebra of order p^2, p a prime. If B_1 is equivalent to B we have already shown that $G = B \times B_1$ is a total matric algebra. Consider the algebra $H = B_1 \times A = G \times C$. Now A and G are total matric algebras and B_1 and C are normal division algebras, so that, by Theorem 2, B_1 is equivalent to C. It follows that C is equivalent to B.

5. Applications to the theory of pure Riemann matrices. Let ω be a pure Riemann matrix over a real field K. It is known that the algebra of multiplications of ω is a division algebra D of order $h \leqq 2p$ over K and that D has a representation as an algebra of $2p$-rowed square matrices with elements in K. Let D be expressed as an algebra which is a normal division algebra of order n^2 over its central field $K(q)$ of order t, so that $h = n^2 t$. The minimum equation of q is irreducible in K since D is a division algebra and there exists a representation of q as a $2p$-rowed square matrix whose elements on the diagonal are the same t-rowed square matrix Q with elements in F and whose elements off the diagonal are zero matrices, while $2p = mt$. Any two representations of q as a $2p$-rowed square matrix with elements in K are similar in K so that we may take the representation of D such that q has the above representation. It follows that D has a representation as an algebra of m-rowed square matrices with elements in $K(Q)$, and by applying Theorem 11, with $F = K(Q)$, we have immediately m divisible by n^2. This gives $m = n^2 r$, $2p = hr$, and

THEOREM 22. *The multiplication index h of a pure Riemann matrix ω of genus p over a real field K is a divisor of $2p$.*

The only normal division algebras of order n^2 which are known are the so-called algebras of type R_n. By this we mean that the algebra A which is a normal division algebra in n^2 units over F contains a quantity a whose minimum equation $\phi(\xi)=0$ with respect to F has degree n and roots $\theta_j(a)$, where $\theta_1(a)=a$, $\theta_j(a)$ is in $F(a)$. Also $\theta_j[\theta_k(a)]$ is a root of $\phi(\xi)=0$ so that there exists a set of integers $t_{j,k}$ such that

$$\theta_j[\theta_k(a)] = \theta_{t_{j,k}}(a) \qquad (j, k = 1, \cdots, n).$$

Algebra A has a basis

$$a^{i-1}y_k \qquad (j, k = 1, \cdots, n),$$

where $y_1 = 1$ and

$$y_j a = \theta_j(a) y_j, \quad y_k y_j = g_{j,k} y_{t_{j,k}},$$

with the $g_{j,k}$ in $F(a)$. The author has studied the case where such algebras A are the multiplication algebras of pure Riemann matrices of genus p over a real field F in great detail. Assume first that the equation $\phi(\xi)=0$ has a real root α_1 so that all of its roots are real when F is a real field. The author has shown (loc. cit. *On the structure of pure Riemann matrices with non-commutative multiplication algebras*) that there must necessarily exist n numbers $\beta_j(\alpha_1)$ in $F(\alpha_1)$ and positive, such that if $\alpha_j=\theta_j(\alpha_1)$, then

$$\beta_j(\alpha_k)\beta_k(\alpha_1) = [g_{j,k}(\alpha_1)]^2 \beta_{t_{j,k}}(\alpha_1).$$

We shall assume that n has an odd prime factor p. Then, since the Galois group of $\phi(\xi)=0$ has order n, it contains at least one substitution of order p, since it is known that if a prime divides the order of a group there is a sub-group G_1 of order the prime in the original group. This sub-group of order p is necessarily cyclical and generated by a single substitution of order p. Without loss of generality we may take $\theta_2(a)$ to be the quantity by which a is replaced by the above substitution, so that $y_2^r a = \theta_2^r(a) y_2^r$ $(r=0, 1, \cdots, p-1)$, and we may take $y_{r+1}=y_2^r$ $(r=0, 1, \cdots, p-1)$, $y_2^p = g(a)$. Let v be a polynomial in a belonging to the sub-group G_1. Then A contains a cyclic sub-algebra

$$H = (a^r y_2^s) \qquad (r, s = 0, 1, \cdots, p-1)$$

over the field $F(v)$ and, since $g(a)$ is a power of y_2, therefore commutative with y_2, therefore unaltered when we replace a by $\theta_2(a)$, $g(a)$ is in $F(v)$. Assume now that α_1, a root of the minimum equation of a, is real. Since $F(a)$ and $F(\alpha_1)$ are equivalent and $\alpha_j=\theta_j(\alpha_1)$ there must exist polynomials $\beta_j(a)$ in $F(a)$ such that

$$\beta_j[\theta_k(a)]\beta_k(a) = [g_{j,k}(a)]^2\beta_{t_{j,k}}(a).$$

We have taken

$$\theta_2^k(a) = \theta_{k+1}(a), \quad y_2^k = y_{k+1} \qquad (k = 0, 1, \cdots, p-1)$$

whence

$$g_{j,2}(a) = 1\,(j < p), \quad t_{j,2} = j+1 \qquad (j = 2, \cdots, p-1),$$
$$\beta_1(a) = t_{p,2} = 1, \quad g_{p,2} = g(a).$$

But then if $\theta(a) = \theta_2(a)$

$$\beta_j[\theta(a)]\beta_2(a) = \beta_{j+1}(a) \qquad (j = 2, \cdots, p-1),$$

so that

$$\beta_3(a) = \beta_2[\theta(a)]\beta_2(a),$$
$$\beta_4(a) = \beta_3[\theta(a)]\beta_2(a) = \beta_2[\theta^2(a)]\beta_2[\theta(a)]\beta_2(a),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\beta_p(a) = \beta_{p-1}[\theta(a)]\beta_2(a) = \beta_2[\theta_2^{p-2}(a)]\cdots\beta_2[\theta(a)]\beta_2(a),$$

and we obtain

$$\beta_p[\theta(a)]\cdot\beta_2(a) = N_H[\beta_2(a)] = [g(a)]^2\cdot 1$$

where $N_H[\beta_2(a)]$ is the norm in the cyclic algebra H of the polynomial $\beta_2(a)$. But $g(a)$ is the γ of Theorem 20 and p is not a power of 2, a contradiction. Hence n contains no odd prime factor p and is a power of 2.

The only other case where it is possible to make a detailed study of the structure of the Riemann matrices is the case where the minimum equation of a has all imaginary roots but now such that the unique substitution replacing each root by its complex conjugate is commutative with all of the substitutions of the Galois group of this minimum equation. Let P_0 be this unique substitution, and as before assume that n has an odd prime factor p so that the Galois group G of the minimum equation $\phi(\xi) = 0$ of a has a cyclic substitution P of order p. Now $PP_0 = P_0P$ and $P_2 = PP_0$ obviously generates a cyclic substitution of order $2p$ such that $P_2{}^p = P_0$. We may rearrange the roots of $\phi(\xi) = 0$ so that $\theta_2(a)$ is the polynomial into which a is carried by P_2,

$$\theta_2{}^j(a) = \theta_{j+1}(a)\ (j = 0, 1, \cdots, 2p-1), \quad y_2{}^{2p} = g(a) = g_{2p,2}(a),$$
$$g_{j,2}(a) = 1 \quad (j < 2p), \quad t_{2p,2} = 1, \quad t_{j,2} = j+1 \quad (j = 2, \cdots, 2p-1).$$

The author has shown that there exist positive numbers $\beta_j\ (\alpha_1)$ such that if α_1 is an ordinary complex root of $\phi(\xi) = 0$ and $\alpha_j = \theta_j(\alpha_1)$, then $\beta_k = \beta_k(\alpha_1)$ in $F(\alpha_1)$ and $n = 2n_1$,

$$\beta_j[\alpha_k]\beta_k(\alpha_1) = g_{j,k}(\bar{\alpha}_1)\cdot g_{jk}(\alpha_1)\cdot\beta_{t_{j,k}}(\alpha_1).$$

The above equations were proved true under the assumptions that

$$\bar{\alpha}_1 = \theta_{1+n_1}(\alpha_1), \quad \bar{\alpha}_j = \theta_{1+n_1}(\alpha_j) = \theta_j[\theta_{1+n_1}(\alpha_1)],$$

but, since they hold for any subscripts, they will hold after our permutation of the subscripts which makes $\theta_2^k(a) = \theta_{k+1}(a)$ $(k=0, 1, \cdots, 2p-1)$. Now $\theta_2^p(\alpha_j) = \bar{\alpha}_j$ and we have, from the correspondence between $F(a)$ and $F(\alpha_1)$,

$$\beta_j[\theta_k(a)]\beta_k(a) = g_{j,k}[\theta_2^p(a)]\cdot g_{j,k}(a)\cdot \beta_{t_{j,k}}(a) \qquad (j, k = 0, 1, \cdots, n).$$

But $g(a) = y_2^{2p}$ is unaltered by transforming by the quantity y_2^p, so that $g(a) = g[\theta_2^p(a)]$ and the above equations make the square of $g(a)$, as for the case of real α_1 but with p now replaced by $2p$, equal to the norm, with respect to the substitution P_2, of a polynomial in a. As before this would make a division sub-algebra of A not a division algebra unless $2p$ is a power of two. This contradicts our hypothesis that p is odd and we have

THEOREM 23. *Let ω be a pure Riemann matrix over a real field F and with D its multiplication algebra. Let D be a normal division algebra in n^2 units over F, and suppose that D contains a quantity a whose minimum equation has degree n, n complex roots all polynomials in one of them with coefficients in F, and the property that these roots are all real, or all imaginary such that the substitution carrying each root to its complex conjugate is commutative with all of the substitutions of the Galois group of the equation. Then n is a power of two.*

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ON NORMAL DIVISION ALGEBRAS OF TYPE R IN THIRTY-SIX UNITS*

BY

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1. Introduction. A normal division algebra in n^2 units over a non-modular field F is of type R if it contains a quantity i whose minimum equation with respect to F, $\phi(\omega)=0$, has degree n and n distinct roots which are polynomials in i with coefficients in F. Algebras of type R occupy a central position in the theory of division algebras as they are the only normal division algebras whose structure is known, and all division algebras of order less than twenty-five are expressible as algebras of type R.

The normal division algebras D whose structure is the simplest are those for the case where $\phi(\omega)=0$ has the cyclic group with respect to F. When n is six and $\phi(\omega)=0$ is cyclic, D is expressible as the direct product of a generalized quaternion division algebra and a cyclic division algebra of order nine, while conversely every such direct product is a cyclic division algebra of order thirty-six. The group of $\phi(\omega)=0$ is evidently regular and hence the only other type of equation to be considered for algebras of order thirty-six and type R is one which has the single non-cyclic, non-abelian regular group on six letters, a case giving a very complicated algebra.

It has never been demonstrated that there exist normal division algebras which are not cyclic algebras. The author showed, in a recent paper,† that the algebras which had been constructed by F. Cecioni‡ and which were based on a non-cyclic quartic were cyclic algebras. We show here that *all normal division algebras of type R in thirty-six units are cyclic algebras.*

2. Algebras based on a non-cyclic sextic with regular group. Let D be an associative normal division algebra of order thirty-six and type R, and let i be the quantity of D which defines the type of D. If $\phi(\omega)=0$, the minimum equation of i, is a cyclic sextic, D is called a cyclic algebra. There remains to be considered the case where the group of $\phi(\omega)=0$ is non-cyclic. The author has shown‖ that $\phi(\omega)$ may be taken to have only even powers of the indeterminate ω and that there exists a polynomial $\theta(i)$ in $F(i)$ such that

* Presented to the Society, October 25, 1930; received by the editors in August, 1930.

† These Transactions, vol. 32 (1930), pp. 171–195.

‡ Rendiconti del Circolo Matematico di Palermo, vol. 47 (1923), pp. 209–254.

‖ See Theorem 12 of the author's paper, American Journal of Mathematics, vol. 52 (1930), pp. 283–292.

$$(1) \quad \phi(\omega) \equiv [\omega + \theta^2(i)][\omega - \theta^2(i)][\omega + \theta(i)][\omega - \theta(i)](\omega + i)(\omega - i),$$

while for the non-cyclic case

$$(2) \qquad \theta^3(i) = i, \ \theta(-i) = -\theta^2(i), \ \theta^2(-i) = -\theta(i).$$

Evidently i^2 satisfies a cubic equation irreducible in F, and $F(i^2)$ is a cubic field over F. The set of all quantities in $F(i)$ which are symmetric in i, $\theta(i)$, $\theta^2(i)$, form a quadratic sub-field

$$(3) \qquad K = F(v), \ v^2 = \rho \text{ in } F,$$

of $F(i)$. A cubic field contains no quadratic sub-field so v is not in $F(i^2)$. Hence $1, v$ are linearly independent with respect to $F(i^2)$, and every quantity in $F(i)$ is expressible in the form

$$(4) \qquad a = a(i) = a_1 + a_2 v \qquad (a_1 \text{ and } a_2 \text{ in } F(i^2)).$$

But then

$$i = p_1 + pv$$

with p and p_1 in $F(i^2)$, so that

$$i^2 = (p_1^2 + p^2\rho) + 2p_1pv, \ 0 = (p_1^2 + p^2\rho - i^2) + 2p_1pv.$$

It follows that $2p_1p=0$. If p were zero then i would be in $F(i^2)$, a cubic field, contrary to the fact that $F(i)$ is a field of order six. Hence p_1 is zero and

$$(5) \qquad i = pv, \ p \text{ in } F(i^2).$$

It is known* that D has a basis

$$(6) \qquad i^s j^t, \ i^s j^t z \qquad (s = 0, 1, \cdots, 5; t = 0, 1, 2),$$

and a multiplication table

$$(7) \qquad \begin{aligned} &\phi(i) = 0, && j^t a = a[\theta^t(i)]j^t && (t = 0, 1, \cdots), \\ &za(i) = a(-i)z, && zj = \alpha j^2 z, \ zj^2 = \alpha\alpha[\theta^2(i)]gjz, \\ &j^3 = g, && z^2 = \gamma, \end{aligned}$$

for every a in $F(i)$ where g, α, γ are in $F(i)$. Since $zz^2 = z^2z$ we have $\gamma = \gamma(-i)$ is in $F(i^2)$. Similarly $jg = gj$ gives $g = g[\theta(i)]$ is in $F(v)$. Write

$$\gamma = \gamma_1 + \gamma_2 i^2 + \gamma_3 i^4 \qquad (\gamma_1, \gamma_2, \gamma_3 \text{ in } F),$$

and suppose that $\gamma_3 \neq 0$. We can then define scalars γ_5, γ_6 in F by $\gamma_1 = \gamma_3\gamma_5$, $\gamma_2 = 2\gamma_3\gamma_6$ and

* See L. E. Dickson, *Algebren und ihre Zahlentheorie*, pp. 75–79, where $q=2$, $z \equiv j_q$, $j \equiv j_1$, $\theta_q(i) \equiv -i$, $\theta_1(i) \equiv \theta(i)$.

$$\gamma = \gamma_3(i^4 + 2\gamma_6 i^2 + \gamma_6^2 + \gamma_5 - \gamma_6^2) = \gamma_3(i^2 + \gamma_6)^2 + \gamma_3(\gamma_5 - \gamma_6^2).$$

Consider the quantity

$$i_1 = (i^2 + \gamma_6)v.$$

Since $v \neq 0$ we have $v^2 = \rho \neq 0$ in a division algebra and

$$i_1^2 = (i^2 + \gamma_6)^2\rho = \rho i^4 + 2\rho i^2\gamma_6 + \rho\gamma_6^2$$

is in $F(i^2)$ but not in F, since in particular the coefficient of i^4 is not zero. But $F(i^2)$ has no proper sub-field other than F, so that $F(i_1^2) = F(i^2)$. The quantities

$$(8)\qquad \begin{aligned}&(i^2 + \gamma_6)v,\ -(i^2 + \gamma_6)v,\ [\theta(i)^2 + \gamma_6]v,\\ &\qquad -[\theta^2(i)^2 + \gamma_6]v,\ [\theta^2(i)^2 + \gamma_6]v,\ [\theta(i)^2 + \gamma_6]v\end{aligned}$$

are transforms

$$i_1, \quad zi_1z^{-1}, \quad ji_1j^{-1}, \quad zji_1(zj)^{-1}, \quad j^2i_1j^{-2}, \quad zj^2i_1(zj^2)^{-1}$$

of i_1 and are roots of its minimum equation. If they were not distinct, two of

$$\pm (i^2 + \gamma_6), \ \pm [\theta(i)^2 + \gamma_6], \ \pm [\theta^2(i) + \gamma_6]$$

would be equal, which is impossible since those with plus signs are the distinct roots of the irreducible cubic minimum equation of $i^2 + \gamma_6$, while this cubic has not the negative of any one of its roots as a root since *it has not even powers only*. The minimum equation of i_1 has thus six distinct roots in $F(i)$ so that its degree is six, $F(i_1)$ contained in $F(i)$ has order six, and $F(i_1) = F(i)$. Evidently $zi_1 = -i_1z$, while j transforms i_1 into a quantity in $F(i_1)$, that is a polynomial in i_1. We may thus replace i by i_1 in the basis of D without loss of generality, and, since $\gamma = (\gamma_3\rho^{-1})i_1^2 + \gamma_3(\gamma_5 - \gamma_6^2)$, *for this new i* we have γ expressed as a linear combination with coefficients in F of 1 and i^2. When $\gamma_3 = 0$ we also have immediately such an expression, so that we have proved

LEMMA 1. *The quantity i may be so chosen that, without altering any other property of D,*

$$(9)\qquad \gamma = \gamma_1 + \gamma_2 i^2 \qquad (\gamma_1 \text{ and } \gamma_2 \text{ in } F).$$

We shall utilize the notations

$$(10)\qquad a' = a(-i), \quad a_\theta = a[\theta(i)], \quad a_{\theta\theta} = (a_\theta)_\theta,$$

so that from (2) we immediately have

$$(11)\qquad \begin{aligned}&(a')' = a, \qquad (a_\theta)_{\theta\theta} = (a_{\theta\theta})_\theta = a,\\ &(a')_\theta = (a_{\theta\theta})', \quad (a')_{\theta\theta} = (a_\theta)',\end{aligned}$$

for every a of $F(i)$. Also

(12) $$ja = a_\theta j, \quad j^2 a = a_{\theta\theta} j^2, \quad za = a'z,$$

from (7), while

(13) $$i' = -i, \quad v' = -v, \quad v_\theta = v, \quad (i^2)' = i^2, \quad g = g_\theta = g_{\theta\theta}, \quad \gamma = \gamma'.$$

Consider the quantities

(14) $$d = \lambda_1 + \lambda_4 i, \quad e = \lambda_2 + \lambda_3 i,$$

where

(15) $$2\lambda_1 = 1 + \gamma_1, \quad 2\lambda_2 = 1 - \gamma_1, \quad 2\lambda_3 = 1 + \gamma_2, \quad 2\lambda_4 = 1 - \gamma_2,$$

so that $\lambda_1, \cdots, \lambda_4$ are in F, $\lambda_1^2 - \lambda_2^2 = \gamma_1$, $\lambda_3^2 - \lambda_4^2 = \gamma_2$. Then

$$dd' - ee' = \lambda_1^2 - \lambda_4^2 i^2 - (\lambda_2^2 - \lambda_3^2 i^2) = \gamma_1 + \gamma_2 i^2 = \gamma.$$

But $\gamma = \gamma'$ and if we put $f = d\gamma^{-1}$, $h = e\gamma^{-1}$, we have

$$(ff' - hh')\gamma = (dd' - ee')\gamma^{-1} = 1,$$

and obtain

LEMMA 2. *There exist polynomials f and h in $F(i)$ such that*

(16) $$(ff' - hh')\gamma = 1.$$

Let now r and s be defined by

(17) $$r = \alpha f, \quad s = \alpha h,$$

where α is the quantity of (7) such that $zj = \alpha j^2 z$ and α is in $F(i)$. Then r and s are in $F(i)$, and

(18) $$(rr' - ss')\gamma(\alpha^{-1})(\alpha^{-1})' = 1.$$

But if $\delta = (jz)^2 = jzjz = j\alpha j^2 zz = \alpha_\theta g\gamma$, then $\delta_\theta = j(jz)^2 j^{-1} = (jjzj^{-1})^2 = (\alpha^{-1}\alpha j^2 zj^{-1})^2 = (\alpha^{-1}zjj^{-1})^2 = (\alpha^{-1}z)^2 = (\alpha^{-1})(\alpha^{-1})'\gamma$, so that (18) gives

LEMMA 3. *There exist quantities r and s in $F(i)$ such that, if*

(19) $$\delta = (jz)^2 = \alpha_\theta g\gamma,$$

then

(20) $$(rr' - ss')\delta_\theta = 1.$$

If a and b are defined by

(21) $$a = s\alpha^{-1}, \quad b = r_{\theta\theta},$$

so that $b_\theta = r$, then

(22) $$Q \equiv [b_\theta(b_\theta)' - a\alpha a'\alpha'], \quad Q\delta_\theta = 1.$$

3. **The cyclic property.** We shall now proceed to prove that D is a cyclic algebra by the use of our fundamental existence theorem, Lemma 3. Consider the quantity

$$X = a + bj + cj^2, \tag{23}$$

where we take a and b to be the polynomials of Lemma 3 which satisfy (22) and where c will be chosen to be a polynomial in i with coefficients in F. For every c in $F(i)$, we have

$$zXz^{-1} \equiv X' = a' + c'\alpha\alpha_{\theta\theta}gj + b'\alpha j^2, \tag{24}$$

so that

$$XX' = (a + bj + cj^2)(a' + c'\alpha\alpha_{\theta\theta}gj + b'\alpha j^2) = A + Bj + Ej^2, \tag{25}$$

where A is a polynomial in i and

$$B = ac'\alpha\alpha_{\theta\theta}g + b(a')_\theta + c(b'\alpha)_{\theta\theta}g = Rc + Sc' + T, \tag{26}$$

$$E = ab'\alpha + b(c'\alpha\alpha_{\theta\theta}z)_\theta + c(a')_{\theta\theta} = Gc + H(c')_\theta + K. \tag{27}$$

The quantities B and E are polynomials in i and we have defined above

$$R \equiv (b'\alpha)_{\theta\theta}g = (b_\theta)'\alpha_{\theta\theta}g, \quad S \equiv a\alpha\alpha_{\theta\theta}g, \quad T \equiv b(a')_\theta = b(a_{\theta\theta})', \tag{28}$$

$$G = (a')_{\theta\theta} = (a_\theta)', \; H = b\alpha_\theta\alpha g, \; K = ab'\alpha, \tag{29}$$

all in $F(i)$. Now

$$(g')_\theta = (g_{\theta\theta})' = g', \; (g')_{\theta\theta} = [(g')_\theta]_\theta = (g')_\theta = g. \tag{30}$$

Transforming B by z we have

$$B' = R'c' + S'c + T', \tag{31}$$

whence

$$\begin{aligned} R'B - SB' &= R'Rc + R'Sc' + R'T - SR'c' - SS'c - ST' \\ &= (RR' - SS')c - (ST' - R'T). \end{aligned} \tag{32}$$

But

$$\begin{aligned} RR' - SS' &= (b_\theta)'\alpha_{\theta\theta}gb_\theta(\alpha_{\theta\theta})'g' - a\alpha\alpha_{\theta\theta}ga'\alpha'(\alpha_{\theta\theta})'g' \\ &= gg'\alpha_{\theta\theta}(\alpha_{\theta\theta})'[b_\theta(b_\theta)' - a\alpha a'\alpha'] = gg'\alpha_{\theta\theta}(\alpha_{\theta\theta})'Q. \end{aligned} \tag{33}$$

From (19) $\delta_\theta = \alpha_{\theta\theta}g\gamma_\theta$, so that, utilizing the relation $Q\delta_\theta = 1$, we have

$$RR' - SS' = gg'\alpha_{\theta\theta}(\alpha_{\theta\theta})'\delta_\theta^{-1} = g'(\alpha_{\theta\theta})'(\gamma_\theta)^{-1} \neq 0, \tag{34}$$

since g, α and γ are all not zero in a division algebra. Hence $RR' - SS'$ has an inverse $(RR' - SS')^{-1}$ in $F(i)$, and if we define the quantity c by

$$c = (ST' - R'T)(RR' - SS')^{-1}, \tag{35}$$

then

$$R'B - SB' = (ST' - R'T) - (ST' - R'T) = 0. \tag{36}$$

We shall henceforth consider the quantity X *as completely defined in* (23) *with the* a *and* b *of Lemma* 3 *and the* c *of* (35), *so that* (36) *is satisfied.* Transforming (36) by z we have

$$RB' - BS' = 0, \tag{37}$$

whence

$$R(R'B - SB') + S(-BS' + RB') = B(RR' - SS') = 0. \tag{38}$$

But $RR'-SS'$ has an inverse in $F(i)$, whence $B=0$.

We consider now the polynomial E. We first compute

$$ST' - R'T = a\alpha\alpha_{\theta\theta}gb'a_{\theta\theta} - b_\theta(\alpha_{\theta\theta})'g'b(a_{\theta\theta})'. \tag{39}$$

Next

$$\begin{aligned} H(K')_\theta - (G')_\theta K &= b\alpha_\theta\alpha g(a')_\theta b_\theta(\alpha')_\theta - a_{\theta\theta}ab'\alpha \\ &= -\left[\alpha a a_{\theta\theta}b' - \alpha\alpha_\theta gbb_\theta(\alpha_{\theta\theta})'(a_{\theta\theta})'\right], \end{aligned} \tag{40}$$

so that

$$-\alpha_{\theta\theta}g[H(K')_\theta - (G')_\theta K] = a\alpha a_{\theta\theta}\alpha_{\theta\theta}b'g - bb_\theta(\alpha\alpha_\theta\alpha_{\theta\theta}g^2)(a_{\theta\theta})'(\alpha_{\theta\theta})'. \tag{41}$$

But $j^3=g$, $g'=zgz^{-1}=zj^3z^{-1}=(zjz^{-1})^3=(\alpha j^2zz^{-1})^3=(\alpha j^2)^3=\alpha j^2\alpha j^2\alpha j^2=\alpha\alpha_{\theta\theta}\alpha_\theta g^2$, and we have the relations

$$g' = \alpha\alpha_\theta\alpha_{\theta\theta}g^2, \tag{42}$$

$$g = \alpha'(\alpha_{\theta\theta})'(\alpha_\theta)'(g')^2 = \alpha'(\alpha')_\theta(\alpha')_{\theta\theta}(g')^2. \tag{43}$$

Substituting (42) in (41) and comparing with (39) we write immediately

$$ST' - R'T = -\alpha_{\theta\theta}g[H(K')_\theta - (G')_\theta K]. \tag{44}$$

We also have, by the use of (11), $[(\alpha_\theta)']_\theta = [(\alpha')_{\theta\theta}]_\theta = \alpha'$, and

$$G(G')_\theta - H(H')_\theta = (a_\theta)'a_{\theta\theta} - b\alpha\alpha_\theta g(b')_\theta(\alpha')_\theta\alpha'g'. \tag{45}$$

But then

$$\begin{aligned} &-g\alpha_{\theta\theta}(\alpha')_{\theta\theta}[G(G')_\theta - H(H')_\theta] \\ &\qquad = (\alpha\alpha_\theta\alpha_{\theta\theta}g^2)\left[\alpha'(\alpha')_\theta(\alpha')_{\theta\theta}g'\right]b(b')_\theta - ga_{\theta\theta}(a')_{\theta\theta}\alpha_{\theta\theta}(\alpha')_{\theta\theta}, \end{aligned} \tag{46}$$

which by (42) and (43) has the value

$$g[b(b')_\theta - a_{\theta\theta}\alpha_{\theta\theta}(a')_{\theta\theta}(\alpha')_{\theta\theta}]. \tag{47}$$

But if

$$Q = b_\theta(b_\theta)' - a\alpha a'\alpha' = b_\theta(b')_{\theta\theta} - a\alpha a'\alpha', \tag{48}$$

then from (22), $\delta_\theta Q = 1$, $Q \neq 0$,

$$Q_{\theta\theta} = b(b')_\theta - a_{\theta\theta}\alpha_{\theta\theta}(a')_{\theta\theta}(\alpha')_{\theta\theta}. \tag{49}$$

Hence

$$-\alpha_{\theta\theta}(\alpha')_{\theta\theta}g[G(G')_\theta - H(H')_\theta] = gQ_{\theta\theta} \neq 0, \tag{50}$$

so that

$$\frac{H(K')_\theta - (G')_\theta K}{G(G')_\theta - H(H')_\theta} = \frac{-\alpha_{\theta\theta}g[H(K')_\theta - (G')_\theta K]}{-\alpha_{\theta\theta}g[G(G')_\theta - H(H')_\theta]}\cdot\frac{(\alpha')_{\theta\theta}}{(\alpha')_{\theta\theta}} = \frac{(\alpha')_{\theta\theta}[ST' - R'T]}{gQ_{\theta\theta}}. \tag{51}$$

From (22) we have $\delta_\theta Q = 1$, so that

$$j^2(\delta_\theta Q)j^{-2} = 1 = \delta Q_{\theta\theta} = \delta_\theta Q, \tag{52}$$

whence

$$\frac{\delta_\theta Q}{\gamma_\theta} = \frac{\delta Q_{\theta\theta}}{\gamma_\theta}. \tag{53}$$

Now $\delta = \gamma\alpha_\theta g$ from (19), while $\delta = (jz)^2$ is commutative with jz and equals its transform by jz, the quantity $(\delta')_\theta$. Hence

$$g = g_\theta, \quad \alpha_\theta g = \frac{\delta}{\gamma}, \quad \alpha_{\theta\theta}g = \frac{\delta_\theta}{\gamma_\theta}, \quad \alpha g = \frac{\delta_{\theta\theta}}{\gamma_{\theta\theta}}, \tag{54}$$

and

$$\alpha' g' = \frac{(\delta_{\theta\theta})'}{(\gamma_{\theta\theta})'} = \frac{(\delta')_\theta}{(\gamma')_\theta} = \frac{\delta}{\gamma_\theta}, \tag{55}$$

since $\gamma = \gamma'$. Equation (53) becomes

$$\alpha_{\theta\theta}gQ = \alpha' g' Q_{\theta\theta}. \tag{56}$$

From (43) $g = \alpha'(\alpha')_\theta(\alpha')_{\theta\theta}(g')^2$, so that

$$\alpha' g' Q_{\theta\theta} = (\alpha' g')[(\alpha')_\theta(\alpha')_{\theta\theta}g']\alpha_{\theta\theta}Q,$$

and

$$Q_{\theta\theta} = (\alpha')_{\theta\theta}(\alpha_{\theta\theta})'\alpha_{\theta\theta}g'Q. \tag{57}$$

It follows now that

$$gQ_{\theta\theta}/(\alpha')_{\theta\theta} = gg'\alpha_{\theta\theta}(\alpha_{\theta\theta})'Q = RR' - SS', \tag{58}$$

by (33). Using (35) and (51) we have

(59) $c=(ST'-R'T)(RR'-SS')^{-1}=[H(K')_\theta-(G')_\theta K][G(G')_\theta-H(H')_\theta]^{-1}$.

We have now demonstrated that

(60) $$[G(G')_\theta - H(H')_\theta]c - [H(K')_\theta - (G')_\theta K] = 0,$$

a relation very similar to (36). In fact, since E is given by (27), and $[(c')_\theta]' = c_{\theta\theta}$, $(c_{\theta\theta})_\theta = c$,

(61) $$(E')_\theta = (G')_\theta(c')_\theta + (H')_\theta c + (K')_\theta,$$

so that, by (60),

(62) $$\begin{aligned}(G')_\theta E - H(E')_\theta &= (G')_\theta Gc + (G')_\theta H(c')_\theta + (G')_\theta K\\ &\qquad - H(G')_\theta(c')_\theta - H(H')_\theta c - H(K')_\theta\\ &= [G(G')_\theta - H(H')_\theta]c - [H(K')_\theta - (G')_\theta K] = 0.\end{aligned}$$

Transforming (62) by jz we have

(63) $$G(E')_\theta - (H')_\theta E = 0,$$

and

(64) $0=G[(G')_\theta E - H(E')_\theta] + H[G(E')_\theta - (H')_\theta E] = [G(G')_\theta - H(H')_\theta]E = 0.$

It follows from (50) that $E=0$ and that (25) becomes

(65) $$XX' = A \text{ in } F(i).$$

But then

(66) $$(Xz)^2 = XzXz = XX'\gamma = A\gamma = t \text{ in } F(i),$$

since γ is in $F(i)$, and X' was defined so that $zX = X'z$ in (24).

Let first b and a be both not zero, so that since Xz is commutative with its square,

$$tXz = (ta + tbj + tcj^2)z = Xzt = (a + bj + cj^2)t'z.$$

Hence

$$ta + tbj + tcj^2 = t'a + (t')_\theta bj + (t')_{\theta\theta}cj^2,$$

and since (6) are a basis of D, $ta=t'a$, $tb=(t')_\theta b$. Since a is not zero, $t=t'$ is in $F(i^2)$. Since also b is not zero, $t=(t')_\theta=t_\theta$ is in F. It follows that when $ab\neq 0$ we have shown that there exists a quantity X in the algebra

$$\Sigma = (i^s j^r) \qquad (s = 0, 1, \cdots, 5; r = 0, 1, 2),$$

such that $X\neq 0$ and

$$(Xz)^2 = \lambda \text{ in } F. \tag{67}$$

Suppose next that a were zero so that from its origin (21) we have $s=0$ and $h=0$ in (17). Then (16) becomes

$$ff'\gamma = (fz)^2 = 1,$$

while then obviously f cannot be zero and $f=\alpha^{-1}r=\alpha^{-1}b_\theta$ is in $F(i)$ and in Σ. Again we have (67) for $X\neq 0$ in Σ. Finally the only remaining case is b zero. Then $r_{\theta\theta}=0$ so that $r=0$ in (17), and hence the quantity f is zero. Equation (16) now becomes $hh'\gamma=(hz)^2=-1$, and since then $h=\alpha^{-1}s=a$ in $F(i)$ cannot be zero when $(hz)^2=-1$, we have again proved the existence of $X\neq 0$ in Σ and satisfying (67). Hence in all cases we have

LEMMA 4. *There exists a quantity X in Σ such that $X\neq 0$, and if $y=Xz$ then*

$$y^2 = \lambda \textit{ in } F. \tag{68}$$

The quantities $1, v, y, vy$ are linearly independent with respect to F, for otherwise (6) could not be a basis of D when $X\neq 0$. A relation of the form

$$\xi_1 + \xi_2 v + (\xi_3 + \xi_4 v)Xz = 0,$$

with $\xi_1, \xi_2, \xi_3, \xi_4$ not all zero and in F, would then evidently express z as a quantity of Σ. Also $v^2=\rho$, $y^2=\lambda$, $yv=Xzv=-Xvz=-vXz=-vy$, since v, a polynomial in i commutative with j, is commutative with X. But the linear set

$$\Gamma = (1, v, y, vy)$$

is evidently a generalized quaternion algebra over F, and is a normal division cyclic algebra over F. Hence D, containing Γ, is the direct product* of Γ and another algebra Ω of order nine over F. Since D is a normal algebra, so is necessarily Ω,† so that Ω is a cyclic algebra‡ of order nine. Hence D, the direct product of algebra Γ and algebra Ω, is a cyclic algebra.†

THEOREM. *Every normal division algebra of type R in thirty-six units is a cyclic algebra.*

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* A theorem of Wedderburn; cf. *Algebras and their Arithmetics*, p. 237.

† For the first and second of the above references respectively see Theorems 7 and 16 of the author's paper, *On direct products, cyclic division algebras, and pure Riemann matrices*, which appears in the present number of these Transactions.

‡ A theorem of Wedderburn, these Transactions, vol. 22 (1921), pp. 129–135.

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INTEGRALS WHOSE EXTREMALS ARE A GIVEN 2n-PARAMETER FAMILY OF CURVES*

BY

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Introduction

It has previously been shown that a necessary† and sufficient‡ condition for a system of second-order differential equations of the form $H_i(x, y_j, y_j', y_j'')=0$ $(i, j=1, \cdots, n)$ to be the Euler equations of an integral

$$I = \int_{x_1}^{x_2} f(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx \tag{1}$$

is that the equations of variation of the system $H_i=0$ form a self-adjoint system along every curve $y_i=y_i(x)$.

The possibility of determining an integral of the above form when given a $2n$-parameter family of curves as its extremal arcs is here discussed. An example illustrating the method of procedure is also given.

I. Properties of given equations

Consider the $2n$-parameter family of arcs

$$y_i = y_i(x, a_1, \cdots, a_n, b_1, \cdots, b_n) \qquad (i = 1, \cdots, n) \tag{2}$$

which have the derivatives

$$\begin{aligned} y_i' &= y_{ix}(x, a_1, \cdots, a_n, b_1, \cdots, b_n), \\ y_i'' &= y_{ixx}(x, a_1, \cdots, a_n, b_1, \cdots, b_n). \end{aligned}$$

If $a_1, \cdots, a_n, b_1, \cdots, b_n$ are eliminated from these equations we obtain n equations of the form

$$y_i'' = F_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n'). \tag{3}$$

The general solution of this system is the system (2) where $a_1, \cdots, a_n, b_1, \cdots, b_n$ are arbitrary constants of integration.

* Presented to the Society, August 29, 1929; received by the editors in May, 1929, and May, 1930.

† For the necessity of this condition see *The inverse problem of the calculus of variations in higher space*, by the author, these Transactions, October, 1928. Also J. Hadamard, *Leçons sur le Calcul des Variations*, p. 156.

‡ The sufficiency of this condition is proved in *The inverse problem of the calculus of variations in a space of* $(n+1)$ *dimensions*, Bulletin of the American Mathematical Society, May–June, 1929.

If the equations (3) are to represent the extremal arcs of an integral I of the form (1), then, by the necessary condition stated in the Introduction, the functions F_i of (3) must be the solutions for y_i'' of a system of equations

$$H_i(x, y_i, y_i', y_i'') = 0$$

whose equations of variation are self-adjoint. Hence, there must exist a set of multipliers P_{ij} of non-zero determinant such that*

$$H_i \equiv P_{ij}(y_j'' - F_j) = 0 \qquad (i, j = 1, \cdots, n) \tag{4}$$

where the H_i have the form indicated in the following theorem.†

THEOREM. *If a system of equations $H_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n') = 0$ is to have equations of variation*

$$H_{iy_j}u_j + H_{iy_j'}u_j' + H_{iy_j''}u_j'' = 0 \qquad (i, j = 1, \cdots, n)$$

which are self-adjoint along every curve $y_i = y_i(x)$, then it must have the form

$$H_i \equiv M_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n') + P_{ij}(x, y_1, \cdots, y_n, y_1', \cdots, y_n')y_j'', \tag{4a}$$

where the functions M_i and P_{ij} satisfy the following relations identically in x, y_i, y_i':

$$\begin{aligned} P_{ij} &= P_{ji}, \quad P_{jky_i'} = P_{iky_j'}, \\ M_{iy_i'} + M_{iy_j'} &= 2(P_{ijx} + P_{ijy_k}y_k'), \\ M_{jy_i} - M_{iy_j} &= \tfrac{1}{2}(M_{jy_i'} - M_{iy_j'})_x + \tfrac{1}{2}(M_{jy_i'} - M_{iy_j'})_{y_k}y_k'. \end{aligned} \tag{5}$$

From the first two of (5) it follows that the expression $P_{jky_i'}$ remains unchanged for all permutations of the indices i, j, k.

By comparing expressions (4) and (4a), it is evident from the form of (4a) that the desired multipliers have $P_{ij} = P_{ji}$, and consequently, that the first two of relations (5) remain the same, namely,

$$P_{ij} = P_{ji}, \quad P_{jky_i'} = P_{iky_j'}. \tag{6}$$

There is also

$$M_i = - P_{ik}F_k,$$

the substitution of which in (5_3) with the aid of (6_2) gives

$$P_{ijx} + P_{ijy_k}y_k' + P_{ijy_k'}F_k = - \tfrac{1}{2}(P_{ik}F_{ky_j'} + P_{jk}F_{ky_i'}). \tag{7}$$

---

* In this and following notation where the indices in two factors of a term of the form $P_{ij}\, F_j$ are alike it represents a sum with respect to the repeated index.

† See Theorem I of *The inverse problem of the calculus of variations in a space of* $(n+1)$ *dimensions*, loc. cit.

From the last of the self-adjoint relations (5), one obtains

$$
\begin{aligned}
(P_{ik}F_k)_{y_j} - (P_{jk}F_k)_{y_i} &= \tfrac{1}{2}[(P_{ik}F_k)_{y_j'} - (P_{jk}F_k)_{y_i'}]_x \\
&\quad + \tfrac{1}{2}[(P_{ik}F_k)_{y_j'} - (P_{jk}F_k)_{y_i'}]_{y_k}y_k' \\
&= \tfrac{1}{2}(P_{ik}F_{ky_j'} - P_{jk}F_{ky_i'})_x \\
&\quad + \tfrac{1}{2}(P_{ik}F_{ky_j'} - P_{jk}F_{ky_i'})_{y_k}y_k'.
\end{aligned} \tag{8}
$$

As a consequence of relations (7) and (6_2) we have

$$P_{iky_j} - P_{jky_i} = \tfrac{1}{2}(P_{j\alpha}F_{\alpha y_i'} - P_{i\alpha}F_{\alpha y_j'})_{y_k'}$$

which may be readily verified by writing system (7) with subscripts i, j, k replaced first by i, k, α, secondly by j, k, α respectively; then, after differentiating the first system with respect to y_j' and the second with respect to y_i' and subtracting, the result readily reduces to the above system. The expressions on the left are now replaced by those on the right in the first member of (8), which with the use of (6) readily reduces to

$$P_{i\alpha}(F_{\alpha y_j} + \tfrac{1}{4}F_{\alpha y_k'}F_{ky_j'} - \tfrac{1}{2}F'_{\alpha y_j'}) - P_{j\alpha}(F_{\alpha y_i} + \tfrac{1}{4}F_{\alpha y_k'}F_{ky_i'} - \tfrac{1}{2}F_{\alpha y_i'}) \equiv 0, \tag{9}$$

where it is understood that

$$
\begin{aligned}
F'_{\alpha y_j'} &\equiv F_{\alpha y_j' x} + F_{\alpha y_j' y_k}y_k' + F_{\alpha y_j' y_k'}F_k, \\
F'_{\alpha y_i'} &\equiv F_{\alpha y_i' x} + F_{\alpha y_i' y_k}y_k' + F_{\alpha y_i' y_k'}F_k.
\end{aligned}
$$

The above results may be summarized in the following theorem:

THEOREM I. *If the solutions $y_i = y_i(x)$ of the differential equations*

$$y_i'' = F_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n') \qquad (i = 1, \cdots, n)$$

are to be the totality of extremal arcs for an integral of the form (1), *then there must exist a set of multipliers P_{ij} of non-zero determinant which are functions of $x, y_1, \cdots, y_n, y_1', \cdots, y_n'$ such that the functions*

$$H_i = P_{ij}(y_j'' - F_j)$$

have expressions of variation which are self-adjoint along every arc $y_i = y_i(x)$. Necessary and sufficient conditions for such multipliers to exist are

$$
\begin{aligned}
&P_{ij} = P_{ji}, \quad P_{iky_i'} = P_{iky_j'}, \\
&P_{ijx} + P_{ijy_k}y_k' + P_{ijy_k'}F_k = -\tfrac{1}{2}(P_{ik}F_{ky_j'} + P_{jk}F_{ky_i'}), \\
&P_{i\alpha}(F_{\alpha y_j} + \tfrac{1}{4}F_{\alpha y_k'}F_{ky_j'} - \tfrac{1}{2}F'_{\alpha y_j'}) - P_{j\alpha}(F_{\alpha y_i} + \tfrac{1}{4}F_{\alpha y_k'}F_{ky_i'} - \tfrac{1}{2}F'_{\alpha y_i'}) = 0,
\end{aligned} \tag{10}
$$

which must be identities in $x, y_1, \cdots, y_n, y_1', \cdots, y_n'$.

For a given set of functions F_i of the form indicated in the above theorem the theory of partial differential equations assures us that there exist solu-

tions of the system (10_3). Hence, if these solutions $P_{ij}(i, j=1, \cdots, n)$ are such that they satisfy the remaining self-adjoint relations, namely, the first two and the last of (10), then the primitives of the given equations give the extremal arcs for an integral I of the form (1).

In order to obtain the solutions of the system (10_3) let us introduce the new variables $x, a_1, \cdots, a_n, b_1, \cdots, b_n$ in place of $x, y_1, \cdots, y_n, y_1', \cdots, y_n'$ by means of the equations

$$
(11)\qquad \begin{aligned} y_i &= y_i(x, a_1, \cdots, a_n, b_1, \cdots, b_n), \\ y_i' &= y_{ix}(x, a_1, \cdots, a_n, b_1, \cdots, b_n), \end{aligned}
$$

which have solutions of the form

$$
(12)\qquad \begin{aligned} a_i &= A_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n'), \\ b_i &= B_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n'). \end{aligned}
$$

Since each of these functions is a solution of the homogeneous equation

$$
A_x + A_{y_k}y_k' + A_{y_k'}F_k \equiv 0,
$$

it follows that every set of functions $P_{ij}(x, a_1, \cdots, a_n, b_1, \cdots, b_n)$ which satisfy the system (10_3) when $a_1, \cdots, a_n, b_1, \cdots, b_n$ are replaced by the expressions (12) must satisfy the equations

$$
(13)\qquad \frac{d}{dx}(P_{ij}) = -\frac{1}{2}(P_{ik}F_{ky_j'} + P_{jk}F_{ky_i'}),
$$

where the variables $x, y_1, \cdots, y_n, y_1', \cdots, y_n'$ which occur in the expressions on the right are everywhere to be replaced by $x, a_1, \cdots, a_n, b_1, \cdots, b_n$ by means of equations (11).

The form of the second members of (13) shows that there are only $\frac{1}{2}n(n+1)$ distinct equations and that we shall have $P_{ij}=P_{ji}$.

According to the theory of ordinary differential equations, if $P_{ij}^{(r)}(i\leqq j, r=1, 2, \cdots, \frac{1}{2}n(n+1))$ are a set of $\frac{1}{2}n(n+1)$ independent particular solutions of the system (13), then every solution can be expressed in the form

$$
(14)\qquad \sum_r C_r P_{ij}^{(r)} = P_{ij}(x, a_1, \cdots, a_n, b_1, \cdots, b_n) \qquad (i \leqq j, r = 1, \cdots, \tfrac{1}{2}n(n+1)),
$$

where the C_r are arbitrary functions of $a_1, \cdots, a_n, b_1, \cdots, b_n$. If the C_r are determined in any manner, and the functions $a_1, \cdots, a_n, b_1, \cdots, b_n$ are replaced by their respective values given in (12), the resulting expressions for the P_{ij} are solutions of the system (10_3). Conversely, every solution of the system (10_3) can be so obtained.

It now remains to determine the functions C_r so that the relations (10_2) and (10_4) are satisfied. The possibility of doing this depends upon the nature of the given functions F_i which we have not been able to define completely. The following example, however, will illustrate the above theory and also a successful method of procedure that may be applied to special problems.

II. Integrals whose extremals are linear functions of one independent variable

Consider the system of linear functions

$$y_i = a_i x + b_i \qquad (i = 1, \cdots, n), \tag{15}$$

which are the primitives of the differential equations

$$y_i'' = 0.$$

If these are the respective solutions for y_i'' of n equations of the form

$$H_i(x, y_1, \cdots, y_n, y_1', \cdots, y_n', y_1'', \cdots, y_n'') = 0,$$

then, according to the above theory, there exist a set of multipliers P_{ij} such that the functions H_i take the form

$$H_i = P_{ij} y_j'' = 0. \tag{16}$$

The self-adjoint relations enumerated in Theorem I when applied to this system, since the $F_i = 0$, become

$$(17)\qquad \begin{aligned} &P_{ij} = P_{ji}, \quad P_{jky_i'} = P_{iky_j'}, \\ &P_{ijx} + P_{ijy_k} y_k' = 0 \qquad (i, j = 1, 2, \cdots, n), \end{aligned}$$

which must be identities in $x, y_1, \cdots, y_n, y_1', \cdots, y_n'$. If from the partial derivative of the last system of equations (17_3) with respect to y_i' we subtract its partial derivative with respect to y_j', we obtain the additional relations

$$P_{jky_i} = P_{iky_j}. \tag{18}$$

From the given functions (15) and their first derivatives with respect to x are obtained the following values of the $2n$ parameters a_i and b_i:

$$a_i = y_i', \quad b_i = y_i - y_i' x. \tag{19}$$

The corresponding total differential equations for the system (17_3) are

$$\frac{d}{dx} P_{ij} = 0,$$

whose solutions are arbitrary functions of a_i, b_i, namely,

$$P_{ij} = P_{ij}(a_1, \cdots, a_n, b_1, \cdots, b_n) \quad (i \leqq j = 1, \cdots, n).$$

Therefore, necessary and sufficient conditions for the system (16) to have self-adjoint equations of variations are

$$(20) \qquad \begin{aligned} &P_{ij} = P_{ji}, \quad P_{jky_i'} = P_{iky_j'}, \\ &P_{ij} = P_{ij}(a_1, \cdots, a_n, b_1, \cdots, b_n), \end{aligned}$$

where the P_{ij} are arbitrary functions of the $2n$ parameters which have the values given in (19).

It will now be shown that the functions P_{ij} can be so chosen that relations (20_2) can be satisfied. Let the P_{ij} be differentiated with respect to $a_1, \cdots, a_n$, $b_1, \cdots, b_n$ and these in turn with respect to y_i' and y_j' as indicated in (20_2); we thus obtain

$$P_{jka_i} - P_{jkb_i}x = P_{ika_j} - P_{ikb_j}x.$$

Since these equations are identities in x, we must have

$$(21) \qquad P_{jka_i} = P_{ika_j}, \quad P_{jkb_i} = P_{ikb_j}.$$

This system of partial differential equations of the first order is compatible and its general solution will be of the form

$$(22) \qquad P_{ij} = P_{ij}(a_1, \cdots, a_n, b_1, \cdots, b_n).$$

With the use of the expressions (22) for the P_{ij} we find a function $g(x, y_1, \cdots, y_n, y_1', \cdots, y_n')$ which will be a solution of the system

$$(23) \qquad g_{y_i'y_j'} = P_{ij}(y_1', \cdots, y_n', y_1 - y_1'x, \cdots, y_n - y_n'x).$$

The required conditions of integrability for this system are the first two sets of relations (20). The value of g is given by the integral

$$g = \int_{y_{10}', \ldots, y_{n0}'}^{y_1', \ldots, y_n'} L_1 dy_1' + L_2 dy_2' + \cdots + L_n dy_n',$$

where

$$L_i = \int_{y_{10}', \ldots, y_{n0}'}^{y_1', \ldots, y_n'} P_{i1}dy_1' + P_{i2}dy_2' + \cdots + P_{in}dy_n' \qquad (i = 1, \cdots, n).$$

If g is a particular solution of (23) then the most general solution is given by the formula

$$f = g(x, y_1, \cdots, y_n, y_1', \cdots, y_n') + A(x, y_1, \cdots, y_n) + B_k(x, y_1, \cdots, y_n)y_k', \tag{24}$$

where $A, B_1, \cdots, B_n$ are arbitrary functions of $x, y_1, \cdots, y_n$.

Relations (18) applied to the above integrals give

$$g_{y_i' y_j} = g_{y_j' y_i}. \tag{25}$$

The Euler-Lagrange conditions are now applied to the above value of f and the condition imposed that the resulting expressions be identically equal to the functions (16). With the aid of relations (18) and the fact that the given functions F_i of (3) in this particular case are each equal to zero we readily find that the functions $A, B_1, \cdots, B_n$ of (24) must satisfy the following conditions:*

$$\begin{aligned} B_{jy_i} - B_{i_j y} &= 0, \\ B_{ix} - A_{y_i} &= g_{y_i} - g_{y_i' x} - g_{y_i y_k'} y_k'. \end{aligned} \tag{26}$$

Due to relations (25) the second member of the last of these equations is identically zero. For we have

$$\begin{aligned} g_{y_i} - g_{y_i' x} - g_{y_i' y_k} y_k' = & \int_{y_{10}', \cdots, y_{n0}'}^{y_1', \cdots, y_n'} L_{1y_i} dy_1' + L_{2y_i} dy_2' + \cdots + L_{ny_i} dy_n' \\ & - \int_{y_{10}', \cdots, y_{n0}'}^{y_1', \cdots, y_n'} P_{i1x} dy_1' + \cdots + P_{inx} dy_n' \\ & - y_k' \int_{y_{10}', \cdots, y_{n0}'}^{y_1', \cdots, y_n'} P_{i1y_k} dy_1' + \cdots + P_{iny_k} dy_n'. \end{aligned}$$

But the second integral of this equation by means of equations (17_3) may be replaced by the integral

$$\begin{aligned} & \int_{y_{10}', \cdots, y_{n0}'}^{y_1', \cdots, y_n'} (P_{i1y_k} dy_1' + \cdots + P_{iny_k} dy_n') y_k' \\ & \quad = y_k' \int_{y_{10}', \cdots, y_{n0}'}^{y_1', \cdots, y_n'} P_{i1y_k} dy_1' + \cdots + P_{iny_k} dy_n' \\ & \quad - \int_{y_{10}', \cdots, y_{n0}'}^{y_1', \cdots, y_n'} \int_{y_{10}', \cdots, y_{n0}'}^{y_1', \cdots, y_n'} (P_{i1y_k} dy_1' + \cdots + P_{iny_k} dy_n') dy_k'. \end{aligned}$$

* Cf. Equations (19), *The inverse problem of the calculus of variations in as pace of* $(n+1)$ *dimensions*, loc. cit.

When this value is substituted in the previous equation we see that because of relations (18) the second member vanishes identically in $x, y_1, \cdots, y_n, y_1', \cdots, y_n'$. Hence, equations (26) may be written

$$B_{iy_i} - B_{iy_j} = 0, \quad A_{y_j} - B_{jx} = 0,$$

which are necessary and sufficient conditions for the expression

$$A + B_k y_k'$$

to be the total derivative of a function $t(x, y_1, \cdots, y_n)$.

THEOREM II. *The most general integral whose extremals are the 2n-parameter family of arcs*

$$y_i = a_i x + b_i \qquad (i = 1, \cdots, n)$$

has an integrand f of the form

$$f = g(x, y_1, \cdots, y_n, y_1', \cdots, y_n') + (d/dx)t(x, y_1, \cdots, y_n)$$

where g is a particular solution of the system (23) *and t is an arbitrary function of* $x, y_1, \cdots, y_n$.

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ON THE REGULAR POINTS OF A CONTINUUM*

BY

W. L. AYRES

I. Introduction

1. We consider a compact, connected metric space M which we shall call the continuum M. A point p of M is said to be a *regular point* if for each $\epsilon>0$ there exists a neighborhood U_p of p (i.e., an open subset of M containing p) such that $d(U_p)<\epsilon$ and $F(U_p)$ consists of a finite number of points. The point p is said to be a *point of order* α if (1) for each $\epsilon>0$ there exists a neighborhood U_p such that $d(U_p)<\epsilon$ and the cardinal number of $F(U_p)\leqq\alpha$, and (2) α is the smallest cardinal number for which (1) is true. Regular points of no finite order are called points of order ω. Let M^α denote the set of all points of M of order α. Then

$$M = M^1 + M^2 + \cdots + M^\omega + M^{\aleph_0} + M^c.$$

These definitions were introduced by Urysohn and Menger† several years ago and since that time have been studied in a number of papers.

One of the interesting studies in this theory is that of the distribution and structure of the various sets M^α. It is of course quite obvious that M may be composed entirely of points of order 2 or entirely of points of order c. Urysohn‡ has given examples to show that M may be composed entirely of points of order ω or entirely of points of order $\aleph_0$. Except for these four orders this is not possible, a consequence of a theorem proved independently by Urysohn,§ G. T. Whyburn‖ and H. Künneth¶ that if all points of M are of order $\leqq n$, then the points of order $\leqq\frac{1}{2}n+1$ are dense in M.

Since $M\neq M^n(n\neq 2)$, it would be interesting to know more of the distribution of the points of M^n. Whyburn** has shown that M^n is punctiform and has raised the question as to whether it is of dimension zero. In the present

* Presented to the Society, September 9, 1930; received by the editors September 9, 1930.

† P. Urysohn, Comptes Rendus, vol. 175 (1922), pp. 481–483; and K. Menger, Monatshefte für Mathematik und Physik, vol. 33 (1923), pp. 148–160.

‡ *Mémoire sur les multiplicités cantoriennes*, 2ème Partie, Verhandelingen, Akademie van Wetenschappen, Amsterdam, vol. 13 (1927), No. 4, pp. 109–115.

§ Ibid., pp. 105–9.

‖ *On regular points of continua* etc., Bulletin of the American Mathematical Society, vol. 35 (1929), pp. 218–224.

¶ *Ein Theorem der Kurventheorie*, Monatshefte für Mathematik und Physik, vol. 36 (1929), pp. 149–152.

** Loc. cit.

paper we will answer this question in the affirmative as a special case of our theorem that $M^n+M^{n+1}+\cdots+M^{2n-3}$ is a set of dimension zero. As corollaries of this theorem we obtain most of the previously known results concerning the distribution of the points of finite orders. In the last section of the paper we complete our study of the structure of the sets of various finite orders by examining the set M^2. Here we prove that M^2 is composed of a set of dimension zero plus a countable set of arcs.

2. **Notation.** Capitals will denote sets, lower case letters denote individual elements which are either points or numbers. The usual notation of the theory of sets will be employed. Below we will list some special notation which, while not new, is not universally employed by writers in this field and thus needs definition.

$p\in N \equiv p$ is an element of the set N.

p non-$\in N \equiv p$ is not an element of the set N.

$\rho(p, q)\equiv$distance between the points p and q.

$\rho(M, N)\equiv$greatest lower bound of $\rho(p, q)$ for $p\in M$ and $q\in N$.

$d(M)\equiv$diameter of $M\equiv$least upper bound of $\rho(p, q)$ for $p+q\subset M$.

$S(p, \eta)\equiv$set of all points q such that $\rho(p, q)<\eta$.

$\mathcal{C}(p, N)\equiv$component of set N containing the point p.

$F(N)\equiv \overline{N}\cdot(M-N)+\overline{M-N}\cdot N\equiv$frontier or boundary of N.

$C(N)\equiv M-N\equiv$complement of N.

$\dim_p E$, $\text{order}_p E\equiv$dimension of set E at p, order of E at p.

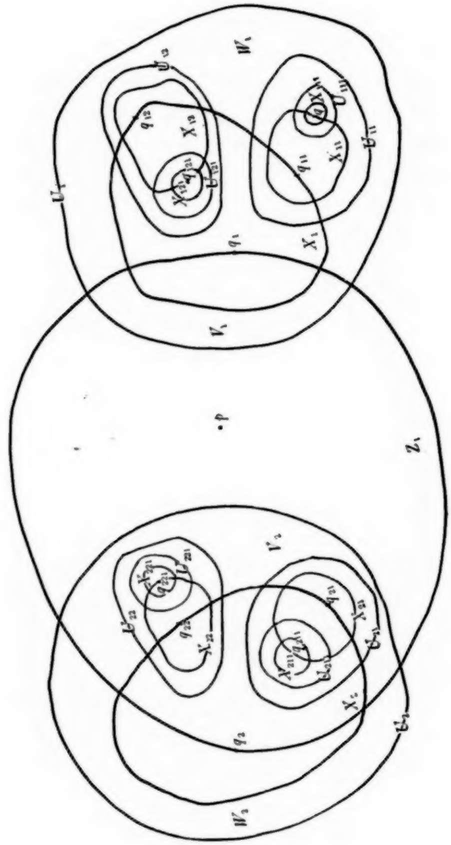
II. The structure of M^n, $n\neq 2$

3. Theorem. *For any integer $n>2$, the set of all points p of a continuum M such that $n\leqq \text{order}_p M\leqq 2n-3$ is a zero-dimensional set (or vacuous).**

Let E denote the set of points p such that $n\leqq \text{order}_p M\leqq 2n-3$. Given $\epsilon>0$ and $p\in E$, we shall show the existence of a neighborhood $U_p\subset S(p, \epsilon)$ such that $F(U_p)\cdot E=0$, i.e., $\dim E=0$. If p is a point of order m, by an *order neighborhood* of p we shall mean a neighborhood of p whose boundary consists of exactly m points.

Let $Z_1\subset S(p, \frac{1}{2}\epsilon)$ be an order neighborhood of p. If $E\cdot F(Z_1)=0$, our proof is complete. If not, let $E\cdot F(Z_1)=q_1+q_2+\cdots+q_{s_1}$ $(s_1\leqq 2n-3)$. Let U_k be an order neighborhood of $q_k(1\leqq k\leqq s_1)$ such that $\overline{U}_k\subset S(q_k, r_k)$, where r_k is the smaller of the numbers $\epsilon/4$ and $\frac{1}{2}\rho(q_k, F(Z_1)+p-q_k)$. We see that $\overline{U}_i\cdot\overline{U}_j=0$ if $i\neq j$. Let $V_k=Z_1\cdot U_k$ and $W_k=C(\overline{Z}_1)\cdot U_k$. Since $q_k\in E$, $F(U_k)$

* Professor K. Menger has called attention to the fact that no use is made of the condition that M be connected and compact in the proof of this theorem. Hence the theorem is true for *any* metric space M.


W_1
U_1
U_{12}
q_{12}
X_{12}
U_{121}
X_{121}
q_{121}
U_{11}
U_{111}
X_{111}
q_{11}
X_{11}
q_1
X_1
V_1
p
Z_1
V_2
U_{22}
U_{221}
X_{221}
q_{221}
q_{22}
X_{22}
q_{21}
X_{21}
U_{21}
X_{211}
q_{211}
U_{211}
X_2
U_2
q_2
W_2

consists of $\leqq 2n-3$ points. And since $F(V_k)+F(W_k)=F(U_k)+q_k$ and $F(V_k)\cdot F(W_k)=q_k$, either $F(V_k)$ or $F(W_k)$ consists of $\leqq n-1$ points. Let I_1 be the set of all integers i such that $F(W_i)$ consists of $\leqq n-1$ points, and let J_1 be the set of all integers $1\leqq j\leqq s_1$ not in I_1. Let X_k be an order neighborhood of q_k such that $\overline{X}_k\subset U_k$. Now let

$$Z_2 = Z_1 + \sum X_i - \sum \overline{X}_j,$$

where the first summation extends over I_1 and the second over J_1.

If $F(Z_2)\cdot E=0$, then Z_2 is the desired neighborhood U_p. If not, we have $F(Z_2)\cdot E\subset\sum_k F(X_k)$. Let $F(X_k)\cdot F(Z_2)\cdot E=q_{k1}+q_{k2}+\cdots+q_{ks_{2}k}$. For each point $q_{im}(i\in I_1)$, let U_{im} be an order neighborhood of q_{im} such that $\overline{U}_{im}\subset W_i\cdot S(q_{im}, r_{im})$, where r_{im} is the smaller of the numbers $\epsilon/8$ and $\frac{1}{2}\rho(q_{im}, F(X_i)-q_{im})$. For each point $q_{jm}(j\in J_1)$, let U_{jm} be an order neighborhood of q_{jm} such that $\overline{U}_{jm}\subset V_{jm}\cdot S(q_{jm}, r_{jm})$, where r_{jm} is the smaller of the numbers $\epsilon/8$ and $\frac{1}{2}\rho(q_{jm}, F(X_j)-q_{jm})$. We see that $\overline{U}_{k_1m_1}\cdot\overline{U}_{k_2m_2}=0$ unless $k_1=k_2$, $m_1=m_2$. Let $V_{km}=Z_2\cdot U_{km}$ and $W_{km}=C(\overline{Z}_2)\cdot U_{km}$. Since $q_{km}\in E$, $F(U_{km})$ consists of $\leqq 2n-3$ points. Then either $F(W_{km})$ or $F(V_{km})$ consists of $\leqq n-1$ points. Let I_2 be the set of all pairs (k, m) such that $F(W_{km})$ consists of $\leqq n-1$ points. Let J_2 be the set of all pairs (k, m) for which U_{km} is defined that are not in I_2. Let X_{km} be an order neighborhood of q_{km} such that $\overline{X}_{km}\subset U_{km}$. Now let

$$Z_3 = Z_2 + \sum X_{km} - \sum \overline{X}_{km},$$

where in the first summation $(k, m)\in I_2$ and in the second $(k, m)\in J_2$.

If $F(Z_3)\cdot E=0$, Z_3 is the desired neighborhood U_p. If not, we repeat this process on the points of $F(Z_3)\cdot E=F(Z_3)\cdot E\cdot\sum_k\sum_m F(X_{km})$. Continuing this process, at some stage we reach a neighborhood Z_i such that $F(Z_i)\cdot E=0$ or the process continues indefinitely.

In case the process continues indefinitely, we define a monotonic increasing sequence of neighborhoods of p as follows:

$$Y_1 = Z_1 - \sum_{k=1}^{s_1}\overline{U}_k, \quad Y_2 = Z_2 - \sum_{k=1}^{s_1}\sum_{m=1}^{s_{2k}}\overline{U}_{km},$$

and similarly we define Y_t for each positive integer t. Now let

$$U_p = \sum_{t=1}^{\infty} Y_t.$$

Since $U_p\subset Z_1+\sum_k U_k\subset S(p, \epsilon)$, then U_p is the desired neighborhood if $F(U_p)\cdot E=0$. Every point of $F(Z_t)$ not belonging to E is a point of $F(Y_t)$ and of every $F(Y_s)$ for $s\geqq t$. Then the points of $F(U_p)$ are of two classes:

(a) points which belong to $F(Y_s)$ for every s greater than some fixed integer, and (b) $\Pi_{t=1}^{\infty}(F(U_p)-F(Y_t))$. Just above it was seen that no point of the class (a) is a point of E. Now as

$$F(U_p) - F(Y_t) \subset \sum U_{k_1k_2\cdots k_t},$$

we have that every point of $F(U_p)$ of class (b) belongs to the set

$$H = \prod_{t=1}^{\infty} \sum U_{k_1k_2\cdots k_t}.$$

Now consider any point y of H. Each neighborhood U_{km} is a subset of V_k or W_k according as $k\mathcal{E}I_1$ or J_1, i.e. U_{km} is a subset of a neighborhood of the first stage whose boundary contains $\leqq n-1$ points. Similarly for the neighborhoods $U_{k_1k_2\cdots k_t}$ of any stage. Then at each stage y belongs to a neighborhood whose boundary contains $\leqq n-1$ points. And as the diameters of the neighborhoods approach zero, it follows that order$_y$ $M\leqq n-1$. Hence y non-$\mathcal{E}E$. Then U_p is the desired neighborhood of p as

$$U_p \subset S(p, \epsilon) \text{ and } F(U_p)\cdot E = 0.$$

4. This section proves corollaries to the preceding theorem.

COROLLARY 1. *For each positive integer $n\neq 2$, the set of all points of M of order n is zero-dimensional (or vacuous).*

Since a subset of a vacuous or zero-dimensional set is necessarily of the same type, for $n>2$ this follows from our theorem. That the set M^1 is of this type has been shown by Menger and Urysohn.*

COROLLARY 2.† *There exists no continuum all of whose points are of order $n\neq 2$.*

If M is a continuum, dim $M\geqq 1$. Hence $M-M^n\neq 0$ for any $n\neq 2$.

COROLLARY 3. *The simple closed curve is the only (compact) continuum all of whose points are of the same finite order.*

COROLLARY 4. *If the order of every point of the continuum M is $\leqq m$, then the points of order $\leqq \frac{1}{2}m+1$ are dense in M.*

From our theorem it follows that the set of all points p such that $\frac{1}{2}m+1<$order$_p$ $M\leqq m$ is zero-dimensional. Hence it contains no open subset and the remaining points are dense in M.

5. **Remarks.** It may be noticed in the proof of the preceding theorem

* K. Menger, Mathematische Annalen, vol. 95 (1925), p. 293; and P. Urysohn, Second reference, p. 79.

† The corollaries 2, 3 and 4 are all known results. See the papers cited in the Introduction.

that, while at each stage the neighborhoods have only finite boundaries, the final neighborhood U_p may have an infinite boundary. This raises the question as to whether it is possible to select U_p with only a finite boundary. This is not always true. If we join two of the Sierpinski triangle curves* at the vertices we have a continuum containing only points of orders 3 and 4. Now if we take $n=4$ the only neighborhoods U_p such that $F(U_p)\cdot E=0$ have infinite boundaries.

We have seen that dim $M^n=0$ for $n>2$. It would be interesting now to determine the conditions under which dim $\sum_{n>2} M^n=0$.

Urysohn has constructed very interesting examples of continua containing points of orders n and $2n-2$ only for any $n>2$. This should lead to a study of continua containing points of orders m and n only $(m>n>2)$. From our theorem or Corollary 4, it follows that such a continuum can exist only if $m\geqq 2n-2$. It would be interesting to determine whether it can exist if $m\neq k(n-1)$. Also it seems likely that in such a continuum the points of order m are countable.

III. The structure of M^2

6. Lemma. *If p is a point of M^2 and a cut point im kleinen of M and* dim$_p$ $M^2>0$, *then M contains an arc one of whose end points is p such that the arc-segment*† *is an open subset of M.*

As p is a cut point im kleinen of M, there is a neighborhood Z_p such that $\mathcal{C}(p, \overline{Z}_p)=R+S$, where R and S are continua and $R\cdot S=p$. Then either dim$_p$ $R\cdot M^2>0$ or dim$_p$ $S\cdot M^2>0$ and we suppose the former. There exists a neighborhood U_p such that if the neighborhood $V_p\subset U_p$, then

$$F(V_p)\cdot M^2\cdot R\neq 0.$$

As $p\,\mathcal{E}M^2$ there is a neighborhood W_p such that $\overline{W}_p\subset U_p$ and $F(W_p)\cdot R$ is a single point r and $R-\overline{W}_p\neq 0$. Let $G=\overline{W}_p\cdot R$. As $r\,\mathcal{E}M^2$ and $R-\overline{W}_p\neq 0$, the point $r\,\mathcal{E}G^1$, i.e. the points of G of order 1 with respect to G. As $p\,\mathcal{E}G^1$ and $M^2\cdot G=G^2+p+r$, we will complete our proof by establishing the next lemma.

7. Lemma. *If G is a continuum and* (a) *p and r are end points of G, and* (b) *for any neighborhood U_p such that r non-$\mathcal{E}\overline{U}_p$, the set $F(U_p)\cdot G^2\neq 0$, then G is an arc with end points p and r.*

Let Q denote the set consisting of $p+r+$those points of G^2 which separate p and r in G. We shall prove that Q is a closed set. Let $[q_i]$ be a sequence of

* Comptes Rendus, vol. 160 (1915), pp. 302–5; and Prace Matematyczno-Fizyczne, vol. 27 (1915), pp. 77–86.

† By the *arc-segment* is meant the arc minus its end points.

distinct points of Q such that $\lim q_i=q$. We shall show that $q\mathcal{E}Q$. This is true if $p=q$ or $r=q$. If $p\neq q\neq r$, there are two cases:

Case I. There is an infinite subsequence $[q_{i_k}]$ such that

$$G - q_{i_k} = G_{pk} + G_{rk},\ G_{pk}\cdot\overline{G}_{rk} + \overline{G}_{pk}\cdot G_{rk} = 0,$$
$$p\mathcal{E}G_{pk},\ q + q_{i_{k+1}} + r \subset G_{rk}.$$

Since $q_{i_k}\mathcal{E}G^2$, G_{pk} and G_{rk} are connected and $G_{pk}\subset G_{pk+1}$. Let

$$X_p = \sum_{k=1}^{\infty} G_{pk}.$$

The set X_p is a neighborhood of p* and q non-$\mathcal{E}X_p$. Suppose $F(X_p)\supset t\mathcal{E}G^2+r$, $t\neq q$. Let $[t_n]$ be a sequence such that $t_n\mathcal{E}G_{pn}$, $\lim t_n=t$. As $t\mathcal{E}\Pi G_{rk}$ it follows that if N is any subcontinuum of G containing t and t_n, then $q_{i_j}\mathcal{E}N$ for $j\geqq n$. Then N contains q. For this reason G is not locally connected at t. But $t\mathcal{E}G^2+r$ and a continuum is locally connected at every point of finite order. Hence t cannot exist. By condition (b), $F(X_p)\cdot G^2\neq 0$. Then $q\mathcal{E}G^2$.

Suppose there exists a neighborhood U_q of q such that both p and r belong to one component L of $G-U_q$. Then $L\supset\sum q_{i_k}$. But as $\lim q_{i_k}=q$, for k sufficiently large $q_{i_k}\subset U_q$. But this is absurd as $L\cdot U_q=0$. Hence, for any neighborhood U_q such that $p+r\subset G-U_q$, p and r belong to different components of $G-U_q$. Now let $[U_n]$ be a set of neighborhoods of q such that $\overline{U}_{n+1}\subset U_n$, $d(U_n)<1/n$, $F(U_n)$ consists of two points, $G-U_1\supset p+r$. It is easily seen that

$$G - U_n = \mathcal{C}(p, G - U_n) + \mathcal{C}(r, G - U_n);$$

and thus

$$\mathcal{C}(p, G - U_{n+1}) \supset \mathcal{C}(p, G - U_n),$$
$$\mathcal{C}(r, G - U_{n+1}) \supset \mathcal{C}(r, G - U_n).$$

Then

$$G - q = \sum_n \mathcal{C}(p, G - U_n) + \sum_n \mathcal{C}(r, G - U_n).$$

Hence $q\mathcal{E}Q$.

Case II. There is an infinite subsequence $[q_{i_k}]$ such that

$$G - q_{i_k} = G_{pk} + G_{rk},\ \overline{G}_{pk}\cdot G_{rk} + G_{pk}\cdot\overline{G}_{rk} = 0,$$
$$p + q + q_{i_{k+1}} \subset G_{pk},\ r\mathcal{E}G_{rk}.$$

Let

$$P = \sum_{k=1}^{\infty} G_{rk}.$$

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* For this lemma we consider G as our space. Relative to G alone, the set X_p is open and hence is a neighborhood.

Exactly as in Case I we may show that if $F(P)\supset t\varepsilon G^2+p$, $t\neq q$, then G is not locally connected at t. We have thus

$$(1)\qquad F(P)\cdot(G^2-q+p)=0.$$

The set $G-\overline{P}$ is a neighborhood of p, and r non-$\varepsilon F(G-\overline{P})$. We have $F(G-\overline{P})\subset F(P)$. From (1) then,

$$F(G-\overline{P})\cdot(G^2-q)=0;$$

and, from condition (b), we must have $q\varepsilon G^2$. As in Case I we may show that q separates p and r in G.

We have shown that the set Q is closed. Let H be a subcontinuum of G irreducible between p and r. Suppose there exists a point $x\varepsilon H-Q$, and let $G_x=\mathcal{C}(x, H-Q)$. Let $y\varepsilon\overline{G}_x\cdot Q$. Since $p\varepsilon G^1$, there is a neighborhood Z_p such that $F(Z_p)=u$ (a single point) and $(x+r)\cdot\overline{Z}_p=0$. The point u separates p and r, and $u\varepsilon Q$ from condition (b). Then $G_x\subset G-\overline{Z}_p$ and hence $y\neq p$. Similarly $y\neq r$. Let U_y be a neighborhood of y such that $(x+p+r)\cdot\overline{U}_y=0$. As $y\varepsilon G^2$ there is a neighborhood $V_y\subset U_y$ such that $F(V_y)=u_1+u_2$. From the fact that H is an irreducible continuum it follows that both u_1 and u_2 separate p and r in G. Then there exists a neighborhood Y_{pi} $(i=1, 2)$ of p such that $F(Y_{pi})=u_i$ and thus $u_i\varepsilon Q$ from (b). The connected set G_x contains a point in V_y and a point x non-εV_y, so either u_1 or u_2 belongs to G_x. But $G_x\cdot Q=0$ and $u_i\varepsilon Q$. This is a contradiction and hence $H-Q=0$.

From this it follows that $H=Q=G$. Then $p+r=G^1$ and $G-p-r=G^2$. By a result due to Urysohn and Menger* the set G is an arc with end points p and r.

8. LEMMA. *If $p\varepsilon M^2$ and* $\dim_p M^2>0$, *then p is a cut point im kleinen of M.*

There exists a neighborhood U_p such that if $V_p\subset U_p$ then

$$(2)\qquad F(V_p)\cdot M^2\neq 0,$$

and

$$(3)\qquad F(V_p)\supset \text{at least two points}.$$

There exists a sequence of neighborhoods $V_1, V_2, \cdots$ of p such that $V_i\subset U_p$, $\overline{V}_{i+1}\subset V_i$, $d(V_i)<1/i$, $F(V_i)=u_i+v_i$. Consider the set $\overline{V}_{i-1}-V_i$. Obviously the set $u_{i-1}+v_{i-1}+u_i+v_i=G\subset\overline{V}_{i-1}-V_i$. Let $x\varepsilon\overline{V}_{i-1}-V_i$. Then $\mathcal{C}(x, \overline{V}_{i-1}-V_i)\cdot G\neq 0$, for otherwise there would be a separation of $\overline{V}_{i-1}-V_i$ into mutually separated sets containing $\mathcal{C}(x, \overline{V}_{i-1}-V_i)$ and G respectively. But this separation would effect a separation of M, which was connected. Then $\overline{V}_{i-1}-V_i$ contains at most four components. But $\mathcal{C}(u_i, \overline{V}_{i-1}-V_i)$ must con-

* P. Urysohn, loc. cit., and K. Menger, loc. cit., p. 303.

tain u_{i-1} or v_{i-1} for otherwise $V_i+\mathcal{C}(u_i, \overline{V}_{i-1}-V_i)$ is a neighborhood $V_p \subset U_p$, but $F(V_p)=v_i$ contrary to (3). Similarly for $\mathcal{C}(v_i, \overline{V}_{i-1}-V_i)$. Also $\mathcal{C}(u_{i-1}, \overline{V}_{i-1}-V_i)\cdot(u_i+v_i)\neq 0$, for otherwise $V_{i-1}-\mathcal{C}(u_{i-1}, \overline{V}_{i-1}-V_i)$ is a neighborhood V_p and (3) is not true. Similarly for $\mathcal{C}(v_{i-1}, \overline{V}_{i-1}-V_i)$. Thus $\overline{V}_{i-1}-V_i$ has at most two components and each contains a point of u_i+v_i and a point of $u_{i-1}+v_{i-1}$. From (2) we have that $u_i \mathcal{E} M^2$ or $v_i \mathcal{E} M^2$. Further at least one of these points is a point of M^2 and is such that dim $M^2\cdot(\overline{V}_{i-1}-V_i)>0$ at the point. For suppose $u_i+v_i \subset M^2$, dim $M^2\cdot(\overline{V}_{i-1}-V_i)=0$ at both points. There exists a neighborhood $U_{u_i}\subset V_{i-1}$ such that $F(U_{u_i})\cdot M^2\cdot(\overline{V}_{i-1}-V_i)=0$, and similarly a neighborhood $U_{v_i}\subset V_{i-1}$. Then $W_p=V_i+U_{u_i}+U_{v_i}$ is a neighborhood of p such that $W_p\subset U_p$ and $F(W_p)\cdot M^2=0$ contrary to (2). In case $u_i \mathcal{E} M^2$, v_i non-$\mathcal{E} M^2$, $\dim_{u_i} M^2\cdot(\overline{V}_{i-1}-V_i)=0$, we define $W_p=V_i+U_{u_i}$. The only other possibilities are the interchange of u_i and v_i.

Case I. Suppose (a) $u_i \mathcal{E} M^2$, (b) $\dim_{u_i} M^2\cdot(\overline{V}_{i-1}-V_i)>0$, (c) v_i non-$\mathcal{E} M^2$. Since $u_i \mathcal{E} M^2$ there is a neighborhood $U_{u_i}\subset V_{i-1}-\overline{V}_{i+1}-v_i$ such that $F(U_{u_i})=x+y$. Since there is a component of $\overline{V}_i-V_{i+1}$ containing u_i and a point of $u_{i+1}+v_{i+1}$, either x or y, let us suppose y, belongs to $V_i-\overline{V}_{i+1}$. Then $x \mathcal{E} M^2$, since $\dim_{u_i} M^2\cdot(\overline{V}_{i-1}-V_i)>0$. And $\mathcal{C}(x, \overline{U}_{u_i}\cdot(\overline{V}_{i-1}-V_i))=H$ is a continuum such that $u_i+x\subset H^1$. Now if W_{u_i} is any neighborhood of u_i such that x non-$\mathcal{E} W_{u_i}$, then $W_{u_i}\cdot U_{u_i}$ is a neighborhood such that $F(W_{u_i}\cdot U_{u_i})\cdot(\overline{V}_{i-1}-V_i)\subset F(W_{u_i})\cdot H$. Then from (b) we have that $F(W_{u_i})\cdot H\cdot M^2\neq 0$. Thus by the lemma of §7 the continuum H is an arc from x to u_i. Now let N_i denote u_i plus all points of $\overline{V}_{i-1}-V_i$ that can be joined to u_i by an *open* arc-segment of $\overline{V}_{i-1}-V_i$, i.e., a point $z \mathcal{E} \overline{V}_{i-1}-V_i$ is a point of N_i if there is an arc $A\subset\overline{V}_{i-1}-V_i$ with end points z and u_i such that $M-A+u_i+z$ is closed. Evidently $H\subset N_i$. Since u_i is a point of order 1 of $\overline{V}_{i-1}-V_i$, we see that N_i is an arc or homeomorphic with an arc minus one end point. Consider the second possibility. As V_i+N_i is a neighborhood of p contained in U_p, it follows from (2) and (c) that one point of $\overline{N}_i-N_i$ is a point $q \mathcal{E} M^2$. Since M is locally connected at q we have that q is the only point of $\overline{N}_i-N_i$. Then N_i+q is an arc and q may be joined to u_i by the open arc-segment N_i-u_i. Thus $q \mathcal{E} N_i$ which is a contradiction. Hence N_i is an arc and let u_i and q be its end points. As V_i+N_i-q is a neighborhood of p, $q \mathcal{E} M^2$ from (2) and (c). If q non-$\mathcal{E} u_{i-1}+v_{i-1}$, there exists a neighborhood $U_q\subset V_{i-1}-\overline{V}_i$ such that $F(U_q)$ consists of just two points. Then just as was the case with H, we may show that $\overline{U}_q-N_i+q$ is an arc by using the lemma of §7. Then any point of this new arc belongs to N_i by definition, a contradiction.

Hence $q \mathcal{E} u_{i-1}+v_{i-1}$, say $q=u_{i-1}$. As $q \mathcal{E} M^2$ and q is a limit point of $M-\overline{V}_{i-1}$, $q=u_{i-1}$ is a point of order 1 of $\overline{V}_{i-1}-V_i$. Hence N_i is a component of $\overline{V}_{i-1}-V_i$ and $\overline{V}_{i-1}-V_i-N_i$ is closed. Then

$$\overline{V}_{i-1} - V_i = N_i + M_i,$$

where M_i is a continuum containing v_i and v_{i-1}.

Case II. Suppose (a) $u_i+v_i \subset M^2$, (b) $\dim_{u_i} M^2\cdot(\overline{V}_{i-1}-V_i)>0$, (c) $\dim_{v_i} M^2\cdot(\overline{V}_{i-1}-V_i)=0$. From (c) there exists a neighborhood U_{v_i} such that $\overline{U}_{v_i} \subset V_{i-1}-\overline{V}_{i+1}-u_i$ and $F(U_{v_i})\cdot(\overline{V}_{i-1}-V_i)\cdot M^2=0$. The proof in Case II is exactly the same as Case I except that where a neighborhood of p is formed by taking V_i plus some open set, we take $V_i+U_{v_i}$ plus the open set.

Case III. Suppose (a) $u_i+v_i \subset M^2$, (b) $\dim_{u_i} M^2\cdot(\overline{V}_{i-1}-V_i)>0$, (c) $\dim_{v_i} M^2\cdot(\overline{V}_{i-1}-V_i)>0$. Let N_{u_i} and N_{v_i} denote the sets consisting of u_i and v_i respectively together with all points of $\overline{V}_{i-1}-V_i$ that can be joined to u_i or v_i, as the case may be, by an arc of $\overline{V}_{i-1}-V_i$ such that the arc-segment is an open subset of M. The sets N_{u_i} and N_{v_i} are either arcs or homeomorphic with an arc minus one end point. Either N_{u_i} or N_{v_i} is an arc, for otherwise $V_i+N_{u_i}+N_{v_i}$ is a neighborhood of p and we have a contradiction exactly as in Case I. Also we may prove, similar to the proof of Case I, that one of these must be an arc with u_{i-1} or v_{i-1} as one end point and this arc is a component of $\overline{V}_{i-1}-V_i$ and also an open subset of it. Hence

$$\overline{V}_{i-1} - V_i = N_i + M_i, \quad N_i\cdot M_i = 0,$$

where each is a continuum joining a point of u_i+v_i to a point of $u_{i-1}+v_{i-1}$.

Thus we have seen in any case that each set $\overline{V}_{i-1}-V_i$ consists of two components N_i and M_i, and suppose the components are so lettered that $N_i\cdot N_{i+1}\neq 0 \neq M_i\cdot M_{i+1}$. Then

$$\overline{V}_1 = \left(p + \sum_{i=2}^{\infty} N_i\right) + \left(p + \sum_{i=2}^{\infty} M_i\right)$$

is the sum of two continua having only p in common. Hence p is a cut point im kleinen of M.

9. THEOREM. *If M is any continuum, then $M^2=H+K$, where* (a) *H is vacuous or* $\dim H=0$, (b) *if $p \in H$ then* $\dim_p M^2=0$, (c) *K is vacuous or a countable set of arcs A_i,* (d) *each arc-segment A_i is an open subset of M,* (e) $A_i\cdot A_j=0$, $A_i \subset A_j$, *or* $A_j \subset A_i$.

Let K denote the set of all points p such that $p \in M^2$ and $\dim_p M^2>0$. Then if $q \in M^2-K$, $\dim_q M^2=0$. Then as $H=M^2-K \subset M^2$, $\dim_q H=0$ for each $q \in H$. Now let $p \in K$. By the lemmas of §§6 and 8, there is an arc B_p one of whose end points is p such that $B_p \subset M^2$ and $<B_p>$, i.e. B_p minus its end points, is an open subset of M. And the arcs B_p may be chosen so that if x is any point of an open arc-segment of M, there is some point $p \in K$

such that $x \mathcal{E} \langle B_p \rangle$. Then the set $[\langle B_p \rangle]$ for all points $p \mathcal{E} K$ is a set of open subsets of M covering all points of open arc-segments of M. By the Lindelöf property there is a countable subset, $\langle B_1 \rangle, \langle B_2 \rangle, \cdots$, which covers the same set. Since each $B_i \subset M^2$ it follows that if the sum of any finite number of the B_i's is connected, then it is an arc. From this we find that

$$N = \sum_i B_i$$

consists of a countable number of maximal connected subsets, $N_1, N_2, \cdots$, each of which is homeomorphic with a closed, half-open, or an open interval. Take the interval $(0, 1)$ and let Φ_i be the homeomorphism which carries this interval (closed, half-open, or open) into N_i. There are three cases.

Case I. N_i is homeomorphic with the closed interval. In this case N_i is an arc and let $A_{i1} = N_i$, $A_{ij} = 0$ for $j > 1$.

Case II. N_i is homeomorphic with the half-open interval $(0, 1\rangle$. In case $\lim_{n\to\infty} \Phi_i(n/(n+1))$ exists and is a point $x_i \mathcal{E} M^2$, then $N_i + x_i$ is an arc $\subset M^2$ and we define $A_{i1} = N_i + x_i$, $A_{ij} = 0$ for $j > 1$. In case the limit does not exist or x_i non-$\mathcal{E} M^2$, we define $A_{ij} = \Phi_i(I_j)$, where I_j is the closed interval $(0, j/(j+1))$.

Case III. N_i is homeomorphic with the open interval $\langle 0, 1 \rangle$. Suppose (a) $\lim_{n\to\infty} \Phi_i(1/n)$ exists and is a point $x_i \mathcal{E} M^2$, (b) $\lim_{n\to\infty} \Phi_i(n/(n+1))$ exists and is a point $y_i \mathcal{E} M^2$. Then $N_i + x_i + y_i$ is an arc $\subset M^2$ and we define $A_{i1} = N_i + x_i + y_i$, $A_{ij} = 0$ for $j > 1$. If (a) is true and (b) false, we define $A_{ij} = x_i + \Phi_i(I_j)$, where I_j is the half-open interval $\langle 0, j/(j+1))$. If (b) is true and (a) false, we define $A_{ij} = y_i + \Phi_i(I_j)$, where I_j is the half-open interval $(1/(j+1), 1\rangle$. If both (a) and (b) are false, we define $A_{ij} = \Phi_i(I_j)$, where I_j is the closed interval $(1/(j+2), (j+1)/(j+2))$.

We shall show now that $[A_{ij}]$ is the required countable set of arcs, i.e., for every i and j, $A_{ij} \subset K$, and if $p \mathcal{E} K$, then $p \mathcal{E} A_{ij}$ for some i and j. The first part is obvious from the definition of A_{ij}. Now if $p \mathcal{E} K$, there is an arc $B_p \subset M^2$ with p as one end point such that $\langle B_p \rangle$ is an open subset of M. Now as $[B_i]$ covers all such open subsets, there exists an integer i such that $\langle B_p \rangle \subset N_i$. If $p \mathcal{E} N_i$, then $p \mathcal{E} A_{ij}$ for some j. If p non-$\mathcal{E} N_i$, since p is an end point of B_p, it follows that either $\lim_{n\to\infty} \Phi_i(1/n)$ or $\lim_{n\to\infty} \Phi_i(n/(n+1))$ exists and is the point p. In this case $p \mathcal{E} A_{i1}$.

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SOLUTION OF THE PROBLEM OF PLATEAU*

BY

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1. **Introduction.** The problem of Plateau is to prove the existence of a minimal surface bounded by a given contour. This memoir presents the first solution of this problem for the most general kind of contour: an arbitrary Jordan curve in n-dimensional euclidean space. Topological complications in the contour, as well as the dimensionality n of the containing space, are without consequence for either method or result. Naturally, an arrangement of knots in the contour will produce corresponding complications in the minimal surface, such as self-intersections and branch points.

The method used is entirely novel, representing a complete departure from the classical modes of attack hitherto employed. In this introduction we shall outline three of the classic methods (wherein n is always 3) and, fourth, the method of the present paper, which we believe to furnish the key to the problem. That this is the fact will become even clearer when, in future papers, we apply this method to the case of several contours and of various topological structures of the minimal surface,† for instance, a Möbius leaf with a prescribed boundary.

It is to be signalized that the solution here given is strictly elementary, employing only the most simple and usual parts of analysis, and that the presentation is self-sufficient, requiring no special preliminary knowledge.

(1) First to be considered, in this introductory survey, is the method based on the ideas of Riemann, Weierstrass and Schwarz.‡ Here the given

* This work, in successive stages of its development, was presented to the Society at various meetings from December, 1926, to December, 1929. Abstracts appear in the Bulletin of the American Mathematical Society as follows: vol. 33 (1927), pp. 143, 259; vol. 34 (1928), p. 405; vol. 35 (1929), p. 292; vol. 36 (1930), pp. 49–50, 189–190. Received by the editors, Parts I–IV in August and Part V in December, 1930.

Various phases of the work were also presented in the seminars of Hadamard at Paris (January 18 and December 17, 1929), Courant and Herglotz at Göttingen (June 4, 11, 18; 1929), and Blaschke at Hamburg (July 26, 1929). The second presentation in Hadamard's seminar was in the present form.

† See the abstract *A general formulation of the problem of Plateau*, Bulletin of the American Mathematical Society, vol. 36 (1930), p. 50, which gives methods adequate to solve this most general form of the problem. The cases of two contours, a Möbius leaf, and three contours have already been worked out by the author and the results presented to this Society; abstracts are in the Bulletin of the American Mathematical Society, vol. 36 (1930), p. 797, and vol. 37 (1931).

‡ Riemann, Werke, Leipzig, 1892, pp. 301–337, 445–454; Weierstrass, Werke, Berlin, 1903, vol. 3, pp. 39–52, 219–220, 221–238; Schwarz, Gesammelte Mathematische Abhandlungen, Berlin, 1890, vol. 1. See also Darboux, *Théorie des Surfaces* (2d edition, Paris, 1914), pp. 490–601.

contour is a polygon Π. The problem is made to depend on a linear differential equation of second order

$$\frac{d^2\theta}{dw^2} + p\frac{d\theta}{dw} + q\theta = 0 \tag{1.1}$$

where the coefficients p, q are rational functions of the complex variable w with, at first, undetermined coefficients. The monodromy group G of this equation (this is the group of linear transformations undergone by a fundamental set of solutions $\theta_1(w)$, $\theta_2(w)$ when w performs circuits about the singular points of the equation) is known as soon as the polygonal contour is given. The monodromy group problem of Riemann is to determine the coefficients in p, q so that (1.1) shall have the prescribed monodromy group G. But the solution of this problem is not all that is required, for that gives only a minimal surface whose polygonal boundary Π_1 has its sides parallel to those of Π. It remains further to arrange that the sides of Π_1 shall have the same lengths as those of Π. All this is reduced by Riemann and Weierstrass to a complicated system of transcendental equations in the coefficients of p, q, which they and Schwarz succeed in solving only in special cases.

To these ideas attaches the solution given by R. Garnier for the problem of Plateau. In a preliminary memoir* he first gives his form of solution of the Riemann monodromy group problem, previously solved by Hilbert, Plemelj and Birkhoff. In a following memoir† he deals with the supplementary conditions relating to lengths of sides of the polygonal boundary, and concludes the existence of a solution of the above mentioned system of transcendental equations. The Plateau problem being thus solved for a polygon, Garnier passes to the case of a more general contour Γ by regarding Γ as a limit of polygons Π. He shows that the solution of the Riemann group problem with the supplementary conditions varies continuously with the data, so that the minimal surface determined by Π approaches to a minimal surface bounded by Γ. To insure the validity of the limit process, Γ is restricted to have bounded curvature by segments.

Subsequent to the presentation by the present writer to this Society‡ of a series of papers containing the substance of the memoir at hand, T. Radó published a note§ showing how the part of Garnier's work concerned with the passage from polygons to more general contours could be materially simpli-

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* Annales Scientifiques de l'Ecole Normale Supérieure, (3), vol. 43 (1926), pp. 177–307.

† *Le problème de Plateau*, ibid., vol. 45 (1928), pp. 53–144.

‡ Annual meeting, at Bethlehem, Pa., December 27, 1929.

§ Proceedings of the National Academy of Sciences, vol. 16 (1930), pp. 242–248.

fied, and rendering the restriction on Γ less stringent by requiring only rectifiability.

*(1′) Later, after the dispatch to the editors of this journal of the manuscript of the present paper, two other papers by Radó† appeared in which he gives a solution of the Plateau problem, in the first for a rectifiable contour, and in the second for any contour capable of spanning a finite area (three-dimensional space). As its author states, this work is a continuation of the classic ideas, being based on the least-area characterization of a minimal surface and the theory of conformal mapping, especially the Koebe theory—relating to abstract Riemann manifolds—in the form of the theorem that it is possible to map any simply-connected polyhedral surface conformally on the interior of a circle, the map remaining one-one and continuous as between the boundary of the polyhedral surface and the circumference of the circle.

The present work, on the other hand, apart from its advantage of complete generality of the contour, breaks completely with the hitherto classic methods, replacing the area functional by an entirely new and much simpler functional, and carrying through the existence proof without assuming any of the theory of conformal mapping even for ordinary plane regions; on the contrary, in Part IV our results are applied to give new proofs of the classic theorems concerning conformal mapping essentially simpler than the classic proofs, a demonstration of the superior fundamental character of the present mode of attack.

In Part V, after the existence of the minimal surface has been established, the Koebe mapping theorem plays a rôle beside the formulas of Part III in a brief proof of the least-area property. However, we regard this treatment only as a stop-gap, having under development a disposal of the least-area part of the problem not using the Koebe or any other conformal mapping theorem. Such an independent treatment is desirable because the Koebe theory of conformal mapping is comparable in difficulty with the Plateau problem. The avoidance of the former will bring the solution of the least area problem to rest directly on the axioms of analysis, as has already been done in this paper with the proof of the existence of the minimal surface.

(2) A second class of methods is based on the partial differential equation of minimal surfaces (Lagrange)

$$(1.2)\qquad (1+q^2)r - 2pqs + (1+p^2)t = 0.$$

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* Article (1′) added in proof.

† Annals of Mathematics, (2), vol. 31 (1930), pp. 457–469. Mathematische Zeitschrift, vol. 32 (1930), pp. 762–796.

Here belongs the work of S. Bernstein* and Ch. H. Müntz.† The surface is assumed in the restricted form $z=f(x, y)$, and the problem is regarded as a generalized Dirichlet problem, with (1.2) replacing Laplace's equation. Besides the restriction on the representation of the surface, it is assumed that the contour has a convex projection on the xy-plane. The work of Müntz has been criticized by Radó.‡

(3) Minimal surfaces first presented themselves in mathematics, and were named, by their property of having least area among all surfaces bounded by a given contour:

$$\iint (1 + p^2 + q^2)^{1/2} dx dy = \text{minimum}. \tag{1.3}$$

It is in this way that minimal surfaces appear in the pioneer memoir of Lagrange§ on the calculus of variations for double integrals.

In recent years A. Haar‖ has treated the Plateau problem from this point of view, using the direct methods of the calculus of variations introduced by Hilbert. Haar assumes the surface in the form $z=f(x, y)$ and the contour subject to the following restriction: any plane containing three points of the contour has a slope with respect to the xy-plane that is less than a fixed finite upper bound, an assumption occurring first in the work of Lebesgue.¶

(4) The method of the present memoir is as follows. The contour Γ being taken as any Jordan curve in euclidean space of n dimensions, we consider the class of all possible ways of representing Γ as topological image of the unit circle C:

$$x_i = g_i(\theta) \qquad (i = 1, 2, \cdots, n).$$

This class forms an L-set in the sense of Fréchet's thesis,** and is compact and closed after "improper" topological representations of Γ have been adjoined: these are limits of proper ones and cause arcs of Γ to correspond to single points of C, or vice-versa (§3). The principal idea is then to introduce the functional (§5)

* Mathematische Annalen, vol. 69 (1910), pp. 82–136, especially § 18.

† Mathematische Annalen, vol. 94 (1925), pp. 53–96.

‡ Mathematische Annalen, vol. 96 (1927), pp. 587–596.

§ Miscellanea Taurinensia, vol. 2 (1760–61); also Oeuvres, vol. 1, p. 335.

‖ *Ueber das Plateausche Problem*, Mathematische Annalen, vol. 97 (1927), pp. 124–158.

¶ *Intégrale, longueur, aire*, Annali di Matematica, (3), vol. 7, pp. 231–359; see chapter VI, especially p. 348.

** *Sur quelques points du calcul fonctionnel*, Rendiconti del Circolo Matematico di Palermo, vol. 22 (1906), pp. 1–74.

$$(1.4)\qquad A(g) = \frac{1}{4\pi}\int_C\int_C \frac{\sum_{i=1}^{n}[g_i(\theta) - g_i(\phi)]^2}{4\sin^2\dfrac{\theta-\phi}{2}}d\theta d\phi,$$

where the integrand has the following simple geometric interpretation: square of chord of contour divided by square of corresponding chord of the unit circle. This improper double integral has a determinate positive value, finite or $+\infty$, for every representation g. It is readily shown (§9) that $A(g)$ is lower semi-continuous; therefore, by a theorem of Fréchet† to the effect that a lower semi-continuous functional on a compact closed set attains its minimum value, the minimum of $A(g)$ is attained for a certain representation $x_i = g_i^*(\theta)$. If

$$(1.4_1)\qquad x_i = H_i(u, v) = \Re F_i(w) \qquad (w = u + iv)$$

are the harmonic functions in the interior of the unit circle determined according to Poisson's integral by the boundary functions

$$(1.4_2)\qquad x_i = g_i^*(\theta)$$

it is then proved (§§11–16) that

$$(1.4_3)\qquad \sum_{i=1}^{n} F_i'^2(w) = 0;$$

briefly speaking, this condition expresses that the first variation of $A(g)$ vanishes for the minimizing $g = g^*$. According to the formulas of Weierstrass, the condition (1.4_3) expresses that (1.4_1) defines a minimal surface. After it has been shown (§§17, 18) that g^* is a proper representation of Γ, it follows that this minimal surface is bounded by Γ, since by the properties of Poisson's integral the functions (1.4_1) then attach continuously to the boundary values (1.4_2).

One consideration is necessary to validate the preceding argument: we must be sure that $A(g)$ is not identically $= +\infty$, that it takes a finite value for some g. This is what makes it necessary to divide the discussion into two parts. In Part I we assume that there exists a parametric representation g of the given contour such that $A(g)$ is finite. It will be seen from Parts III and V that this means, more concretely, that it is possible to span some surface of finite area in the given contour. A sufficient condition for the property "there exists a g for which $A(g)$ is finite" (which, in anticipation of the discussion of Parts III and V, we will call the finite-area-spanning property) is that

† Loc. cit., §11, p. 9.

the contour be rectifiable. For if the contour have length L, and we choose as parameter $\theta=2\pi s/L$, s being arc-length reckoned from any fixed initial point, then it will be readily seen from the fact that a chord is not greater than its arc that the integrand in $A(g)$ stays bounded, hence $A(g)$ is finite for this parameter. In particular, every polygon has the finite-area-spanning property. That, however, a finite-area-spanning contour is superior in generality to a rectifiable contour may be seen by taking any simply-connected portion of a surface, having finite area, and drawing upon it any non-rectifiable Jordan curve (e.g., a non-rectifiable Jordan curve on a sphere).

Part II deals with the case of an arbitrary Jordan contour, where generally $A(g)\equiv+\infty$, meaning that no finite area whatever can be spanned in the given contour; an example of such a contour is given in §27. The existence theorem is extended to a contour of this type by an easy limit process, wherein the given contour is regarded as a limit of polygons. In this case the minimal surface can be defined only by the Weierstrass equations, the least-area characterization becoming meaningless.†

The distinctive feature of the present work is the determination of the minimal surface by the minimizing of the functional $A(g)$, decidedly simpler of treatment than the classic area functional. $A(g)$ has a simple relation to the area functional, dealt with in Part III. If $S(g)$ denote the area of the surface $x_i=H_i(u, v)$, these being the harmonic functions in $u^2+v^2<1$ determined by the boundary functions $x_i=g_i(\theta)$, then $A(g)\geqq S(g)$, and the relation of equality holds when and only when the surface is minimal. Thus $A(g)$, not equal to area in general, is capable of giving information about area in the case of a minimal surface. Part III, moreover, provides the basis for the easy proof in Part V that the minimal surface whose existence is proved in Parts I and II has the least area of any surface bounded by Γ.

An interesting and important consideration, unremarked before the writer's work, is that *the Riemann conformal mapping problem is included as the special case $n=2$ in the problem of Plateau.* The Riemann mapping theorem relating to the interiors of two Jordan regions is supplemented by the theorem of Osgood‡ and Carathéodory§ to the effect that the conformal correspondence between the interiors induces a topological correspondence between the boundaries. In Part IV a proof is given of the combined theorems of Riemann and Osgood-Carathéodory, independent of any previous treatment, and more elementary and perspicuous.

† But see the footnote at the end of this paper.

‡ Osgood and E. H. Taylor, *Conformal transformations on the boundaries of their regions of definition*, these Transactions, vol. 14 (1913), pp. 277–298.

§ Carathéodory, *Ueber die gegenseitige Beziehung der Ränder bei der konformen Abbildung des Innern einer Jordanschen Kurve auf einen Kreis*, Mathematische Annalen, vol. 73 (1913), pp. 305–320.

Examples show that the solution of the Plateau problem may not be unique.† The question of the degree of multiplicity of the solution is not dealt with here. As indicated above, the minimal surface whose existence is here assured is the one which furnishes an absolute minimum for the area.

2. **Formulation.** For definition of a minimal surface we adopt the formulas given, for $n=3$, by Weierstrass:

$$x_i = \Re F_i(w) \qquad (i = 1, 2, \cdots, n) \tag{2.1}$$

with

$$\sum_{i=1}^{n} F_i'^2(w) = 0. \tag{2.2}$$

The problem of Plateau may then be formulated precisely as follows.

Given any contour Γ in the form of a Jordan curve in euclidean space of n dimensions. To prove the existence of n functions $F_1, F_2, \cdots, F_n$ of the complex variable w, holomorphic in the interior of the unit circle C, satisfying there the condition

$$\sum_{i=1}^{n} F_i'^2(w) = 0$$

identically, and whose real parts

$$x_i = \Re F_i(w)$$

attach continuously to boundary values on C

$$x_i = g_i(\theta)$$

which represent Γ as a topological image of C.

As defined by (2.1), (2.2), the minimal surface appears in a representation on the circular region $|w|<1$ which is conformal except at those (necessarily isolated) points where simultaneously

$$F_1'(w) = 0, \quad F_2'(w) = 0, \quad \cdots, \quad F_n'(w) = 0.$$

† The following example was communicated to the writer by N. Wiener. Two co-axial circles may be so placed that the area of the catenoidal segment determined by them is greater than the sum of the areas of the two circles (Goldschmidt discontinuous solution). Consider the contour formed by two meridians of the catenoid, very close together, and the arcs remaining on the circles after the small arcs intercepted between the meridians have been removed. One solution of the Plateau problem for this contour is the intercepted part of the catenoid. But the surface formed of the two circles and the narrow catenoidal strip between the meridians has a smaller area. Consequently, there will be a second minimal surface bounded by the given contour, varying slightly from the surface just described; this second surface will have the absolutely least area.

On this remark is based the inclusion of the Riemann mapping problem in the Plateau problem as the special case $n=2$. We show that for $n=2$ the conformality is free of singular points, but for $n>2$ their absence cannot be guaranteed.†

I. A FINITE-AREA-SPANNING CONTOUR

Hypothesis. Part I is based on the hypothesis that there exists a parametric representation g of the given contour for which $A(g)$ is finite.

3. Topological correspondences between Γ and C. Γ may be supposed

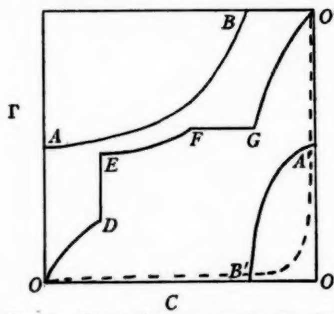

FIG. 1. Torus of representation $R=\Gamma C$

given in some initial representation $x_i=f_i(t)$, from which its most general representation may be derived by a relation $t=t(\theta)$ defining a one-one continuous transformation of C into itself. The two-dimensional manifold (t, θ) of pairs of points one on Γ, one on C, forms a torus ΓC, which will be called the *torus of representation* and denoted by R. This torus is depicted in the annexed figure as a square where points opposite one another on parallel sides, such as A and A', B and B', are to be regarded as identical. Rectilinear transversals of the square parallel to C will be termed *parallels*, those parallel to Γ *meridians*. A topological correspondence between Γ and C is represented by a continuous closed curve, such as $ABB'A'$, which is intersected in one and only one point by each parallel and by each meridian; such a curve may be described as *cyclically monotonic*. We will denote by $\mathfrak{P}$ the totality of these curves, which, we will say, represent *proper* topological correspondences between Γ and C. In the corresponding equations $x_i=g_i(\theta)$ of Γ the functions g_i are continuous, and are not all constant on any arc of C.

† Example: $x_1=\Re w^2$, $x_2=\Re -iw^2$, $x_3=\Re w^3$, $x_4=\Re -iw^3$; $|w|\leqq 1$. This is a minimal surface bounded by the contour $x_1=\cos 2\theta$, $x_2=\sin 2\theta$, $x_3=\cos 3\theta$, $x_4=\sin 3\theta$. Neither the minimal surface nor the contour is self-intersecting. The representation on $|w|<1$ is conformal except at the origin, where angles are multiplied by 2.

A disadvantage in dealing with $\mathfrak{P}$ is that it is not a closed set: a sequence of curves of $\mathfrak{P}$ may converge to a limit not belonging to $\mathfrak{P}$. For instance, we may obtain as limit of curves of $\mathfrak{P}$ a curve such as $ODEFGO''$, containing a segment of meridian DE or a segment of parallel FG, as well as properly monotonic arcs. An extreme case is that indicated by the dotted curve, where the limit is $OO'O''$, consisting of a parallel together with a meridian.

A correspondence between Γ and C whose graph contains, besides properly monotonic arcs, a meridian-segment less than an entire meridian, or a parallel-segment less than an entire parallel, will be called an *improper* topological correspondence.

In the graph $ODEFGO''$, the meridian-segment DE represents an arc $Q'Q''$ of Γ which corresponds to a single point P of C. In the corresponding equations of Γ, the functions g_i have one-sided limits at P equal respectively to the coördinates of Q' and Q''; and at least one of these functions is discontinuous (the vector g is discontinuous) since, Γ having by hypothesis no double points, Q' and Q'' are distinct. A monotonic function has at most a denumerable infinity of discontinuities, each in the form of distinct one-sided limits; therefore in an improper representation the functions g_i have at most a denumerable infinity of discontinuities, all of the so-called first kind, that is, where one-sided limits exist but are unequal. This observation will assure us in §5 of the existence for an improper representation of the Riemann integrals used in defining $A(g)$.

The parallel-segment FG represents an arc $P'P''$ of C which corresponds to a single point Q of Γ. The functions g_i are constant on the arc $P'P''$, where they are equal to the coördinates of Q.

The class of all improper representations of Γ will be denoted by $\mathfrak{J}$, and will be divided according to the above description into the two sub-classes $\mathfrak{J}_1$ and $\mathfrak{J}_2$, *not mutually exclusive*:

$\mathfrak{J}_1$, *improper representations of the first kind*, in which an arc of Γ less than all of Γ corresponds to a single point of C.

$\mathfrak{J}_2$, *improper representations of the second kind*, in which an arc of C less than all of C corresponds to a single point of Γ.

Special attention must now be given to the correspondences between Γ and C whose graph consists of a parallel together with a meridian, such as $OO'O''$. Here the whole of Γ corresponds to a single point of C, and the whole of C to a single point of Γ. Such a representation will be termed *degenerate*; there are evidently ∞^2 degenerate representations, obtained by varying the distinguished points on Γ and C. In the corresponding equations of Γ the functions g_i reduce to constants. *The functional $A(g)$ will not be defined for the degenerate representations.*

Three fixed points. After having established in §6 a certain invariance property of $A(g)$, we shall be led to consider the class of those proper or improper representations of Γ wherein three distinct fixed points P_1, P_2, P_3, of C, correspond to three distinct fixed points Q_1, Q_2, Q_3, of Γ. These representations are pictured on the torus $R=\Gamma C$ by proper or improper cyclically monotonic curves passing through three fixed points no two of which lie on the same parallel or the same meridian.

The preceding discussion leads us to distinguish the following classes of representations of Γ on C.

(1) The class of all representations: proper, improper, and degenerate,

$$\mathfrak{R} = \mathfrak{P} + \mathfrak{J} + \mathfrak{D}.$$

(2) The class of all proper and improper representations:

$$\mathfrak{M} = \mathfrak{P} + \mathfrak{J}.$$

This is not a closed set, since a sequence of representations of $\mathfrak{M}$ may tend to a degenerate representation as limit. $\mathfrak{M}$ will serve as the range of the argument in the functional $A(g)$.

(3) The class of all proper and improper representations whereby three distinct fixed points of Γ correspond to three distinct fixed points of C:

$$\mathfrak{M}' = \mathfrak{P}' + \mathfrak{J}'.$$

It is important to observe the following two properties of $\mathfrak{M}'$: it is closed; it does not contain any degenerate representation.

4. **Harmonic surfaces.** Each representation $x_i=g_i(\theta)$ of Γ determines a surface $x_i=\Re F_i(w)$, where the harmonic functions $\Re F_i(w)$ are those defined by Poisson's integral based on the respective boundary functions $g_i(\theta)$. We will refer to this surface as *the harmonic surface determined by the representation g.*

The limit of Poisson's integral when w approaches to a point θ of C where $g_i(\theta)$ is continuous is $g_i(\theta)$. If $g_i(\theta)$ has unequal one-sided limits at the point θ, then the limiting value of Poisson's integral in the approach of w to θ varies between these one-sided limits in a manner that depends linearly on the angle made by the direction of approach with the radius to the point θ.†

It follows that the harmonic surface determined by any proper representation of Γ is bounded by Γ. For an improper representation of the first kind, where the point P of C corresponds to the arc $Q'Q''$ of Γ, it is evident that the boundary points of the harmonic surface obtained by allowing w to approach

† A result due to Schwarz; cf. Picard, *Traité d'Analyse*, vol. 1 (3d edition, Paris, 1922), pp. 315–319.

to P in all the possible directions form the *chord* $Q'Q''$. In an improper representation of the second kind, the point w approaching to any point of an arc $P'P''$ of C gives the same boundary point Q for the harmonic surface.

It is to be observed from this that in the case of an improper representation of the first kind the corresponding harmonic surface will not be bounded by Γ, but by a curve derived from Γ by replacing certain of its arcs (at most a denumerable infinity) by their chords, which chords will correspond to single points of C. In the case of an improper representation of the second kind, the boundary point corresponding to each arc $P'P''$ lies on Γ, but then Γ is not in one-one relation with C.

Example 1. The graph of the correspondence between Γ and C may consist of k parallel-segments alternating with k meridian-segments. The boundary of the corresponding harmonic surface is a polygon of k sides and k vertices inscribed in Γ. The sides of the polygon correspond respectively to k points of C, and the vertices to the k arcs into which these points divide C. If $k=2$, the harmonic surface reduces to a chord of Γ. If $k=1$, the case of a degenerate representation, the harmonic surface reduces to a point of Γ.

Example 2. The correspondence $t=t(\theta)$ between Γ and C may be defined by the frequently cited monotonic function based on Cantor's perfect set.† Here the boundary of the harmonic surface consists of a denumerable infinity of chords of Γ together with the nowhere-dense perfect set of points of Γ which remain after the arcs of these chords have been removed. On C we have an everywhere-dense denumerable infinity of points of discontinuity of $x_i=g_i(\theta)$, corresponding respectively to the above-mentioned chords of Γ.

It will be seen from these examples that the harmonic surface determined by a given representation of Γ cannot be regarded as bounded by Γ unless this representation is proper. It is for this reason that after establishing the existence of a representation $x_i=g_i^*(\theta)$ such that the corresponding harmonic surface obeys the condition $\sum_{i=1}^n F_i'^2(w)=0$, it is necessary (as is done in §§17, 18) to prove that the representation g^* is proper before we can say we have a minimal surface bounded by Γ.

5. **The fundamental functional** $A(g)$. The functional $A(g)$ is defined on the set $\mathfrak{M}=\mathfrak{P}+\mathfrak{J}$ of all proper and improper representations of Γ by the formula

$$A(g) = \frac{1}{16\pi}\int_C \int_C \frac{\sum_{i=1}^n [g_i(\theta) - g_i(\phi)]^2}{\sin^2 \dfrac{\theta-\phi}{2}} d\theta d\phi. \tag{5.1}$$

† See Carathéodory, *Vorlesungen über reelle Funktionen*, Leipzig, 1918, §156, p. 159.

The domain of integration CC is a torus, which will be denoted by T. This *torus of integration* is to be carefully distinguished from the *torus of representation* R of §3.

The integrand is defined everywhere on T except on the *diagonal* $\theta=\phi$, where it takes the indeterminate form 0/0. Let us isolate the diagonal from the rest of the torus by means of two regular curves† symmetrically disposed

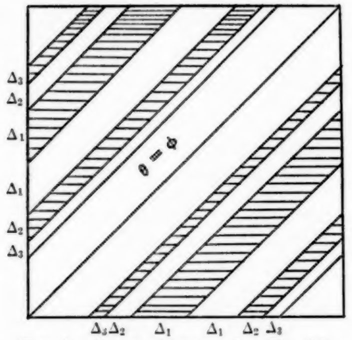

FIG. 2. Torus of integration $T=CC$

on either side of it, enclosing a region τ_1 which we delete from the torus, leaving $T_1=T-\tau_1$.

In case g is proper, or improper only of the second kind, the integrand of (5.1) is defined and continuous on T_1.

If g is improper of the first kind, the integrand is discontinuous at the points of certain parallels and meridians, symmetrically disposed with respect to the diagonal, and at most denumerably infinite in number. At a point belonging to a parallel but not to a meridian of discontinuity (or vice-versa), the discontinuity is in the form of distinct limits‡ according as the point is approached from one side or the other of the parallel (or meridian). At a point of intersection of a parallel of discontinuity with a meridian of discontinuity, there are four distinct limits‡ according to the quadrant within which the point is approached.

In any event, the discontinuities of the integrand in T_1 form at most a set of zero measure.

Let d denote the diameter (greatest chord) of Γ, and $\delta>0$ the smallest value of $|\theta-\phi|$ in T_1; then the integrand of (5.1) is bounded on T_1, being

† The only curves which will come into consideration in this connection will be straight lines parallel to the diagonal and images of them by the regular analytic transformations (6.1), (12.1).

‡ Some of these may accidentally be equal.

$$\leq \frac{d^2}{\sin^2 \dfrac{\delta}{2}}\,.$$

These two remarks insure the existence of the Riemann integral taken over T_1 of the integrand of (5.1).

Imagine now an infinite sequence of regions

$$\tau_1, \tau_2, \cdots, \tau_r, \cdots$$

each contained in the preceding and shrinking to the diagonal as limit, so that the complementary regions

$$T_1, T_2, \cdots, T_r, \cdots$$

swell continually and tend to the entire torus T as limit. Then, because every element of the integral (5.1) is positive (wide sense) the proper Riemann integrals

$$\iint_{T_1}, \iint_{T_2}, \cdots, \iint_{T_r}, \cdots$$

form a continually increasing† sequence of positive† numbers. Hence they approach either to a finite positive limit or to $+\infty$; and *this limit is by definition* $A(g)$, which thus appears as an improper integral. The same fact of the positivity of each element proves easily that the value obtained for $A(g)$ is independent of the particular sequence of regions τ used in its definition; in fact, $A(g)$ may be defined uniquely as the upper bound of the integral over any region of T to which the diagonal is exterior.

$A(g)$ as an infinite series. For greater definiteness in determining $A(g)$, we proceed to divide the torus T into an infinite number of strips (Fig. 2) by means of the lines

$$|\theta - \phi| = \frac{\pi}{r} \qquad (r = 1, 2, 3, \cdots).‡$$

The region defined by the inequality

$$\frac{\pi}{r+1} \leq |\theta - \phi| \leq \frac{\pi}{r}, \tag{5.2}$$

† That "increasing" and "positive" may here be taken in the strict sense follows by the same proof given a little later on to show that $\Delta_r(g)$ is strictly positive. The only assumption to be made is that g is not a degenerate representation.

‡ By $|\theta-\phi|$ we shall understand the minor arc intercepted between the points θ and ϕ on the unit circle C.

consisting of a pair of strips symmetric with respect to the diagonal, will be denoted by Δ_r. We then define the functional

$$(5.3) \qquad \Delta_r(g) = \frac{1}{16\pi}\iint_{\Delta_r} \frac{\sum_{i=1}^{n}[g_i(\theta) - g_i(\phi)]^2}{\sin^2 \dfrac{\theta - \phi}{2}} d\theta d\phi.$$

This is a proper Riemann integral since the integrand stays bounded on Δ_r, being

$$\leqq \frac{d^2}{\sin^2 \dfrac{\pi}{2(r+1)}}.$$

[Certainly $\Delta_r(g) \geqq 0$; but it is interesting (though not necessary for the sequel) to prove the strict inequality $\Delta_r(g) > 0$.

First we see that we could have $\Delta_r(g) = 0$ only by having $\sum_{i=1}^{n}[g_i(\theta) - g_i(\phi)]^2$ identically zero in Δ_r. For if $\sum_{i=1}^{n}[g_i(\theta) - g_i(\phi)]^2 = p > 0$ at some interior point† of Δ_r, it would be $> p/2$ in at least a sufficiently small square in the corner of one of the quadrants about this point; therefore this square makes a contribution >0 to the value of the integral, which cannot be neutralized by the non-negative contribution of the other elements; hence $\Delta_r(g) > 0$, contrary to the hypothesis $\Delta_r(g) = 0$.

Thus $\Delta_r(g) = 0$ implies $g_i(\theta) = g_i(\phi)$ $(i = 1, 2, \cdots, n)$ for all (θ, ϕ) obeying (5.2). Now it is easy to see that, for every ϕ_1, ϕ_2 such that

$$(5.4) \qquad |\phi_1 - \phi_2| \leqq \frac{\pi}{r} - \frac{\pi}{r+1},$$

a θ exists such that (θ, ϕ_1) and (θ, ϕ_2) obey (5.2). On account of the transitivity of the relation $g_i(\theta) = g_i(\phi)$, it follows that $\Delta_r(g) = 0$ implies $g_i(\phi_1) = g_i(\phi_2)$ for all (ϕ_1, ϕ_2) obeying (5.4).

Now any two points whatever θ, ϕ of C can be made the first and last of a finite sequence $\theta, \phi_1, \phi_2, \cdots, \phi_m, \phi$ any two consecutive terms of which obey (5.4). Consequently $\Delta_r(g) = 0$ implies $g_i(\theta) = g_i(\phi)$ for all θ, ϕ; but then g would be a degenerate representation; however, such are excluded from the range of the argument in $A(g)$.]

The functional $A(g)$ may now be defined as the finite or positively infinite sum of the infinite series of positive terms

† At a point of discontinuity this will mean that one of the two or four limiting values is equal to p.

$$A(g) = \Delta_1(g) + \Delta_2(g) + \cdots + \Delta_r(g) + \cdots . \tag{5.5}$$

6. **The invariance of $A(g)$.** This section is devoted to proving the important observation that $A(g)$ is invariant under the transformation

$$\tan\frac{\bar\theta}{2} = \frac{a\tan\dfrac{\theta}{2} + b}{c\tan\dfrac{\theta}{2} + d} \qquad (a, b, c, d \text{ real};\ ad - bc \neq 0) \tag{6.1}$$

of the circumference of the unit circle into itself.

In the expression (5.1) for $A(g)$ make the substitution

$$x = \tan\frac{\theta}{2}, \quad y = \tan\frac{\phi}{2}\,.\dagger \tag{6.2}$$

We have

$$\frac{d\theta d\phi}{\sin^2\dfrac{\theta-\phi}{2}} = \frac{d\theta d\phi}{\left(\sin\dfrac{\theta}{2}\cos\dfrac{\phi}{2} - \cos\dfrac{\theta}{2}\sin\dfrac{\phi}{2}\right)^2}$$

$$= \frac{\sec^2\dfrac{\theta}{2}d\theta\cdot\sec^2\dfrac{\phi}{2}d\phi}{\left(\tan\dfrac{\theta}{2} - \tan\dfrac{\phi}{2}\right)^2} = \frac{4dxdy}{(x-y)^2}\,.$$

When θ and ϕ vary independently over C from $-\pi$ to $+\pi$, x and y vary independently from $-\infty$ to $+\infty$; so that, denoting by $h_i(x)$, $h_i(y)$ the functions which result from $g_i(\theta)$, $g_i(\phi)$ by the substitutions inverse to (6.2):

$$h_i(x) = g_i(2\arctan x), \quad h_i(y) = g_i(2\arctan y),$$

we have as transformed expression for $A(g)$:

$$A(g) = \frac{1}{4\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\sum_{i=1}^{n}[h_i(x) - h_i(y)]^2\frac{dxdy}{(x-y)^2}\,. \tag{6.3}$$

In terms of the variables (6.2), the transformation (6.1) is

$$\bar x = \frac{ax+b}{cx+d}, \quad \bar y = \frac{ay+b}{cy+d}\,. \tag{6.4}$$

† This may be interpreted as replacing the unit circle by a half-plane, its circumference by the edge of the half-plane.

Let $\bar{g}_i(\bar{\theta})$, $\bar{h}_i(\bar{x})$ be the functions that result from $g_i(\theta)$, $h_i(x)$ by the respective transformations (6.1), (6.4); then

$$A(\bar{g}) = \frac{1}{4\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\sum_{i=1}^{n}[\bar{h}_i(\bar{x}) - \bar{h}_i(\bar{y})]^2\frac{d\bar{x}d\bar{y}}{(\bar{x}-\bar{y})^2}. \tag{6.5}$$

The domain of integration in (6.3) is the entire real plane (x, y), and the entire real plane $(\bar{x}, \bar{y})$ is the domain of integration in (6.5), because (6.4) sets up a one-one correspondence between all finite and infinite real values of x and of $\bar{x}$, and the same for y and $\bar{y}$.

The following simple calculations now lead to the desired result:

$$d\bar{x} = \frac{ad-bc}{(cx+d)^2}dx, \quad d\bar{y} = \frac{ad-bc}{(cy+d)^2}dy;$$

$$\bar{x}-\bar{y} = \frac{(ad-bc)(x-y)}{(cx+d)(cy+d)};$$

$$\frac{d\bar{x}d\bar{y}}{(\bar{x}-\bar{y})^2} = \frac{dxdy}{(x-y)^2}.$$

Therefore, by comparison of (6.5) with (6.3),

$$A(\bar{g}) = A(g).$$

Equivalent representations. Three fixed points. Let us designate as *equivalent* any two representations g, $\bar{g}$ which are related to one another by a transformation (6.1). On account of the presence of three essential parameters in the transformation, it is possible to find for every non-degenerate representation g an equivalent $\bar{g}$ which causes any chosen three distinct points of C to correspond to any chosen three distinct points of Γ. In the notation of §3: every element of $\mathfrak{M}$ has an equivalent in $\mathfrak{M}'$.

Since $A(g)$ has the same value for two equivalent g's, it follows that the lower bound M of $A(g)$ on $\mathfrak{M}$ is equal to the lower bound M' of $A(g)$ on $\mathfrak{M}'$; and if we can prove that M' is attained for a certain $\bar{g}$, then obviously M is attained for every g equivalent to $\bar{g}$. As already pointed out, the advantage of referring from $\mathfrak{M}$ to $\mathfrak{M}'$ is that the latter is a closed set not containing any degenerate representation, while the former is an open set, having the degenerate representations among its limit elements.

7. **Fréchet's thesis; compact closed sets.** We now have the problem of proving that $A(g)$ attains its minimum on $\mathfrak{M}'$. This will be accomplished by means of the simple and general ideas of Fréchet's thesis,† which concerns real-valued functions on sets of a very general nature.

† For reference, see the Introduction.

Suppose we have a set $\mathfrak{A}$ of elements a of any nature. Fréchet terms this an L-set under the following conditions. Every infinite sequence

$$a_1, a_2, \cdots, a_m, \cdots$$

of elements of $\mathfrak{A}$ is definitely designated as *convergent* or *divergent*, and with each convergent sequence there is associated a unique element of $\mathfrak{A}$ called the *limit* of the sequence. Every sub-sequence of a convergent sequence must itself be convergent and have the same limit as the original sequence. Every infinite sequence all of whose elements are identical with the same element of $\mathfrak{A}$ is convergent and has this element for limit.

An L-set is termed *compact* if it obeys the Bolzano-Weierstrass theorem: every infinite sub-set of the given set contains a convergent sequence of distinct elements, or has a limit element.

An L-set is termed *closed* provided every limit of an infinite sequence of elements of the set belongs to the set. Evidently the notion of closure has meaning only when the given set is considered as part of a larger set.

An L-set both compact and closed is termed by Fréchet an *extremal* set, because the fundamental theorem of Weierstrass that a continuous function on a closed interval in the ordinary real domain attains its extreme values, maximum and minimum, applies to such a set. This is easily proved in the cited thesis after the following definition of continuous function on an L-set has been given.

A real-valued function $U(a)$ of the elements of a set $\mathfrak{A}$ is termed *continuous* if whenever the sequence of elements

$$a_1, a_2, \cdots, a_m, \cdots$$

converges to the element a as limit, then the sequence of functional values

$$U(a_1), U(a_2), \cdots, U(a_m), \cdots$$

converges in the ordinary sense to the value $U(a)$. This means, of course, that for every $\epsilon > 0$ there is an index m_ϵ such that, for every $m > m_\epsilon$,

$$U(a) - \epsilon < U(a_m) < U(a) + \epsilon.$$

If we require only the first of these two inequalities, we have the notion of lower semi-continuity: a function $U(a)$ is *lower semi-continuous* if, with the conventions of the preceding paragraph, we have

$$U(a_m) > U(a) - \epsilon \tag{7.1}$$

for $m > m_\epsilon$.

An alternative statement is:

(7.2) if $$a_1, a_2, \cdots, a_m, \cdots \to a,$$
and $$U(a_1), U(a_2), \cdots, U(a_m), \cdots \to L,$$
then $$U(a) \leqq L,$$

a condition also expressed in the form

$$U(a) \leqq \liminf_{m\to\infty} U(a_m).$$

It is shown by Fréchet† that a lower semi-continuous function on a compact closed L-set attains its minimum value. Our proof that $A(g)$ attains its minimum will be a particular application of this general theorem. However, for the sake of completeness, we will not assume this easily proved theorem, but shall establish it in §10 with the actual set $\mathfrak{M}'$ and functional $A(g)$ here under consideration.

8. **The topological correspondences between Γ and C as an L-set.** With a natural definition of limit, the set $\mathfrak{R}$ of all (proper, improper, and degenerate) representations of Γ as topological image of C is an L-set, as are also its sub-sets $\mathfrak{M}$ and $\mathfrak{M}'$. We will say, namely, that a sequence of representations

$$g^{(1)}, g^{(2)}, \cdots, g^{(m)}, \cdots$$

converges to a certain representation g as limit when the graphs of $g^{(1)}$, $g^{(2)}, \cdots, g^{(m)}, \cdots$ on the torus $R=\Gamma C$ (Fig. 1) converge in the ordinary sense to the graph of g; this will mean that if R_ϵ denote the region covered by a circle of radius ϵ whose center describes the graph of g, then for all sufficiently large values of m the graph of $g^{(m)}$ lies within R_ϵ.

The L-set $\mathfrak{R}$ (and, automatically, its sub-sets $\mathfrak{M}$ and $\mathfrak{M}'$) has the important property of *compactness*, the proof of which results directly from the following theorem:‡

An infinite set of curves contained in a finite domain is compact if the curves are rectifiable and their lengths less than a fixed finite upper bound.

The torus R which contains all the graphs of $\mathfrak{R}$ is a finite domain.

Suppose a rectilinear polygon inscribed in the graph of any representation g. Then, resolving each side into its projections along a parallel and along a meridian, and adding, we obtain

length of inscribed polygon $\leqq$ length of parallel + length of meridian,

where the cyclically monotonic character of the graph insures that each projection is counted once and only once. It follows from this inequality that each

† Loc. cit., §11, p. 9.

‡ Fréchet, loc. cit., p. 65.

curve of $\mathfrak{R}$ is rectifiable and has a length $\leqq$ the finite upper bound: length of parallel + length of meridian.

Thus $\mathfrak{R}$ obeys all the conditions of the above theorem, and is compact.

$\mathfrak{M}'$ is an extremal set: compact and closed.

9. **The lower semi-continuity of $A(g)$.** In (5.5) we have expressed $A(g)$ as the sum of an infinite series of positive terms. Defining the partial sums

$$(9.1)\quad A_r(g) = \Delta_1(g) + \Delta_2(g) + \cdots + \Delta_r(g) = \frac{1}{16\pi}\int\int_{T_r} \frac{\sum_{i=1}^{n}[g_i(\theta) - g_i(\phi)]^2}{\sin^2 \frac{\theta - \phi}{2}} d\theta d\phi,$$

where T_r denotes the domain

$$\frac{\pi}{r+1} \leqq |\theta - \phi| \leqq \pi,$$

we may also express $A(g)$ as the limit of the sequence

$$A_1(g), A_2(g), \cdots, A_r(g), \cdots,$$

which, since each $\Delta_r(g) > 0$, is continually increasing:

$$A_1(g) < A_2(g) < \cdots < A_r(g) < \cdots.$$

It is easily seen that *each $A_r(g)$ is a continuous functional of g*: if a sequence of representations

$$g^{(1)}, g^{(2)}, \cdots, g^{(m)}, \cdots$$

tends to g as limit, then

$$A_r(g^{(1)}), A_r(g^{(2)}), \cdots, A_r(g^{(m)}), \cdots$$

tends to $A_r(g)$. For the integrand in (9.1) is *uniformly bounded*, being

$$\leqq \frac{d^2}{\sin^2 \frac{\pi}{2(r+1)}},$$

where d is the diameter of Γ, and under this condition it is permissible to pass to the limit under the sign of integration.

The lower semi-continuity of $A(g)$ now results from the following general theorem.

THEOREM. *If a sequence of continuous*† *functions on an L-set tend, in increasing (wide sense), to a limit function (finite or infinite valued), then this limit function is lower semi-continuous.*

Let a denote an arbitrary element of the L-set, and

$$U_1(a) \leqq U_2(a) \leqq \cdots \leqq U_r(a) \leqq \cdots, \tag{9.2}$$

the increasing sequence of functions tending to the limit $U(a)$. Let

$$a_1, a_2, \cdots, a_m, \cdots$$

be any sequence of elements converging to a as limit.

Case 1: $U(a)$ *finite.* If $\epsilon > 0$ be assigned arbitrarily, there exists an r_ϵ such that for $r > r_\epsilon$

$$U_r(a) > U(a) - \epsilon/2\,\cdot \tag{9.3}$$

We suppose that in this inequality r has a *fixed* value $> r_\epsilon$, for instance, $r = r_\epsilon + 1$.

By hypothesis, the function U_r is continuous; this implies the existence of an m_ϵ such that for $m > m_\epsilon$

$$U_r(a_m) > U_r(a) - \epsilon/2\,\cdot \tag{9.4}$$

Combining the inequalities (9.3) and (9.4), we have

$$U_r(a_m) > U(a) - \epsilon \tag{9.5}$$

for $m > m_\epsilon$.

Now, by (9.2), each of the functions U_r is, for any fixed value of the argument, not greater than the limit function U; thus

$$U(a_m) \geqq U_r(a_m) \tag{9.6}$$

for every m, in particular for $m > m_\epsilon$.

From (9.5) and (9.6) it follows that

$$U(a_m) > U(a) - \epsilon$$

for $m > m_\epsilon$; but this is the definition of lower semi-continuity, according to (7.1).

Case 2: $U(a) = +\infty$. Here lower semi-continuity becomes identical with continuity: if $a_1, a_2, \cdots, a_m, \cdots$ is any sequence of elements converging to a, then

$$U(a_1), U(a_2), \cdots, U(a_m), \cdots$$

tends to $+\infty$.

† The theorem still remains valid if the functions of the sequence are merely lower semi-continuous. Cf. Carathéodory, *Vorlesungen über reelle Funktionen*, Leipzig, 1918, p. 175, where this theorem is proved for functions of n real variables.

For the proof, let G be an arbitrarily assigned finite positive number; then since by hypothesis

$$\lim_{r\to\infty} U_r(a) = +\infty,$$

an index r exists such that

$$U_r(a) > 2G. \tag{9.7}$$

Because of the continuity of the function U_r, there exists an index m_G such that for $m>m_G$

$$U_r(a_m) > U_r(a) - G. \tag{9.8}$$

Combining (9.7) and (9.8), we have

$$U_r(a_m) > G$$

for $m>m_G$. From this and (9.6), it follows that

$$U(a_m) > G$$

for $m>m_G$, that is,

$$\lim_{m\to\infty} U(a_m) = +\infty,$$

which was to be proved.

Since $A(g)$ obeys all the conditions of the theorem just proved, its lower semi-continuity is established.†

10. **$A(g)$ attains its minimum.** In this section we prove that $A(g)$, as a lower semi-continuous function on the compact closed set $\mathfrak{M}'$, must attain its minimum value on $\mathfrak{M}'$.

Since the values of $A(g)$ are all positive, and some are finite, they have a finite lower bound $M \geqq 0$.‡ By definition of lower bound, $A(g)$ cannot take any value less than M, but can approach to M from above as closely as we please. On this basis we can construct a *minimizing sequence*

$$g^{(1)}, g^{(2)}, \cdots, g^{(m)}, \cdots, \tag{10.1}$$

that is, one such that

$$A(g^{(1)}), A(g^{(2)}), \cdots, A(g^{(m)}), \cdots \tag{10.2}$$

tends to the limit M; the construction of such a sequence is the first step in the direct treatment of any calculus of variations problem.

† It is easy to prove that by making g approach suitably to g^0, any number whatever $\geqq A(g^0)$ (including $+\infty$) can be made the limit of $A(g)$.

‡ After it has been proved that M is attained, it results that $M>0$.

Now the sequence (10.1) may not converge to a limit, but since the set $\mathfrak{M}'$ is compact, we can select a sub-sequence

$$(10.1') \qquad g^{(m_1)}, g^{(m_2)}, \cdots, g^{(m_k)}, \cdots$$

which converges to a limit g^*; on account of the closure of $\mathfrak{M}'$, g^* belongs to $\mathfrak{M}'$. The sequence of corresponding functional values

$$(10.2') \qquad A(g^{(m_1)}), A(g^{(m_2)}), \cdots, A(g^{(m_k)}), \cdots,$$

being a sub-sequence of (10.2) with the limit M, must tend to the same limit M.

Using the lower semi-continuity of $A(g)$ as expressed in (7.2), we therefore have

$$A(g^*) \leqq M;$$

but the definition of lower bound makes $A(g^*) < M$ impossible, consequently

$$A(g^*) = M,$$

that is: the minimum of $A(g)$ on $\mathfrak{M}'$ is attained for g^*.

By the discussion at the end of §6, it follows that the minimum of $A(g)$ on $\mathfrak{M}$ is attained for g^* and all its equivalents.

11. **Calculation of the power series $\sum_{i=1}^{n} F_i'^2(w)$.** The rest of our argument is concerned with showing that the harmonic surface

$$x_i = \Re F_i(w)$$

determined by the minimizing representation g^* is minimal:

$$\sum_{i=1}^{n} F_i'^2(w) = 0.$$

This will be done by showing that, in a sense whose precise meaning appears in the sequel, the last condition expresses the vanishing of the first variation of $A(g)$ for $g = g^*$.

The functions $F_i(w)$ determined by the representation

$$x_i = g_i(\theta)$$

of Γ are given by the power series, convergent (at least) in the interior of the unit circle C,

$$(11.1) \qquad F_i(w) = \frac{a_{i0}}{2} + \sum_{p=1}^{\infty} (a_{ip} - ib_{ip}) w^p, \dagger$$

† Throughout, the symbol i will be used in two senses: i the index running from 1 to n, and i the square root of -1. This notation should not lead to any confusion.

where a_{ip}, b_{ip} are the Fourier coefficients of $g_i(\theta)$:

$$(11.2)\qquad a_{ip} = \frac{1}{\pi}\int_C g_i(\theta)\cos p\theta d\theta,\quad b_{ip} = \frac{1}{\pi}\int_C g_i(\theta)\sin p\theta d\theta.$$

Instead of $\sum_{i=1}^n F_i'^2(w)$ we shall find it more convenient to work with $w^2\sum_{i=1}^n F_i'^2(w)$; it will be easy to dispose of the factor w^2 when we wish. The latter expression is representable in the interior of C by a power series, derivable from (11.1) by performing formally the operations indicated:

$$(11.3)\qquad w^2\sum_{i=1}^n F_i'^2(w) = \sum_{m=2}^\infty (A_m - iB_m)w^m.$$

The special object of this section is *to calculate the coefficients* A_m, B_m; the results appear in (11.15).

Since the power series (11.1) is convergent in the interior of C, we may differentiate termwise, and find

$$(11.4)\qquad wF_i'(w) = \sum_{p=1}^\infty p(a_{ip} - ib_{ip})w^p.$$

Rewriting this as

$$(11.4')\qquad wF_i'(w) = \sum_{q=1}^\infty q(a_{iq} - ib_{iq})w^q,$$

we obtain, on multiplying together (11.4) and (11.4′) and summing as to i from 1 to n, the formula (11.3) with

$$(11.5)\qquad A_m - iB_m = \sum_{i=1}^n \sum_{p,q} p(a_{ip} - ib_{ip})\cdot q(a_{iq} - ib_{iq})$$

where the range of the summation indices p, q is

$$(11.6)\qquad p \geqq 1,\quad q \geqq 1,\quad p + q = m.$$

From (11.2),

$$a_{ip} - ib_{ip} = \frac{1}{\pi}\int_C g_i(\theta)e^{-pi\theta}d\theta,\quad a_{iq} - ib_{iq} = \frac{1}{\pi}\int_C g_i(\phi)e^{-qi\phi}d\phi.$$

The product of the two parts of this formula, after multiplying the first by p, the second by q, may be expressed as a double integral:

$$(11.7)\qquad \begin{aligned} &p(a_{ip} - ib_{ip})\cdot q(a_{iq} - ib_{iq}) \\ &\qquad = \frac{1}{\pi^2}\int_C\int_C g_i(\theta)g_i(\phi)\cdot pe^{-pi\theta}\cdot qe^{-qi\phi}\cdot d\theta d\phi. \end{aligned}$$

We have obviously

$$(11.8)\quad \int_C \int_C g_i^2(\theta)\cdot pe^{-pi\theta}\cdot qe^{-qi\phi}\cdot d\theta d\phi = \int_C g_i^2(\theta)\cdot pe^{-pi\theta}d\theta\cdot \int_C qe^{-qi\phi}d\phi = 0$$

since the second factor vanishes; and similarly

$$(11.8')\qquad \int_C \int_C g_i^2(\phi)\cdot pe^{-pi\theta}\cdot qe^{-qi\phi}\cdot d\theta d\phi = 0.$$

By the last three equations we may write

$$(11.9)\qquad \begin{aligned} &p(a_{ip} - ib_{ip})\cdot q(a_{iq} - ib_{iq}) \\ &\qquad = -\frac{1}{2\pi^2}\int_C \int_C [g_i(\theta) - g_i(\phi)]^2\cdot pe^{-pi\theta}\cdot qe^{-qi\phi}\cdot d\theta d\phi, \end{aligned}$$

for, on expanding the bracket squared, this reduces to (11.7) when account is taken of (11.8), (11.8′).

Substituting (11.9) in (11.5), we have

$$(11.10)\ A_m - iB_m = -\frac{1}{2\pi^2}\int_C \int_C \sum_{i=1}^{n}[g_i(\theta) - g_i(\phi)]^2\cdot \sum_{p,q} pe^{-pi\theta}\cdot qe^{-qi\phi}\cdot d\theta d\phi.$$

Accordingly, we have to calculate

$$(11.11)\qquad \sum_{p,q} pe^{-pi\theta}\cdot qe^{-qi\phi}$$

where the range of p, q is as defined in (11.6). Let

$$(11.12)\qquad e^{-i\theta} = \zeta,\quad e^{-i\phi} = z,$$

so that (11.11) becomes

$$(11.11')\qquad \begin{aligned} \sum_{p,q} p\zeta^p\cdot qz^q &= (m-1)\zeta^{m-1}z + (m-2)\zeta^{m-2}\cdot 2z^2 + \cdots \\ &\qquad + 2\zeta^2\cdot(m-2)z^{m-2} + \zeta\cdot(m-1)z^{m-1}. \end{aligned}$$

The value of the last expression can be found by starting with the geometric progression

$$\zeta^m + \zeta^{m-1}z + \zeta^{m-2}z^2 + \cdots + \zeta^2 z^{m-2} + \zeta z^{m-1} + z^m = \frac{\zeta^{m+1} - z^{m+1}}{\zeta - z}.$$

Applying to this the operator

$$\zeta z\frac{\partial^2}{\partial\zeta\partial z},$$

we get

$$\sum_{p,q} p\zeta^p \cdot qz^q = (m+1)\zeta z(\zeta^m + z^m)(\zeta - z)^{-2} - 2\zeta z(\zeta^{m+1} - z^{m+1})(\zeta - z)^{-3}. \tag{11.13}$$

The calculation of the first and second terms of this expression is as follows.

First term. Let

$$\sigma = \frac{\theta + \phi}{2}, \qquad \delta = \frac{\theta - \phi}{2}\,. \tag{11.14}$$

Then

$$\begin{aligned}
\zeta z &= e^{-i(\theta+\phi)} = e^{-2i\sigma};\\
\zeta^m + z^m &= (\cos m\theta + \cos m\phi) - i(\sin m\theta + \sin m\phi)\\
&= 2\cos m\sigma \cos m\delta - 2i \sin m\sigma \cos m\delta\\
&= 2\cos m\delta e^{-mi\sigma};\\
\zeta - z &= (\cos\theta - \cos\phi) - i(\sin\theta - \sin\phi)\\
&= -2\sin\sigma\sin\delta - 2i\cos\sigma\sin\delta\\
&= -2i\sin\delta e^{-i\sigma};\\
(\zeta - z)^{-2} &= -\frac{1}{4\sin^2\delta}e^{2i\sigma};\\
\text{First term} &= -\frac{(m+1)\cos m\delta}{2\sin^2\delta}e^{-mi\sigma}.
\end{aligned}$$

Second term.

$$\begin{aligned}
\zeta z &= e^{-2i\sigma};\\
\zeta^{m+1} - z^{m+1} &= -2i\sin(m+1)\delta e^{-(m+1)i\sigma},
\end{aligned}$$

found from the above formula for $\zeta - z$ by replacing θ, ϕ by $(m+1)\theta$, $(m+1)\phi$;

$$(\zeta - z)^{-3} = \frac{1}{8i\sin^3\delta}e^{3i\sigma};$$

$$\text{Second term} = \frac{\sin(m+1)\delta}{2\sin^3\delta}e^{-mi\sigma}.$$

Substituting these results in (11.13), we obtain

$$\sum_{p,q} p\zeta^p \cdot qz^q = \left\{ -\frac{(m+1)\cos m\delta}{2\sin^2\delta} + \frac{\sin(m+1)\delta}{2\sin^3\delta}\right\} e^{-mi\sigma}.$$

The bracket is equal to

$$\frac{-(m+1)\cdot 2\cos m\delta\sin\delta+2\sin(m+1)\delta}{4\sin^3\delta}$$
$$=\frac{-(m+1)\{\sin(m+1)\delta-\sin(m-1)\delta\}+2\sin(m+1)\delta}{4\sin^3\delta}$$
$$=-\frac{(m-1)\sin(m+1)\delta-(m+1)\sin(m-1)\delta}{4\sin^3\delta};$$

so that, referring to the notation (11.12), we have finally for the expression (11.11):

$$\sum_{p,q} pe^{-pi\theta}\cdot qe^{-qi\phi}=-\frac{(m-1)\sin(m+1)\delta-(m+1)\sin(m-1)\delta}{4\sin^3\delta}e^{-mi\sigma}.$$

Substituting this in (11.10), we get

$$A_m-iB_m=\frac{1}{8\pi^2}\int_C\int_C\sum_{i=1}^n[g_i(\theta)-g_i(\phi)]^2$$
$$\cdot\frac{(m-1)\sin(m+1)\delta-(m+1)\sin(m-1)\delta}{\sin^3\delta}e^{-mi\sigma}d\theta d\phi.$$

Writing $e^{-mi\sigma}=\cos m\sigma-i\sin m\sigma$, separating the real and imaginary parts, and referring to the notation (11.14), we arrive at the final expressions for A_m, B_m:

(11.15)

$$A_m=\frac{1}{8\pi^2}\int_C\int_C\sum_{i=1}^n[g_i(\theta)-g_i(\phi)]^2$$
$$\cdot\frac{(m-1)\sin\left[(m+1)\frac{\theta-\phi}{2}\right]-(m+1)\sin\left[(m-1)\frac{\theta-\phi}{2}\right]}{\sin^3\frac{\theta-\phi}{2}}\cos\left[m\frac{\theta+\phi}{2}\right]d\theta d\phi,$$

$$B_m=\frac{1}{8\pi^2}\int_C\int_C\sum_{i=1}^n[g_i(\theta)-g_i(\phi)]^2$$
$$\cdot\frac{(m-1)\sin\left[(m+1)\frac{\theta-\phi}{2}\right]-(m+1)\sin\left[(m-1)\frac{\theta-\phi}{2}\right]}{\sin^3\frac{\theta-\phi}{2}}\sin\left[m\frac{\theta+\phi}{2}\right]d\theta d\phi.$$

It may be observed that these are *proper* Riemann integrals, for the fraction, which takes for $\theta=\phi$ the indeterminate form 0/0, has the limiting value $-(2/3)m(m^2-1)$.

12. The functions $C_m(\lambda)$, $S_m(\lambda)$; $C_{mr}(\lambda)$, $S_{mr}(\lambda)$. *Hypothesis.* The work of §§12–15 is valid on the sole hypothesis that g is a fixed representation for which the functional $A(g)$ has a finite value.† In particular, we may put $g=g^*$, the minimizing representation of $A(g)$, since $A(g^*)$ is finite.

Consider the transformations

$$\bar{\theta} = \theta + \lambda \cos m\theta = c(\theta), \quad \bar{\theta} = \theta + \lambda \sin m\theta = s(\theta), \tag{12.1}$$

m any fixed positive integer,

of the unit circle C into itself $\bar{C}$. If λ is a real parameter restricted to the interval

$$-\frac{1}{m} < \lambda < \frac{1}{m}, \tag{12.2}$$

then each of these transformations is one-one and continuous. This results from the fact that the respective derivatives

$$\frac{d\bar{\theta}}{d\theta} = 1 - m\lambda \sin m\theta, \quad \frac{d\bar{\theta}}{d\theta} = 1 + m\lambda \cos m\theta$$

are positive for all θ under the condition (12.2). Consequently the transformations (12.1) have one-one continuous inverses

$$\theta = c^{-1}(\bar{\theta}), \quad \theta = s^{-1}(\bar{\theta}), \tag{12.1'}$$

and applying these to the representation g, we get the family of representations $g(c^{-1}(\bar{\theta}))$, $g(s^{-1}(\bar{\theta}))$, depending on the parameter λ.

The values of $A(g)$ for these representations are respectively

$$\frac{1}{16\pi}\iint_{\bar{T}} \frac{\sum_{i=1}^{n}[g_i(c^{-1}(\bar{\theta})) - g_i(c^{-1}(\bar{\phi}))]^2}{\sin^2 \frac{\bar{\theta}-\bar{\phi}}{2}} d\bar{\theta}d\bar{\phi},$$

$$\frac{1}{16\pi}\iint_{\bar{T}} \frac{\sum_{i=1}^{n}[g_i(s^{-1}(\bar{\theta})) - g_i(s^{-1}(\bar{\phi}))]^2}{\sin^2 \frac{\bar{\theta}-\bar{\phi}}{2}} d\bar{\theta}d\bar{\phi},$$

† It will follow from §17 that, in particular, g cannot be improper of the first kind.

where the torus $\bar{T}=\bar{C}\bar{C}$ corresponds to the torus $T=CC$ by the equations (12.1). Making the change of variables (12.1) in these double integrals, we obtain two functions of λ, which we will denote respectively by $C_m(\lambda)$, $S_m(\lambda)$:

$$(12.3)\qquad \begin{aligned} C_m(\lambda) &= \frac{1}{16\pi}\iint_T \sum_{i=1}^{n}[g_i(\theta) - g_i(\phi)]^2 \frac{(1 - m\lambda \sin m\theta)(1 - m\lambda \sin m\phi)}{\sin^2\left(\dfrac{\theta-\phi}{2} + \lambda\dfrac{\cos m\theta - \cos m\phi}{2}\right)} d\theta d\phi, \\ S_m(\lambda) &= \frac{1}{16\pi}\iint_T \sum_{i=1}^{n}[g_i(\theta) - g_i(\phi)]^2 \frac{(1 + m\lambda \cos m\theta)(1 + m\lambda \cos m\phi)}{\sin^2\left(\dfrac{\theta-\phi}{2} + \lambda\dfrac{\sin m\theta - \sin m\phi}{2}\right)} d\theta d\phi. \end{aligned}$$

These are *improper* Riemann integrals (singular locus $\theta=\phi$).

As in §5, we express the torus T as the sum of an infinite number of strips:

$$T = \Delta_1 + \Delta_2 + \cdots + \Delta_r + \cdots,$$

defined by (5.2). Replacing the domain of integration T in (12.3) by Δ_r, we derive the functions

$$(12.4)\qquad \begin{aligned} C_{mr}(\lambda) &= \frac{1}{16\pi}\iint_{\Delta_r} \sum_{i=1}^{n}[g_i(\theta) - g_i(\phi)]^2 \frac{(1 - m\lambda \sin m\theta)(1 - m\lambda \sin m\phi)}{\sin^2\left(\dfrac{\theta-\phi}{2} + \lambda\dfrac{\cos m\theta - \cos m\phi}{2}\right)} d\theta d\phi, \\ S_{mr}(\lambda) &= \frac{1}{16\pi}\iint_{\Delta_r} \sum_{i=1}^{n}[g_i(\theta) - g_i(\phi)]^2 \frac{(1 + m\lambda \cos m\theta)(1 + m\lambda \cos m\phi)}{\sin^2\left(\dfrac{\theta-\phi}{2} + \lambda\dfrac{\sin m\theta - \sin m\phi}{2}\right)} d\theta d\phi. \end{aligned}$$

These are *proper* Riemann integrals since the domain Δ_r does not contain any singular points $\theta=\phi$.

In this way we have for definition of $C_m(\lambda)$, $S_m(\lambda)$ the infinite series

$$(12.5)\qquad \begin{aligned} C_m(\lambda) &= C_{m1}(\lambda) + C_{m2}(\lambda) + \cdots + C_{mr}(\lambda) + \cdots, \\ S_m(\lambda) &= S_{m1}(\lambda) + S_{m2}(\lambda) + \cdots + S_{mr}(\lambda) + \cdots. \end{aligned}$$

Complex values of λ. We shall wish to make use of certain classic convergence theorems valid only in the complex domain; for this reason we now allow λ to be a complex variable:

$$\lambda = \mu + \nu i,$$

subject to the restriction

$$(12.6)\qquad |\lambda| < 1/m\,.$$

The (open) circle in the complex plane defined by this inequality will be denoted by $\mathcal{C}_m$. Evidently $\mathcal{C}_m$ contains the real interval (12.2).

The expressions (12.4) are still proper Riemann integrals defining the now complex-valued functions $C_{mr}(\lambda)$, $S_{mr}(\lambda)$. To be assured of this, we must show that the denominators of the respective integrands do not vanish in the domain Δ_r. These denominators are

$$\sin^2\left(\frac{\theta-\phi}{2}+\mu\frac{\cos m\theta-\cos m\phi}{2}+i\nu\frac{\cos m\theta-\cos m\phi}{2}\right),$$

$$\sin^2\left(\frac{\theta-\phi}{2}+\mu\frac{\sin m\theta-\sin m\phi}{2}+i\nu\frac{\sin m\theta-\sin m\phi}{2}\right).$$

Since the function $\sin^2(z/2)$ vanishes when and only when $z=2k\pi$ $(k=0, \pm 1, \pm 2, \cdots)$, it is evident that for these expressions to vanish we must have respectively

$$\begin{aligned} &\theta+\mu\cos m\theta=\phi+\mu\cos m\phi, \\ &\theta+\mu\sin m\theta=\phi+\mu\sin m\phi \qquad (\mathrm{mod}\ 2\pi). \end{aligned} \tag{12.7}$$

But $|\mu|\leqq|\lambda|<1/m$, and under this condition on the real parameter μ the equations (12.1), with μ in place of λ, define a one-one transformation. Hence each equation (12.7) implies

$$\theta=\phi \qquad (\mathrm{mod}\ 2\pi);$$

but this condition is never satisfied in the domain Δ_r.

The proof is thus complete that $C_{mr}(\lambda)$, $S_{mr}(\lambda)$ are well-defined by the formulas (12.4) for all values of λ in the interior of the circle $\mathcal{C}_m$.

We now define $C_m(\lambda)$, $S_m(\lambda)$ for complex λ by the infinite series (12.5) provided these series are absolutely convergent. We shall prove that the series in question are indeed absolutely convergent for all λ in $\mathcal{C}_m$, and then (what is of paramount importance) *that $C_m(\lambda)$, $S_m(\lambda)$ are analytic functions of λ in $\mathcal{C}_m$.*

Plan. The proof of the analyticity of $C_m(\lambda)$, $S_m(\lambda)$ will consist of a direct application of the Weierstrass double-series theorem.† We will show in §13 that each term $C_{mr}(\lambda)$, $S_{mr}(\lambda)$ of the series (12.5) is an analytic function of λ in $\mathcal{C}_m$, and then in §14 that the series (12.5) is uniformly convergent in every circle concentric with and smaller than $\mathcal{C}_m$. The result is to justify the formal operation of expanding in powers of λ the integrands in the formulas (12.3) for $C_m(\lambda)$, $S_m(\lambda)$, and then performing termwise the double integration as to θ, ϕ.

† Cf. Knopp, *Funktionentheorie*, Berlin and Leipzig, 1918, vol. 1, p. 89.

13. **Analyticity of $C_{mr}(\lambda)$, $S_{mr}(\lambda)$.** Considering the formulas (12.4) for $C_{mr}(\lambda)$, $S_{mr}(\lambda)$, we have just shown that the denominators of the fractions

$$(13.1)\qquad \frac{(1 - m\lambda \sin m\theta)(1 - m\lambda \sin m\phi)}{\sin^2\left(\dfrac{\theta - \phi}{2} + \lambda\dfrac{\cos m\theta - \cos m\phi}{2}\right)},\quad \frac{(1 + m\lambda \cos m\theta)(1 + m\lambda \cos m\phi)}{\sin^2\left(\dfrac{\theta - \phi}{2} + \lambda\dfrac{\sin m\theta - \sin m\phi}{2}\right)}$$

do not vanish for (θ, ϕ) in Δ_r and λ in $\mathcal{C}_m$; therefore these fractions are holomorphic functions of λ in $\mathcal{C}_m$, depending on the parameters θ, ϕ. They are developable in a series of powers of λ convergent in $\mathcal{C}_m$, with coefficients functions of θ, ϕ. The developments may be obtained in the usual way as Taylor series, and are given by the following calculations, where we retain terms only as far as the first power of λ.

Returning to the notation (11.14),

$$(13.2)\qquad \sigma = \tfrac{1}{2}(\theta + \phi), \quad \delta = \tfrac{1}{2}(\theta - \phi),$$

we have

$$\frac{\theta - \phi}{2} + \lambda\frac{\cos m\theta - \cos m\phi}{2} = \delta - \lambda \sin m\sigma \sin m\delta.$$

Making in the Taylor expansion

$$\frac{1}{\sin^2(x + h)} = \frac{1}{\sin^2 x} - \frac{2\cos x}{\sin^3 x}h + \cdots$$

the substitutions

$$x = \delta, \quad h = -\lambda \sin m\sigma \sin m\delta,$$

we get

$$(13.3)\qquad \begin{aligned} &\frac{1}{\sin^2\left(\dfrac{\theta - \phi}{2} + \lambda\dfrac{\cos m\theta - \cos m\phi}{2}\right)} \\ &\qquad = \frac{1}{\sin^2 \delta} + \lambda\,\frac{2\cos\delta \sin m\,\sigma \sin m\,\delta}{\sin^3\delta} + \cdots. \end{aligned}$$

For the numerator of (13.1) we have

$$(13.4)\qquad \begin{aligned} &(1 - m\lambda \sin m\theta)(1 - m\lambda \sin m\phi) \\ &\qquad = 1 + \lambda(-2m \sin m\sigma \cos m\delta) + \text{term in } \lambda^2. \end{aligned}$$

Multiplying together (13.3) and (13.4), we get for the development of the first of (13.1)

$$\frac{1}{\sin^2\delta} + \lambda\left\{\frac{2\cos\delta \sin m\sigma \sin m\delta}{\sin^3\delta} - \frac{2m \sin m\sigma \cos m\delta}{\sin^2\delta}\right\} + \cdots.$$

The bracket reduces to

$$\frac{2\cos\delta\sin m\delta - 2m\cos m\sigma\sin\delta}{\sin^3\delta}\sin m\sigma$$

$$= \frac{\{\sin(m+1)\delta + \sin(m-1)\delta\} - m\{\sin(m+1)\delta - \sin(m-1)\delta\}}{\sin^3\delta}\sin m\sigma$$

$$= -\frac{(m-1)\sin(m+1)\delta - (m+1)\sin(m-1)\delta}{\sin^3\delta}\sin m\sigma.$$

Thus we have finally, referring to (13.2),

$$(13.5)\quad \frac{(1 - m\lambda\sin m\theta)(1 - m\lambda\sin m\phi)}{\sin^2\left(\dfrac{\theta-\phi}{2} + \lambda\dfrac{\cos m\theta - \cos m\phi}{2}\right)} = \frac{1}{\sin^2\dfrac{\theta-\phi}{2}}$$

$$-\lambda\frac{(m-1)\sin\left[(m+1)\dfrac{\theta-\phi}{2}\right] - (m+1)\sin\left[(m-1)\dfrac{\theta-\phi}{2}\right]}{\sin^3\dfrac{\theta-\phi}{2}}\sin\left[m\frac{\theta+\phi}{2}\right] + \cdots.$$

An entirely similar calculation gives for the second fraction (13.1):

$$(13.5')\quad \frac{(1 + m\lambda\cos m\theta)(1 + m\lambda\cos m\phi)}{\sin^2\left(\dfrac{\theta-\phi}{2} + \lambda\dfrac{\sin m\theta - \sin m\phi}{2}\right)} = \frac{1}{\sin^2\dfrac{\theta-\phi}{2}}$$

$$+\lambda\frac{(m-1)\sin\left[(m+1)\dfrac{\theta-\phi}{2}\right] - (m+1)\sin\left[(m-1)\dfrac{\theta-\phi}{2}\right]}{\sin^3\dfrac{\theta-\phi}{2}}\cos\left[m\frac{\theta+\phi}{2}\right] + \cdots.$$

The following observation is now of prime importance: if λ is given any fixed value in $\mathcal{C}_m$, so that

$$(13.6)\qquad |\lambda| = \rho_0 < \frac{1}{m},$$

then the series (13.5), (13.5′) are *uniformly convergent when* (θ, ϕ) *varies over* Δ_r. To prove this, denote by $F(\theta, \phi, \lambda)$ the first member of (13.5), and let ρ be any fixed positive number such that

$$(13.7)\qquad \rho_0 < \rho < \frac{1}{m}\cdot$$

Suppose θ, ϕ, λ to vary arbitrarily subject to the conditions

$$(\theta, \phi) \text{ in } \Delta_r, \quad |\lambda| = \rho. \tag{13.8}$$

The three-dimensional domain so defined is evidently closed and bounded, and the positive real-valued function $|F(\theta, \phi, \lambda)|$ is finite and continuous on this domain, for it has been shown that the denominator of $F(\theta, \phi, \lambda)$ is never zero in (13.8). By a fundamental theorem, $|F(\theta, \phi, \lambda)|$ has therefore a finite upper bound on (13.8):

$$|F(\theta, \phi, \lambda)| \leqq B,$$

B being a positive real number independent of θ, ϕ, λ.

We now apply the appraisal formula of Cauchy† for the coefficients in the power series expansion of an analytic function. According to this formula, if $A_k(\theta, \phi)$ denote the coefficient of λ^k in the power series (13.5), then

$$|A_k(\theta, \phi)| \leqq \frac{B}{\rho^k}$$

for all (θ, ϕ) in Δ_r. Hence, with (13.6),

$$|A_k(\theta, \phi)\lambda^k| \leqq B\left(\frac{\rho_0}{\rho}\right)^k.$$

Since the series of constant positive terms

$$\sum_{k=0}^{\infty} B\left(\frac{\rho_0}{\rho}\right)^k$$

is convergent, being a geometric progression of ratio <1 (by (13.7)), the conditions of a standard uniform convergence test of Weierstrass are satisfied, and we have proved the uniform convergence of (13.5) for λ fixed in $\mathcal{C}_m$ and (θ, ϕ) varying over Δ_r. The same argument applies to (13.5′).

That this uniform convergence is not disturbed when, to obtain the integrands in (12.4) of $C_{mr}(\lambda)$, $S_{mr}(\lambda)$, we multiply (13.5), (13.5′) by $\sum_{i=1}^{n}[g_i(\theta) - g_i(\phi)]^2$, results from the fact that the multiplying factor is bounded:

$$\sum_{i=1}^{n}[g_i(\theta) - g_i(\phi)]^2 \leqq d^2,$$

where d is the diameter of Γ.

Consequently, after introducing this factor in (13.5), (13.5′), *we may integrate as to θ, ϕ over Δ_r term by term*, and find that $C_{mr}(\lambda)$, $S_{mr}(\lambda)$ are equal to power series in λ:

† Cf. Knopp, loc. cit., vol. 1, p. 84.

$$(13.9)\qquad \begin{aligned} C_{mr}(\lambda) &= \Delta_r(g) + C'_{mr}(0)\lambda + \cdots,\\ S_{mr}(\lambda) &= \Delta_r(g) + S'_{mr}(0)\lambda + \cdots, \end{aligned}$$

convergent for λ in $\mathcal{C}_m$.

We are especially interested in the values of the coefficients of the first power of λ, which, referring to (13.5), (13.5′), are seen to be

$$C'_{mr}(0) = -\frac{1}{16\pi}\iint_{\Delta_r}\sum_{i=1}^{n}[g_i(\theta) - g_i(\phi)]^2$$

$$\cdot\frac{(m-1)\sin\left[(m+1)\dfrac{\theta-\phi}{2}\right]-(m+1)\sin\left[(m-1)\dfrac{\theta-\phi}{2}\right]}{\sin^3\dfrac{\theta-\phi}{2}}\sin\left[m\frac{\theta-\phi}{2}\right]d\theta d\phi,$$

(13.10)

$$S'_{mr}(0) = \frac{1}{16\pi}\iint_{\Delta_r}\sum_{i=1}^{n}[g_i(\theta) - g_i(\phi)]^2$$

$$\cdot\frac{(m-1)\sin\left[(m+1)\dfrac{\theta-\phi}{2}\right]-(m+1)\sin\left[(m-1)\dfrac{\theta-\phi}{2}\right]}{\sin^3\dfrac{\theta-\phi}{2}}\cos\left[m\frac{\theta+\phi}{2}\right]d\theta d\phi.$$

14. **Analyticity of the functions $C_m(\lambda)$, $S_m(\lambda)$.** The function $C_m(\lambda)$ is defined by the infinite series (12.5), each of whose terms has just been proved analytic in the circle $\mathcal{C}_m$. We proceed to prove that this series is *uniformly convergent* in every smaller concentric circle

$$(14.1)\qquad |\lambda| \leqq \rho,\quad \rho < \frac{1}{m}\cdot$$

The analyticity of $C_m(\lambda)$ will then result immediately by the Weierstrass double-series theorem. The same argument applies to $S_m(\lambda)$.

Considering $C_{mr}(\lambda)$, we have from (12.4)

$$(14.2)\qquad \begin{aligned} |C_{mr}(\lambda)| \leqq \frac{1}{16\pi}\iint_{\Delta_r}&\sum_{i=1}^{n}[g_i(\theta) - g_i(\phi)]^2\\ &\cdot\left|\frac{(1 - m\lambda\sin m\theta)(1 - m\lambda\sin m\phi)}{\sin^2\left(\dfrac{\theta-\phi}{2} + \lambda\dfrac{\cos m\theta - \cos m\phi}{2}\right)}\right| d\theta d\phi. \end{aligned}$$

Comparing this with the formula (5.3) for $\Delta_r(g)$, also a double integral over Δ_r with positive real elements, we wish to prove that the quotient of the integrand of (14.2) by that of $\Delta_r(g)$,

$$(14.3)\qquad Q(\theta, \phi, \lambda) = \left| \frac{(1 - m\lambda \sin m\theta)(1 - m\lambda \sin m\phi)}{\sin^2 \left(\dfrac{\theta - \phi}{2} + \lambda \dfrac{\cos m\theta - \cos m\phi}{2} \right)} \right| \sin^2 \frac{\theta - \phi}{2},$$

has a finite upper bound when (θ, ϕ) varies over the entire torus T (*not merely over* Δ_r) and λ varies independently over the closed circle (14.1).

The expression (14.3) is not defined on the diagonal $\theta = \phi$ of the torus T, taking there the indeterminate form 0/0; but it is seen directly that (14.3) has the limit value 1 for $\theta = \phi$:

$$(14.3')\qquad Q(\phi, \phi, \lambda) = 1.$$

The function $Q(\theta, \phi, \lambda)$ is thus defined on the *closed* domain (of four real dimensions)

$$(14.4)\qquad (\theta, \phi) \text{ on } T, \ |\lambda| \leqq \rho,$$

by (14.3) if $\theta \neq \phi$ and by (14.3′) if $\theta = \phi$. As has been remarked several times, the denominator of (14.3) cannot vanish if $|\lambda| < 1/m$, $\theta \neq \phi$. Hence $Q(\theta, \phi, \lambda)$ is finite and continuous on the closed and bounded domain (14.4). It therefore has a finite positive upper bound K.

Consequently,

$$(14.5)\qquad |C_{mr}(\lambda)| \leqq K\Delta_r(g)$$

where it is to be emphasized that *K is independent of r.*

According to the hypothesis stated at the beginning of §12, g is a fixed representation for which $A(g)$ is finite. It follows that

$$K\Delta_1(g) + K\Delta_2(g) + \cdots + K\Delta_r(g) + \cdots$$

is a series of *constant, positive* terms, *convergent* to the sum $KA(g)$. By the inequality (14.5) this series dominates the series (12.5) for $C_m(\lambda)$. Therefore, by the same uniform convergence test of Weierstrass used in the preceding section, *the series* (12.5) *for $C_m(\lambda)$ is absolutely and uniformly convergent in the circle* (14.1).

The conditions of the Weierstrass double-series theorem being now satisfied, we may assert that $C_m(\lambda)$ *is analytic in the circle* $\mathcal{C}_m$.

According to the same theorem, the coefficients in the power series representing $C_m(\lambda)$ are to be found by adding together the coefficients of like powers of λ in the various terms $C_{mr}(\lambda)$ (summation of the double series by columns). By §13, the coefficient of any given power of λ in the expansion of $C_{mr}(\lambda)$ is a double integral taken over Δ_r of an integrand that is the same for all r. Therefore the coefficient of this power of λ in $C_m(\lambda)$ is the double

integral of the same integrand taken over the entire torus T. Consequently, referring to (13.9), (13.10), we have

$$C_m(\lambda) = A(g) + C_m'(0)\lambda + \cdots \tag{14.6}$$

with

$$C_m'(0) = -\frac{1}{16\pi}\iint_T \sum_{i=1}^{n}[g_i(\theta) - g_i(\phi)]^2 \cdot \frac{(m-1)\sin\left[(m+1)\frac{\theta-\phi}{2}\right] - (m+1)\sin\left[(m-1)\frac{\theta-\phi}{2}\right]}{\sin^3\frac{\theta-\phi}{2}} \sin\left[m\frac{\theta+\phi}{2}\right] d\theta d\phi. \tag{14.7}$$

The entire preceding argument applies to $S_m(\lambda)$, and gives

$$S_m(\lambda) = A(g) + S_m'(0)\lambda + \cdots \tag{14.6$'$}$$

with

$$S_m'(0) = \frac{1}{16\pi}\iint_T \sum_{i=1}^{n}[g_i(\theta) - g_i(\phi)]^2 \cdot \frac{(m-1)\sin\left[(m+1)\frac{\theta-\phi}{2}\right] - (m+1)\sin\left[(m+1)\frac{\theta-\phi}{2}\right]}{\sin^3\frac{\theta-\phi}{2}} \cos\left[m\frac{\theta-\phi}{2}\right] d\theta d\phi. \tag{14.7$'$}$$

15. **Relations between A_m, B_m and $C_m'(0)$, $S_m'(0)$.** The ultimate purpose of the calculations of §§11–14 was to establish the following relations, observable immediately by comparing (11.5) with (14.7), (14.7′):

$$A_m = \frac{2}{\pi}S_m'(0), \quad B_m = -\frac{2}{\pi}C_m'(0). \tag{15}$$

16. **Existence of the minimal surface.** The introduction of complex values of λ having served its purpose of establishing the power series (14.7), (14.7′), we now return to real values of λ in the interval (12.2),

$$-\frac{1}{m} < \lambda < \frac{1}{m}, \tag{16.1}$$

where these power series remain valid, since this interval is part of the circle $\mathcal{C}_m$.

By applying the one-one continuous transformations (12.1) to the fixed representation g, we obtained in §12 a family of representations depending on

the parameter λ, and containing, for $\lambda=0$, the original representation g. Thus the function $C_m(\lambda)$ is a *part* of the functional $A(g)$ in the sense that its values are those of $A(g)$ on a certain part of the total range of g.

Suppose now $g=g^*$, the minimizing representation of $A(g)$. Then, a fortiori, $C_m(\lambda)$, considered as a function on the interval (16.1), has a minimum at $\lambda=0$, the value corresponding to g^*. Consequently,

$$C_m'(0) = 0. \tag{16.2}$$

Analogously,

$$S_m'(0) = 0. \tag{16.2'}$$

Therefore, by (15),

$$A_m = 0, \quad B_m = 0. \tag{16.3}$$

Since m may have any integral value, it follows that every coefficient in $w^2\sum_{i=1}^n F_i'^2(w)$ vanishes, where $F_i(w)$ are the power series, convergent within the unit circle, determined by g^* according to (11.1), (11.2). Hence, dividing out the non-identically vanishing factor w^2, we have

$$\sum_{i=1}^n F_i'^2(w) = 0, \tag{16.4}$$

which expresses that the harmonic surface

$$x_i = \Re F_i(w)$$

determined by g^* is minimal.

That this minimal surface is bounded by Γ will follow after we have shown in the next two sections that g^* is a proper representation of Γ.

17. **g^* cannot be improper of the first kind.** We will rule out the possibility that g^* be improper of the first kind by proving that for a g of this type

$$A(g) = +\infty.$$

Since, by the hypothesis governing Part I, $A(g)$ takes a finite value for at least one representation g, the supposition that g^*, minimizing $A(g)$, could be improper of the first kind is thus reduced to an absurdity.

In an improper representation of the first kind, a point P of C corresponds to an arc $Q'Q''$ of Γ less than all of Γ. Since, by hypothesis, Γ has no double points, the end points Q', Q'' are distinct and their distance l is not equal to zero.

The (vector) function g will have a discontinuity at P, where the distinct one-sided limits Q', Q'' will exist. Therefore if two points of C approach to P

from opposite sides, the distance between the corresponding points of Γ tends to l as limit. Consequently, if f denote any fixed proper fraction,

$$\sum_{i=1}^{n}[g_i(\theta) - g_i(\phi)]^2 > fl^2 \tag{17.1}$$

under the conditions

$$\alpha < \theta \leqq \alpha + \delta, \quad \alpha - \delta \leqq \phi < \alpha, \tag{17.2}$$

where α denotes the angular coördinate of P and $\delta > 0$ is fixed sufficiently small.

The domain (θ, ϕ) defined by (17.2) is a square S on the torus T, with one vertex $\theta = \phi = \alpha$ on the diagonal of T.

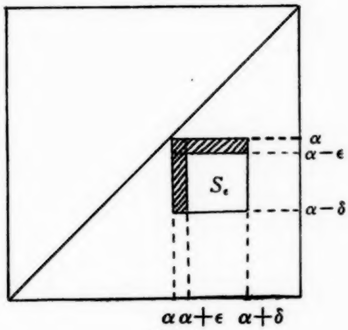

FIG. 3

Since the double integral (5.1) defining $A(g)$ is composed of non-negative elements, its value over T is not less than its value over S:

$$A(g) \geqq \frac{1}{16\pi}\iint_S \frac{\sum_{i=1}^{n}[g_i(\theta) - g_i(\phi)]^2}{\sin^2 \dfrac{\theta - \phi}{2}} d\theta d\phi; \tag{17.3}$$

therefore, by (17.1),

$$A(g) \geqq m \iint_S \frac{d\theta d\phi}{\sin^2 \dfrac{\theta - \phi}{2}} \tag{17.4}$$

where $m = fl^2/(16\pi)$ is a positive constant.

The integral (17.4) is improper, since the integrand becomes infinite at the vertex of S lying on the diagonal $\theta = \phi$; hence this integral must be expressed as a limit:

$$(17.5)\quad \iint_S \frac{d\theta d\phi}{\sin^2 \dfrac{\theta - \phi}{2}} = \lim_{\epsilon \to 0} \iint_{S_\epsilon} \frac{d\theta d\phi}{\sin^2 \dfrac{\theta - \phi}{2}} = \lim_{\epsilon \to 0} \int_{\alpha+\epsilon}^{\alpha+\delta} \int_{\alpha-\delta}^{\alpha-\epsilon} \frac{d\theta d\phi}{\sin^2 \dfrac{\theta - \phi}{2}} .$$

Here S_ϵ is the square indicated in Fig. 3, obtained from S by removing strips of width ϵ along two of its sides.

We have for any double integral between *constant* limits the formula

$$\int_a^b \int_c^d f(\theta, \phi) d\theta d\phi = F(b, d) - F(b, c) - F(a, d) + F(a, c),$$

where $F(\theta, \phi)$ is any primitive of $f(\theta, \phi)$, that is, such that

$$\frac{\partial^2 F}{\partial\theta\partial\phi} = f(\theta, \phi).$$

In the case of (17.5),

$$F = 4 \log \sin \frac{\theta - \phi}{2},$$

and therefore

$$\int_{\alpha+\epsilon}^{\alpha+\delta} \int_{\alpha-\delta}^{\alpha-\epsilon} \frac{d\theta d\phi}{\sin^2 \dfrac{\theta - \phi}{2}} = 4 \log \sin \frac{\delta - \epsilon}{2}$$

$$- 4 \log \sin \delta - 4 \log \sin \epsilon + 4 \log \sin \frac{\delta -}{2} \cdot$$

If now $\epsilon \to 0$, the limit of the third term is $+\infty$, while the other terms stay finite; therefore the double integral over S is equal to $+\infty$; consequently, by (17.4),

$$A(g) = +\infty .$$

18. **g^* cannot be improper of the second kind.** That g^* cannot be improper of the second kind means that an arc $P'P''$ of C, less than all of C, cannot correspond by g^* to a single point Q of Γ. This will be established by showing that g^* cannot convert an arc $P'P''$ of C into a point Q of Γ without converting all of C into the point Q (degenerate representation).

The harmonic functions $\Re F_i(w)$ determined by g^* are given by

$$(18.1)\qquad F_i(w) = \frac{1}{2\pi} \int_C \frac{e^{i\theta} + w}{e^{i\theta} - w} g_i^*(\theta) d\theta$$

(equivalent to the integral of Poisson). It is permissible to differentiate under the integral sign, and we obtain

$$wF_i'(w) = \frac{1}{2\pi}\int_C \frac{2we^{i\theta}}{(e^{i\theta}-w)^2}g_i^*(\theta)d\theta. \tag{18.2}$$

Writing $w=\rho e^{i\alpha}$, and taking the imaginary part of each side, we find

$$\Im wF_i'(w) = -\frac{1}{2\pi}\int_C \frac{2\rho(1-\rho^2)\sin(\theta-\alpha)}{[1-2\rho\cos(\theta-\alpha)+\rho^2]^2}g_i^*(\theta)d\theta. \tag{18.3}$$

Except for the minus sign, the last expression is identical with one studied by Fatou in his thesis.† Fatou shows that at every point where the derivative $g_i^{*\prime}(\theta)$ exists and is continuous the expression in question has a unique limit equal to $g_i^{*\prime}(\theta)$ when $(\rho, \alpha)\to(1, \theta)$.

Since in the present case $g_i^*(\theta)$ is supposed constant on a certain arc $P'P''$, $g_i^{*\prime}(\theta)$ has the continuous value zero on this arc; hence for θ any point interior to $P'P''$,

$$\lim_{w\to e^{i\theta}} \Im wF_i'(w) = 0 \qquad (i = 1, 2, \cdots, n). \tag{18.4}$$

We have proved in §16 that, $F_i(w)$ being determined by g^*,

$$\sum_{i=1}^n w^2F_i'^2(w) = \sum_{i=1}^n [\Re wF_i'(w) + i\Im wF_i'(w)]^2 = 0;$$

taking real parts, we have

$$\sum_{i=1}^n [\Re wF_i'(w)]^2 = \sum_{i=1}^n [\Im wF_i'(w)]^2.$$

Therefore, by (18.4),

$$\lim_{w\to e^{i\theta}} \sum_{i=1}^n [\Re wF_i'(w)]^2 = 0;$$

and since for each value of i

$$|\Re wF_i'(w)| \leqq \left\{\sum_{i=1}^n [\Re wF_i'(w)]^2\right\}^{1/2},$$

we have

$$\lim_{w\to e^{i\theta}} \Re wF_i'(w) = 0 \qquad (i = 1, 2, \cdots, n). \tag{18.4$'$}$$

By (18.4) and (18.4′),

$$\lim_{w\to e^{i\theta}} wF_i'(w) = 0 \qquad (i = 1, 2, \cdots, n), \tag{18.5}$$

for θ any interior point of $P'P''$.

† P. Fatou, *Séries trigonométriques et séries de Taylor*, Acta Mathematica, vol. 30 (1906), pp. 347–348.

Since the limiting value of the function $wF_i'(w)$ when w approaches to any point of the circular arc $P'P''$ is real, we may apply the symmetry principle of Riemann-Schwarz† to prolong the function analytically across this arc, attaching conjugate imaginary values to points inverse with respect to the arc. The function $wF_i'(w)$ will then be equal to zero on an arc *interior* to a domain of regularity, and is therefore identically equal to zero. Hence $F_i(w)$ is identically equal to a constant, $F_i(w)=a_i+ib_i$.

It follows that $x_i=g_i^*(\theta)$, boundary values of $\Re F_i(w)$, makes all of C correspond to the point of coördinates a_i; but this is contrary to hypothesis.

By this and the preceding section, the representation g^* is proper; and therefore the minimal surface of §16, determined by g^*, is bounded by Γ.

With this, we have completed the solution of the problem of Plateau for any finite-area-spanning contour in n-dimensional euclidean space.

II. An arbitrary Jordan contour

19. **The Jordan contour as a limit of polygons.** The case of an arbitrary Jordan contour Γ, for which $A(g)$ is identically $+\infty$, will be dealt with by regarding Γ as the limit of a sequence of non-self-intersecting polygons‡

$$\Gamma^{(1)}, \Gamma^{(2)}, \cdots, \Gamma^{(m)}, \cdots. \tag{19.1}$$

Let Γ, referred to a fixed initial parameter t, have the equations

$$x_i = f_i(t). \tag{19.2}$$

Then the polygons can be represented parametrically by equations

$$x_i = f_i^{(1)}(t),\ x_i = f_i^{(2)}(t), \cdots,\ x_i = f_i^{(m)}(t), \cdots, \tag{19.3}$$

so that $f_i^{(m)}(t)$ tends uniformly to $f_i(t)$ when $m\to\infty$.

Each polygon $\Gamma^{(m)}$ has, by Part I, a parameter θ such that the corresponding representation

$$x_i = g_i^{(m)}(\theta) \tag{19.4}$$

† Cf. G. Julia, *Principes Geométriques d'Analyse*, Paris, 1930, pp. 44–48, especially paragraph e.

‡ See C. Jordan, *Cours d'Analyse* (2d edition, Paris, 1893), p. 93. The polygons may be only partially inscribed in Γ, being derived from inscribed polygons by removing possible loops.

We could also proceed by expressing the continuous functions $x_i=f_i(t)$ representing Γ as the limits of their Fejér trigonometric polynomials $x_i=S_{im}(t)$. The contours $\Gamma^{(m)}$ thus represented might have multiple points. By referring back to Part I, it will be seen that this does not prevent the existence of a representation $t=\omega_m(\theta)$ of $\Gamma^{(m)}$ on C such that the parameter θ determines a minimal surface; only this representation might be improper in that a loop of $\Gamma^{(m)}$ could correspond to a single point of C, for the argument of §17 breaks down when $l=0$, and only then. The reader will readily see that the use of an improper representation $t=\omega_m(\theta)$ of $\Gamma^{(m)}$ would not at all complicate the proof which follows.

determines a minimal surface bounded by $\Gamma^{(m)}$:

$$(19.5) \qquad \begin{gathered} x_i = \Re F_i^{(m)}(w), \\ \sum_{i=1}^{n} [F_i^{(m)\prime}(w)]^2 = 0. \end{gathered}$$

Each parameter θ of (19.4) is related to the corresponding parameter t of (19.3) by a proper topological transformation

$$(19.6) \qquad t = \omega_1(\theta), \quad t = \omega_2(\theta), \quad \cdots, \quad t = \omega_m(\theta), \quad \cdots$$

of the unit circle C (considered as θ-locus) into itself (considered as t-locus). By adjoining a linear fractional transformation (6.1), which, according to §6, changes nothing essential in the preceding relations, we can secure that each transformation (19.6) converts three distinct fixed points θ_1, θ_2, θ_3 into three distinct fixed points t_1, t_2, t_3.

The sequence (19.6) is thus part of the set $\mathfrak{M}'$ (§3) of topological transformations of C into itself. Since $\mathfrak{M}'$ is compact and closed, we can select from (19.6) a sub-sequence converging to a limit

$$(19.7) \qquad t = \omega(\theta),$$

which belongs to $\mathfrak{M}'$, and may be proper or improper but not degenerate, since $\mathfrak{M}'$ contains no degenerate representations. To avoid complicating the notation, we will suppose that (19.6) already represents this convergent sub-sequence, and similarly in the formulas (19.1) to (19.5).

Let the (proper or improper) representation of Γ derived by applying the transformation (19.7) to (19.2) be

$$(19.8) \qquad x_i = g_i(\theta) = f^i(\omega(\theta));$$

then we have

$$(19.9) \qquad \lim_{m\to\infty} g_i^{(m)}(\theta) = g_i(\theta),$$

abstraction being made, in case the representation g is improper of the first kind, of the values of θ, at most denumerably infinite in number, where $g_i(\theta)$ is discontinuous. (19.9) rests on the fact (whose proof is trivial) that if $f_i^{(m)}(t)$ tends uniformly to the continuous $f_i(t)$ when $m\to\infty$, then if $t_m\to t$ as $m\to\infty$, we have $\lim f_i^{(m)}(t_m) = f_i(t)$ for $m\to\infty$.

The assertion is now easily proved that if

$$(19.10) \qquad x_i = \Re F_i(w)$$

are the harmonic functions determined by $g_i(\theta)$, then

$$\sum_{i=1}^{n} F_i'^2(w) = 0, \tag{19.11}$$

so that the surface (19.10) is minimal.

For consider (18.2) without the factor w:

$$F_i^{(m)\prime}(w) = \frac{1}{2\pi}\int_C \frac{2e^{i\theta}}{(e^{i\theta} - w)^2} g_i^{(m)}(\theta)d\theta. \tag{19.12}$$

Since all the polygons $\Gamma^{(m)}$ are contained in a finite region of space, the functions $g_i^{(m)}(\theta)$ are uniformly bounded; and if w is any fixed point interior to the unit circle, the denominator $(e^{i\theta}-w)^2$ remains superior in absolute value to a fixed positive quantity when $e^{i\theta}$ describes C. Therefore the integrand in (19.12) remains uniformly bounded during the limit process (19.9); consequently the limit of the integral is equal to the integral of the limit:

$$\lim_{m\to\infty} F_i^{(m)\prime}(w) = F_i'(w). \tag{19.13}$$

It is evident that in case g is improper of the first kind this result is not affected by the circumstance that the points of discontinuity of $g_i(\theta)$ are not considered in the limit relation (19.9), since these points, being at most denumerably infinite in number, form a set of zero measure.

The result (19.11) now follows from (19.13) and the subsistence of (19.5) for every m.

20. **The minimal surface is bounded by Γ.** To show that the minimal surface whose existence is proved in the preceding section is bounded by Γ, we must prove that the representation (19.8) of Γ is proper. That it cannot be improper of the second kind is proved in §18, which, being based on the relation (19.11), applies here with full validity. We cannot however apply the argument of §17 to prove that (19.8) cannot be improper of the first kind. For although we would still have for a g of this kind $A(g)=+\infty$, it would not be true in the case of a general Jordan contour that $A(g)$ sometimes takes finite values.

We therefore use the following argument, based on the relation (19.11), to obtain the desired result. Suppose that under g the point P of C corresponds to the arc $Q'Q''$ of Γ. Since Γ is a Jordan curve, Q' and Q'' are distinct: and if a_i denote the coördinates of Q', b_i of Q'', the distance $Q'Q''$ or l with

$$l^2 = \sum_{i=1}^{n}(b_i - a_i)^2 \tag{20.1}$$

is not equal to zero.

There is no loss of generality in supposing P to be at $w=1$, for this may be achieved by a rotation of the unit circle, which changes nothing essential.

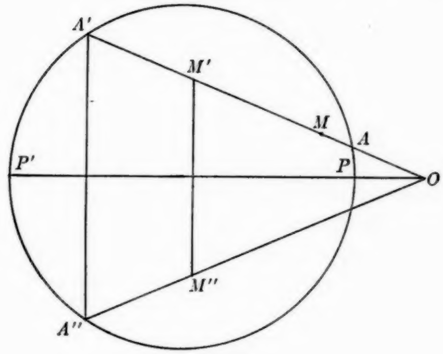

FIG. 4

Let O be any point on the diameter $P'P$ prolonged; then we define the point transformation $M \to M'$ (Fig. 4) by the condition

$$OM \cdot OM' = OA \cdot OA'. \tag{20.2}$$

If this be combined with a reflection $M' \to M''$ in $P'P$, the resulting transformation $M \to M''$, to be called $\mathfrak{T}$, is a conformal transformation converting the interior and circumference of the unit circle into themselves respectively. $\mathfrak{T}$ has the linear fractional form

$$w' = \frac{aw + b}{cw + d}. \tag{20.3}$$

Suppose $\mathfrak{T}$ to act on the functions $F_i(w)$ of (19.10) and on the boundary values $g_i(\theta)$ of $\Re F_i(w)$. Then $F_i(w)$ is transformed into

$$G_i(w) = F_i\left(\frac{aw + b}{cw + d}\right), \tag{20.4}$$

and the boundary values of $\Re G_i(w)$ are $h_i(\theta)$ defined by

$$h_i(A'') = h_i(A). \tag{20.5}$$

Differentiating (20.4), we find

$$G_i'(w) = \frac{ad - bc}{(cw + d)^2} F_i'\left(\frac{aw + b}{cw + d}\right),$$

whence

$$\sum_{i=1}^{n} G_i'^2(w) = \frac{(ad-bc)^2}{(cw+d)^4} \sum_{i=1}^{n} F_i'^2\left(\frac{aw+b}{cw+d}\right).$$

Since, according to (19.11), $\sum_{i=1}^{n} F_i'^2(w)=0$ identically in the interior of the unit circle, we have, also identically in the interior of the unit circle,

$$(20.6) \qquad \sum_{i=1}^{n} G_i'^2(w) = 0.$$

Imagine now that O *takes a sequence of positions tending to* P *as limit.* Then we obtain a sequence of functions $G_i(w)$ constantly obeying (20.6). Consider the boundary values $h_i(\theta)$ of $\Re G_i(w)$ as defined by (20.5). It is evident that if A'' is any fixed point of the circumference, its image A by $\mathfrak{T}^{-1}$ tends to P from below or above according as A'' lies on the upper or lower semi-circumference PP'. Hence, on account of the distinct limiting values, a_i and b_i, of $g_i(\theta)$ on the different sides of P, it follows that $h_i(\theta)$ tends to a function equal to the constant a_i on the upper semi-circumference, and to the constant b_i on the lower semi-circumference.

We have by (19.12)

$$G_i'(w) = \frac{1}{\pi}\int_C \frac{e^{i\theta}}{(e^{i\theta}-w)^2} h_i(\theta)d\theta.$$

Since $h_i(\theta)$, because it always represents Γ, remains uniformly bounded, we may, as in connection with (19.12), pass to the limit under the integral sign; and so find that with the approach of O to P,

$$\begin{aligned} G_i'(w) &\to \frac{1}{\pi}\int_0^{\pi} \frac{e^{i\theta}}{(e^{i\theta}-w)^2} a_i d\theta + \frac{1}{\pi}\int_{\pi}^{2\pi} \frac{e^{i\theta}}{(e^{i\theta}-w)^2} b_i d\theta \\ &= \frac{2i(b_i-a_i)}{\pi}\cdot\frac{1}{1-w^2}. \end{aligned}$$

Hence, with the notation (20.1),

$$(20.7) \qquad \sum_{i=1}^{n} G_i'^2(w) \to \frac{-4l^2}{\pi^2}\cdot\frac{1}{(1-w^2)^2}.$$

Here we have contradiction, for since $l\neq 0$, the last condition is incompatible with the fact that in the passage to the limit (20.6) is constantly obeyed.

Hence the representation $x_i = g_i(\theta)$ of Γ is proper, and the minimal surface determined by it is bounded by Γ.

III. The relation of $A(g)$ to the area functional

It is customary to characterize a minimal surface by the property of rendering area a minimum. Instead, we have minimized the simpler functional $A(g)$. This leads to the question of the relationship between these two functionals, with which this Part is occupied. We shall develop certain formulas interesting in themselves, and useful in Parts IV and V. The reader who is interested in the application of the present theory to conformal mapping and in the proof of the least-area property of the minimal surface whose existence has just been established may go on at once to Parts IV and V, referring back to this Part for the necessary results.

21. **Other forms of $A(g)$.** In this section we derive two further formulas for $A(g)$. The first of these is

$$A(g) = \iint_D \tfrac{1}{2} \sum_{i=1}^{n} | F_i'(w) |^2 d\sigma, \tag{21.1}$$

where D denotes the interior of the unit circle, of which $d\sigma$ is the element of area. The second expresses $A(g)$ in terms of the Fourier coefficients of g:

$$A(g) = \frac{\pi}{2} \sum_{m=1}^{\infty} m \sum_{i=1}^{n} (a_{im}^2 + b_{im}^2). \tag{21.2}$$

Proof of (21.1). Since the integrand of (21.1) may not be bounded in D, we need the following limit process to define the double integral. Let D_ρ denote the interior of the circle C_ρ of radius $\rho < 1$ concentric with C. Then

$$\iint_{D_\rho} \tfrac{1}{2} \sum_{i=1}^{n} | F_i'(w) |^2 d\sigma$$

is evidently an increasing function of ρ, since the element of integration is positive. Hence the limit as $\rho \to 1$ of this double integral exists, finite or infinite, and we define

$$\iint_D \tfrac{1}{2} \sum_{i=1}^{n} | F_i'(w) |^2 d\sigma = \lim_{\rho \to 1} \iint_{D_\rho} \tfrac{1}{2} \sum_{i=1}^{n} | F_i'(w) |^2 d\sigma. \tag{21.3}$$

Suppose $F_i(w)$ separated into its real and imaginary parts:

$$F_i(w) = U_i(u, v) + iV_i(u, v) \qquad (w = u + iv).$$

Then

$$F_i'(w) = \frac{\partial U_i}{\partial u} - i \frac{\partial U_i}{\partial v}. \tag{21.4}$$

Therefore

$$\sum_{i=1}^{n} | F_i'(w) |^2 = \sum_{i=1}^{n} \left[\left(\frac{\partial U_i}{\partial u} \right)^2 + \left(\frac{\partial U_i}{\partial v} \right)^2 \right]. \tag{21.5}$$

Since U_i is a harmonic function, we have by Green's formula†

$$\iint_{D_\rho} \left[\left(\frac{\partial U_i}{\partial u} \right)^2 + \left(\frac{\partial U_i}{\partial v} \right)^2 \right] d\sigma = \int_{C_\rho} U_i \frac{\partial U_i}{\partial \rho} ds.$$

Whence, applying (21.5),

$$\iint_{D_\rho} \frac{1}{2} \sum_{i=1}^{n} | F_i'(w) |^2 d\sigma = \frac{1}{2} \int_0^{2\pi} \sum_{i=1}^{n} U_i \frac{\partial U_i}{\partial \rho} \rho d\theta. \tag{21.6}$$

For U_i there is the formula

$$U_i = \frac{a_{i0}}{2} + \sum_{p=1}^{\infty} \rho^p (a_{ip} \cos p\theta + b_{ip} \sin p\theta), \tag{21.7}$$

derived by taking the real part of (11.1) after writing $w = \rho e^{i\theta}$. This series is uniformly convergent for all θ and for $\rho \leqq \rho_0$, with ρ_0 fixed and <1. This observation makes it legitimate to perform term by term the operations of differentiation, multiplication and integration indicated in the second member of (21.6). We get first

$$\rho \frac{\partial U_i}{\partial \rho} = \sum_{q=1}^{\infty} q \rho^q (a_{iq} \cos q\theta + b_{iq} \sin q\theta), \tag{21.8}$$

and then, observing the relations

$$\int_0^{2\pi} \cos p\theta \sin q\theta d\theta = 0, \int_0^{2\pi} \cos p\theta \cos q\theta d\theta = \int_0^{2\pi} \sin p\theta \sin q\theta d\theta = \begin{cases} \pi \text{ if } p = q, \\ 0 \text{ if } p \neq q, \end{cases}$$

we obtain from (21.6), (21.7), (21.8):

$$\iint_{D_\rho} \frac{1}{2} \sum_{i=1}^{n} | F_i'(w) |^2 d\sigma = \frac{\pi}{2} \sum_{m=1}^{\infty} m \rho^{2m} \sum_{i=1}^{n} (a_{im}^2 + b_{im}^2). \tag{21.9}$$

This infinite series may be expressed as a double integral (formula (21.12)) in the following way. From the expressions (11.2) of the Fourier coefficients we easily derive in a manner analogous to (11.9):‡

† See Picard, *Traité d'Analyse*, vol. 2 (3d edition, Paris, 1925), p. 22.

‡ Cf. Picard, *Traité d'Analyse*, vol. 1 (3d edition, Paris, 1922), p. 341.

$$(21.10)\qquad \begin{aligned} &m\rho^{2m}\sum_{i=1}^{n}(a_{im}^2+b_{im}^2)\\ &\quad = -\frac{1}{2\pi^2}\int_0^{2\pi}\int_0^{2\pi}\sum_{i=1}^{n}[g_i(\theta)-g_i(\phi)]^2\cdot m\rho^{2m}\cos m(\theta-\phi)\,d\theta d\phi. \end{aligned}$$

Now by writing

$$z = \rho^2 e^{i(\theta-\phi)}$$

in the formula

$$\sum_{m=1}^{\infty} m z^m = \frac{z}{(1-z)^2}$$

and taking real parts, we obtain after some reduction

$$(21.11)\qquad \sum_{m=1}^{\infty} m\rho^{2m}\cos m(\theta-\phi) = -\rho^2\frac{(1+\rho^2)^2\sin^2\dfrac{\theta-\phi}{2}-(1-\rho^2)^2\cos^2\dfrac{\theta-\phi}{2}}{\left[(1+\rho^2)^2\sin^2\dfrac{\theta-\phi}{2}+(1-\rho^2)^2\cos^2\dfrac{\theta-\phi}{2}\right]^2}.$$

The convergence, moreover, is uniform for all θ, ϕ if ρ be regarded as fixed <1, since the convergent series of constant positive terms

$$\sum_{m=1}^{\infty} m\rho^{2m}$$

is majorant for (21.11). Hence, after multiplying (21.11) by the bounded factor

$$-\frac{1}{2\pi^2}\sum_{i=1}^{m}[g_i(\theta)-g_i(\phi)]^2,$$

which does not disturb the uniform convergence, we may integrate term by term, and get, with attention to (21.10), (21.9),

$$(21.12)\qquad \begin{aligned} &\iint_{D_\rho}\frac{1}{2}\sum_{i=1}^{n}|F_i'(w)|^2 d\sigma\\ &\quad = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\sum_{i=1}^{n}[g_i(\theta)-g_i(\phi)]^2\,\rho^2\\ &\quad\cdot\frac{(1+\rho^2)^2\sin^2\dfrac{\theta-\phi}{2}-(1-\rho^2)^2\cos^2\dfrac{\theta-\phi}{2}}{\left[(1+\rho^2)^2\sin^2\dfrac{\theta-\phi}{2}+(1-\rho^2)^2\cos^2\dfrac{\theta-\phi}{2}\right]^2}\,d\theta d\phi. \end{aligned}$$

If in the last expression we write formally $\rho=1$, there results

$$\frac{1}{4\pi}\int_0^{2\pi}\int_0^{2\pi}\frac{\sum_{i=1}^{n}[g_i(\theta)-g_i(\phi)]^2}{4\sin^2\dfrac{\theta-\phi}{2}}\,d\theta\,d\phi = A(g). \tag{21.13}$$

To prove that (21.13) is the limit of (21.12) when $\rho\to 1$, we first observe that if the second member of (21.12) and the first of (21.13) be abbreviated respectively as

$$\frac{1}{4\pi}\iint J(\rho;\theta,\phi)d\theta\,d\phi,\qquad \frac{1}{4\pi}\iint I(\theta,\phi)d\theta\,d\phi,$$

then for all ρ and $\theta\neq\phi$

$$|J(\rho;\theta,\phi)| < I(\theta,\phi), \tag{21.14}$$

since the ratio

$$\frac{|J(\rho;\theta,\phi)|}{I(\theta,\phi)} = \frac{4\rho^2\sin^2\dfrac{\theta-\phi}{2}}{(1+\rho^2)^2\sin^2\dfrac{\theta-\phi}{2}+(1-\rho^2)^2\cos^2\dfrac{\theta-\phi}{2}}\cdot\frac{\left|(1+\rho^2)^2\sin^2\dfrac{\theta-\phi}{2}-(1-\rho^2)^2\cos^2\dfrac{\theta-\phi}{2}\right|}{(1+\rho^2)^2\sin^2\dfrac{\theta-\phi}{2}+(1-\rho^2)^2\cos^2\dfrac{\theta-\phi}{2}}$$

is less than 1 (each denominator minus the corresponding numerator gives a positive quantity).

Suppose, as first case, that $A(g)$ is finite. Then (21.14) expresses that in the approach of ρ to 1 the absolute value of $J(\rho;\theta,\phi)$ remains less than a summable function, and hence a theorem of Lebesgue† permits us to pass to the limit under the integral sign:

$$\lim_{\rho\to 1}\iint J(\rho;\theta,\phi)d\theta d\phi = \iint I(\theta,\phi)d\theta d\phi,$$

† Lebesgue, *Leçons sur l'Intégration* (2d edition, Paris, 1928), p. 131. In the present case all the functions involved are Riemann integrable (proper or improper), and the result in question may easily be established without recourse to the notion of Lebesgue integral.

that is,

$$\lim_{\rho\to 1}\iint_{D_\rho}\frac{1}{2}\sum_{i=1}^{n}|F_i'(w)|^2 d\sigma \tag{21.15}$$
$$=\iint_{D}\frac{1}{2}\sum_{i=1}^{n}|F_i'(w)|^2 d\sigma = A(g).$$

Thus (21.1) is proved in the case $A(g)$ finite. If $A(g)=+\infty$, a simple way of seeing that the $\iint_D$ is then also $=+\infty$ is to observe that $\iint_D$, as the limit of $\iint_{D_\rho}$ which tends to it in increasing, is a lower semi-continuous functional of g, like $A(g)$. This remains true if g is not restricted to represent Γ but may be any parameterized contour. With this understanding, a g such that $A(g)=+\infty$ may always be approached by a sequence of g's such that $A(g)$ is finite. It is to be seen immediately that two lower semi-continuous functionals which coincide whenever the first has a finite value must also coincide in case this value is $+\infty$.

Proof of (21.2). Returning to (21.4) and (21.9), we have

$$\iint_D \frac{1}{2}\sum_{i=1}^{n}|F_i'(w)|^2 = \lim_{\rho\to 1}\frac{\pi}{2}\sum_{m=1}^{\infty} m\rho^{2m}\sum_{i=1}^{n}(a_{im}^2+b_{im}^2).$$

Now

$$\lim_{\rho\to 1}\frac{\pi}{2}\sum_{m=1}^{\infty} m\rho^{2m}\sum_{i=1}^{n}(a_{im}^2+b_{im}^2) = \frac{\pi}{2}\sum_{m=1}^{\infty} m\sum_{i=1}^{n}(a_{im}^2+b_{im}^2).$$

If the latter series is convergent, the justification for this is Abel's theorem asserting the continuity of the power series in the first member at the point of convergence $\rho=1$. If the second member equals $+\infty$, it is easy to show, by taking account of the positive nature of all the terms, that this is also the value of the first member.

Combining the last two equations with (21.1), we have the desired result:

$$A(g) = \frac{\pi}{2}\sum_{m=1}^{\infty} m\sum_{i=1}^{n}(a_{im}^2+b_{im}^2).$$

22. **The area functional $S(g)$ and its relation to $A(g)$.** We have seen how every representation g of Γ determines a harmonic surface

$$x_i = \Re F_i(w) = U_i(u, v).$$

The linear element of this surface is

$$ds^2 = E\,du^2 + 2F\,du\,dv + G\,dv^2$$

with

$$E = \sum_{i=1}^{n}\left(\frac{\partial U_i}{\partial u}\right)^2,\quad F = \sum_{i=1}^{n}\frac{\partial U_i}{\partial u}\frac{\partial U_i}{\partial v},\quad G = \sum_{i=1}^{n}\left(\frac{\partial U_i}{\partial v}\right)^2, \tag{22.1}$$

and its area is a functional of g which we denote by $S(g)$:

$$S(g) = \iint_D (EG - F^2)^{1/2} d\sigma. \tag{22.2}$$

We have from (21.5)

$$\sum_{i=1}^{n} |F_i'(w)|^2 = \sum_{i=1}^{n} \left(\frac{\partial U_i}{\partial u}\right)^2 + \sum_{i=1}^{n} \left(\frac{\partial U_i}{\partial v}\right)^2 = E + G,$$

so that by (21.1)

$$A(g) = \iint_D \tfrac{1}{2}(E + G) d\sigma. \tag{22.3}$$

To the end of comparing the integrands in (22.2), (22.3), we observe

$$\tfrac{1}{4}(E + G)^2 - (EG - F^2) = \tfrac{1}{4}(E - G)^2 + F^2 \geqq 0;$$

therefore

$$\tfrac{1}{2}(E + G) \geqq (EG - F^2)^{1/2}, \tag{22.4}$$

the equality holding when and only when

$$E - G = 0, \quad F = 0. \tag{22.5}$$

Since by (21.4) and (22.1),

$$\sum_{i=1}^{n} F_i'^2(w) = (E - G) - 2iF,$$

the conditions (22.5) are equivalent to

$$\sum_{i=1}^{n} F_i'^2(w) = 0, \tag{22.6}$$

characteristic of a minimal surface.

Consequently,

$$A(g) \geqq S(g), \tag{22.7}$$

and the equality holds when and only when the harmonic surface determined by g is minimal.

IV. The Riemann mapping theorem and the theorem of Osgood and Carathéodory

23. **The case $n=2$ of the problem of Plateau.** Let the contour Γ be any Jordan curve in the plane (x_1, x_2), Δ the region bounded by Γ, C the unit

circle, and D the interior of the unit circle. Then the classic Riemann mapping theorem states the existence of a one-one continuous and conformal correspondence between D and Δ. According to a theorem of Osgood† and Carathéodory,† it is possible to supplement this conformal map with a one-one continuous correspondence between C and Γ, so that the combination is one-one and continuous between $D+C$ and $\Delta+\Gamma$.

We show in this Part how by merely writing $n=2$ in the preceding work we have an immediate proof of the theorem of Riemann together with the theorem of Osgood-Carathéodory.

In the cited papers of the last two authors a sharp distinction is drawn between the "interior problem" and the "boundary problem." The existence of a conformal map of the interiors is supposed already established by the classic methods, and these authors then proceed to prove that this map of the interiors induces by continuity a topological correspondence between the boundaries.

It is characteristic of the method of the present paper to follow a directly opposite procedure: namely, we first distinguish a certain topological correspondence between the boundaries by the property of rendering $A(g)$ a minimum; this topological correspondence found, the conformal map of the interiors can be expressed immediately (see the theorem stated at the end of this Part).

The work of Parts I and II, with $n=2$, assures us of the existence of a certain proper representation of Γ,

$$x_1 = g_1^*(\theta), \quad x_2 = g_2^*(\theta), \tag{23.1}$$

such that if

$$x_1 = \Re F_1(w), \quad x_2 = \Re F_2(w) \tag{23.2}$$

are the harmonic functions determined by the boundary values (23.1), we have

$$F_1'^2(w) + F_2'^2(w) = 0. \tag{23.3}$$

The functions F_1, F_2 are given by the formula

$$F_1(w) = \frac{1}{2\pi}\int_C \frac{e^{i\theta}+w}{e^{i\theta}-w} g_1^*(\theta)d\theta, \quad F_2(w) = \frac{1}{2\pi}\int_C \frac{e^{i\theta}+w}{e^{i\theta}-w} g_2^*(\theta)d\theta. \tag{23.4}$$

From (23.3),

$$F_1'(w) = \pm\, iF_2'(w),$$

† Reference in the Introduction.

and choosing the + sign (the − sign will lead to an inversely conformal transformation, easily discussed), we have by integration

$$F_1(w) = iF_2(w) + a + ib$$

where a, b are real constants. Separating F_1, F_2 into their real and imaginary parts:

$$F_1 = U_1 + iV_1, \qquad F_2 = U_2 + iV_2,$$

this gives

$$U_1 = - V_2 + a, \qquad V_1 = U_2 + b.$$

Consequently,

$$x_1 + ix_2 = U_1 + iU_2 = U_1 + iV_1 - ib = F_1(w) - ib = iF_2(w) + a.$$

Denote by $F(w)$ the common value of the last two expressions:

$$F(w) = F_1(w) - ib, \quad F(w) = iF_2(w) + a; \tag{23.5}$$

then

$$x_1 + ix_2 \equiv W = F(w), \tag{23.6}$$

a holomorphic function of w in the interior D of the unit circle. It will therefore be proved that the transformation defined by (23.6) or (23.2) is conformal in the domain D after we have shown a little later that $F'(w) \neq 0$ at any point of D.

We will first prove that (23.6) maps D in a one-one way on Δ. To this end, let W_0 be any point in the complex plane $x_1+ix_2 \equiv W$ not on Γ; what has to be shown is that the equation

$$F(w) = W_0 \tag{23.7}$$

has exactly one solution w in D if W_0 belongs to Δ, and no solution w in D if W_0 does not belong to Δ.

Certainly there are only a finite number of solutions of (23.7) in any circle concentric with and smaller than C; therefore we can construct a sequence of circles C_ρ concentric with C and with radii ρ increasing to 1 as limit, such that no solution of (23.7) lies on a circumference C_ρ. The number of solutions of (23.7) in the interior of C_ρ is given by the formula of Cauchy:

$$N_\rho = \frac{1}{2\pi i}\int_{C_\rho} \frac{F'(w)dw}{F(w) - W_0}, \tag{23.8}$$

applicable here with full validity because C_ρ is *interior* to a simply connected domain of regularity of $F(w)$. The number of solutions of (23.7) in the interior of C is evidently

$$N = \lim_{\rho \to 1} N_\rho. \tag{23.9}$$

With $W = F(w)$, formula (23.8) gives

$$N_\rho = \frac{1}{2\pi i}[\log (W - W_0)]_{\Gamma_\rho} = \text{order of } W_0 \text{ with respect to } \Gamma_\rho. \tag{23.10}$$

Here Γ_ρ denotes the closed analytic curve† which is the image of C_ρ by the transformation $W = F(w)$, the bracket denotes the variation of $\log (W - W_0)$ when W describes Γ_ρ, and the order of W_0 with respect to Γ_ρ is an integer equal to $1/(2\pi)$ times the variation in the angle made by the vector W_0W with a fixed direction, followed continuously while W describes Γ_ρ.

Now when $\rho \to 1$, Γ_ρ tends uniformly to Γ, for the formulas (23.2), (23.4) are equivalent to Poisson's integral, and the boundary functions (23.1) are continuous. Evidently then, the order of W_0 with respect to Γ_ρ tends to the order of W_0 with respect to Γ; indeed, for ρ near enough to 1 the former remains equal to the latter. Hence by (23.9), (23.10),

$$N = \text{order of } W_0 \text{ with respect to } \Gamma = \begin{cases} 1 \text{ if } W_0 \text{ is interior to } \Gamma, \\ 0 \text{ if } W_0 \text{ is exterior to } \Gamma. \end{cases} \tag{23.11}$$

According to this, the image of a point w of D is never a point exterior to Γ. But neither can it be a point Q of Γ. For then a neighborhood of w would go over into a neighborhood of Q;‡ now every neighborhood of Q contains points exterior to Γ, so that we would have contradiction with the first statement of this paragraph. Hence the image of any point w of D is a point interior to Γ. Furthermore, by (23.11) every point interior to Γ is obtained, and exactly once, as image of a point w. Therefore the transformation $W = F(w)$ is one-one as between D and Δ.

To prove that this transformation is conformal without singular points, we must show that we cannot have $F'(w) = 0$ at any point w of D. If $F'(w) = 0$, then a neighborhood of w is mapped on a multiply-covered neighborhood of W,§ but this contradicts the proof just given of the one-one nature of the correspondence between D and Δ.

That the conformal correspondence thus established between D and Δ attaches continuously to the topological correspondence (23.1) between C and Γ is an immediate consequence of the remark, already made, that the

† It will result from the sequel that Γ_ρ does not intersect itself but this fact is not necessary for the present argument.

‡ This is by the region-preserving property (Gebietstreu) of transformations $W = F(w)$; cf. L. Bieberbach, *Lehrbuch der Funktionentheorie*, Leipzig and Berlin, 1921, pp. 187–188.

§ Bieberbach, loc. cit., p. 188.

formulas (23.2), (23.4) are equivalent to Poisson's integral based on the continuous boundary functions (23.1).

In sum, we have proved the combined theorems of Riemann and Osgood-Carathéodory.

Expression for $F(w)$. An expression for $F(w)$ in the Cauchy form, more elegant than (23.2), (23.4), where real and imaginary parts are separated, may be obtained as follows.

Let w be a fixed point of D; take $\rho > |w|$ and <1; then by the formula of Cauchy,

$$F(w) = \frac{1}{2\pi i}\int_{C_\rho} \frac{F(z)dz}{z - w}\,.$$

If now $\rho \to 1$, then $F(z)$ tends uniformly to

$$f^*(z) = g_1^*(\theta) + ig_2^*(\theta)$$

and $1/(z-w)$ (z on C_ρ) tends uniformly to $1/(z-w)$ (z on C), wherein corresponding points on C_ρ and C are those with the same angular coördinate. Therefore

$$(23.12) \qquad F(w) = \frac{1}{2\pi i}\int_{C} \frac{f^*(z)dz}{z - w}\,.$$

24. **Range of values of** $A(g)$. For a Jordan curve *in the plane*, we can easily obtain the exact range of values of $A(g)$, and see that finite values always occur among them.

For since the functions $F_1(w)$, $F_2(w)$ determined by the representation g^* (23.1) obey the condition (23.3) $F_1'^2(w)+F_2'^2(w)=0$, we have by the final statement of Part III:

$$A(g^*) = \text{inner area of } \Gamma.$$

The *inner* area† must be taken because $S(g)$, as defined by (22.2), is the limit of the area bounded by Γ_ρ, which approaches to Γ from its interior.

To see that $A(g)$ takes every value in the interval

$$\text{inner area of } \Gamma \leqq A(g) \leqq +\infty$$

(and, of course, no other values), consider a continuous series of representations g connecting g^* with a representation g^0 such that $A(g^0) = +\infty$ (example: g^0 improper of the first kind); it is easy to arrange that $A(g)$ be continuous on this series of g's.

† The region bounded by a Jordan curve has in general distinct inner and outer areas, differing by an amount called the exterior content of the curve. The first example of a Jordan curve of positive exterior content was given by Osgood, these Transactions, vol. 4 (1903), pp. 107–112.

25. The combined interior-boundary conformal mapping theorem. The results of this Part are summarized in the following theorem, combining the theorems of Riemann and Osgood-Carathéodory.

THEOREM. *Let* Γ *denote any Jordan curve in the plane, and*

$$x_1 = g_1(\theta), \quad x_2 = g_2(\theta),$$

or

$$Z = f(z)$$

with

$$Z = x_1 + ix_2, \quad z = e^{i\theta},$$

an arbitrary representation of Γ *as topological image of the unit circle* C.

The range of values of the functional

$$A(g) = \frac{1}{16\pi}\int_C \int_C \frac{\sum_{i=1}^{2}[g_i(\theta) - g_i(\phi)]^2}{\sin^2 \dfrac{\theta - \phi}{2}} d\theta d\phi$$

$$= \frac{1}{4\pi}\int_C \int_C \frac{|f(z) - f(\zeta)|^2}{|z - \zeta|^2} |dz||d\zeta|$$

when g *or* f *varies over all possible representations of* Γ *consists of all positive real numbers (including* $+\infty$*) greater than or equal to the inner area of the region enclosed by* Γ. *This minimum value is attained for a certain proper representation*

$$Z = f^*(z)$$

(determined up to linear fractional transformation of C *into itself,*

$$z' = (az+b)/(\bar{b}z+\bar{a})).$$

Then the transformation $w \to W$ *defined by*

$$W = \frac{1}{2\pi i}\int_C \frac{f^*(z)dz}{z - w}$$

produces a one-one, continuous and conformal map of the interior of Γ *on the interior of* C, *which attaches continuously to the topological correspondence* $Z = f^*(z)$ *between* Γ *and* C.

The implications of this theorem for the Dirichlet problem are apparent. The Dirichlet problem for any continuous distribution of assigned values on a Jordan boundary is reduced to the same problem for a circle, and therefore solved by Poisson's integral. The Dirichlet functional

$$D(\phi) = \iint_\Delta \left[\left(\frac{\partial\phi}{\partial x_1}\right)^2 + \left(\frac{\partial\phi}{\partial x_2}\right)^2\right] dx_1 dx_2$$

is replaced by the functional $A(g)$, much simpler to deal with chiefly because the range of g is a compact closed set while the range of ϕ is not. It was the latter fact alone which rendered valid the criticism of Weierstrass against Riemann's treatment of the Dirichlet problem based on minimizing the Dirichlet functional.

V. Absolute minimum of area

26. **Proof of the least-area property.** We conclude this paper with the following brief proof, based on the formulas of Part III, that the minimal surface whose existence is proved in Part I has the least area of any surface bounded by the given contour.

Let the contour Γ be, first, a polygon. Then let Π be any polyhedral surface bounded by Γ. By the conformal mapping theorem of Koebe,† Π can be mapped conformally on the interior of the unit circle, the map attaching continuously to a topological correspondence between Γ and the circumference. Let the parametric equations of Π determined by this map be

$$x_i = x_i(u, v)$$

and of Γ

$$x_i = g_i(\theta). \tag{26.1}$$

The element of length of Π is

$$ds^2 = Edu^2 + 2F\,du\,dv + G\,dv^2$$

with

$$E = \sum_{i=1}^n \left(\frac{\partial x_i}{\partial u}\right)^2, \quad F = \sum_{i=1}^n \left(\frac{\partial x_i}{\partial u}\right)\left(\frac{\partial x_i}{\partial v}\right), \quad G = \sum_{i=1}^n \left(\frac{\partial x_i}{\partial v}\right)^2;$$

and by the conformality

$$E = G, \quad F = 0;$$

so that the area of Π is

$$S(\Pi) = \iint (EG - F^2)^{1/2} du dv = \frac{1}{2}\iint (E + G) du dv \tag{26.2}$$

$$= \frac{1}{2}\sum_{i=1}^n \iint \left\{\left(\frac{\partial x_i}{\partial u}\right)^2 + \left(\frac{\partial x_i}{\partial v}\right)^2\right\} du dv.$$

† Cf. T. Radó, Annals of Mathematics, (2), vol. 31 (1930), pp. 458–460.

Consider the harmonic surface determined by the representation (26.1) of Γ, and denote by $\bar{E}$, $\bar{F}$, $\bar{G}$ its fundamental quantities; then since a harmonic function gives the least value to the Dirichlet functional for fixed boundary values, we have

$$(26.3)\qquad \frac{1}{2}\iint(\bar{E}+\bar{G})dudv \leqq \frac{1}{2}\iint(E+G)dudv.$$

By formula (22.3) of Part III,

$$(26.4)\qquad A(g) = \frac{1}{2}\iint(\bar{E}+\bar{G})dudv;$$

and by the minimum property of g^*,

$$(26.5)\qquad A(g^*) \leqq A(g).$$

By the chain (26.2)–(26.5),

$$(26.6)\qquad A(g^*) \leqq S(\Pi):$$

the minimum value of $A(g) \leqq$ the area of any polyhedral surface bounded by the polygon Γ.

Let now Γ denote an arbitrary Jordan curve and Σ any simply-connected surface bounded by Γ. According to the Lebesgue definition, the area $S(\Sigma)$ of Σ is the lower limit (finite or $+\infty$) of the areas of polyhedral surfaces which tend to Σ. There exists, then, a sequence of polyhedral surfaces Π_m tending to Σ such that

$$(26.7)\qquad S(\Pi_m) \to S(\Sigma);$$

the bounding polygons $\Gamma^{(m)}$ tend to Γ.

Each polygon $\Gamma^{(m)}$ has a representation

$$(26.8)\qquad x_i = g_i^{(m*)}(\theta)$$

minimizing $A(g)$ for that polygon, and by the procedure of §19 we can select a sub-sequence

$$(26.9)\qquad m = m_1,\ m_2,\ \cdots,\ m_k,\ \cdots$$

so that (26.8) tends to a proper representation of Γ

$$(26.10)\qquad x_i = g_i^*(\theta).$$

The harmonic surface determined by g^* will be minimal; consequently, by (22.7),

$$(26.11)\qquad S(g^*) = A(g^*).$$

From the sequence (26.9) we can in turn select a sub-sequence

$$(26.12)\qquad m = m_1',\ m_2',\ \cdots,\ m_k',\ \cdots$$

so that $A(g^{(m)*})$ tends to a limit (finite or $+\infty$):

(26.13) $$A(g^{(m)*}) \to L.$$

By the lower semi-continuity of $A(g)$, which holds for any sort of approach of one parameterized contour to another, as well when the contour itself is allowed to vary as when we have merely different parameterizations of the same contour, it follows (see (7.2)) that

(26.14) $$A(g^*) \leqq L.$$

By (26.6),

$$A(g^{(m)*}) \leqq S(\Pi_m);$$

hence

$$\lim A(g^{(m)*}) \leqq \lim S(\Pi_m),$$

that is

(26.15) $$L \leqq S(\Sigma).$$

Combining this with (26.14) and (26.11), we obtain

(26.16) $$S(g^*) \leqq S(\Sigma),$$

which was to be proved.

It is easy to see that the g^* here obtained from approaching polygons is the same as the g^* which minimizes $A(g)$ for Γ. For let g^{**} minimize $A(g)$ for Γ; then

(26.17) $$A(g^{**}) \leqq A(g^*).$$

By (26.16),

(26.18) $$S(g^*) \leqq S(g^{**}),$$

and by (22.7)

$$A(g^{**}) = S(g^{**}) \text{ as well as } A(g^*) = S(g^*),$$

so that (26.18) may be expressed as

(26.19) $$A(g^*) \leqq A(g^{**}).$$

From (26.19) and (26.17) it follows that

$$A(g^*) = A(g^{**}).$$

27. **A non-finite-area-spanning Jordan contour.** It is important to note that if the Jordan contour is sufficiently tortuous, the least-area property of the minimal surface may become vacuous through the impossibility of spanning any finite area whatever within the given contour. The following example was constructed by the author in collaboration with P. Franklin.

Consider a broken line, broken at right angles and in the form of a spiral, whose successive segments have the lengths $1, 1/2^{1/2}, 1/3^{1/2}, \cdots$. On each segment construct a square lying towards the outside of the spiral. In each square let there be a Peano curve starting at the initial point and ending at the terminal point of the corresponding segment. Let the unit interval $0 \leqq t \leqq 1$ be divided by the points $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots$ into a denumerable infinity of sub-intervals, and let the nth Peano curve be represented as continuous image of the nth interval. If to the equations $x=\phi(t)$, $y=\psi(t)$, representing all these Peano curves laid end to end, we adjoin $z=t$, then we have a Jordan arc in space, being a one-one continuous image of the unit t-interval, end point $t=1$ included. The desired example is formed of the four Jordan arcs $x=\phi(t)$, $y=\psi(t)$, $z=t$; $x=\phi(t)$, $y=\psi(t)$, $z=2-t$; $x=-\phi(t)$, $y=-\psi(t)$, $z=t$; $x=-\phi(t)$, $y=-\psi(t)$, $z=2-t$.

The proof results from the fact that the content of the orthogonal projection of this Jordan curve on the xy-plane, counting each point with its proper multiplicity, is four times the sum of the harmonic series. A fortiori, the content of the orthogonal projection of any surface spanned with the curve is $+\infty$, and this is, a fortiori, the area of the surface.

It thus appears that the separation of the existence proof into Parts I, II is inherent in the very nature of the problem, since it is futile to try to create distinctions with a functional which is identically $+\infty$. The limit process is absolutely essential for a non-finite-area-spanning Jordan contour, and the minimal surface can then be characterized only by the Weierstrass equations, the minimum-area property becoming meaningless.†

The corresponding situation in the Dirichlet problem is well known, having been pointed out by Hadamard‡ with the following example. If boundary values on the unit circle are defined by

$$f(\theta) = \sum_{p=1}^{\infty} \frac{\cos 2^{2p}\theta}{2^p},$$

then the Dirichlet functional is identically $+\infty$, but a harmonic function as defined by Laplace's equation exists, being

$$\sum_{p=1}^{\infty} \rho^{2^{2p}} \frac{\cos 2^{2p}\theta}{2^p}.$$

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† Bulletin de la Société Mathématique de France, vol. 34 (1906), pp. 135–139.

‡ However, a good sense in which the least-area property continues to hold will be given in a supplementary note.

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ON ORTHOGONAL POLYNOMIALS*

BY

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1. Let $f(z)$ be a function which is analytic inside and on the ellipse C, having the points ± 1 for its foci. We suppose also that, for real x in the interval $(-1, +1)$, $f(x)$ is real. We have

$$f(y) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{z-y}\cdot \tag{1}$$

Consider now the function $p(x)$, summable and not negative in the interval $(-1, +1)$ and satisfying the condition that

$$\int_{-1}^{1}\frac{\log p(x)dx}{(1-x^2)^{1/2}}$$

exists.

Consider also the normal orthogonal polynomials $P_0, P_1(x), P_2(x), \cdots,$

$$P_k(x) = d_0^{(k)}x^k + d_1^{(k)}x^{k-1} + \cdots + d_k^{(k)} \qquad [d_0^{(k)} > 0],$$

corresponding to the characteristic function $p(x)$, i.e.

$$\int_{-1}^{1} p(x)P_k(x)P_s(x)dx = \begin{cases} 0, & k \neq s, \\ 1, & k = s. \end{cases}$$

Then the series

$$\frac{1}{z-y} = \sum_{k=0}^{\infty} a_k P_k(y) \qquad \left(a_k = \int_{-1}^{1}\frac{p(y)P_k(y)}{z-y}dy\right)$$

converges absolutely and uniformly with respect to y, if y lies in any domain lying wholly inside the ellipse C, which passes through the point z and has the points ± 1 for its foci,† i.e.,

$$\left|\frac{y+(y^2-1)^{1/2}}{z+(z^2-1)^{1/2}}\right| < 1-\epsilon \qquad (\epsilon > 0 \text{ arbitrarily small}),$$

$$|z+(z^2-1)^{1/2}| > 1, \ |y+(y^2-1)^{1/2}| \geqq 1.$$

* Presented to the Society, December 30, 1930; received by the editors in June, 1929.

† G. Szegö, *Über die Entwickelung einer analytischen Funktion nach den Polynomen eines Orthogonalsystems*, Mathematische Annalen, vol. 82 (1920), p. 209.

Hence, introducing the functions $Q_k(z)$ of the second kind*

$$a_k = Q_k(z) \qquad (k = 0, 1, 2, 3, \cdots), \tag{2}$$

$$\frac{1}{z-y} = \sum_{k=0}^{\infty} P_k(y)Q_k(z). \tag{3}$$

Inserting this value of $1/(z-y)$ in (1) we get

$$\begin{aligned} f(y) &= \frac{1}{2\pi i}\int_C \left\{ \sum_{k=0}^{\infty} P_k(y)Q_k(z) \right\} f(z)dz = \sum_{k=0}^{\infty} C_k P_k(y), \\ C_k &= \frac{1}{2\pi i}\int_C f(z)Q_k(z)dz \qquad (k = 0, 1, 2, 3, \cdots). \end{aligned} \tag{4}$$

The coefficients C_k may be found independently:

$$C_k = \int_{-1}^{1} p(x)f(x)P_k(x)dx \qquad (k = 0, 1, 2, 3, \cdots), \tag{5}$$

whence we find the relation

$$\frac{1}{2\pi i}\int_C f(z)Q_k(z)dz = \int_{-1}^{1} p(x)f(x)P_k(x)dx \quad (k = 0, 1, 2, 3, \cdots). \tag{6}$$

Put $k=0$ *and* $f(x) = P_s(x)P_r(x)$. Then

$$\begin{aligned} &\frac{1}{2\pi i P_0}\int_C Q_0(z)P_s(z)P_r(z)dz \\ &\qquad = \int_{-1}^{1} p(x)P_s(x)P_r(x)dx = \begin{cases} 0, & s \neq r, \\ 1, & s = r. \end{cases} \end{aligned} \tag{7}$$

This formula shows that *the polynomials* $P_k(x)$, *which are orthogonal and normal in the interval* $(-1, +1)$ *with the characteristic function* $p(x)$, *have the same property* (x *being replaced by the complex variable* z) *on the contour* C *with the characteristic function*

$$\frac{Q_0(z)}{P_0} = \int_{-1}^{1} \frac{p(x)dx}{z-x}.\dagger \tag{8}$$

In particular, we find that *the normalized trigonometric polynomials*

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* G. Darboux, *Mémoire sur l'approximation des fonctions de très grands nombres*, Journal de Mathématiques Pures et Appliquées, (3), vol. 4 (1878), p. 414.

† Cf. J. Sokhotzki, *The Theory of Integral Residues with Applications* (Thesis in Russian), St. Petersburg, 1868, p. 59, where formula (7) was established in a different way.

$$T_k(z) = \left(\frac{2}{\pi}\right)^{1/2} \cos k \arc\cos z$$

$$(9) \qquad = \left(\frac{2}{\pi}\right)^{1/2} \frac{(z + (z^2 - 1)^{1/2})^k + (z - (z^2 - 1)^{1/2})^k}{2}$$

$$\left(k = 1, 2, \cdots; T_0 = \left(\frac{1}{\pi}\right)^{1/2}\right),$$

orthogonal on $(-1, 1)$ *with the characteristic function* $p(z) = 1/(1-z^2)^{1/2}$, *are orthogonal on the contour* C *with the characteristic function* $\pi/(z^2-1)^{1/2}$.

In fact, according to S. Bernstein,* we have

$$(10) \qquad Q_0(z) = \left(\frac{\pi}{z^2 - 1}\right)^{1/2}, \quad Q_k(z) = \frac{(2\pi)^{1/2}}{(z^2 - 1)^{1/2}\{z + (z^2 - 1)^{1/2}\}^k}$$

$$(k = 1, 2, \cdots).$$

2. A well known property of the polynomials (9) is the following. The formal developments

$$f(x) \sim \sum_{k=0}^{\infty} A_k T_k(x), \quad \phi(x) \sim \sum_{k=0}^{\infty} B_k T_k(x)$$

$$\left(A_k = \int_{-1}^{1} \frac{f(x) T_k(x)}{(1 - x^2)^{1/2}} dx, \qquad B_k = \int_{-1}^{1} \frac{\phi(x) T_k(x)}{(1 - x^2)^{1/2}} dx\right)$$

imply, provided the integrals $\int_{-1}^{1} (f^2(x)/(1-x^2)^{1/2})dx$ and $\int_{-1}^{1} (\phi^2(x)/(1-x^2)^{1/2})dx$ exist,

$$(11) \qquad \int_{-1}^{1} \frac{f(x)\phi(x)}{(1-x^2)^{1/2}} dx = \sum_{k=0}^{\infty} A_k B_k.$$

Apply (11) to $f(x) = p(x)(1-x^2)^{1/2}$, $\phi(x) = 1/(z-x)$, assuming that

$$(12) \qquad \int_{-1}^{1} p^2(x)(1 - x^2)^{1/2} dx = \int_{0}^{\pi} p^2(\cos \phi) \sin^2 \phi d\phi$$

exists. Thus we get, writing

$$(13) \qquad p(x)(1 - x^2)^{1/2} \sim \sum_{k=0}^{\infty} c_k T_k(x) \qquad \left(c_k = \int_{-1}^{1} p(x) T_k(x) dx\right),$$

$$(14) \qquad \frac{1}{z - x} = \sum_{k=0}^{\infty} T_k(x) Q_k(z),$$

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* S. Bernstein, *Sur la valeur asymptotique de la meilleure approximation* (in Russian), Proceedings of the Kharkow Mathematical Society, 1913.

$$\frac{Q_0(x)}{P_0(x)} = \int_{-1}^{1} \frac{p(x)(1-x^2)^{1/2}}{z-x} \frac{dx}{(1-x^2)^{1/2}} = \sum_{k=0}^{\infty} c_k Q_k(z) \tag{15}$$

(see (3), $Q_k(z)$ given by (10)).

Hence, the formal trigonometric expansion

$$\begin{gathered} p(\cos\phi)\sin\phi \sim \left(\frac{2}{\pi}\right)^{1/2}\left\{\frac{a_0}{2^{1/2}} + \sum_{k=1}^{\infty} a_k \cos k\phi\right\} \\ \left(a_k = \int_0^{\pi} p(\cos\phi)\sin\phi\cos k\phi d\phi\right) \end{gathered} \tag{16}$$

yields at once, under condition (12), the expansion for $Q_0(z)/P_0$ with the same coefficients:

$$\frac{Q_0(z)}{P_0} = \left(\frac{2\pi}{z^2-1}\right)^{1/2} \cdot \left\{\frac{a_0}{2^{1/2}} + \sum_{k=1}^{\infty} a_k [z-(z^2-1)^{1/2}]^k\right\}. \tag{17}$$

If $p(\cos\phi)\sin\phi$ is a finite trigonometric sum, then $Q_0(z)/P_0$ is also a finite sum. For example, taking

$$p(x) = (1-x^2)^{1/2}, \quad p(\cos\phi)\sin\phi = \sin^2\phi = \tfrac{1}{2} - \tfrac{1}{2}\cos 2\phi, \tag{18}$$

we find

$$\frac{Q_0(z)}{P_0} = \pi\{z-(z^2-1)^{1/2}\}. \tag{19}$$

In other words, *the polynomials*

$$P_0 = \left(\frac{2}{\pi}\right)^{1/2}, \quad P_k(z) = \frac{\{z+(z^2-1)^{1/2}\}^{k+1} - \{z-(z^2-1)^{1/2}\}^{k+1}}{(2\pi)^{1/2}(z^2-1)^{1/2}} \tag{20}$$

$$(k = 1, 2, 3, \cdots)$$

are orthogonal and normal on the contour C with the characteristic function $\pi\{z-(z^2-1)^{1/2}\}$.

3. We proceed to derive some interesting properties of the functions $Q_n(z)$. Darboux has shown* that they satisfy the same recurrence relation as the $P_n(x)$:

$$\begin{gathered} A_{n+1}Q_{n+1}(z) + A_nQ_{n-1}(z) = (B_n + z)Q_n(z) \quad (n = 1, 2, 3, \cdots), \\ A_1Q_1(z) = (B_0 + z)Q_0(z) - \frac{1}{P_0} \qquad (A_i = \text{const.}). \end{gathered} \tag{21}$$

* Loc. cit., p. 415.

We multiply both members of Darboux's formula*

$$(22)\quad \sum_{k=0}^{n} P_k(x)P_k(y) = A_{n+1}\frac{P_{n+1}(x)P_n(y) - P_n(x)P_{n+1}(y)}{x-y} \qquad \left(A_{n+1} = \frac{d_0^{(n)}}{d_0^{(n+1)}}\right)$$

by $p(x)$ and $p(x)/(z-x)$, and integrate between -1 and 1. We get

$$Q_n(z)P_{n+1}(z) - Q_{n+1}(z)P_n(z) = \frac{1}{A_{n+1}}, \dagger$$

$$(23)\quad \sum_{k=0}^{n} P_k(y)Q_k(z) = A_{n+1}\frac{Q_{n+1}(z)P_n(y) - Q_n(z)P_{n+1}(y)}{z-y} + \frac{1}{z-y}. \ddagger$$

Suppose now that $p(-x) \equiv p(x)$ ("symmetric" orthogonal polynomials). Then, as is known,

$$P_k(-x) \equiv (-1)^k P_k(x),$$

and we get from (22), denoting by $[m]$ the greatest integer $\leqq m$,

$$\sum_{s=0}^{[n/2]} P_{2s}(x)P_{2s}(y) = A_{n+1}\frac{xP_{n+1}(x)P_n(y) - yP_n(x)P_{n+1}(y)}{x^2-y^2},$$

which, combined with the recurrence relation for $P_k(x)$, gives

$$(24)\quad \sum_{s=0}^{[n/2]} P_{n-2s}(x)P_{n-2s}(y) = A_{n+1}A_{n+2}\frac{P_{n+2}(x)P_n(y) - P_n(x)P_{n+2}(y)}{x^2-y^2}. \S$$

The same method applied to (23) gives

$$(25)\quad \sum_{s=0}^{n/2} P_{n-2s}(y)Q_{n-2s}(z) = A_{n+1}A_{n+2}\frac{Q_{n+2}(z)P_n(y) - Q_n(z)P_{n+2}(y)}{z^2-y^2} + \frac{z}{z^2-y^2} \qquad (n \text{ even}),$$

$$(26)\quad \sum_{s=0}^{(n-1)/2} P_{n-2s}(y)Q_{n-2s}(z) = A_{n+1}A_{n+2}\frac{Q_{n+2}(z)P_n(y) - Q_n(z)P_{n+2}(y)}{z^2-y^2} + \frac{y}{z^2-y^2} \qquad (n \text{ odd}).$$

4. Assume now again that the integral

$$\int_{-1}^{1} \frac{\log p(x)}{(1-x^2)^{1/2}}dx$$

* Loc. cit., p. 413.

† F. Neumann, *Beiträge zur Theorie der Kugelfunktionen*, 1878, p. 71 ($p(x) \equiv 1$).

‡ Darboux, loc. cit., p. 415.

§ Cf. ($p(x) \equiv 1$) C. Neumann, *Über einige Reihenentwickelungen die nach Produkten von Kugelfunktionen fortschreiten*, Journal für Mathematik, vol. 135 (1909), p. 165.

exists, and use the results of §1, concerning the expansion

$$(3)\qquad \frac{1}{z-y} = \sum_{k=0}^{\infty} P_k(y)Q_k(z).$$

Combining (23), (3), we get the expansion*

$$(27)\qquad \sum_{k=1}^{\infty} P_{n+k}(y)Q_{n+k}(z) = A_{n+1}\frac{P_{n+1}(y)Q_n(z) - P_n(y)Q_{n+1}(z)}{z-y}.$$

Multiplying (26), (27) by $p(y)/(x-y)$ and integrating between -1 and 1, we get the expansions

$$(28)\qquad \sum_{k=0}^{\infty} Q_k(x)Q_k(z) = \frac{Q_0(x) - Q_0(z)}{P_0(z-x)},$$

$$(29)\qquad \sum_{k=1}^{\infty} Q_{n+k}(x)Q_{n+k}(z) = A_{n+1}\frac{Q_n(z)Q_{n+1}(x) - Q_{n+1}(z)Q_n(x)}{z-x},$$

which are valid for

$$(30)\qquad |x+(x^2-1)^{1/2}| > 1+\epsilon,\ |z+(z^2-1)^{1/2}| > 1+\epsilon_1,$$

where ϵ and ϵ_1 are arbitrarily small but fixed positive constants.

In particular, for $z=x$, we derive from (28), (29),

$$(31)\qquad \sum_{k=0}^{\infty} Q_k^2(x) = -\frac{Q_0'(x)}{P_0},$$

$$(32)\qquad \sum_{k=1}^{\infty} Q_{n+k}^2(x) = A_{n+1}\ \{Q_{n+1}(x)Q_n'(x) - Q_n(x)Q_{n+1}'(x\}).$$

In the case of symmetric orthogonal polynomials, we get from (26)

$$(33)\qquad \sum_{k=0}^{\infty} P_{2k}(y)Q_{2k}(z) = \frac{z}{z^2-y^2}.$$

Similarly,

$$(34)\qquad \sum_{k=0}^{\infty} P_{2k+1}(y)Q_{2k+1}(z) = \frac{y}{z^2-y^2}.$$

From (25) and (33) we find†

$$(35)\qquad \sum_{k=1}^{\infty} P_{n+2k}(y)Q_{n+2k}(z) = A_{n+1}A_{n+2}\frac{P_{n+2}(y)Q_n(z) - P_n(y)Q_{n+2}(z)}{z^2-y^2}.$$

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* Neumann, Journal für Mathematik, vol. 135, p. 174.

† Neumann, Journal für Mathematik, vol. 135, p. 171.

All these expansions are valid for

$$\left| z + (z^2 - 1)^{1/2} \right| > 1, \ \left| y + (y^2 - 1)^{1/2} \right| \geqq 1,$$

$$\left| \frac{y + (y^2 - 1)^{1/2}}{z + (z^2 - 1)^{1/2}} \right| < 1 - \epsilon.$$

Multiplying (33), (34), (35) by $p(y)/(x-y)$ and integrating between -1 and 1, we get the expansions

$$\sum_{k=0}^{\infty} Q_{2k}(x)Q_{2k}(z) = \frac{xQ_0(z) - zQ_0(x)}{P_0(x^2 - z^2)}, \tag{36}$$

$$\sum_{k=0}^{\infty} Q_{2k+1}(x)Q_{2k+1}(z) = \frac{zQ_0(z) - xQ_0(x)}{P_0(x^2 - z^2)}, \tag{37}$$

$$\sum_{k=1}^{\infty} Q_{n+2k}(x)Q_{n+2k}(z) = A_{n+1}A_{n+2} \frac{Q_{n+2}(z)Q_n(x) - Q_n(z)Q_{n+2}(x)}{x^2 - z^2}, \tag{38}$$

which are valid under condition (30). Putting $z=x$, we get

$$\sum_{k=0}^{\infty} Q_{2k}^2(x) = \frac{Q_0(x) - xQ_0'(x)}{2P_0(x)}, \tag{39}$$

$$\sum_{k=0}^{\infty} Q_{2k+1}^2(x) = -\frac{Q_0(x) + xQ_0'(x)}{2P_0(x)}, \tag{40}$$

$$\sum_{k=1}^{\infty} Q_{n+2k}^2(x) = A_{n+1}A_{n+2} \frac{Q_{n+2}(x)Q_n'(x) - Q_n(x)Q_{n+2}'(x)}{2x}. \tag{41}$$

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MODERATELY THICK CIRCULAR PLATES WITH PLANE FACES*

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Part I. Theory

1. **Introduction.** Immediately after the propounding of the modern theory of elasticity by Navier in the first quarter of the nineteenth century, a number of mathematicians turned their attention to problems in elasticity. Before long, Poisson and Cauchy had found the differential equations for the displacements of plates which are infinitesimally thin, but it was not until 1883 that a solution for a moderately thick plate was obtained. This solution, found by de Saint-Venant,† involves rational integral functions of the cylindrical coördinates, r and z, where r is measured from the axis of the plate and z from the middle surface. His method consists in finding values for the displacements for several cases of loading of simple type; these solutions are then combined so as to give more complicated loading situations.

In 1887 C. Chree‡ found the solution for a rotating plate. He obtained sets of complementary solutions for the differential equations which must be satisfied by the displacements. With each complementary solution is associated an arbitrary constant; these constants are determined for any particular problem by the loading conditions on the outer surface of the plate.

The problem of a moderately thick circular plate under uniform load was not solved correctly by de Saint-Venant; it was not until after 1900 that the correct solution was obtained by A. E. H. Love.§ Love's solution is an extension of the method developed by J. H. Mitchell in 1900,|| and gives the displacements in terms of rational integral powers of r and z.

Subsequent to Love's solution for uniform load, the first important contribution to the literature of moderately thick circular plates was made by A. Nádai¶ in 1920. He obtained, in terms of Bessel functions, the solution for the bending of a moderately thick circular plate under a concentrated load at

* Presented to the Society, September 12, 1930; received by the editors in May, 1930.

† Final note of § 45 of his translation of Clebsch, 1883.

‡ Transactions of the Cambridge Philosophical Society, vol. 14.

§ Love, *The Mathematical Theory of Elasticity*, 4th edition, 1927, p. 465.

|| Proceedings of the London Mathematical Society, vol. 31 (1900), p. 100.

¶ Schweizerische Bauzeitung, vol. 76, No. 22, pp. 257–260.

the center. Several years later, C. A. Garabedian* found correct solutions for a circular plate of constant thickness in terms of polynomials in r and z. The next year, A. Timpe† found the same solutions by an entirely different method. In 1926, C. A. Clemmow‡ obtained solutions for the bending of a circular plate by adding solutions involving polynomials in r and z to the solution in terms of Bessel functions which had been obtained previously by Nádai.

Both Nádai's and Clemmow's solutions are very cumbersome, even for the case of uniform load. The extension of either of these methods to more complicated types of loading would be almost an impossibility on account of the extremely difficult computations which would arise. The methods used by de Saint-Venant, Love, and Timpe could be extended to more complicated types of loading without giving rise to very difficult computations, but all three of these methods have the disadvantage that, for each new type of loading, the problem must be solved from the very beginning by the process of "trial and error." Garabedian's method is much less involved than those of de Saint-Venant, Love, or Timpe; and, in addition, it gives the necessary machinery for solving any type of loading by means of a single set of equations.

Garabedian's method is based on the assumption that the displacements can be expanded in rational integral powers of a parameter, an assumption which had previously been used by G. D. Birkhoff§ in an attempt to solve problems in circular plates by the use of the calculus of variations.

Although Garabedian was the first to solve successfully problems in moderately thick circular plates by a method involving the assumption that the displacements can be expanded in convergent series, he was not the first to make use of such a postulation. Before 1827, Cauchy|| had made use of this hypothesis; he had solved problems by assuming that the stresses and displacements could be expressed as convergent series in ascending powers of z. But Cauchy concerned himself only with plates which were infinitesimally thin, and neglected all powers of z higher than the second. In 1877, M. Lévy,¶ in his study of a thick circular plate having no load on either base, made the assumption that the displacements could be expanded in ascending powers of z. He did not attempt to find the solution for any given loading condition, but he was able to prove that the displacements could not contain powers of z

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* These Transactions, vol. 25 (1923), pp. 343–398.

† Zeitschrift für Angewandte Mathematik und Mechanik, vol. 4 (1924), pp. 361–376.

‡ Proceedings of the Royal Society, London, (A), vol. 112 (1926), pp. 559–598.

§ Philosophical Magazine, London, Edinburgh, and Dublin, (6), vol. 43 (1922).

|| Cauchy's *Exercices de Mathématiques*, vol. II, pp. 330–348, 1827.

¶ Liouville's Journal, (3), vol. 3 (1877), p. 219.

greater than the third when the cylinder is weightless and its bases are free from load.

Garabedian, at the close of his paper, sketched a physical argument to show that his series were convergent. Subsequent to the work done on this paper, Garabedian has found the general term of his series, and has been able to establish convergence for a certain class of loading functions. Moreover, he has found connected with his series an infinite set of constants which turn out to be the same set exhibited in this paper.

The present paper was inspired directly by the above-mentioned paper by Garabedian, and hence, indirectly, by the work done by Birkhoff. Although Garabedian's method and the method used in this paper lead to the same results for any given loading condition, they are quite distinct. On the other hand, it should be said of the two methods that, precisely because of the difference in approach, each method sheds light on the other. Indeed, the two methods, in a sense, complement each other and eventually completely clarify a problem which has waited a full century for solution.

The method of solution employed in this paper is based on the assumption that the components of displacement can be developed in positive integral powers of z. In §9 this assumption will be justified by proving that the series defining the displacements are convergent for a certain class of loading functions.

The nature of the problem makes the employment of cylindrical coördinates desirable. The axis of the plate is taken as the z-axis and the middle plane of the plate as the plane $z=0$. Let the upper and lower faces be $z=h$ and $z=-h$, respectively; thus the thickness of the plate is $2h$. The plate is taken to be homogeneous, isotropic, and only slightly bent; moreover, the plate must be thin enough so that de Saint-Venant's principle of the elastic equivalence of statically equipollent systems of load can be used at the edge (cf. §4). For the sake of simplicity, all stresses and displacements will be assumed to be independent of θ; the advantage of this assumption is that the differential equations which determine the coefficients of z will be ordinary instead of partial.

Love's notation† with some slight modifications will be used. To obtain results in compact form, the star operator introduced by Garabedian (loc. cit.) will be employed; this operator is defined as follows:

$$A^* = \frac{1}{r}\frac{\partial(rA)}{\partial r} = A' + \frac{A}{r}\cdot$$

† In this paper, all references to Love are to the fourth edition of his *Mathematical Theory of Elasticity*, 1927.

Although this method is applicable to problems involving displacements in the direction of θ, lack of space makes it desirable to develop the theory only for the case of displacements in the directions of r and z. Moreover, only those problems in which the surface tractions are known and the displacements are to be found will be considered. In this type of problem the displacements must satisfy

(1) the stress equations of motion throughout the body,

(2) the surface traction conditions on the upper and lower faces,

(3) the boundary conditions at the edge.

These three requirements, in the order just indicated, will now be discussed.

2. **Stress equations of motion.** Let U and w denote the displacements in the directions of r and z, respectively. The stress equations of motion may be written in the form (Love, pp. 56, 75, 78, 90, 102)

$$\text{(1a)} \qquad \frac{\partial^2 U}{\partial z^2} + \frac{1}{1-2\sigma}\frac{\partial w'}{\partial z} + \frac{2(1-\sigma)}{1-2\sigma}U^{*\prime} = \frac{2(1+\sigma)}{E}\rho(f_r - F_r),$$

$$\text{(1b)} \qquad \frac{\partial^2 w}{\partial z^2} + \frac{1}{2(1-\sigma)}\frac{\partial U^*}{\partial z} + \frac{1-2\sigma}{2(1-\sigma)}w'^* = \frac{(1+\sigma)(1-2\sigma)}{(1-\sigma)E}\rho(f_z - F_z),$$

in which, as usual, ρ, σ, and E represent, respectively, density, Poisson's ratio, and the modulus of elasticity; while f and F are the acceleration and body force, respectively. All important and familiar applications involving accelerations or body forces will be provided for if it is assumed that $\rho(f_r - F_r)$ is proportional to r and $\rho(f_z - F_z)$ is a constant. Thus

$$\text{(1c)} \qquad \rho(f_r - F_r) = c_r r,$$

$$\text{(1d)} \qquad \rho(f_z - F_z) = c_z$$

where c_r and c_z are constants. The quantities $c_r r$ and c_z will be called the radial and axial mass forces, respectively. Observe that these mass forces may be due to accelerations or to body forces.

3. **Surface-traction conditions.** The surface tractions applied to the upper and lower faces can be resolved into radial, tangential, and normal components. The positive direction of the normal component will be taken for both the upper and lower faces to be that of the outward drawn normal (Love, p. 75). The positive direction of the radial component will be taken outward on the upper face. It will be taken inward on the lower face, since the axes for the radial, tangential, and normal components must form either a right-handed or a left-handed system on both faces, and it is desirable that the axes for the tangential component should point in the same direction for both faces. The radial and normal components of the surface tractions will

be designated by J_1 and L_1 on the upper face and by J_2 and L_2 on the lower face. Note that L_1 and L_2 are tensions when positive and pressures when negative.

In order to satisfy the surface-traction conditions on a face, it is necessary that the components of the internal stress at every point of the face should be equal to the corresponding component of the surface traction at that point. Since the displacements in the direction of θ are not being considered in this paper, the tangential components of the surface tractions and of the stress will be taken to be zero. Thus the surface-traction conditions on the two faces are

$$\widehat{zz}\,\Big|_{z=h} = L_1, \quad \widehat{zz}\,\Big|_{z=-h} = L_2; \tag{2a}$$

$$\widehat{rz}\,\Big|_{z=h} = J_1, \quad \widehat{rz}\,\Big|_{z=-h} = J_2. \tag{2b}$$

4. **Boundary conditions at an edge.** In applying boundary conditions at an edge, it is important to distinguish two cases:

(a) the stresses or displacements may be assigned values for every z in the interval from $z=-h$ to $z=h$;

(b) the displacements may have prescribed values at only a limited number of points, or values may be assigned to certain resultant stresses and to certain resultant stress moments taken along a vertical element of an edge.

In the first case, the solution obtained is exact; moreover, this solution applies to a plate of any thickness and may be called a three-dimensional solution. In the second case, the solution is mathematically rigorous, but is only approximate in the physical sense unless for every value of z at the edge the surface tractions are precisely in agreement with the corresponding internal stresses as calculated from the values obtained for the displacements. This type of solution, when not exact, is essentially two-dimensional in character, since this type requires that the thickness of the plate be small as compared with the diameter.

The two-dimensional type of solution is the one used in this paper. The resultant stresses and the resultant stress moments mentioned in case (b) above are (Love, p. 465)

$$T_r = \int_{-h}^{h} \widehat{rr}\,dz, \tag{3a}$$

$$N_r = \int_{-h}^{h} \widehat{zr}\,dz, \tag{3b}$$

$$G_r = \int_{-h}^{h} \widehat{rr}\,z\,dz. \tag{3c}$$

For moderately thick plates, de Saint-Venant's principle states that the actual distribution of the tractions applied to an edge is of no practical importance. Therefore, instead of dealing with the tractions themselves, their force and couple resultants estimated per unit length of the edge-line are considered. Let the components of these resultants be T, N, G in the sense previously assigned to T_r, N_r, G_r. It is necessary that the applied tractions be statically equivalent to the stress resultants and the resultant stress moments at the edge; hence, so far as the stresses are concerned, the boundary conditions at an edge are given by the equations

$$T_r = T, \quad N_r = N, \quad G_r = G. \tag{4}$$

It is true that the solutions obtained by means of these equations may not be exact, but, if not exact, they will sufficiently approximate the exact solutions for all points which are not too close to the edge of the plate (Love, pp. 131, 132).

5. **The U and w systems.** It is now possible to determine formally the values of U and w which satisfy equations (1), (2), (4). It is convenient to postpone for the present the consideration of mass forces. Thus the right hand members of formulas (1a) and (1b) are set equal to zero.

The fundamental idea underlying this method is the assumption that U and w can be expanded in powers of z; that is,

$$U = \sum_{n=0}^{\infty} U_n \frac{z^n}{n!}, \quad w = \sum_{n=0}^{\infty} w_n \frac{z^n}{n!}, \tag{5}$$

in which U_n and w_n are functions of r only. Substitute (5) in formulas (1), and equate to zero the coefficients of like powers of z. The result is

$$U_n = -\frac{1}{1-2\sigma}\left\{w'_{n-1} + 2(1-\sigma)U^{*\prime}_{n-2}\right\}, \tag{6a}$$

$$w_n = -\frac{1}{2(1-\sigma)}\left\{U^{*}_{n-1} + (1-2\sigma)w'^{*}_{n-2}\right\}. \tag{6b}$$

By successive applications of the recurrence relations (6a) and (6b), it is possible to express U_n and w_n directly in terms of U_0, U_1, w_0, w_1. A material simplification of these formulas is obtained by the introduction of two new functions defined as follows:

$$\bar{U}^{*}_{0} = \frac{U^{*}_{0} + w_1}{1-2\sigma}, \quad \bar{w}'_0 = \frac{U_1 - w'_0}{2(1-\sigma)}. \tag{7}$$

The final results are

$$U_{2i} = (-1)^i \{ i\bar{U}_0 + U_0 \}^{(*')^i}, \tag{8a}$$

$$U_{2i+1} = (-1)^i \{ (i + 2 - 2\sigma)\bar{w}_0 + w_0 \}^{'(*')^i}, \tag{8b}$$

$$w_{2i} = (-1)^i \{ i\bar{w}_0 + w_0 \}^{('*)^i}, \tag{8c}$$

$$w_{2i+1} = -(-1)^i \{ (i - 1 + 2\sigma)\bar{U}_0 + U_0 \}^{*('*)^i}. \tag{8d}$$

The substitution of (8) in (5) results in the following formulas for U and w:

$$U = \sum_{i=0}^{\infty} (-1)^i \{ i\bar{U}_0 + U_0 \}^{(*')^i} \frac{z^{2i}}{(2i)!} + \sum_{i=0}^{\infty} (-1)^i \{ (i + 2 - 2\sigma)\bar{w}_0 + w_0 \}^{'(*')^i} \frac{z^{2i+1}}{(2i+1)!}, \tag{9a}$$

$$w = \sum_{i=0}^{\infty} (-1)^i \{ i\bar{w}_0 + w_0 \}^{('*)^i} \frac{z^{2i}}{(2i)!} - \sum_{i=0}^{\infty} (-1)^i \{ (i - 1 + 2\sigma)\bar{U}_0 + U_0 \}^{*('*)^i} \frac{z^{2i+1}}{(2i+1)!}. \tag{9b}$$

These expressions for the displacements satisfy formally the stress equations of motion, (1a) and (1b), when U and w are infinite series. It can be shown also that they satisfy (1a) and (1b) when U and w are finite series; in this connection, it is natural to seek necessary and sufficient conditions for U and w to terminate.

Consider first the expression for U. If U is to have a finite number of terms, both $\{i\bar{U}_0+U_0\}^{(*')^i}$ and $\{(i+2-2\sigma)\bar{w}_0+w_0\}^{'(*')^i}$ must eventually vanish and, moreover, independently of each other, since the former is associated with even powers of z and the latter with odd powers of z. For $\{i\bar{U}_0+U_0\}^{(*')^i}$ to vanish eventually, there must exist a smallest integer α for which $\{i\bar{U}_0+U_0\}^{(*')^i}=0$, $i \geqq \alpha$.

Let the anti-prime-star operation be defined as that operation which must be performed upon $P^{*'}$ in order to change it to P. If this operation is performed upon $\{\alpha\bar{U}_0+U_0\}^{(*')^\alpha}$, it turns out that $\{\alpha\bar{U}_0+U_0\}^{(*')^{\alpha-1}}$ must be of the form $\{C_1 r+C_2/r\}$, where C_1 and C_2 are constants. Moreover, since α is a constant, it follows that both $\bar{U}_0^{(*')^{\alpha-1}}$ and $U_0^{(*')^{\alpha-1}}$ must also be of this form. By performing alternately integrations and inverse-star operations, it may be shown that U_0 can contain no terms which are not of the form $\{Cr^{2m-3} + Kr^{2n-1}\log r\}$ and that $\bar{U}_0^*$ can contain no terms which are not of the form $\{C'r^{2p-2}+K'r^{2q-2}\log r\}$, m, n, p, q being any positive integers. By a similar argument, it may be demonstrated that $\bar{w}_0'$ and w_0 must have the form speci-

fied above for U_0 and for $\overline{U}_0^*$, respectively. Furthermore, it is not difficult to show that the above restrictions on U_0, $\overline{U}_0^*$, w_0, $\overline{w}_0'$ constitute not only a sufficient condition for U to terminate but also a necessary and sufficient condition for w to have a finite number of terms. Hence it is evident that if U terminate, so also will w; and vice versa.

It is now possible to show that U and w satisfy the stress equations of motion when U and w are finite series. The work is the same as for the case when U and w are infinite series except that now the upper limits for each summation must be determined. All of these limits are readily found, since the operation by which a term becomes eventually zero is differentiation or the star operation according as the term has the form Cr^m or $Kr^n \log r$, respectively.

It is now necessary to impose upon U and w the requirement that they satisfy the surface-traction conditions on the upper and lower faces. By means of formulas (9) the stresses $\widehat{zz}$ and $\widehat{rz}$ may be written in the following form (Love, pp. 56, 102):

$$\text{(10a)} \quad \begin{aligned} \widehat{zz} = &-\frac{E}{1+\sigma} \sum_{i=0}^{\infty}(-1)^i \{(i-1+\sigma)\overline{U}_0 + U_0\}^{*('*)^i} \frac{z^{2i}}{(2i)!} \\ &- \frac{E}{1+\sigma} \sum_{i=0}^{\infty}(-1)^i \{(i+1-\sigma)\overline{w}_0 + w_0\}^{('*)^{i+1}} \frac{z^{2i+1}}{(2i+1)!}, \end{aligned}$$

$$\text{(10b)} \quad \begin{aligned} \widehat{rz} = &\frac{E}{1+\sigma} \sum_{i=0}^{\infty}(-1)^i \{(i+1-\sigma)\overline{w}_0 + w_0\}^{'(*')^i} \frac{z^{2i}}{(2i)!} \\ &- \frac{E}{1+\sigma} \sum_{i=0}^{\infty}(-1)^i \{(i+\sigma)\overline{U}_0 + U_0\}^{(*')^{i+1}} \frac{z^{2i+1}}{(2i+1)!}. \end{aligned}$$

Substitute (10) in (2), and take the sum and difference not only of the two resulting values of $\widehat{zz}$ but also of the resulting values of $\widehat{rz}$. A simplification will be obtained by the introduction of four new quantities defined as follows:

$$\text{(11a)} \qquad L = L_1 + L_2, \quad l = L_1 - L_2;$$

$$\text{(11b)} \qquad J = J_1 + J_2, \quad j = J_1 - J_2.$$

The final result is

$$\text{(12)} \qquad \sum_{i=0}^{\infty}(-1)^i \{(i-1+\sigma)\overline{U}_0 + U_0\}^{*('*)^i} \frac{h^{2i}}{(2i)!} = -\frac{1+\sigma}{2E} L,$$

$$\text{(13)} \qquad \sum_{i=0}^{\infty}(-1)^i \{(i+1-\sigma)\overline{w}_0 + w_0\}^{('*)^{i+1}} \frac{h^{2i}}{(2i+1)!} = -\frac{1+\sigma}{2Eh} l,$$

$$\text{(14)} \qquad \sum_{i=0}^{\infty}(-1)^i \{(i+1-\sigma)\overline{w}_0 + w_0\}^{'(*')^i} \frac{h^{2i}}{(2i)!} = \frac{1+\sigma}{2E} J,$$

$$\sum_{i=0}^{\infty}(-1)^i\{(i+\sigma)\bar{U}_0+U_0\}^{(*')^{i+1}}\frac{h^{2i}}{(2i+1)!}=-\frac{1+\sigma}{2Eh}j. \tag{15}$$

These four systems of ordinary linear differential equations determine the displacements save for the arbitrary constants of integration; the latter are to be fixed subsequently by the edge conditions (§12).

6. The determination of $\bar{w}_0'$ and w_0. Since $\bar{w}_0$ does not appear in the formulas for U and w, it is necessary to find only $\bar{w}_0'$ and w_0. These two functions may be obtained by an indirect process from equations (13) and (14). It turns out that it is necessary to break up $\bar{w}_0'$ and w_0' into terms ordered according to the powers of the ratio h/r. For this purpose, the following convenient definition is introduced. If X and Y are two polynomials in r which contain the same number of terms, Y is defined to be of the nth order of magnitude as compared with X if each term of Y is proportional to $(h/r)^n$ times the corresponding term in X. It is easy to demonstrate that both $X^{(*')^n}h^{2n}$ and $X^{('*)^n}h^{2n}$ are of the $(2n)$th order of magnitude as compared with X.

In order to apply the method of this paper, certain assumptions must be made with regard to $\bar{w}_0'$ and w_0'; namely, that these functions are polynomials in r, and that they involve h in such a manner that their terms may be grouped and arranged in ascending order of magnitude. Since only even powers of h enter in (13) and (14), it is clear that only even orders of magnitude need be provided for in the developments for $\bar{w}_0'$ and w_0'. Thus it is assumed that $\bar{w}_0'$ and w_0' may be written in the form

$$\bar{w}_0'=\sum_{n=0}^{\infty}\bar{w}_{2n,0}', \quad w_0'=\sum_{n=0}^{\infty}w_{2n,0}', \tag{16}$$

where $\bar{w}_{2n,0}'$ and $w_{2n,0}'$ are of the $(2n)$th order of magnitude as compared with either $\bar{w}_{00}'$ or w_{00}'. The only assumption made with reference to the leading terms $\bar{w}_{00}'$ and w_{00}' is that they include, in the case that $\bar{w}_0'$ and w_0' do not vanish identically, the terms of lowest order of magnitude occurring in either development. Furthermore, it is assumed that $\bar{w}_{00}^{'*'*}$ and $w_{00}^{'*'*}$ cannot both be identically zero unless

$$\bar{w}_{2n,0}^{'*'*}\equiv w_{2n,0}^{'*'*}\equiv 0 \qquad (n=0,1,2,\cdots).$$

If (16) is substituted in (13) and (14), it follows that

$$\sum_{i=0}^{\infty}(-1)^i\left\{(i+1-\sigma)\sum_{n=0}^{\infty}\bar{w}_{2n,0}+\sum_{n=0}^{\infty}w_{2n,0}\right\}^{('*)^{i+1}}\frac{h^{2i}}{(2i+1)!}=-\frac{1+\sigma}{2Eh}l, \tag{17}$$

$$\sum_{i=0}^{\infty}(-1)^i\left\{(i+1-\sigma)\sum_{n=0}^{\infty}\bar{w}_{2n,0}+\sum_{n=0}^{\infty}w_{2n,0}\right\}^{'(*')^{i}}\frac{h^{2i}}{(2i)!}=\frac{1+\sigma}{2E}J. \tag{18}$$

By means of equations (17) and (18), it will be possible to solve for $\bar{w}_{2n,0}^{\prime*\prime*}$ and $w_{2n,0}^{\prime*\prime*}$, $n=0, 1, 2, \cdots$.

In obtaining $\bar{w}_{2n,0}^{\prime*\prime*}$ and $w_{2n,0}^{\prime*\prime*}$, it is convenient to treat the cases of normal load and shearing traction separately; by the principle of superposition, it will be possible to solve more complicated problems by a synthesis of the separate solutions. The case of normal load will be considered first.

Before the two systems of differential equations obtainable from (17) and (18) can be written out, it is necessary to determine the order of magnitude of $\{-(1+\sigma)l/(2Eh)\}$. If this quantity is assumed to be of the same order of magnitude as $\bar{w}_{00}^{\prime*}$ or $w_{00}^{\prime*}$, the two equations of lowest order of magnitude in (17) and (18) are, respectively, $(1-\sigma)\bar{w}_{00}^{\prime*}+w_{00}^{\prime*}=-(1+\sigma)l/(2Eh)$ and $(1-\sigma)\bar{w}_{00}'+w_{00}'=0$. If each term of the latter is starred, an equation is obtained which is inconsistent with the former; hence $\{-(1+\sigma)l/(2Eh)\}$ cannot be of the same order of magnitude as $\bar{w}_{00}^{\prime*}$ or $w_{00}^{\prime*}$. For the present, it will be assumed that this quantity is of the second order of magnitude as compared with $\bar{w}_{00}^{\prime*}$ or $w_{00}^{\prime*}$.

The two systems of equations which result from equating all terms of the same order of magnitude in (17) and (18) may now be written out. They are, respectively,

$$(19.0)\quad \{(1-\sigma)\bar{w}_{00}+w_{00}\}^{\prime*}=0,$$

$$(19.2)\quad \{(1-\sigma)\bar{w}_{20}+w_{20}\}^{\prime*}-\{(2-\sigma)\bar{w}_{00}+w_{00}\}^{(\prime*)^3}\frac{h^2}{3!}=-\frac{1+\sigma}{2Eh}l,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(20.0)\quad \{(1-\sigma)\bar{w}_{00}+w_{00}\}'=0,$$

$$(20.2)\quad \{(1-\sigma)\bar{w}_{20}+w_{20}\}'-\{(2-\sigma)\bar{w}_{00}+w_{00}\}^{\prime*\prime}\frac{h^2}{2!}=0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

It is not possible to obtain $\bar{w}_{00}^{\prime*}$ and $w_{00}^{\prime*}$ directly from (19.0) and (20.0) since the latter equation, when starred, becomes identical with the former. It will be possible, though, to find $\bar{w}_{00}^{\prime*\prime*}$ and $w_{00}^{\prime*\prime*}$; and, this done, $\bar{w}_{00}^{\prime*}$ and $w_{00}^{\prime*}$ may be found by quadratures.

The first step in finding $\bar{w}_{00}^{\prime*\prime*}$ and $w_{00}^{\prime*\prime*}$ is the formation of a third system of equations by starring (20.$2n$) and subtracting from (19.$2n$); the equations thus obtained will be in simpler form if each is multiplied by $(3/h^2)$. The final result is

$$(21.0)\qquad 0=0,$$

$$(21.2)\qquad \{(2-\sigma)\bar{w}_{00}+w_{00}\}^{('*)^2}=-\frac{3(1+\sigma)}{2Eh^3}l,$$

$$(21.4)\qquad \{(2-\sigma)\bar{w}_{20}+w_{20}\}^{('*)^2}-\{(3-\sigma)\bar{w}_{00}+w_{00}\}^{('*)^2}\frac{6\cdot 2h^2}{5!}=0,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Equations (19) and (21) were obtained on the assumption that $\{-(1+\sigma)l/(2Eh)\}$ was of the second order of magnitude as compared with $\bar{w}_{00}^{\prime *}$ or $w_{00}^{\prime *}$. It has already been proved that this quantity is not of the same order of magnitude as $\bar{w}_{00}^{\prime *}$ or $w_{00}^{\prime *}$; it is now possible to show that it cannot be of order higher than the second. If it were of the fourth or higher order, equation (21.2) would be $\{(2-\sigma)\bar{w}_{00}+w_{00}\}^{('*)^2}=0$. The result of solving this equation simultaneously with either (19.0) or (20.0) is $\bar{w}_{00}^{\prime *\prime *}\equiv w_{00}^{\prime *\prime *}\equiv 0$. Since l must appear on the right hand side of some one of the equations (21), it is not possible that $\bar{w}_{2n,0}^{\prime *\prime *}\equiv w_{2n,0}^{\prime *\prime *}\equiv 0$, $n=0, 1, 2, \cdots$. But by a previous assumption, this implies that $\bar{w}_{00}^{\prime *\prime *}$ and $w_{00}^{\prime *\prime *}$ cannot both be identically zero. From this contradiction, it follows that $\{-(1+\sigma)l/(2Eh)\}$ is of the second order of magnitude.

By means of (19), (20), and (21), it will be possible to obtain $\bar{w}_{2n,0}^{\prime *\prime *}$ and $w_{2n,0}^{\prime *\prime *}$ for $n=0, 1, 2, \cdots$. The procedure consists in solving $(21.2n+2)$ simultaneously with $(20.2n)$. Accordingly, the star-prime-star operation is performed upon each equation of (20); the resulting set of equations is designated by $(20)^{*\prime *}$. In order to secure a notation appropriate to the discussion which follows, $(20.0)^{*\prime *}$ and (21.2) are written in the following form:

$$(22.0)\qquad \{(1-\sigma)\bar{w}_{00}+w_{00}\}^{('*)^2}=-\frac{3(1+\sigma)}{2Eh^3}d_0 l,$$

$$(23.2)\qquad \{(2-\sigma)\bar{w}_{00}+w_{00}\}^{('*)^2}=-\frac{3(1+\sigma)}{2Eh^3}b_0 l,$$

where $b_0=1$, $d_0=0$. The result of solving these two equations simultaneously is

$$\bar{w}_{00}^{\prime *\prime *}=-\frac{3(1+\sigma)}{2Eh^3}\{b_0-d_0\}l,$$

$$w_{00}^{\prime *\prime *}=\frac{3(1+\sigma)}{2Eh^3}\{(1-\sigma)b_0-(2-\sigma)d_0\}l.$$

If these values of $\bar{w}_{00}^{\prime *\prime *}$ and $w_{00}^{\prime *\prime *}$ are substituted in (20.2)$^{*\prime *}$ and (21.4), we get

$$(22.2) \qquad \{(1-\sigma)\bar{w}_{20} + w_{20}\}^{(\prime *)^2} = -\frac{3(1+\sigma)}{2Eh^3} d_1 h^2 l^{\prime *},$$

$$(23.4) \qquad \{(2-\sigma)\bar{w}_{20} + w_{20}\}^{(\prime *)^2} = -\frac{3(1+\sigma)}{2Eh^3} b_1 h^2 l^{\prime *},$$

where

$$b_1 = \frac{6\cdot 2(2\cdot b_0 - 1\cdot d_0)}{5!}, \quad d_1 = \frac{1\cdot b_0 - 0\cdot d_0}{2!}.$$

If this pair of equations is solved simultaneously, the result is

$$\bar{w}_{20}^{\prime *\prime *} = -\frac{3(1+\sigma)}{2Eh^3}\{b_1 - d_1\} h^2 l^{\prime *},$$

$$w_{20}^{\prime *\prime *} = \frac{3(1+\sigma)}{2Eh^3}\{(1-\sigma)b_1 - (2-\sigma)d_1\} h^2 l^{\prime *}.$$

By continuing this process, the following general formulas are obtained:

$$(24a) \quad \bar{w}_{2n,0}^{\prime *\prime *} = -\frac{3(1+\sigma)}{2Eh^3}\{b_n - d_n\} h^{2n} l^{(\prime *)^n} \qquad (n = 0, 1, 2, \cdots),$$

$$(24b) \quad w_{2n,0}^{\prime *\prime *} = \frac{3(1+\sigma)}{2Eh^3}\{(1-\sigma)b_n - (2-\sigma)d_n\} h^{2n} l^{(\prime *)^n} \qquad (n = 0, 1, 2, \cdots),$$

where b_n and d_n are given by the formulas

$$(24c) \qquad b_n = 6\sum_{i=0}^{n-1}(-1)^i \frac{(i+2)\{(i+2)b_{n-1-i} - (i+1)d_{n-1-i}\}}{(2i+5)!} \qquad (n = 1, 2, 3, \cdots);$$

$$(24d) \qquad d_n = \sum_{i=0}^{n-1}(-1)^i \frac{(i+1)b_{n-1-i} - i d_{n-1-i}}{(2i+2)!} \qquad (n = 1, 2, 3, \cdots).\dagger$$

† $\bar{w}_{2n,0}^{\prime *\prime *}$ and $w_{2n,0}^{\prime *\prime *}$ can also be found by solving equations $(21.2n+2)$ simultaneously with equations $(19.2n)$. Again let $b_0 = 1$, $d_0 = 0$; then, by a process similar to that employed above, the following values for b_n and d_n may be deduced:

$$(24'c) \qquad b_n = 6\sum_{i=0}^{n-1}(-1)^i \frac{(i+2)\{(i+2)b_{n-1-i} - (i+1)d_{n-1-i}\}}{(2i+5)!} \qquad (n = 1, 2, 3, \cdots);$$

$$(24'd) \qquad d_n = \sum_{i=0}^{n-1}(-1)^i \frac{(i+1)b_{n-1-i} - i d_{n-1-i}}{(2i+3)!} \qquad (n = 2, 3, 4, \cdots);$$

$$(24'dd) \qquad d_1 = \frac{1\cdot b_0 - 0\cdot d_0}{3!} + \frac{1}{3} = \frac{1}{2}.$$

It may be proved without difficulty that the above values of b_n and d_n are precisely those given by formulas (24c) and (24d).

Star-prime-star (16), and make use of (24); the result is

$$\bar{w}_0'^{*'*} = -\frac{3(1+\sigma)}{2Eh^3}\sum_{n=0}^{\infty}\{b_n - d_n\}\,h^{2n}l^{('*)^n}, \tag{24e}$$

$$w_0'^{*'*} = \frac{3(1+\sigma)}{2Eh^3}\sum_{n=0}^{\infty}\{(1-\sigma)b_n - (2-\sigma)d_n\}\,h^{2n}l^{('*)^n}. \tag{24f}$$

It is not yet possible to solve for $\bar{w}_0'$ and w_0, since these two quantities are not independent of each other. The relation between them is found as follows. Transfer all except the first two terms of equations (20.2*n*) to the right hand side; by means of (24a), (24b), and (16), there results

$$\begin{aligned}(1-\sigma)\bar{w}_0' + w_0' - \{(2-\sigma)\bar{w}_0' + w_0\}'^{*'}\frac{h^2}{2!} \\ = -\frac{3(1+\sigma)}{2Eh^3}\sum_{n=0}^{\infty}\left\{d_{n+1} - \frac{b_n}{2}\right\}h^{2n+2}l^{'(*')^{n-1}}.\end{aligned} \tag{24g}$$

This equation may be obtained in more convenient form. Star-prime each term, and make use of (24e) and (24f); the result is

$$\{(1-\sigma)\bar{w}_0 + w_0\}'^{*'} = -\frac{3(1+\sigma)}{2Eh^3}\sum_{n=0}^{\infty}d_n h^{2n}l^{'(*')^{n-1}}. \tag{24h}$$

Substitute in (24g) the value of $\bar{w}_0'^{*'}$ obtained from (24h); the final result is

$$\begin{aligned}(1-\sigma)\bar{w}_0' + w_0' + \frac{h^2}{2(1-\sigma)}w_0'^{*'} = -\frac{3(1+\sigma)}{2Eh^3}\sum_{n=0}^{\infty}d_n h^{2n}l^{'(*')^{n-2}} \\ + \frac{3(1+\sigma)}{2Eh^3}\sum_{n=0}^{\infty}\frac{(1-\sigma)b_n - (2-\sigma)d_n}{2(1-\sigma)}h^{2n+2}l^{'(*')^{n-1}}.\end{aligned} \tag{24i}$$

It is now possible to solve for $\bar{w}_0'$ and w_0. There are two possible methods of procedure. The simpler of the two is first to find w_0 from (24f) and then obtain $\bar{w}_0'$ by means of (24i). Let $\bar{w}_{0c}'$ and w_{0c} designate the complementary solutions of $\bar{w}_0'$ and w_0, respectively; and let $\bar{w}_{0p}'$ and w_{0p} be the corresponding particular solutions. The complementary solutions are obtained by solving the homogeneous equations associated with equations (24e), (24f), (24i). From (24f), there results

$$w_{0c} = K_1r^2\log r + K_2r^2 + K_3\log r + K_4, \tag{24j}$$

where K_1, K_2, K_3, K_4 are arbitrary constants. The substitution in (24i) of the values of w_{0c}' and $w_{0c}'^{*'}$ obtained from (24j) results in

(24k) $$\bar{w}'_{0c} = -\frac{1}{1-\sigma}\left[K_1\left\{2r\log r + r + \frac{2h^2}{(1-\sigma)r}\right\} + 2K_2r + \frac{K_3}{r}\right].$$

It turns out that the complementary solutions for all types of loading are given by (24j) and (24k).

The particular solutions when $\bar{w}_0^{\prime *\prime *}$ and $w_0^{\prime *\prime *}$ are infinite series will now be found; the case when they terminate will be considered in §10. Let the anti-star-prime-star-prime operation be designated by the symbol $('*)^{-2}$, and let $l^{('*)^{n-2}}$ be the result obtained when $l^{('*)^n}$ is anti-star-prime-star-primed without introduction of arbitrary constants. From (24f), there results

(24l) $$w_{0p} = \frac{3(1+\sigma)}{2Eh^3}\sum_{n=0}^{\infty}\{(1-\sigma)b_n - (2-\sigma)d_n\}h^{2n}l^{('*)^{n-2}}.$$

Substitute the above value of w_{0p} in (24i); the result is

(24m) $$\bar{w}'_{0p} = -\frac{3(1+\sigma)}{2Eh^3}\sum_{n=0}^{\infty}\{b_n - d_n\}h^{2n}l^{'(*')^{n-2}}.$$

It is evident that this value of $\bar{w}'_{0p}$ satisfies (24e).

For the case in which there is shearing traction only, equations (21) are solved simultaneously with (20). If $\bar{b}_0=1$ and $\bar{d}_0=0$, the following general formulas are obtained:

(25c) $$\bar{b}_n = 6\sum_{i=0}^{n-1}(-1)^i\frac{(i+2)\{(i+2)\bar{b}_{n-1-i} - (i+1)\bar{d}_{n-1-i}\}}{(2i+5)!} \quad (n = 1, 2, 3, \cdots);$$

(25d) $$\bar{d}_n = \sum_{i=0}^{n-1}(-1)^i\frac{(i+1)\bar{b}_{n-1-i} - i\bar{d}_{n-1-i}}{(2i+2)!} \quad (n = 2, 3, 4, \cdots);$$

(25dd) $$\bar{d}_1 = \frac{1\cdot\bar{b}_0 - 0\cdot\bar{d}_0}{2!} - \frac{1}{3} = \frac{1}{6};$$

(25e) $$\bar{w}_0^{\prime *\prime *} = \frac{3(1+\sigma)}{2Eh^2}\sum_{n=0}^{\infty}\{b_n - d_n\}h^{2n}J^{*('*)^n};$$

(25f) $$w_0^{\prime *\prime *} = \frac{3(1+\sigma)}{2Eh^2}\sum_{n=0}^{\infty}\{(1-\sigma)\bar{b}_n - (2-\sigma)\bar{d}_n\}h^{2n}J^{*('*)^n};†$$

† Observe that $\bar{w}_{2n,0}^{\prime *\prime *}$ and $w_{2n,0}^{\prime *\prime *}$ might also be found by solving equations (21) simultaneously with (19). Since $\bar{b}_0=1$ and $\bar{d}_0=0$, the results are

(25′c) $$\bar{b}_n = 6\sum_{i=0}^{n-1}(-1)^i\frac{(i+2)\{(i+2)\bar{b}_{n-1-i} - (i+1)\bar{d}_{n-1-i}\}}{(2i+5)!} \quad (n = 1, 2, 3, \cdots);$$

(25′d) $$\bar{d}_n = \sum_{i=0}^{n-1}(-1)^i\frac{(i+1)\bar{b}_{n-1-i} - i\bar{d}_{n-1-i}}{(2i+3)!} \quad (n = 1, 2, 3, \cdots).$$

It may readily be proved that the above values of $\bar{b}_n$ and $\bar{d}_n$ are the same as those given by (25c), (25d), (25dd).

$$(25\text{i})\quad (1-\sigma)\overline{w}_0' + w_0' + \frac{h^2}{2(1-\sigma)} w_0{}'^{*\prime} = -\frac{3(1+\sigma)}{2Eh^2}\sum_{n=0}^{\infty}\bar{d}_n h^{2n} J^{(*\prime)^{n-1}}$$

$$+\frac{3(1+\sigma)}{2Eh^2}\sum_{n=0}^{\infty}\frac{(1-\sigma)\bar{b}_n-(2-\sigma)\bar{d}_n}{2(1-\sigma)} h^{2n+2} J^{(*\prime)^{n}}.$$

When $\overline{w}_0'^{*\prime *}$ and $w_0'^{*\prime *}$, as defined by (25e) and (25f), are infinite series, it turns out that

$$(25)\qquad w_{0p} = \frac{3(1+\sigma)}{2Eh^2}\sum_{n=0}^{\infty}\left\{(1-\sigma)\bar{b}_n-(2-\sigma)\bar{d}_n\right\} h^{2n} J^{*(\prime *)^{n-2}},$$

$$(25\text{m})\qquad \overline{w}_{0p}' = -\frac{3(1+\sigma)}{2Eh^2}\sum_{n=0}^{\infty}\left\{\bar{b}_n-\bar{d}_n\right\} h^{2n} J^{(*\prime)^{n-1}}.$$

7. **The determination of $\overline{U}_0^*$ and U_0.** The function $\overline{U}_0$ does not appear in the formulas for U and w; hence, it is necessary to find only $\overline{U}_0^*$ and U_0. Equations (12) and (15), when considered simultaneously, will suffice for the determination of $\overline{U}_0^*$ and U_0. The procedure is analogous to that already given for $\overline{w}_0'$ and w_0; but it is much simpler, since, in the determination of $\overline{U}_0^*$ and U_0, it is not necessary to form a third system of equations corresponding to equations (21). For the case of normal load only, the final results are

$$(26\text{c})\qquad a_n = \sum_{i=0}^{n-1}(-1)^i\frac{(i+2)a_{n-1-i}-(i+1)c_{n-1-i}}{(2i+3)!}\qquad (n=1,2,3,\cdots);$$

$$(26\text{d})\qquad c_n = \sum_{i=0}^{n-1}(-1)^i\frac{(i+1)a_{n-1-i}-ic_{n-1-i}}{(2i+2)!}\qquad (n=1,2,3,\cdots);$$

$$(26\text{e})\qquad \overline{U}_0^{*\prime} = -\frac{1+\sigma}{2E}\sum_{n=0}^{\infty}\left\{a_n-c_n\right\} h^{2n} L'^{(*\prime)^n};$$

$$(26\text{f})\qquad \overline{U}_0^{*\prime} = -\frac{1+\sigma}{2E}\sum_{n=0}^{\infty}\left\{(1-\sigma)a_n+\sigma c_n\right\} h^{2n} L'^{(*\prime)^n};$$

$$(26\text{g})\qquad -(1-\sigma)\overline{U}_0^* + U_0^* = -\frac{1+\sigma}{2E}\sum_{n=0}^{\infty} c_n h^{2n} L^{(\prime *)^n};$$

$$(26\text{h})\qquad U_{0c} = C_1 r + \frac{C_2}{r};$$

$$(26\text{i})\qquad \overline{U}_{0c}^* = \frac{2C_1}{1-\sigma}.$$

The complementary solutions for all other types of loading are also given by

(26h) and (26i). If $\overline{U}_0^{*\prime}$ and $U_0^{*\prime}$, as given by (26e) and (26f), are infinite series, the values of U_{0p} and $\overline{U}_{0p}^{*}$ are

$$U_{0p} = -\frac{1+\sigma}{2E} \sum_{n=0}^{\infty} \{(1-\sigma)a_n + \sigma c_n\} h^{2n} L'^{(*\prime)^{n-1}}, \tag{26j}$$

$$\overline{U}_{0p}^{*} = -\frac{1+\sigma}{2E} \sum_{n=0}^{\infty} \{a_n - c_n\} h^{2n} L^{(\prime *)^{n}}. \tag{26k}$$

It should be observed that for the case of normal load, $a_0 = 0$, $c_0 = 1$.

For the case of shearing force only, $\bar{a}_0 = 1$, $\bar{c}_0 = 0$. The general formulas are

$$\bar{a}_n = \sum_{i=0}^{n-1} (-1)^i \frac{(i+2)\bar{a}_{n-1-i} - (i+1)\bar{c}_{n-1-i}}{(2i+3)!} \qquad (n = 1, 2, 3, \cdots); \tag{27c}$$

$$\bar{c}_n = \sum_{i=0}^{n-1} (-1)^i \frac{(i+1)\bar{a}_{n-1-i} - i\bar{c}_{n-1-i}}{(2i+2)!} \qquad (n = 1, 2, 3, \cdots); \tag{27d}$$

$$\overline{U}_0^{*\prime} = -\frac{1+\sigma}{2Eh} \sum_{n=0}^{\infty} \{\bar{a}_n - \bar{c}_n\} h^{2n} j^{(*\prime)^{n}}; \tag{27e}$$

$$U_0^{*\prime} = -\frac{1+\sigma}{2Eh} \sum_{n=0}^{\infty} \{(1-\sigma)\bar{a}_n + \sigma\bar{c}_n\} h^{2n} j^{(*\prime)^{n}}; \tag{27f}$$

$$-(1-\sigma)\overline{U}_0^{*} + U_0^{*} = -\frac{1+\sigma}{2Eh} \sum_{n=0}^{\infty} \bar{c}_n h^{2n} j^{*(\prime *)^{n-1}}. \tag{27g}$$

When $\overline{U}_0^{*\prime}$ and $U_0^{*\prime}$, as defined by (27e) and (27f), are infinite series, it turns out that

$$U_{0p} = -\frac{1+\sigma}{2Eh} \sum_{n=0}^{\infty} \{(1-\sigma)\bar{a}_n + \sigma\bar{c}_n\} h^{2n} j^{(*\prime)^{n-1}}, \tag{27j}$$

$$\overline{U}_{0p}^{*} = -\frac{1+\sigma}{2Eh} \sum_{n=0}^{\infty} \{\bar{a}_n - \bar{c}_n\} h^{2n} j^{*(\prime *)^{n-1}}. \tag{27k}$$

8. **Upper bounds of the constants.** Before convergence of the series for U and w can be established, upper bounds must be determined for the eight constants which enter in the formulas for $\overline{U}_0^{*\prime}$, $U_0^{*\prime}$, $\overline{w}_0^{\prime *\prime *}$, $w_0^{\prime *\prime *}$.

Consider first the constants $\bar{a}_n$ and $\bar{c}_n$. It may be proved that

$$|\bar{a}_n| < 1/2^n, \quad |\bar{c}_n| < (5/9)/2^n \qquad (n = 2, 3, \cdots). \tag{28a}$$

If $\bar{a}_1$, $\bar{a}_2$, $\bar{c}_1$, $\bar{c}_2$ are calculated from (27c) and (27d), it turns out that (28a) is true for $n = 2$. The proof, therefore, consists in showing that if

$$(28b) \quad |\bar{a}_{k-1-i}| < \frac{1}{2^{k-1-i}}, \quad |\bar{c}_{k-1-i}| < \frac{5}{9}\,\frac{1}{2^{k-1-i}} \qquad (i = 0, 1, \cdots, (k-3)),$$

then

$$(28c) \qquad |\bar{a}_k| < 1/2^k, \quad |\bar{c}_k| < (5/9)/2^k \qquad (k = 3, 4, \cdots).$$

This proof is not difficult to carry through for both $\bar{a}_n$ and $\bar{c}_n$. In the proof for $\bar{c}_n$, it is necessary to use the relation

$$(28d) \quad \bar{c}_n = \sum_{i=0}^{n-2}(-1)^i \frac{(i+1)\{(i+2)\bar{a}_{n-2-i} - (i+1)\bar{c}_{n-2-i}\}}{(2i+4)!} \quad (n = 2, 3, 4, \cdots),$$

a formula readily derived from (27c) and (27d).

In a closely analogous manner, it may be shown that

$$(28e) \qquad |a_n| < 1/2^n, \quad |c_n| < (5/9)/2^n \qquad (n = 1, 2, \cdots);$$

$$(28f) \qquad |b_n| < 1/3^n, \quad |d_n| < (3/4)/3^n \qquad (n = 2, 3, \cdots);$$

$$(28g) \qquad |\bar{b}_n| < 1/3^n, \quad |\bar{d}_n| < (3/4)/3^n \qquad (n = 1, 2, \cdots).$$

In obtaining the upper bounds for d_n and $\bar{d}_n$, it is necessary to use formulas (24′d) and (25′d), respectively.

The above upper bounds seem to be the strongest that can be found by the method of absolute values. By means of these inequalities it will be shown in the next article that the U and w series are uniformly convergent for a limited class of load functions. It would be desirable to include in this class of load functions all powers of r for which the series do not terminate, but the present inequalities are not strong enough for this purpose. On the other hand, if a few values of the constants are calculated, their absolute values are found to be considerably smaller than the above upper bounds would indicate. This suggests the possibility of securing, by a more refined analysis, the stronger upper bounds desired.

Since in the proof of convergence there is no particular advantage in using stronger upper bounds for the c's, b's, and d's than for the a's, it will be sufficient, in preparation for the following article, to write

$$(28h) \qquad |a_n|,\ |\bar{a}_n|,\ |c_n|,\ |\bar{c}_n|,\ |b_n|,\ |\bar{b}_n|,\ |d_n|,\ |\bar{d}_n| \leqq 1/2^n \qquad (n = 0, 1, 2, \cdots).$$

From formulas (28a), (28e), (28f), (28g), it is evident that the above relations hold for $n=2, 3, 4, \cdots$. By calculating the values of these constants for $n=1$ from the original formulas, it turns out that (28h) is also valid for $n=0, 1$.

9. **Convergence of the U and w series.** Let U_c and w_c be the complementary solutions of U and w, respectively, and let U_p and w_p be the corresponding particular solutions. There is no need to examine U_c and w_c, since the consideration of those cases in which U and w terminate will be postponed until §10. For the same reason, only those values of U_{0p}, $\overline{U}^*_{0p}$, w_{0p}, $\overline{w}'_{0p}$ which are defined by infinite series will be considered in this article.

First consider U_p when the plate is under normal load only. U_p is found by substituting (26j), (26k), (24l), (24m) in (9a). It turns out that U_p is composed of four iterated series, the first of which is

$$-\frac{1+\sigma}{2E}\sum_{i=0}^{\infty}(-1)^i i\left[\sum_{n=0}^{\infty}\{a_n - c_n\}h^{2n}L'^{(*')^{n-1}}\right]^{(*')^i}\frac{z^{2i}}{(2i)!}. \tag{29a}$$

Since it will be shown that each of the above series is absolutely convergent for a certain class of load functions, it will be convenient in the discussion which follows to employ the double series rather than the given iterated series.

Let A_1 be the double series corresponding to (29a), and let A be the series formed from the absolute values of the terms of A_1. If (28h) is employed, the result is

$$A \leqq \sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\frac{i}{(2i)!}\frac{2}{2^n}h^{2n}\left|L'^{(*')^{n-1+i}}\right|z^{2i}. \tag{29b}$$

Since $i/(2i)! \leqq 1/2^i$ $(i=0, 1, 2, \cdots)$ and $|z| \leqq h$, (29b) becomes

$$A \leqq 2\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\frac{h^{2(n+i)}}{2^{n+i}}\left|L'^{(*')^{n-1+i}}\right|. \tag{29c}$$

Let r_0 be the outer radius of the plate, and let q, q_1, q_2 be constants such that $0<q<q_1<q_2=r_0/h$. Then

$$A \leqq 2\sum_{i=0}^{\infty}\sum_{n=0}^{\infty}\left(\frac{r_0}{2^{1/2}q}\right)^{2(n+i)}\left|L'^{(*')^{n-1+i}}\right|\left(\frac{qh}{r_0}\right)^{2(n+i)}. \tag{29d}$$

Since $0<qh/r_0<q_1h/r_0<1$, the above double series of positive terms will be uniformly convergent if L is such a function of r that a constant M can be found for which

$$\left(\frac{r_0}{2^{1/2}q}\right)^{2(n+i)}\left|L'^{(*')^{n-1+i}}\right| < M,\quad 0 \leqq r \leqq r_0, \tag{29e}$$

for all values of n and i.

It may be shown that if $L=pJ_0(\pm 2^{1/2}qr/r_0)$, where p is a constant, then (29e) reduces to $\{pr_0/(2^{1/2}q)\}J_1(\pm 2^{1/2}qr/r_0)$ for all values of n and i. Hence a class of Bessel functions have actually been exhibited for which the iterated

series (29a) is both absolutely and uniformly convergent in the interval $0 \leqq r \leqq r_0$.

By a procedure similar to the foregoing, it may be shown that the series defining either U_p or w_p will be absolutely and uniformly convergent for normal load or shearing traction under hypotheses analogous to (29e).

In the case of the example just given, it is readily shown that the series U_p and w_p may be primed, starred, or differentiated with respect to z, term by term, as many times as may be desired. Thus examples have now been exhibited in which all the operations that have been hitherto applied formally are seen to be justified.

10. **Finite series.** In §5, necessary and sufficient conditions were given for the U and w series to terminate. From formulas (24j), (24k), (26h), (26i) it is apparent that the complementary solutions have the forms which ensure terminating series for the displacements. It remains only to determine the admissible forms of particular solutions in the cases of normal load and shearing traction.

When the plate is subjected to normal load or shearing traction, it is easy to show that a condition sufficient to meet the requirements on U_0, $\overline{U}_0^*$, w_0, $\overline{w}_0'$ given in §5 is that L and l shall contain no terms which are not of the form

$$C' r^{2p-2} + K' r^{2q-2} \log r$$

and that J and j shall contain no terms which are not of the form

$$C r^{2m-3} + K r^{2n-1} \log r,$$

m, n, p, q being any positive integers. Moreover, it may be shown by an argument similar to that employed in §5 that, assuming U_0, $\overline{U}_0^*$, w_0, $\overline{w}_0'$ given by series which terminate, the above conditions are also necessary; no attempt will be made in this paper to find necessary conditions in the case in which U_0, $\overline{U}_0^*$, w_0, $\overline{w}_0'$ are defined by infinite series.

In order to conserve space, only the case in which L, l, J, j contain no logarithmic terms will be considered; in addition, the case in which the shear is proportional to $1/r$ will also be neglected since for this case the particular solutions involve logarithmic terms.

It is first desirable to study the effect of the prime-star operator on powers of r. It may readily be shown that

(30a) $$(r^{2m})^{('*)^n} \neq 0, \quad = 0; \quad n \leqq m, \; n > m,$$

(30b) $$(r^{2m})^{('*)^n\,'} \neq 0, \quad = 0; \quad n < m, \; n \geqq m,$$

where $m = 0, 1, 2, \cdots$, and n is any integer, or zero.

No loss in generality will ensue if it is assumed that L, l, J, j contain but one term, since the more complicated types of loading may be considered as the sum of simple loadings which contain one term only. Thus L and l may be written in the form

$$L = e_1 r^{2m}, \ l = e_2 r^{2m} \qquad (m = 0, 1, 2, \cdots), \tag{31a}$$

where e_1 and e_2 are constants. Substitute (31a) in (26e), (26f), (26g), and make use of (30); the result is

$$\overline{U}_0^{*\prime} = -\frac{1+\sigma}{2E} \sum_{n=0}^{m-1} \{a_n - c_n\} h^{2n} L^{\prime(*\prime)^n}, \tag{31b}$$

$$U_0^{*\prime} = -\frac{1+\sigma}{2E} \sum_{n=0}^{m-1} \{(1-\sigma)a_n + \sigma c_n\} h^{2n} L^{\prime(*\prime)^n}, \tag{31c}$$

$$-(1-\sigma)\overline{U}_0^* + U_0^* = -\frac{1+\sigma}{2E} \sum_{n=0}^{m} c_n h^{2n} L^{(\prime *)^n}. \tag{31d}$$

U_{0p} is found by performing the inverse prime-star operation upon (31c); the result is

$$U_{0p} = -\frac{1+\sigma}{2E} \sum_{n=0}^{m} \{(1-\sigma)a_n + \sigma c_n\} h^{2n} L^{\prime(*\prime)^{n-1}}. \tag{31e}$$

It should be observed that $m-1$ could have been used for the upper limit in the summation defining U_{0p}, but the use of m makes it possible to express the subsequent formulas in more concise and elegant form. Substitute (31e) in (31d); the resulting value of $\overline{U}_{0p}^*$ is

$$\overline{U}_{0p}^* = -\frac{1+\sigma}{2E} \sum_{n=0}^{m} \{a_n - c_n\} h^{2n} L^{(\prime *)^n}. \tag{31f}$$

It is evident that this value of $\overline{U}_{0p}^*$ satisfies (31b).

In an analogous manner, w_{0p} and $\overline{w}_{0p}'$ may be written as

$$w_{0p} = \frac{3(1+\sigma)}{2Eh^3} \sum_{n=0}^{m+1} \{(1-\sigma)b_n - (2-\sigma)d_n\} h^{2n} l^{(\prime *)^{n-2}}, \tag{31g}$$

$$\overline{w}_{0p}' = -\frac{3(1+\sigma)}{2Eh^3} \sum_{n=0}^{m+1} \{b_n - d_n\} h^{2n} l^{\prime(*\prime)^{n-2}}. \tag{31h}$$

For the case of shear, J and j may be written in the form

$$J = s_1 r^{2m+1}, \ j = s_2 r^{2m+1} \qquad (m = 0, 1, 2, \cdots), \tag{32a}$$

where s_1 and s_2 are constants. Observe that

(32a′) $$J^* = 2(m+1)s_1 r^{2m}, \quad j^* = 2(m+1)s_2 r^{2m} \quad (m = 0, 1, 2, \cdots);$$

that is, J^* and j^* are proportional to r^{2m} and, therefore, formulas (30) may be used to determine the upper limits of n. The final results are

(32e) $$U_{0p} = -\frac{1+\sigma}{2Eh}\sum_{n=0}^{m+1}\{(1-\sigma)\bar{a}_n + \sigma\bar{c}_n\} h^{2n} j^{(*\prime)^{n-1}},$$

(32f) $$\overline{U}^*_{0p} = -\frac{1+\sigma}{2Eh}\sum_{n=0}^{m+1}\{\bar{a}_n - \bar{c}_n\} h^{2n} j^{*(\prime *)^{n-1}},$$

(32g) $$w_{0p} = \frac{3(1+\sigma)}{2Eh^2}\sum_{n=0}^{m+1}\{(1-\sigma)\bar{b}_n - (2-\sigma)\bar{d}_n\} h^{2n} J^{*(\prime *)^{n-2}},$$

(32h) $$\overline{w}'_{0p} = -\frac{3(1+\sigma)}{2Eh^2}\sum_{n=0}^{m+1}\{\bar{b}_n - \bar{d}_n\} h^{2n} J^{(*\prime)^{n-1}}.$$

The values of U_{0p}, $\overline{U}^*_{0p}$, w_{0p}, $\overline{w}'_{0p}$ for the cases of radial and axial mass forces may be found by a process similar to that employed for the case of normal load. The formulas for radial mass force only are

(33e) $$U_{0p} = \frac{1-\sigma^2}{8E} c_r r^3,$$

(33f) $$\overline{U}^*_{0p} = \frac{1+\sigma}{2E} c_r r^2,$$

(33g, h) $$w_{0p} = \overline{w}'_{0p} = 0.$$

When there is axial mass force only the formulas are

(34e, f) $$U_{0p} = \overline{U}^*_{0p} = 0,$$

(34g) $$w_{0p} = -\frac{3(1-\sigma^2)}{64Eh^2} c_z r^4,$$

(34h) $$\overline{w}'_{0p} = \frac{3(1+\sigma)}{16Eh^2} c_z r^3 + \frac{3(1+\sigma)}{4(1-\sigma)E} c_z r.$$

The complete formulas for the displacements may now be written out. It is convenient to give in full the portion involving the mass forces and to use the symbols U_{0p}, $\overline{U}^*_{0p}$, w_{0p}, $\overline{w}'_{0p}$ for the particular solutions in the cases of normal load and shearing traction. The displacements take the form

$$\text{(35a)}\quad U = C_1 r + \frac{C_2}{r} - \left[K_1\left\{2r\log r + r + \frac{4h^2}{(1-\sigma)r}\right\} + 2K_2 r + \frac{K_3}{r}\right]z$$

$$+ \frac{2(2-\sigma)}{3(1-\sigma)}\frac{K_1}{r}z^3 + \frac{1-\sigma^2}{8E}c_r r^3 + \frac{3(1-\sigma^2)}{16Eh^2}c_z r^3 z$$

$$+ \frac{3(1+\sigma)}{2E}c_z rz + \frac{\sigma(1+\sigma)}{2E}c_r rz^2 - \frac{(2-\sigma)(1+\sigma)}{4Eh^2}c_z rz^3$$

$$+ \sum_{i=0}(-1)^i\{i\overline{U}_{0p} + U_{0p}\}^{(*\prime)^i}\frac{z^{2i}}{(2i)!}$$

$$+ \sum_{i=0}(-1)^i\{(i+2-2\sigma)\overline{w}_{0p} + w_{0p}\}^{\prime(*\prime)^i}\frac{z^{2i+1}}{(2i+1)!},$$

$$\text{(35b)}\quad w = K_1 r^2\log r + K_2 r^2 + K_3\log r + K_4 - \frac{2\sigma}{1-\sigma}C_1 z$$

$$+ \frac{2\sigma}{1-\sigma}\{K_1(1+\log r) + K_2\}z^2 - \frac{3(1-\sigma^2)}{64Eh^2}c_z r^4$$

$$- \frac{\sigma(1+\sigma)}{2E}c_r r^2 z - \frac{3\sigma(1+\sigma)}{8Eh^2}c_z r^2 z^2$$

$$- \frac{(1+4\sigma)(1+\sigma)}{4(1-\sigma)E}c_z z^2 - \frac{\sigma^2(1+\sigma)}{3(1-\sigma)E}c_r z^3 + \frac{(1+\sigma)^2}{8Eh^2}c_z z^4$$

$$+ \sum_{i=0}(-1)^i\{i\overline{w}_{0p} + w_{0p}\}^{(\prime *)^i}\frac{z^{2i}}{(2i)!}$$

$$- \sum_{i=0}(-1)^i\{(i-1+2\sigma)\overline{U}_{0p} + U_{0p}\}^{*(\prime *)^i}\frac{z^{2i+1}}{(2i+1)!}.$$

The upper limits of i in the above summations are, taken in order, the same as the upper limits of n already found for U_{0p}, $\overline{w}'_{0p}$, w_{0p}, $\overline{U}^*_{0p}$, respectively.

It is now possible to express T_r, N_r, G_r in terms of c_r, c_z, U_{0p}, $\overline{U}^*_{0p}$, w_{0p}, $\overline{w}'_{0p}$, and the arbitrary constants involved in (35). At the same time, two additional quantities needed in Part II will be computed, namely,

$$\text{(36a)}\qquad T_\theta = \int_{-h}^{h}\widehat{\theta\theta}\,dz,$$

$$\text{(36b)}\qquad G_\theta = \int_{-h}^{h}\widehat{\theta\theta}\,z\,dz.$$

The formulas for T_r and T_θ can be simplified by making use of the function $\overline{U}_{0p}$. This function has hitherto been undefined, since it does not appear in the U and w series. $\overline{U}_{0p}$ may be assigned any value which is consistent with the formulas already given for $\overline{U}^*_{0p}$; it will be convenient, in each case, to take for the upper limit of n in $\overline{U}_{0p}$ the value found for U_{0p}. Thus

$$(31\text{f}') \qquad \overline{U}_{0p} = -\frac{1+\sigma}{2E}\sum_{n=0}^{m}\{a_n - c_n\}h^{2n}L'^{(*')^{n-1}},$$

$$(32\text{f}') \qquad \overline{U}_{0p} = -\frac{1+\sigma}{2Eh}\sum_{n=0}^{m+1}\{\bar{a}_n - \bar{c}_n\}h^{2n}j^{(*')^{n-1}}.$$

By means of (31), (32), and the formulas which define the a's, b's, c's, and d's, it may be shown that the following relations are valid for all cases of normal load and shearing traction:

$$(37\text{a}) \quad \sum_{i=0}(-1)^i\{(i+\sigma)\overline{U}_{0p} + U_{0p}\}^{(*')^i}\frac{h^{2i+1}}{(2i+1)!} = -\frac{1+\sigma}{2E}j^{(*')^{-1}},$$

$$(37\text{b}) \quad \sum_{i=0}(-1)^i\{(i+1-\sigma)\overline{w}_{0p} + w_{0p}\}'^{(*')^i}\frac{h^{2i+1}}{(2i+1)!} = -\frac{1+\sigma}{2E}l^{(*)^{-1}},$$

$$(37\text{c}) \quad \sum_{i=0}(-1)^i\{(i+2-\sigma)\overline{w}_{0p} + w_{0p}\}'^{(*')^i}\frac{(i+1)h^{2i+3}}{(2i+3)!} = -\frac{1+\sigma}{4E}hJ^{(*')^{-1}} - \frac{1+\sigma}{4E}l^{(*'*)^{-1}},$$

the upper limits of i in the summations being the same as those for n in U_{0p}, $\overline{w}'_{0p}$, $\overline{w}'_{0p}$, respectively.

Substitute (35) in the formulas for $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{rz}$, and put these values of $\widehat{rr}$, $\widehat{\theta\theta}$, $\widehat{rz}$ in the formulas for T_r, T_θ, G_r, G_θ, N_r, and simplify by means of formulas (37). The final results are

$$(38\text{a}) \quad T_r = \frac{2Eh}{1-\sigma}C_1 - \frac{2Eh}{1+\sigma}\frac{C_2}{r^2} + \frac{3+\sigma}{4}hr^2c_r + \frac{\sigma(1+\sigma)}{3(1-\sigma)}h^3c_r + \frac{2E\sigma}{1+\sigma}\frac{1}{r}\sum_{i=0}(-1)^i\overline{U}_{0p}^{(*')^i}\frac{h^{2i+1}}{(2i+1)!} - j^{(*')^{-1}},$$

$$(38\text{b}) \quad T_\theta = \frac{2Eh}{1-\sigma}C_1 + \frac{2Eh}{1+\sigma}\frac{C_2}{r^2} + \frac{1+3\sigma}{4}hr^2c_r + \frac{\sigma(1+\sigma)}{3(1-\sigma)}h^3c_r + \frac{2E\sigma}{1+\sigma}\sum_{i=0}(-1)^i\overline{U}_{0p}^{(*')^i\,\prime}\frac{h^{2i+1}}{(2i+1)!} - \frac{1}{r}j^{(*')^{-1}},$$

$$\text{(38c)}\quad G_r = -\frac{2Eh^3}{3(1-\sigma)}\left\{2\log r + \frac{3+\sigma}{1+\sigma} - \frac{2(8+\sigma)}{5(1+\sigma)}\frac{h^2}{r^2}\right\}K_1 - \frac{4Eh^3}{3(1-\sigma)}K_2$$
$$+\frac{2Eh^3}{3(1+\sigma)}\frac{K_3}{r^2} + \left\{\frac{3+\sigma}{8}\frac{r^2}{h^2} + \frac{24+23\sigma+3\sigma^2}{30(1-\sigma)}\right\}h^3c_z$$
$$+\frac{4E\sigma}{1+\sigma}\frac{1}{r}\sum_{i=0}(-1)^i \overline{w}'^{(*\prime)^i}_{0p}\frac{i+1}{(2i+3)!}h^{2i+3} - hJ^{(*\prime)^{-1}\prime} - l^{(*\prime *)^{-1}\prime},$$

$$\text{(38d)}\quad G_\theta = -\frac{2Eh^3}{3(1-\sigma)}\left\{2\log r + \frac{1+3\sigma}{1+\sigma} + \frac{2(8+\sigma)}{5(1+\sigma)}\frac{h^2}{r^2}\right\}K_1 - \frac{4Eh^3}{3(1-\sigma)}K_2$$
$$-\frac{2Eh^3}{3(1+\sigma)}\frac{K_3}{r^2} + \left\{\frac{1+3\sigma}{8}\frac{r^2}{h^2} + \frac{24+23\sigma+3\sigma^2}{30(1-\sigma)}\right\}h^3c_z$$
$$+\frac{4E\sigma}{1+\sigma}\sum_{i=0}(-1)^i \overline{w}'^{(*\prime)^i\prime}_{0p}\frac{i+1}{(2i+3)!}h^{2i+3} - \frac{h}{r}J^{(*\prime)^{-1}} - \frac{1}{r}l^{(*\prime *)^{-1}},$$

$$\text{(38e)}\quad N_r = -\frac{8Eh^3}{3(1-\sigma^2)}\frac{K_1}{r} + rhc_z - l^{(*)^{-1}},$$

where the upper limits of i in the summations are the same as those of n for $\overline{U}_{0p}$, $\overline{U}'_{0p}$, $\overline{w}'_{0p}$, $\overline{w}''_{0p}$, respectively.

All the formulas necessary for handling a wide class of problems in moderately thick plates have now been obtained. In Part II, the power of this theoretical machinery will be exhibited by applying it to the solution of certain problems of especial interest.

Part II. Application

11. Introduction to Part II. In applying the theory of Part I, the following problems will be considered:

(i) a complete plate, the load being
   (a) a function of r continuous over the whole plate,
   (b) a pressure concentrated at the center,
   (c) a distribution continuous in each of two concentric zones but discontinuous at their junction—a bizonal problem;

(ii) an incomplete plate, that is, a plate with a concentric hole, the load being
   (a) a function of r continuous over the whole plate,
   (b) a uniform shear distributed over the inner edge of the plate.

Further types of problems to which this method is applicable will be mentioned at the close of the paper.

In every problem the surface traction on the faces will be prescribed; hence the displacements will be completely known as soon as the arbitrary constants in (35) have been determined. It turns out that K_2, K_4, C_1 depend upon conditions at the outer edge of the plate; K_1, K_3, C_2 are determined by conditions at the center or at the inner edge, according as the plate is complete or incomplete.

The outer edge of the plate will always be denoted by r_0; the radius of an inner edge, or of a junction of two zones, will be called r_1. In every case, the plate will be fixed in space by demanding that there shall be no axial displacement at the outer edge of the middle surface of the plate; that is

$$w\big|_{z=0,r=r_0} = 0. \tag{39a}$$

If (39a) is substituted in (35b), the result is

$$K_4 = -K_1 r_0^2 \log r_0 - K_2 r_0^2 - K_3 \log r_0 + \frac{3(1-\sigma^2)}{64Eh^2} c_z r_0^4 - w_{0p}\big|_{r=r_0}. \tag{39b}$$

12. **Preliminary formulas for a complete plate.** Consider a cylindrical section of radius r cut concentrically from the plate. The sum of the z-components of all the external forces acting on this portion of the plate must equal $\rho(f_z - F_z)$ times its volume; that is, must equal $2\pi r^2 h c_z$. Note that the value of N_r is constant along the circumference of this section, since all stresses are assumed to be independent of θ. Since the outer normal on both the upper and lower faces was taken as the positive direction for normal load, the following relation is valid for a continuous loading distribution:

$$2\pi r N_r + \int_0^r 2\pi r l \, dr = 2\pi r^2 h c_z, \quad 0 < r \leqq r_0. \tag{40a}$$

For a plate whose only load is a pressure of $-W$ pounds concentrated at the center, the corresponding relation is

$$2\pi r N_r - W = 0, \quad 0 < r \leqq r_0. \tag{40b}$$

It is evident that, as a point approaches the axis of the plate, the direction of $\widehat{\theta\theta}$ at that point approaches a radial direction. Since all stresses are independent of θ, it is obvious that any two radial stresses at the same point on the axis of the plate are equal. Hence, it is clear that, for any constant value of z,

$$\lim_{r\to 0} (\widehat{rr} - \widehat{\theta\theta}) = 0. \tag{41a}$$

Observe that (41a) implies

$$\lim_{r\to 0}(T_r - T_\theta) = \lim_{r\to 0}(G_r - G_\theta) = 0. \tag{41b}$$

It should be observed that, although (41a) implies (41b), the converse may not be true. If (41a) is not also satisfied, the solution will not be valid in the immediate neighborhood of the axis of the plate; on the other hand, by de Saint-Venant's principle, such a solution would closely approximate the truth for all points whose distance from the axis is at least equal to the thickness of the plate.

All the formulas necessary for the determination of K_1, K_3, K_4, C_2 for the case of a complete plate have now been found.

13. **Determination of K_1, K_3, K_4, C_2 for a complete plate with continuous distribution of load.** The value of K_1 will be found first. Substitute in (40a) the value of l given in (31a), and solve for N_r. The result is

$$N_r = -\frac{e_2 r^{2m+1}}{2(m+1)} + rhc_z = -l^{(*)^{-1}} + rhc_z,\quad 0 < r \leqq r_0. \tag{42a}$$

The result of substituting this value of N_r in (38e) is

$$K_1 = 0. \tag{42b}$$

Observe that (42b) is true for both radial mass force and shearing traction as well as for axial mass-force and normal load, since the two former quantities do not appear in either (38e) or (40a).

The values of C_2 and K_3 will now be found. Substitute $K_1=0$ in (38), and solve for C_2 and K_3. The results ($r\neq 0$) are

$$\begin{aligned}(T_r - T_\theta)r^2 = &-\frac{4Eh}{1+\sigma}C_2 + \frac{1-\sigma}{2}hrc_r + rj^{(*\prime)^{-1}} - r^2 j^{(*\prime)^{-1}\prime}\\ &+\frac{2E\sigma}{1+\sigma}\sum_{i=0}(-1)^i\left\{r\overline{U}_{0p}^{(*\prime)^i} - r^2\overline{U}_{0p}^{(*\prime)^i\prime}\right\}\frac{h^{2i+1}}{(2i+1)!},\end{aligned} \tag{43a}$$

$$\begin{aligned}(G_r - G_\theta)r^2 = &\frac{4Eh^3}{3(1+\sigma)}K_3 + \frac{1-\sigma}{4}hr^4c_z + hrJ^{(*\prime)^{-1}}\\ &- hr^2J^{(*\prime)^{-1}\prime} + rl^{(*\prime *)^{-1}} - r^2 l^{(*\prime *)^{-1}\prime}\\ &+\frac{4E\sigma}{1+\sigma}\sum_{i=0}(-1)^i\left\{r\overline{w}_{0p}^{\prime(*\prime)^i} - r^2\overline{w}_{0p}^{\prime(*\prime)^i\prime}\right\}\frac{i+1}{(2i+3)!}h^{2i+3}.\end{aligned} \tag{43b}$$

It is not difficult to show that r^4 is the lowest power of r appearing in the right hand member of either (43a) or (43b). Let $r\to 0$, and make use of (41b). The result is

$$C_2 = K_3 = 0. \tag{43c}$$

If the values of $K_1=K_3=C_2=0$ are substituted in the formulas for $\widehat{rr}$ and $\widehat{\theta\theta}$, it turns out that (41a) is satisfied for each of the loading conditions under consideration. Hence, for these loading conditions, the solution will be valid in the neighborhood of the axis of the plate.

The value of K_4 may now be found in terms of K_2. Substitute $K_1=K_3=0$ in (39b). The resulting value of K_4 is

$$K_4 = -K_2r_0^2 + \frac{3(1-\sigma^2)}{64Eh^2}c_zr_0^4 - w_{0p}\Big|_{r=r_0}. \tag{44}$$

14. **Determination of K_1, K_3, K_4, C_2 for a complete plate with a central load.** In this article, the only load acting is a downward pressure of $-W$ pounds concentrated at the center of the upper face; that is,

$$L = l = 0,\ 0 < r \leqq r_0;\ L = l = -\infty,\ r = 0. \tag{45}$$

The value of K_1 will be found first. Since $l^{(*)^{-1}}=0$ for every point except the center, formula (38e) reduces to

$$N_r = -\frac{8Eh^3}{3(1-\sigma^2)}\frac{K_1}{r},\quad 0 < r \leqq r_0. \tag{46a}$$

The result of substituting this value of N_r in (40b) is

$$K_1 = -\frac{3(1-\sigma^2)}{16\pi Eh^3}W. \tag{46b}$$

C_2 and K_3 may now be found. If (45) is substituted in (31e), (31f′), (31g), (31h), it turns out that

$$U_{0p} = \overline{U}_{0p} = w_{0p} = \overline{w}'_{0p} = 0,\ 0 < r \leqq r_0. \tag{47a}$$

Substitute (46b) and (47a) in (38), and recall that $c_r=c_z=j\equiv J\equiv 0$. The resulting equations yield for C_2 and K_3 the values

$$C_2 = -\frac{1+\sigma}{4Eh}(T_r - T_\theta)r^2,\ 0 < r \leqq r_0; \tag{47b}$$

$$K_3 = \frac{3(1+\sigma)}{4Eh^3}(G_r - G_\theta)r^2 + \frac{3(1+\sigma)W}{16\pi Eh^3}\left\{(-1+\sigma)r^2 + \frac{2(8+\sigma)}{5}h^2\right\},\ 0 < r \leqq r_0. \tag{47c}$$

Let $r\to 0$, and make use of (41b); the final result is

$$C_2 = 0,\ K_3 = \frac{3(1+\sigma)(8+\sigma)}{40\pi Eh}W.\dagger \tag{47d}$$

It is not difficult to show that (41a) is not satisfied. Hence the solution is not valid in the neighborhood of the axis of the plate. This conclusion can be justified from another standpoint. Observe that $\widehat{zz}$ is infinite at the center of the upper face. It is clear that the stress-equations of motion themselves are not valid at the center of the plate, since they were derived on the assumption that the stress remains within the elastic limit. On the other hand, by de Saint-Venant's principle, the actual stress distribution will be closely approximated by this solution at all points whose distance from the axis exceeds the thickness of the plate.

Substitute in (39b) the values of K_1, K_3, and w_{0p} already found. The result is

$$K_4 = \frac{3(1-\sigma^2)}{16\pi Eh^3}Wr_0^2\log r_0 - \frac{3(1+\sigma)(8+\sigma)}{40\pi Eh}W\log r_0 - K_2r_0^2. \tag{48}$$

15. **Pure stretching of a complete or incomplete plate with continuous distribution of load.** Pure stretching will be defined as that state of strain in which the middle plane is not bent and the deformed plate is symmetrical in that plane. It is convenient to have a criterion for pure stretching in terms of surface tractions and mass forces alone, since they generally constitute the data in any given problem. It is not difficult to show that, for the loading conditions considered in this paper, a sufficient condition for pure stretching is, for the complete plate,

$$G_r\Big|_{r=r_0} = c_2 = l \equiv J \equiv 0, \tag{49a}$$

---

† Love (p. 475) gives, for the constant corresponding to K_1, the same value as that found for K_1 in (46b). But he takes the constant corresponding to K_3 to be zero, since that is the only value of K_3 which permits w_0 to remain finite at the center of the plate. Incidentally, this value of K_3 is the only one which makes w_0' vanish at $r=0$.

It turns out that Love's formula for w (which may be found by substituting his value of w_0 in the general formulas given on p. 473) leads to an impossible situation. In spite of a negatively infinite load at the central point of the upper surface, the displacements are positively infinite at every point of the axis of the plate except at the central point of the middle surface. Moreover, since the deflection of the middle surface is finite throughout the plate, we find that all points on the central axis which, before strain, were below the middle surface assume a position above it after strain. Evidently, Love's solution is incorrect; hence $K_3 \neq 0$ and the displacement at the center of the middle plane is not finite. De Saint-Venant in his "Note du §45" of the translation of Clebsch, using a wholly different method, had previously made essentially the same mistake. In a paper published in the Journal de l'Ecole Polytechnique, cahier 26 (1927), p. 89, Garabedian has shown, by an entirely different argument, that both Love's and de Saint-Venant's solutions are in error, and, in the same paper, he gives for the first time the correct solution for a plate centrally loaded.

and, for the incomplete plate,

$$G_r \big|_{r=r_0} = G_r \big|_{r=r_1} = N_r \big|_{r=r_1} = c_z = l \equiv J \equiv 0. \tag{49b}$$

It may readily be shown, for both the complete and incomplete plate, that (49) is a sufficient condition that

$$K_1 = K_2 = K_3 = c_z = l \equiv J \equiv 0. \tag{49c}$$

The value $K_4 = 0$ is obtained by substituting (49c) in (31g) and (39b). The result of substituting (49c) and this value of K_4 in (35) is

$$U = C_1 r + \frac{C_2}{r} + \frac{1-\sigma^2}{8E} c_r r^3 + \frac{\sigma(1+\sigma)}{2E} c_r r z^2 + \sum_{i=0} (-1)^i \{ i\bar{U}_{0p} + U_{0p} \}^{(*r)^{-1}} \frac{z^{2i}}{(2i)!}, \tag{50a}$$

$$w = -\frac{2\sigma}{1-\sigma} C_1 z - \frac{\sigma(1+\sigma)}{2E} c_r r^2 z - \frac{\sigma^2(1+\sigma)}{3(1-\sigma)E} c_r z^3 - \sum_{i=0} (-1)^i \{ (i-1+2\sigma)\bar{U}_{0p} + U_{0p} \}^{*(r*)^i} \frac{z^{2i+1}}{(2i+1)!}. \tag{50b}$$

In dealing with a complete plate, it should be remembered that $C_2 = 0$. In each of the following problems, it is assumed that the distribution of load is such that either (49a) or (49b) is satisfied.

Problem I: Complete plate whose outer edge is free to expand, that is, $T_r \big|_{r=r_0} = 0$. C_1 is readily obtained from (38a); its value is

$$C_1 = -\frac{(3+\sigma)(1-\sigma)}{8E} r_0^2 c_r - \frac{\sigma(1+\sigma)}{6E} h^2 c_r + \left[\frac{1-\sigma}{2Eh} j^{(*r)^{-1}r} - \frac{\sigma(1-\sigma)}{1+\sigma} \frac{1}{r} \sum_{i=0} (-1)^i \bar{U}_{0p}{}^{(*r)^i} \frac{h^{2i}}{(2i+1)!} \right]_{r=r_0}. \tag{51}$$

If this value of C_1 and the value $C_2 = 0$ are substituted in (50), the displacements will be given for a complete plate under any type of loading which results in pure stretching. From these formulas may be obtained a number of important solutions, a few of which will now be given.

Case Ia: A complete plate acted upon by a radial acceleration due to the rotation of the plate about the z-axis with an angular velocity of ω rad./sec. This well known solution (Love, p. 148) is obtained by setting $L_1 \equiv L_2 \equiv L \equiv j \equiv 0$, $c_r = -\rho\omega^2$.

Case Ib: A complete plate on which the only loads are equal constant tensions of magnitude p on both faces. The well known solution, $U = -\sigma p r/E$, $w = pz/E$, is found by setting $c_r = j \equiv 0$, $L_1 = L_2 = p$, $L = 2p$.

Case Ic: A complete plate with no load other than equal tensions which are proportional to r^2; that is, $c_r = j \equiv 0$, $L_1 = L_2 = pr^2$, $L = 2pr^2$. The final result is

$$U = -\frac{\sigma}{4E} pr\{(1-\sigma)r_0^2 + (1+\sigma)r^2\} + \frac{1-\sigma^2}{3E} pr(h^2 - 3z^2),$$

$$w = \frac{p}{2E} z\{\sigma^2 r_0^2 + 2(1-\sigma^2)r^2\} - \frac{2\sigma(1+\sigma)}{3E} pz(h^2 - z^2).$$

Case Id: A complete plate on which the only loads are equal outward shearing tractions proportional to r; that is, $c_r = L \equiv 0$, $J_1 = -J_2 = pr$, $j = 2pr$. The resulting displacements are

$$U = \frac{1-\sigma}{8Eh} pr\{(3+\sigma)r_0^2 - (1+\sigma)r^2\} - \frac{2 + 5\sigma - \sigma^2}{6E} hpr$$
$$+ \frac{(2-\sigma)(1+\sigma)}{2Eh} prz^2,$$

$$w = -\frac{\sigma p}{4Eh} z\{(3+\sigma)r_0^2 - 2(1+\sigma)r^2\} + \frac{3 + 2\sigma + \sigma^2}{3E} hpz$$
$$- \frac{(1+\sigma)^2}{3Eh} pz^3.$$

It is not difficult to show that only in Case Ib does $\widehat{rr}$ vanish at the edge for all z. Hence, for this case only, the solution is valid throughout the entire plate, and the ratio of h to r_0 need not be small. For the other three cases, the solution is not valid in the neighborhood of the edge, and the ratio of h to r_0 must not be large.

Problem II: An incomplete plate whose outer and inner edges are free to expand, that is, $T_r|_{r=r_0} = T_r|_{r=r_1} = 0$. It would be possible to work out as many cases under this problem as were given under Problem I; it will suffice, however, to consider the first case only. C_1 and C_2 are obtained from (38a); the results are

$$C_1 = -\frac{(3+\sigma)(1-\sigma)}{8E}(r_0^2 + r_1^2)c_r - \frac{\sigma(1+\sigma)}{6E} h^2 c_r,$$

$$C_2 = -\frac{(3+\sigma)(1+\sigma)}{8E} r_0^2 r_1^2 c_r.$$

If these values of C_1 and C_2 and the values $L_1 \equiv L_2 \equiv L \equiv j \equiv 0$, $c_r = -\rho\omega^2$ are substituted in (50), the well known solution for a rotating incomplete plate is obtained (Love, p. 148).

Problem III: A complete plate on which the only loads are equal constant pressures of magnitude $-p$ on both faces; that is, $c_r = j \equiv 0$, $L_1 = L_2 = -p$, $L = -2p$. It turns out that

$$U = C_1 r + \frac{\sigma(1+\sigma)}{2E} pr, \quad w = -\frac{2\sigma}{1-\sigma} C_1 z - \frac{1-\sigma^2}{E} pz,$$

$$\widehat{rr} = \frac{E}{1-\sigma} C_1 - \frac{\sigma}{2} p.$$

Case IIIa. The radius of the outer edge is to remain unchanged for all z; that is, $U|_{r=r_0} = 0$. The results are

$$C_1 = -\frac{\sigma(1+\sigma)}{2E} p, \quad U \equiv 0, \quad w = -\frac{(1-2\sigma)(1+\sigma)}{(1-\sigma)E} pz,$$

$$\widehat{rr} = -\frac{\sigma}{1-\sigma} p.$$

Case IIIb. The thickness of the plate is to remain unchanged, that is,

$$w|_{z=\pm h} = 0.$$

It turns out that

$$C_1 = -\frac{(1-\sigma)^2(1+\sigma)}{2E\sigma} p, \quad U = -\frac{(1-2\sigma)(1+\sigma)}{2E\sigma} pr, \quad w \equiv 0,$$

$$\widehat{rr} = -\frac{p}{2\sigma} \cdot$$

The formulas in Cases IIIa and IIIb are valid throughout the plate; hence there is no limitation on the ratio of thickness to diameter. A similar remark applies to the problem which follows.

Problem IV: An incomplete plate with no loads other than constant pressures, $-p_0$ and $-p_1$, at the outer and inner edges, respectively. If the values $c_r = j \equiv L_1 \equiv L_2 \equiv L \equiv 0$ are substituted in (50) and in the formula for $\widehat{rr}$, the result is

$$U = C_1 r + \frac{C_2}{r}, \quad w = -\frac{2\sigma}{1-\sigma} C_1 z, \quad \widehat{rr} = \frac{E}{1-\sigma} C_1 - \frac{E}{1+\sigma} \frac{C_2}{r} \cdot$$

C_1 and C_2 are obtained by setting

$$\widehat{rr}|_{r=r_0} = -p_0 \text{ and } \widehat{rr}|_{r=r_1} = -p_1;$$

the results are

$$C_1 = -\frac{1-\sigma}{E}\frac{p_0r_0^2 - p_1r_1^2}{r_0^2 - r_1^2}, \quad C_2 = -\frac{1+\sigma}{E}\frac{(p_0 - p_1)r_0^2r_1^2}{r_0^2 - r_1^2}.$$

Most of the solutions just obtained are well known; only cases Ic and Id seem to be given for the first time. The object in finding anew the known solutions has been to exhibit the power and elegance of a method which brings all these solutions, and many others, together under one uniform method of treatment.

16. The bending of a complete plate. In the literature of elasticity there are only a limited number of problems dealing with the bending of moderately thick circular plates. A single solution has been given for a plate bent by its own weight; the problem of the plate loaded uniformly over one face has been solved for several types of edge conditions; a plate whose faces are subjected to shearing traction seems not to have been considered. Also, certain problems involving central load have been given, but some of these have been in error. It would be interesting to treat axial mass force, uniform load, and central load so as to check practically every solution given heretofore; however, lack of space makes it desirable to consider one type of loading only. There is not much difference in the facility with which solutions for these three loading conditions may be found; the case of axial mass force will be chosen since the formulas involved are slightly shorter than those appearing in the two other cases.

Formulas will be needed for T_r and G_r as well as for the displacements. These formulas are readily obtained by substituting formulas (33) and the values $K_1 = K_3 = C_2 = c_r = L \equiv l \equiv j \equiv J \equiv 0$ in (35) and (38). The results are

$$\text{(52a)}\quad U = C_1r - 2K_2rz + \frac{3(1-\sigma^2)c_z}{16Eh^2}\left\{r^2 + \frac{8h^2}{1-\sigma}\right\}rz - \frac{(1+\sigma)(2-\sigma)}{4Eh^2}c_zrz^3,$$

$$\text{(52b)}\quad w = -K_2(r_0^2 - r^2) - \frac{2\sigma}{1-\sigma}C_1z + \frac{2\sigma}{1-\sigma}K_2z^2 + \frac{3(1-\sigma^2)}{64Eh^2}(r_0^4 - r^4)c_z - \frac{3\sigma(1+\sigma)c_z}{8Eh^2}\left\{r^2 + \frac{2(1+4\sigma)h^2}{3\sigma(1-\sigma)}\right\}z^2 + \frac{(1+\sigma)^2}{8Eh^2}c_zz^4,$$

$$\text{(52c)}\quad T_r = \frac{2Eh}{1-\sigma}C_1,$$

$$\text{(52d)}\quad G_r = -\frac{4Eh^3}{3(1-\sigma)}K_2 + \left\{\frac{3+\sigma}{8}\frac{r^2}{h^2} + \frac{24 + 23\sigma + 3\sigma^2}{30(1-\sigma)}h^3c_z\right\}.$$

At this point, it will be interesting to check formulas (52a) and (52b) against the only solution heretofore given for a plate bent by its own weight. This solution was found by G. H. Bryan (Love, p. 486). Except for a term in w, accounted for by an axial translation of the plate as a whole, Bryan's results may be obtained from (52a) and (52b) by setting $c_z=\rho g$, $C_1=\rho g(1-\sigma)h/(2E)$, $K_2=\rho g(3+7\sigma)/(8E)$. His solution would have to be combined with other simpler solutions before it would correspond to any of the commonly used edge conditions.

When the constants C_1 and K_2 have been fixed by the conditions at the outer edge, the displacements will be completely determined. Nine types of conditions at the outer edge will be considered; these will be ordered according to the magnitude of the deflection at the center of the middle plane of the plate. Since the deflection at the center is in the direction of the negative z-axis, the central deflection of the middle plane increases as K_2 increases. In order to conserve space, the physical interpretation of the various edge conditions will not be given. The nine cases follow.

Case S. $G_r\big|_{r=r_0} = T_r\big|_{r=r_0} = 0.$

Case M–I. $\partial^2 w/\partial z^2\big|_{r=r_0,z=0} = 0,\ \partial w/\partial z\big|_{r=r_0,z=0}=0.$

Case M–II. $w\big|_{r=r_0,z=\pm h} = 0.$

Case M–III. $\partial w/\partial z\big|_{r=r_0,z=\pm h} = 0.$

Case M–IV. $\partial^2 w/\partial z^2\big|_{r=r_0,z=\pm h} = 0.$

Case C–I. $U\big|_{r=r_0,z=0} = \partial U/\partial z\big|_{r=r_0,z=0} = 0.$

Case C–II. $U\big|_{r=r_0,z=\pm h} = 0.$

Case C–III. $U\big|_{r=r_0,z=0} = w'\big|_{r=r_0,z=\pm h} = 0.$

Case C–IV. $U\big|_{r=r_0,z=0} = w'\big|_{r=r_0,z=0} = 0.$

In each case the value of C_1 turns out to be zero. The values of K_2 for the nine cases are, respectively,

$$K_2 = \frac{3(1-\sigma^2)}{32Eh^2}c_z\left\{\frac{3+\sigma}{1+\sigma}r_0^2 + \frac{4(24+23\sigma+3\sigma^2)}{15(1-\sigma^2)}h^2\right\}, \tag{53a}$$

$$K_2 = \frac{3(1-\sigma^2)}{16Eh^2}c_z\left\{r_0^2 + \frac{2(1+4\sigma)}{3\sigma(1-\sigma)}h^2\right\}, \tag{53b}$$

$$K_2 = \frac{3(1-\sigma^2)}{16Eh^2}c_z\left\{r_0^2 + \frac{1+8\sigma+\sigma^2}{3\sigma(1-\sigma)}h^2\right\}, \tag{53c}$$

$$K_2 = \frac{3(1-\sigma^2)}{16Eh^2}c_z\left\{r_0^2 + \frac{2(4+\sigma)}{3(1-\sigma)}h^2\right\}, \tag{53d}$$

$$K_2 = \frac{3(1-\sigma^2)}{16Eh^2}c_z\left\{r_0^2 - \frac{2(2-4\sigma-3\sigma^2)}{3\sigma(1-\sigma)}h^2\right\}, \tag{53e}$$

$$K_2 = \frac{3(1-\sigma^2)}{32Eh^2}c_z\left\{r_0^2 + \frac{8h^2}{1-\sigma}\right\}, \tag{53f}$$

$$K_2 = \frac{3(1-\sigma^2)}{32Eh^2}c_z\left\{r_0^2 + \frac{4(4+\sigma)}{3(1-\sigma)}h^2\right\}, \tag{53g}$$

$$K_2 = \frac{3(1-\sigma^2)}{32Eh^2}c_z\left\{r_0^2 + \frac{4\sigma}{1-\sigma}h^2\right\}, \tag{53h}$$

$$K_2 = \frac{3(1-\sigma^2)}{32Eh^2}c_z r_0^2. \tag{53i}$$

Cases S and C-IV are those of classical support and classical clamping, respectively. The thin-plate solutions for Cases M-I, C-I, C-II have been given by C. A. Clemmow (loc. cit.) for a plate having uniform pressure on the upper face. The four remaining cases appear to be new.

Clemmow, in an investigation made upon clamped plates (loc. cit.), attempted to obtain a support as rigid as that of classical clamping by cutting a cylinder and head from a solid piece of metal. His expectation was not realized, since the best agreement with his experimental results was given by Case C-I, even when the ratio of thickness to diameter was small. It thus appears that it is virtually impossible to construct a physical type of clamping which will agree with a set of analytical conditions previously advanced. Hence it becomes necessary to devise new analytical conditions which will approximate the physical situations arising in practice. Herein lies a justification for considering Cases C-I, C-II, C-III. It should be borne in mind, however, that only in exceptional cases is the head of a cylinder an integral part of the cylinder itself. Ordinarily, the head is fastened to the cylinder by means of bolts; and this type of fastening is certainly less rigid than that used by Clemmow. It would appear desirable, therefore, to study also the additional cases, M-I, M-II, M-III, M-IV, intermediate between S and C-I.

17. **The bending of an incomplete plate.** Two problems only will be considered in this article. In each problem, it will be assumed that (i) the inner edge is free and (ii) the mass force is nil and there is no shearing traction on the faces; that is,

$$T_r\big|_{r=r_1} = G_r\big|_{r=r_1} = 0, \tag{54a}$$

$$c_r = c_z = J_1 \equiv J_2 \equiv j \equiv J \equiv 0. \tag{54b}$$

Problem I. The only load is a uniform pressure, $-p$, on the upper face; that is,

$$N_r\big|_{r=r_1} = L_2 \equiv 0, \; L_1 = L = l = -p. \tag{55a}$$

Problem II. The only load is a downward shearing force of $-W$ pounds distributed uniformly along the inner edge; that is,

$$L_1 \equiv L_2 \equiv L \equiv l \equiv 0. \tag{56a}$$

The value of K_1 for each problem will be found first. The substitution of $c_z=0, l=-p$ in (38e) yields the following result for Problem I:

$$N_r = -\frac{8Eh^3}{3(1-\sigma^2)}\frac{K_1}{r} + \frac{pr}{2}\cdot \tag{55b}$$

The corresponding result for Problem II is

$$N_r = -\frac{8Eh^3}{3(1-\sigma^2)}\frac{K_1}{r}\cdot \tag{56b}$$

Consider the inner annular ring cut out from the plate by a concentric cylindrical surface of radius r. The equilibrium of this ring requires that

$$2\pi r N_r - \pi(r^2 - r_1^2)p = 0 \tag{55c}$$

for Problem I, and that

$$2\pi r N_r - W = 0 \tag{56c}$$

for Problem II. The elimination of N_r between (55b) and (55c) yields the following value of K_1 for Problem I:

$$K_1 = \frac{3(1-\sigma^2)}{16Eh^3}pr_1^2. \tag{55d}$$

A corresponding procedure in Problem II yields the value

$$K_1 = -\frac{3(1-\sigma^2)}{16\pi Eh^3}W. \tag{56d}$$

It will be convenient to express C_2 and K_3 in terms of C_1 and K_2, respectively. To accomplish this, T_r and G_r must first be computed. For Problem I, the results of substituting $c_r=c_z=j\equiv J\equiv 0$, $L=l=-p$, $K_1 =3(1-\sigma^2)pr_1^2/(16Eh^3)$ in (38a) and (38c) are

$$T_r = \frac{2Eh}{1-\sigma}C_1 - \frac{2Eh}{1+\sigma}\frac{C_2}{r^2} - \frac{\sigma ph}{2}, \tag{55e}$$

$$G_r = -\frac{4Eh^3}{3(1-\sigma)}K_2 + \frac{2Eh^3}{3(1+\sigma)}\frac{K_3}{r^2} + \frac{3+\sigma}{16}p(r^2 - 2r_1^2) - \frac{1+\sigma}{4}pr_1^2 \log r + \frac{8+\sigma}{20r^2}h^2pr_1^2 - \frac{\sigma p h^2}{5}. \tag{55f}$$

For Problem II, the substitution of

$$c_r = c_z = j \equiv J \equiv L \equiv l \equiv 0,\ K_1 = -3(1-\sigma^2)W/(16\pi Eh^3)$$

in (38a) and in (38c) results in

$$T_r = \frac{2Eh}{1-\sigma}C_1 - \frac{2Eh}{1+\sigma}\frac{C_2}{r^2}, \tag{56e}$$

$$G_r = -\frac{4Eh^3}{3(1-\sigma)}K_2 + \frac{2Eh^3}{3(1+\sigma)}\frac{K_3}{r^2} + \frac{W}{8\pi}\left\{3+\sigma+2(1+\sigma)\log r - \frac{2(8+\sigma)}{5r^2}h^2\right\}. \tag{56f}$$

The substitution of (54a) in (55e) and (55f) yields the following results for Problem I:

$$C_2 = \frac{1+\sigma}{1-\sigma}r_1^2C_1 - \frac{\sigma(1+\sigma)}{4E}pr_1^2, \tag{55g}$$

$$K_3 = \frac{2(1+\sigma)}{1-\sigma}r_1^2K_2 + \frac{3(1+\sigma)}{8Eh^3}pr_1^4\left\{\frac{3+\sigma}{4} + (1+\sigma)\log r_1 - \frac{8-3\sigma}{5r_1^2}h^2\right\}. \tag{55h}$$

By a similar procedure, the following results are obtained for Problem II:

$$C_2 = \frac{1+\sigma}{1-\sigma}r_1^2C_1, \tag{56g}$$

$$K_3 = \frac{2(1+\sigma)}{1-\sigma}r_1^2K_2 - \frac{3(1+\sigma)}{16\pi Eh^3}Wr_1^2\left\{3+\sigma+2(1+\sigma)\log r_1 - \frac{2(8-\sigma)}{5r_1^2}h^2\right\}. \tag{56h}$$

As soon as w_{0p} is known, the value of K_4 may be found in terms of K_2. The values of w_{0p} (formula (31g)) for Problems I and II turn out to be, respectively,

$$w_{0p} = -\frac{3(1+\sigma)}{16Eh^3}p\left(\frac{1-\sigma}{8}r^4 - \frac{8-3\sigma}{5}h^2r^2\right), \tag{55i}$$

$$w_{0p} = 0. \tag{56i}$$

Substitute in (39b) the values of K_1, K_3, w_{0_p} already found; the resulting values of K_4 for Problems I and II are, respectively,

$$
\begin{aligned}
\text{(55j)}\quad K_4 = &-K_2\left\{r_0^2+\frac{2(1+\sigma)}{1-\sigma}r_1^2\log r_0\right\}-\frac{3(1-\sigma^2)}{16Eh^3}pr_1^2r_0^2\log r_0\\
&-\frac{3(1+\sigma)}{8Eh^3}pr_1^4\left\{\frac{3+\sigma}{4}+(1+\sigma)\log r_1-\frac{8-3\sigma}{5r_1^2}h^2\right\}\log r_0\\
&+\frac{3(1+\sigma)}{16Eh^3}p\left(\frac{1-\sigma}{8}r_0^4-\frac{8-3\sigma}{5}h^2r_0^2\right),
\end{aligned}
$$

$$
\begin{aligned}
\text{(56j)}\quad K_4 = &-K_2\left\{r_0^2+\frac{2(1+\sigma)}{1-\sigma}r_1^2\log r_0\right\}+\frac{3(1-\sigma^2)}{16\pi Eh^3}Wr_0^2\log r_0\\
&+\frac{3(1+\sigma)}{16\pi Eh^3}Wr_1^2\left\{3+\sigma+2(1+\sigma)\log r_1-\frac{2(8+\sigma)}{5r_1^2}h^2\right\}\log r_0.
\end{aligned}
$$

The values of K_2 and C_1 for the nine cases of edge conditions given in the last article may now be found. In order to conserve space, these constants will be given only for the case of classical support. The result of setting $T_r|_{r=r_0}=G_r|_{r=r_0}=0$ is

$$
\text{(55k)}\qquad C_1=\frac{\sigma(1-\sigma)}{4E}p,
$$

$$
\begin{aligned}
\text{(55l)}\qquad K_2 = &\frac{3(1-\sigma)}{4Eh^3}p\left\{\frac{3+\sigma}{16}(r_0^2-r_1^2)-\frac{\sigma h^2}{5}\right\}\\
&-\frac{3(1-\sigma^2)pr_1^2}{16Eh^3(r_0^2-r_1^2)}(r_0^2\log r_0-r_1^2\log r_1)
\end{aligned}
$$

for Problem I; the results for Problem II are

$$
\text{(56k)}\qquad C_1=0,
$$

$$
\text{(56l)}\qquad K_2=\frac{3(1-\sigma)}{16\pi Eh^3}W\left\{\frac{3+\sigma}{2}+(1+\sigma)\frac{r_0^2\log r_0-r_1^2\log r_1}{r_0^2-r_1^2}\right\}.
$$

It is now possible to write out the displacements in full for both problems. However, lack of space makes it undesirable to do this. The solution for the displacements in Problems I and II were apparently given for the first time by A. Timpe in 1924 (loc. cit.).

It turns out that when $r_1=0$ the solution given in Problem I becomes the same as that for a complete plate uniformly loaded. Moreover, the solution given in Problem II reduces to that of a complete plate centrally loaded if W

is kept constant while r_1 is set equal to zero. It should be observed, however, that one has no right to assume in advance that the complete plate will be a limiting case of the incomplete plate, since, in the case of a complete plate, the constants C_2 and K_3 were determined from the conditions

$$\lim_{r\to 0}(T_r - T_\theta) = \lim_{r\to 0}(G_r - G_\theta) = 0,$$

while, for the incomplete plate, the constants were fixed by the conditions

$$T_r\big|_{r=r_1} = G_r\big|_{r=r_1} = 0.$$

18. **The bending of a plate under bizonal distribution of load.** Since it would require much space to solve a bizonal problem completely, values will be found for only four of the constants; the eight equations necessary for the solution of the eight remaining constants will be given, but no attempt will be made to solve them. The following discussion is valid for all problems in which the only load is a downward pressure of $-W$ pounds distributed continuously over the inner zone, or distributed uniformly over the junction of the two zones. Let the subscripts i and e indicate the interior and exterior zones, respectively.

The inner zone is a complete plate with continuous distribution of load; hence, from §13, $K_{1i}=K_{3i}=C_{2i}=0$. Since there is no load on the outer zone, the value of N_{re} obtained from (38e) is

$$N_{re} = -\frac{8Eh^3}{3(1-\sigma^2)}\frac{K_{1e}}{r}. \tag{57a}$$

Consider a section of the plate having for its outer edge a cylinder of radius r, $r_1<r<r_0$. The equilibrium of this portion of the plate demands that $2\pi rN_{re}-W=0$. The substitution of this value of N_{re} in (57a) results in

$$K_{1e} = -\frac{3(1-\sigma^2)}{16\pi Eh^3}W. \tag{57b}$$

For the determination of the eight remaining constants, three equations may be obtained from the boundary conditions at the outer edge. Two of these are found by imposing any one of the nine edge conditions given in §16 for the determination of K_2 and C_1. A third condition results from the requirement that there shall be no axial displacement at the outer edge of the middle surface; that is,

$$w_{0i}\big|_{r=r_0} = 0.$$

The five remaining equations must be found from conditions at the junc-

tion of the two zones. The principle of the equality of action and reaction demands that

$$\lim_{r\to r_1} \widehat{rr}_e = \lim_{r\to r_1} \widehat{rr}_i, \tag{58}$$

and hence that

$$\lim_{r\to r_1} T_{re} = \lim_{r\to r_1} T_{ri}, \quad \lim_{r\to r_1} G_{re} = \lim_{r\to r_1} G_{ri}. \tag{59}$$

It should be noted that although (58) implies (59), the converse may not be true. If (58) is not satisfied, the solution is not valid in the immediate neighborhood of the junction; however, if (59) is satisfied, the solution, according to de Saint-Venant's principle, will hold for all points not too near the boundary of the two zones. Formula (59) furnishes two of the five equations. A third equation results from the assumption that the middle surface of the plate is continuous at the junction of the two zones; that is,

$$w_{0i}\Big|_{r=r_1} = w_{0e}\Big|_{r=r_1}.$$

The above, or equivalent, conditions at the junction have been used by each person who has attempted to find the solution for a moderately thick plate under bizonal distribution of load. De Saint-Venant (loc. cit.) demanded, in addition, that w_0' (the slope of the middle surface) should be continuous at the junction. Garabedian, in correcting de Saint-Venant's solution,* demanded that, at the junction, both T_θ and G_θ should be continuous.

It should be noted that one has no more right, a priori, to demand the continuity of T_θ and G_θ at the junction than one has to demand the continuity of w_0', since the law of the equality of action and reaction does not apply in the case of T_θ and G_θ when the direction of $\widehat{\theta\theta}$ is tangent to the junction. Some other test must be found for determining which of these assumptions is incorrect. Such a test may be found in the case of the loading situations considered in this article, since each of them reduces to the problem of the complete plate loaded centrally when r_1 is allowed to approach zero while the total load, W, is kept constant. The assumption made by de Saint-Venant results in a solution for central load in agreement with that given by Love. But it has already been shown that Love's solution is incorrect; consequently, the assumption made by de Saint-Venant is also incorrect. On the other hand, Garabedian's assumption leads to the correct solution for central load. Hence Garabedian's assumption is a sufficient condition for obtaining the correct solution for central load if r_1 is allowed to approach zero. The author

* Journal de l'École Polytechnique, vol. 26 (1927), p. 89.

has used Garabedian's assumption of the equality of T_θ and G_θ at the junction even though he has not been able to prove its necessity; such a proof would seem to involve a difficult problem in the calculus of variations. The employment of the foregoing assumption gives the two remaining equations necessary for the solution of the eight unknown constants.

19. **Conclusion.** The generality of the method developed in this paper has been clearly exhibited in Part II, since all the solutions there obtained were determined from a single formula for the displacements, namely, formula (35). Moreover, the power of the method has been demonstrated by the diversity of the problems solved. These problems have been concerned with both complete and incomplete plates and with an extensive class of loads, namely, radial and axial mass force, uniform load, normal load proportional to r^2, concentrated normal load, and shearing traction proportional to r. Furthermore, with each type of load a large number of edge conditions have been studied.

Further evidence of both the power and generality of this method is seen in the ease and rigor with which it has been possible to find the values of the arbitrary constants K_1, K_3, and C_2 in the case of a complete plate. Certain writers, in attempting to determine these constants, have allowed themselves to be influenced by speculation concerning the physical nature of the problem. For instance, both de Saint-Venant and Love were led to give incorrect solutions in the problem of central load by assuming that $K_3=0$, an error due to the assumption that w must be finite at the center of a complete plate.

Since the solutions given in this paper are of the two-dimensional type, a comparison with solutions of three-dimensional character is natural; there arises at once the question of relative accuracy. It has already been pointed out that Clemmow's experiments show that the best agreement with the actual deflection of a clamped plate is given by Case C-I, which is a two-dimensional solution, and not by either Clemmow's three-dimensional solution or Nádai's. Clemmow has attributed this to a mere coincidence, since he (and Nádai also) has taken the stand that the correct solution for a plate can only be found from a three-dimensional solution, a two-dimensional solution being, of necessity, a less accurate approximation to the true physical situation. The position taken by Clemmow and Nádai would be a justifiable one if it were possible to give a correct mathematical definition of the boundary conditions at an edge and subsequently to find a three-dimensional solution in agreement with this mathematical definition. But Clemmow's experiments have clearly shown that it is not possible to give an accurate mathematical description of the boundary conditions at an edge, even for such a simple case as that of a clamped plate.

Furthermore, there are only two types of boundary conditions which can be handled by Nádai's method; Clemmow is able to go beyond these two types, but only at the expense of extremely complicated computations. On the other hand, there seems to be no limit to the number of boundary conditions which can be studied by means of our two-dimensional method; moreover, the computations are much simplified when a two-dimensional solution is used. Finally, in practice, a two-dimensional solution is actually to be preferred to a three-dimensional one; for not only is the former simpler in structure, but, what is more, the results obtained from it may be truer physically (when a proper mathematical definition of the boundary conditions at the edge is used) than the results obtained from a three-dimensional solution whose boundary conditions do not so closely approximate those actually existing.

The application of our method is not limited to circular plates of constant thickness. The method may also be applied to circular plates of variable thickness, to rectangular plates of either constant or variable thickness, and, under the heading of one-dimensional problems, to moderately thin rods of either constant or variable thickness; in fact, this method may be used in any problem where Garabedian's method is applicable. So far as the author is aware no method other than Garabedian's has been developed which can be applied to such broad classes of problems. Moreover, the author is convinced that his method, or the closely parallel method of Garabedian, affords the most natural and satisfactory machinery for handling the two-dimensional problems of elasticity. They seem to be in every way superior, by virtue of their power, generality, and simplicity, to any two-dimensional method hitherto advanced.

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NOTE ON THE OVERCONVERGENCE OF SEQUENCES OF POLYNOMIALS OF BEST APPROXIMATION*

BY

J. L. WALSH

1. **Introduction.** It frequently occurs that the sequence of polynomials (in the complex variable) of best approximation to a function $f(z)$ analytic on a given point set C converges to that function $f(z)$ (or its analytic extension) not merely on the given point set C but also on a larger point set containing C in its interior. This phenomenon may be called *overconvergence*, and for its occurrence the term "best approximation" may be interpreted in any one of several ways. In a recent paper† the present writer pointed out that many results on overconvergence follow from a single theorem on degree of approximation. The present note is essentially a continuation of that previous paper, and indicates several new ways in which the term "best approximation" may be used so that we still have overconvergence. The principal new results are concerned with (1) approximation on an arbitrary rectifiable Jordan arc, approximation to be measured by a line integral on that arc, and (2) approximation on a more general point set C as measured by integration over the unit circle γ: $|w|=1$ when the exterior of C is mapped conformally onto the exterior of γ; as a further new result we add a single remark (§9) on the exact regions of uniform convergence of the sequences. The previous paper considered the approximation of harmonic functions by harmonic polynomials as well as approximation of analytic functions by polynomials in the complex variable, but the present note deals only (except in §9) with the latter topic.

If C is an arbitrary closed limited point set of the z-plane whose complement (with respect to the entire plane) is a simply connected region D, then we denote by C_R the Jordan curve which is the locus $|\phi(z)|=R>1$, where $w=\phi(z)$ maps D conformally onto the exterior of γ: $|w|=1$ so that the points at infinity correspond to each other. That is, C_R is the transform in the z-plane of the circle $|w|=R$.

We shall have occasion to apply the following theorem:‡

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* Presented to the Society, September 11, 1930; received by the editors November 14, 1930.

† These Transactions, vol. 32 (1930), pp. 794–816.

‡ For the proof and detailed references to the literature, see Walsh, Münchner Berichte, 1926, pp. 223–229.

THEOREM I. *Let C be an arbitrary closed limited point set of the z-plane whose complement (with respect to the entire plane) is simply connected.*

If the function $f(z)$ is analytic on and within C_R, there exist polynomials $\pi_n(z)$ of respective degrees $n=1, 2, \cdots$ such that we have*

$$|f(z) - \pi_n(z)| \leqq \frac{M}{R^n}, \quad z \text{ on } C, \ M \text{ independent of } n \text{ and of } z.$$

If there exist polynomials $\pi_n(z)$, $n=1, 2, \cdots$, such that the inequality

$$|f(z) - \pi_n(z)| \leqq \frac{M}{R^n}$$

is valid for z on C, then the function $f(z)$ is analytic interior to C_R and the sequence $\{\pi_n(z)\}$ converges interior to C_R, uniformly on any closed point set interior to C_R.

In the previous paper we had frequent occasion to apply Theorem I directly; in the present paper we shall ordinarily use rather the method of proof than the theorem itself.

2. **A lemma on polynomials.** The following lemma is to be used in our later work. The lemma itself is analogous to one used by Bernstein,† although the present proof is related to the proof of Bernstein's Lemma which was given by M. Riesz.‡

LEMMA. *If on the rectifiable Jordan arc C we have*

$$\int_C |P_n(z)|^p |dz| \leqq \rho^p, \quad p > 1, \tag{1}$$

where $P_n(z)$ is a polynomial of degree n, then we have

$$|P_n(z)| \leqq M_R \rho R^{n+1}, \qquad z \text{ on or within } C_R, \tag{2}$$

where M_R depends on R but not on ρ or n.

Consider the z-plane cut along the curve C, and let us distinguish between the two banks of the curve. The new curve may be considered a Jordan curve Γ, and the function $\phi(z)$ considered above is, if suitably defined on Γ, continuous and single-valued on the entire cut z-plane. The function $P_n(z)/[\phi(z)]^{n+1}$ is analytic except on Γ and continuous on the entire cut plane and vanishes at infinity. Then we have, for z not on Γ (the proof is easy to

* By a polynomial of degree n is meant an expression of the form $a_0z^n + a_1z^{n-1} + \cdots + a_n$.

† Mémoires, Académie Royale de Belgique, Classe des Sciences, (2), vol. 4 (1912), p. 36.

‡ Acta Mathematica, vol. 40 (1916), pp. 337–347.

give by considering a variable auxiliary Jordan curve Γ_n which encloses and approaches Γ) Cauchy's formula

$$\frac{P_n(z)}{[\phi(z)]^{n+1}} = \frac{1}{2\pi i}\int_\Gamma \frac{P_n(t)/[\phi(t)]^{n+1}}{t-z}\,dt,$$

where the integral is taken in the clockwise sense on Γ. By use of the general inequality

$$\left|\int f^{1/p}g^{(p-1)/p}dx\right| \leqq \left(\int |f|\,dx\right)^{1/p}\left(\int |g|\,dx\right)^{(p-1)/p}, \tag{3}$$

setting $f^{1/p}=P_n(t)$, $g^{(p-1)/p}=1/[2\pi i[\phi(t)]^{n+1}(t-z)]$, and by use of the fact that for t on Γ we have $|\phi(t)|=1$, it follows from (1) that for z on C_R we have

$$\left|\frac{P_n(z)}{[\phi(z)]^{n+1}}\right| \leqq \rho\frac{1}{2\pi}\left(\int_\Gamma \frac{|dt|}{|t-z|^{p(p-1)}}\right)^{(p-1)/p}.$$

The second factor on the right is uniformly bounded for all z on C_R; on C_R we have by definition $|\phi(z)|=R$; hence inequality (2) follows for z on C_R; but since this inequality holds for z on C_R it also holds for z on or within C_R, and the lemma is established.

3. **Approximation on a Jordan arc.** We apply the Lemma in proving

THEOREM II. *Let C be an arbitrary rectifiable Jordan arc, and suppose we have for $n=1, 2, \cdots$ the inequality*

$$\int_C |f(z)-\pi_n(z)|^p n(z)\,|dz| \leqq \frac{M}{R^{np}}, \quad p, R > 1, \tag{4}$$

where $\pi_n(z)$ is a polynomial of degree n and where the function $n(z)$ is continuous and positive on C. Then the sequence $\{\pi_n(z)\}$ converges throughout the interior of C_R, uniformly on any closed point set interior to C_R.

Let N be a positive number less than $n(z)$ on C; we have

$$\int_C |f(z)-\pi_n(z)|^p\,|dz| \leqq \frac{M}{NR^{np}}.$$

From the general inequality

$$\int |\chi_1+\chi_2|^p dx \leqq 2^{p-1}\int |\chi_1|^p dx + 2^{p-1}\int |\chi_2|^p dx. \tag{5}$$

it follows that

$$\int_C |\pi_{n+1}(z) - \pi_n(z)|^p |dz| \leqq 2^{p-1} \int_C |f(z) - \pi_n(z)|^p |dz|$$

$$+ 2^{p-1} \int_C |f(z) - \pi_{n+1}(z)|^p |dz|$$

$$\leqq \frac{2^{p-1}M}{NR^{np}} + \frac{2^{p-1}M}{NR^{(n+1)p}} = \frac{2^{p-1}M(1 + R^p)}{NR^{(n+1)p}}.$$

This inequality is of form (1), for the polynomial $\pi_{n+1}(z) - \pi_n(z)$ is of degree $n+1$. If we choose an arbitrary R_1 less than R, we have by (2) an inequality of the form

$$|\pi_{n+1}(z) - \pi_n(z)| \leqq M_1\left(\frac{R_1}{R}\right)^{n+1}, \quad z \text{ on or within } C_{R_1},$$

from which it follows that the sequence $\{\pi_n(z)\}$ converges interior to C_R, uniformly on any closed point set interior to C_R.

The function $\omega(z)$ which is the limit of the sequence $\{\pi_n(z)\}$ on C naturally coincides with the given function $f(z)$ on C except perhaps on a set of measure zero. For the uniform convergence of the sequence to the function $\omega(z)$ as proved implies

$$\lim_{n\to\infty} \int_C |\omega(z) - \pi_n(z)|^p |dz| = 0.$$

Let an arbitrary positive ϵ be given. Choose a particular n so large that we have

$$\int_C |f(z) - \pi_n(z)|^p |dz| < \epsilon, \quad \int_C |\omega(z) - \pi_n(z)|^p |dz| < \epsilon.$$

It follows from (5) that we have

$$\int_C |f(z) - \omega(z)|^p |dz| < 2^p\epsilon,$$

from which it follows that $f(z)$ and $\omega(z)$ are equal on C except perhaps on a point set of measure zero. If the given function $f(z)$ is analytic on C, it must be identical with $\omega(z)$, for the two functions coincide in an infinity of points of C.

4. **Polynomials of best approximation.** There is a sort of converse (not exact) of Theorem II which can be established quite easily:

THEOREM III. *If the function $f(z)$ is analytic on and within C_R, then there exist polynomials $\pi_n(z)$ of respective degrees $n=1, 2, \cdots$ such that we have*

$$\int_C |f(z) - \pi_n(z)|^p n(z) |dz| \leqq \frac{M}{R^{np}}. \tag{4}$$

In fact by Theorem I we have

$$|f(z) - \pi_n(z)| \leqq \frac{M_1}{R^n}, \quad z \text{ on } C,$$

and this leads directly to (4).

Theorem II applies directly to the polynomial of best approximation to $f(z)$ in the sense of least weighted pth powers measured on C. If the integral in the left-hand member of (4) is not greater when formed for the polynomial $\pi_n(z)$ than when formed for any other polynomial of degree n, then $\pi_n(z)$ is called a polynomial of best approximation to $f(z)$ in the sense of least weighted pth powers measured on C. This polynomial of best approximation exists and is unique; see §8 below.

THEOREM IV. *Let C be an arbitrary rectifiable Jordan arc. If the function $f(z)$ is analytic within C_R, then the sequence of polynomials $\pi_n(z)$ of best approximation to $f(z)$ on C in the sense of least weighted pth powers as measured by integration on C, converges to $f(z)$ within C_R, uniformly on any closed point set interior to C_R.*

Let $R_1 < R$ be arbitrary. Then by Theorem III we have for some sequence of polynomials $\pi_n(z)$ inequality (4) with R replaced by R_1. Since this inequality holds for *some* sequence of polynomials it necessarily holds for the sequence of polynomials of *best approximation*, whence Theorem IV follows by Theorem II.

The special case of Theorem IV in which $n(z) \equiv 1$, $p=2$, and C is a segment of the axis of reals, leads to the expansion of $f(z)$ in a series of Legendre polynomials, and the result in this case is well known.

We have required in the present discussion that $n(z)$ should actually be *greater* than zero. If that requirement is replaced by the one of demanding that $n(z)$ should be merely greater than or equal to zero, the situation is somewhat altered. Let us suppose for instance that we have $n(z)$ greater than zero on a closed subset C' of C; let us take the simple case where C' consists of a finite number of arcs of C. The first part of Theorem I has not been proved* for such sets as C', but the second part has been proved (loc.

* The present writer hopes shortly to publish some results in this connection.

cit.). Thus a more special result than Theorem IV can now be established. We assume the function $f(z)$ to be analytic interior to C_R. Then *the sequence of polynomials $\pi_n(z)$ of best approximation to $f(z)$ on C, in the sense of least weighted pth powers as measured by integration on C, converges to $f(z)$ within C_R', uniformly on any closed point set interior to C_R'.* The point set C_R' is defined as above, namely the curve or curves $|\phi(z)|=R$, where the function $w=\phi(z)$ maps the exterior of C' onto the exterior of γ: $|w|=1$ so that the points at infinity correspond to each other. To be sure, this conformal map is not smooth, but the mapping function nevertheless exists and the point set C_R' is uniquely defined.

The point set C' just treated can readily be generalized to include the new case that the function $n(z)$ is positive on the set C', consisting of a finite number of intervals of some Jordan arc C, except that the function $n(z)$ may vanish at some or all of the end points of those intervals. We consider a new variable closed point set C'' consisting of subintervals of the respective intervals of C belonging to C', and the point set C'' varies monotonically and eventually includes any preassigned point of C' not an end point of an interval. Then the point set C_R'' lies near to C_R' and approaches C_R' uniformly. Hence if the function $f(z)$ is analytic interior to C_R, the sequence of polynomials $\pi_n(z)$ of best approximation to $f(z)$ on C, in the sense of least weighted pth powers as measured by integration on C, converges to $f(z)$ within C_R', uniformly on any closed point set interior to C_R'.

The remark just made on the vanishing of the weight function has obvious application to results of the present paper other than Theorem IV, and also to the study of other methods of approximation, notably that of Tchebycheff. In the remainder of the present paper we assume as before that the weight function is actually greater than zero.

The vanishing of the weight function has been studied recently (before the formulation of the remark just made) by Professor Dunham Jackson, for the case that approximation is on a line segment.*

5. **More general rectifiable boundaries.** In Theorems II–IV we have required that C should be a rectifiable Jordan arc. Inspection of the proof shows, however, that the reasoning is valid in much more general cases, in fact is valid if C is an arbitrary limited point set whose complement is simply connected and which is bounded by a finite number of rectifiable Jordan curves and arcs. Thus in the proof of the Lemma, the Jordan curve Γ may consist of the entire boundary of such a point set C, and the parts of the

* Abstract published in the Bulletin of the American Mathematical Society, vol. 36 (1930), p. 629.

boundary of C which are Jordan arcs not bounding Jordan regions belonging to C are counted twice in considering Γ; the z-plane is cut along those Jordan arcs. The reader can easily write the suitable generalizations of Theorems II and III for this case. We state explicitly the generalization of Theorem IV:

THEOREM V. *Let C be an arbitrary limited point set whose complement is simply connected and which is bounded by a finite number of rectifiable Jordan curves and arcs. If the function $f(z)$ is analytic within C_R, the sequence of polynomials $\pi_n(z)$ of best approximation to $f(z)$ on C, in the sense of least weighted pth powers as measured by integration on the boundary of C, converges to $f(z)$ within C_R, uniformly on any closed point set interior to C_R.*

There is perhaps some doubt as to whether, in the integral which is the measure of the approximation, the single Jordan arcs of C should be counted singly or doubly. The reader will easily see that this is a matter of complete indifference. Moreover, even an infinite number of Jordan curves and arcs are allowable in Theorem V if the sum of their lengths is finite.

Theorem V includes not merely Theorem IV but also the interesting case where C is a region bounded by a single rectifiable Jordan curve.* In the study of such a region, one may also approximate to an arbitrary function analytic in the interior, continuous in the closed region. An easy corollary† of results obtainable in this way is

THEOREM VI. *Let C be an arbitrary limited point set whose complement is simply connected and which is bounded by a finite number of rectifiable Jordan curves and arcs. If the function $f(z)$ is continuous on C, analytic in the interior points of C, then the sequence of polynomials of best approximation to $f(z)$ on C, in the sense of least weighted pth powers as measured by integration on the boundary of C, converges to $f(z)$ in the interior points of C, uniformly on any closed point set composed entirely of interior points of C.*

6. **Approximation on more general point sets: a lemma.** If we study approximation to an arbitrary function $f(z)$ on a *non-rectifiable* Jordan arc C, the measure of approximation used in Theorems II–V may have no meaning. We shall now study the related measure of approximation *after the mapping of D (defined as in §1) onto the exterior of the circle γ: $|w|=1$ so that the points at infinity correspond to each other.* That is, the new measure of approximation is an integral over γ instead of over the boundary of C. This enables us to consider point sets C much more general than Jordan arcs, but nevertheless

* See Walsh, loc. cit., Theorem III.

† Loc. cit., Theorem IX, and also Walsh, these Transactions, vol. 30 (1928), pp. 472–482, Theorem IX.

the results already established are not included in those about to be taken up. Julia has recently* used conformal mapping in connection with the measure of approximation of harmonic functions by harmonic polynomials, but he restricts C to being a Jordan region and maps the *interior* of C onto the interior of γ, which is less favorable for our present purposes than the mapping of the exterior as we study it here.

LEMMA. *Let C be an arbitrary limited point set whose complement D is simply connected. If we have*

$$\int_\gamma |P_n(z)|^p |dw| \leqq \rho^p, \quad p > 1, \tag{6}$$

where $P_n(z)$ is a polynomial in z of degree n, then we have also

$$|P_n(z)| \leqq M_R \rho R^{n+1}, \tag{7}$$

z on or within C_R, where M_R depends on R but not on n nor ρ.

Under the conformal map $w=\phi(z)$, $z=\psi(w)$, of D onto the exterior of γ so that the points at infinity correspond to each other, $\lim_{r\to 1, r>1}\psi(re^{i\phi})$ exists† for almost all values of ϕ. Thus

$$\lim_{r\to 1,\, r>1} P_n[\psi(re^{i\phi})], \qquad \phi \text{ constant},$$

likewise exists for almost all values of ϕ, and these are the values of $P_n(z)$ on γ which we contemplate in (6). The function $P_n(z)/w^{n+1}$ is an analytic function of w exterior to γ and approaches zero when w becomes infinite. Thus we have, for an arbitrary point w exterior to γ,

$$\frac{P_n(z)}{w^{n+1}} = \frac{1}{2\pi i}\int_\gamma \frac{P_n[\psi(t)]/t^{n+1}}{t-w}\,dt, \tag{8}$$

where the integral is taken over the circle γ in the clockwise sense. Equation (8) is proved by considering first the equation corresponding to (8) where the integral is taken over a circle $|w|=1+1/m$ and then by taking the limit as m becomes infinite. If w is fixed exterior to γ, the integrand approaches a limit on almost every ray through the origin, and the integrand is uniformly bounded independently of m, so the limiting process ($m\to\infty$) is justified.

The method of proof used in §2 now yields

$$\left|\frac{P_n(z)}{w^{n+1}}\right| \leqq M_R\cdot\rho, \quad \text{for all } |w|=R, \tag{9}$$

* Acta Litterarum ac Scientiarum (Szeged), vol. 4 (1929), pp. 217–226.

† Fatou's theorem applies directly to the function $\psi(w)/w$.

where M_R depends only on C and R. Transformation back to the z-plane now gives (9) for all z on C_R and hence yields (7) for all z on or within C_R.

The analogue of Theorem II follows by the proof used in that theorem; the function $n(w)$ is supposed positive and continuous on γ:

THEOREM VII. *Let C be an arbitrary limited point set whose complement is simply connected. Then the relation*

$$\int_\gamma |f(z) - \pi_n(z)|^p n(w) |dw| \leqq \frac{M}{R^{np}}, \quad p > 1, \tag{10}$$

where $\pi_n(z)$ *is a polynomial of degree n, implies the convergence of the sequence* $\{\pi_n(z)\}$ *throughout the interior of* C_R, *uniformly on any closed point set interior to* C_R.

The values of the polynomials $\pi_n(z)$ on γ considered in (10) are of course the values considered as in the lemma; we make the definition

$$\pi_n[\psi(e^{i\phi})] = \lim_{r\to 1,\, r>1} \pi_n[\psi(re^{i\phi})], \ \phi \text{ constant}.$$

If the given function $f(z)$ is known to be analytic and bounded in the neighborhood of the boundary of D, a similar definition may be used for $f(z)$ on γ. No matter how $f(z)$ may be given on γ originally, that function can differ on γ from the function $F(z) = \lim_{n\to\infty} \pi_n[\psi(e^{i\phi})]$ at most on a set of measure zero. For the sequence $\{\pi_n(z)\}$ converges uniformly on any closed point set interior to C_R, hence converges uniformly in the neighborhood of the boundary of C, and converges uniformly on C in the point set $z = \psi(e^{i\phi})$ corresponding to almost all points of γ. On this point set of γ (i.e. for which $\lim_{r\to 1, r>1}\psi(re^{i\phi})$ exists), the sequence $\pi_n[\psi(e^{i\phi})]$ converges uniformly, hence, by the reasoning used in §3, the functions $F(z)$ and $f(z)$ differ at most on a set of γ of measure zero. If $f(z)$ is given analytic on C, the two functions $f(z)$ and $F(z)$ are identical.

7. **Convergence of the sequence of polynomials of best approximation.** Results analogous to Theorems III and IV can now be established, by methods similar to those previously used.

THEOREM VIII. *If the function* $f(z)$ *is analytic on and within* C_R, *where C is an arbitrary closed limited point set whose complement D is simply connected, then there exist polynomials* $\pi_n(z)$ *of respective degrees n and a number M such that*

$$\int_\gamma |f(z) - \pi_n(z)|^p n(w) |dw| \leqq \frac{M}{R^{np}}, \quad p > 1; \tag{10}$$

here the correspondence between z and w is found by mapping D onto the exterior of γ: $|w|=1$ so that the points at infinity correspond to each other.

To be sure, this conformal map is strictly defined merely in the open regions D and $|w|>1$ respectively, but we consider as before the map to be defined in the closed regions by continuity whenever this is possible. This extension of the original definition of the map is sufficient to define $f(z)$ and $\pi_n(z)$ on γ and to ensure the validity of (10).

By Theorem I there exist polynomials $\pi_n(z)$ of respective degrees n such that we have for z on C

$$|f(z) - \pi_n(z)| \leqq \frac{M_1}{R^n}\cdot \tag{11}$$

Under the conformal map $w=\phi(z)$, $z=\psi(w)$, the limit

$$\lim_{r\to 1,\, r>1} \psi(re^{i\phi})$$

exists for almost all values of ϕ and is denoted as before by $\psi(e^{i\phi})$. For these same values of ϕ, the limits

$$\lim_{r\to 1,\, r>1} f[\psi(re^{i\phi})], \quad \lim_{r\to 1,\, r>1} \pi_n[\psi(re^{i\phi})]$$

exist and are equal respectively to

$$f[\psi(e^{i\phi})], \quad \pi_n[\psi(e^{i\phi})].$$

Thus inequality (11) obtains for these same values of ϕ, that is, for almost all values of ϕ, and (10) follows at once.

If the integral in the left-hand member of (10) is not greater when formed for a particular polynomial $\pi_n(z)$ of degree n than when formed for any other polynomial $\pi_n(z)$ of degree n, that polynomial is called *a polynomial of best approximation to $f(z)$ on C, in the sense of least weighted pth powers as measured on γ*. This polynomial of best approximation exists and is unique; see §8 below.

THEOREM IX. *Let C be an arbitrary closed limited point set whose complement D is simply connected. If the function $f(z)$ is analytic interior to C_R, then the sequence of polynomials $\pi_n(z)$ of best approximation to $f(z)$ on C in the sense of least weighted pth powers as measured by integration on γ: $|w|=1$ after conformal mapping of D onto the exterior of γ, converges to $f(z)$ within C_R, uniformly on any closed point set interior to C_R.*

The proof of Theorem IX follows directly the proof of Theorem IV, by application of Theorems VII and VIII instead of II and III, and is left to the reader.

So far as the writer is aware, the point set C of Theorem IX is the most general one that has been considered in the literature in connection with *polynomials belonging to a region or to a point set.* For in the case $n(z)\equiv 1$, $p=2$, the polynomials $\pi_n(z)$ are of the form

$$\pi_n(z) = a_1p_1(z) + a_2p_2(z) + \cdots + a_np_n(z),$$

where the polynomials $p_i(z)$ depend only on C and not on $f(z)$ nor on n, and where the coefficients a_i depend on $f(z)$ but not (for $n \geqq i$) on n.

We remark too that all the results of the present paper hold for the case $p=1$, and that many of the results are valid provided merely $p>0$. The requisite modifications in the proofs and in the formulations of the theorems are left to the reader.

8. **Existence and uniqueness of polynomials of best approximation.** It is our purpose now to prove

THEOREM X. *In Theorems* VI *and* IX, *the polynomial $\pi_n(z)$ of best approximation exists and is unique.*

Let us give the proof for the polynomial of best approximation in Theorem IX; the case of Theorem VI is in reality somewhat simpler, and the requisite modifications can be made by the reader.

The measure of approximation

$$\epsilon_n^{(i)} = \int_\gamma |f(z) - \pi_n^{(i)}(z)|^p n(w) \, |dw|$$

is positive or zero, so there exists a greatest lower bound ϵ of the set of all numbers $\epsilon_n^{(i)}$ corresponding to polynomials $\pi_n^{(i)}(z)$ of degree n. There exists a sequence of numbers $\epsilon_n^{(1)}, \epsilon_n^{(2)}, \cdots$ (all belonging to the set just noted) which approaches ϵ, so for the corresponding polynomials $\pi_n^{(i)}(z)$ we have

$$\lim_{i\to\infty} \int_\gamma |f(z) - \pi_n^{(i)}(z)|^p n(w) \, |dw| = \epsilon. \tag{12}$$

By means of (5) we can write

$$\int_\gamma |\pi_n^{(i)}(z)|^p \, |dw| \leqq 2^{p-1}\int_\gamma |f(z)|^p |dw| \\ + 2^{p-1}\int_\gamma |f(z) - \pi_n^{(i)}(z)|^p \, |dw| \leqq M,$$

where M is suitably chosen.

The lemma of §6 now informs us that the sequence of polynomials $\{\pi_n^{(i)}(z)\}$, $i=1, 2, \cdots$, is uniformly bounded in C_R, where R is arbitrary. Hence this family of functions is normal and from it can be extracted a subsequence converging uniformly in an arbitrary closed subregion of C_R, say in C_{R_1}, $R_1<R$. In particular this subsequence converges uniformly on the boundary of C and hence on γ, or at least almost everywhere on γ, in the points of γ which correspond to the values of ϕ for which

$$\lim_{r\to 1,\, r>1} \psi(re^{i\phi}), \qquad \phi \text{ constant},$$

exists. The analytic function which is the limit of the subsequence is naturally a polynomial $\pi_n(z)$ of degree n; this follows as an easy application of Lagrange's interpolation formula.* The uniformity of the convergence yields at once from (12)

$$\int_\gamma | f(z) - \pi_n(z) |^p n(w) \, | dw | = \epsilon,$$

and this is the proof of the existence of the polynomial of best approximation.

The uniqueness of the polynomial of best approximation can be established by known methods.† It will be noticed, however, that Theorems VI and IX are entirely independent of this uniqueness.

9. **Exact region of uniform convergence of sequences of polynomials of best approximation.** We shall now prove the following theorem:

THEOREM XI. *Let the function $f(z)$ be analytic on the point set C and interior to C_R but have a singularity on C_R. Then any sequence of polynomials $\pi_n(z)$ (of respective degrees n) which converges on C to $f(z)$ with the same degree of approximation as the sequence of polynomials of best approximation, converges to $f(z)$ interior to C_R, uniformly on any closed point set interior to C_R, and converges uniformly in no region which contains in its interior an arc of C_R.*

In this theorem the term "best approximation" may be taken in any of the senses we have previously considered: (1) in the sense of Tchebycheff, C being an arbitrary limited closed point set whose complement is simply connected; (2) in the sense of least weighted pth powers ($p \geqq 1$) measured by integration over the boundary of the limited region C, this boundary being an arbitrary rectifiable Jordan curve; (3) in the sense of least weighted pth powers on the circumference $\gamma: |w| = 1$, where C is an arbitrary limited sim-

* See for instance de la Vallée Poussin, *Approximation des Fonctions*, Paris, 1919, §55.

† See for instance Julia, loc. cit., pp. 221–222.

ply connected region and is mapped conformally onto the interior of γ; (4) in the sense of least weighted pth powers as measured by integration over the area of C, the region C being an arbitrary limited region; (5) in the sense of least weighted pth powers as measured by integration over C, the point set C being an arbitrary rectifiable Jordan arc; (6) in the sense of least weighted pth powers as measured by integration on the circumference $\gamma: |w|=1$, where C is an arbitrary limited closed point set whose complement D is simply connected and D is mapped onto the exterior of γ so that the points at infinity correspond to each other.

Only the last clause in the statement of Theorem XI contains anything new. The proofs in all of the cases (1)–(6) are so similar that it is not necessary to give them all in detail. Let us restrict ourselves to a typical case, namely (5), where the point set C is a rectifiable Jordan arc.

The requirement that the sequence $\{\pi_n(z)\}$ should converge to $f(z)$ with the same degree of approximation as the sequence of polynomials of best approximation is taken to mean that for an arbitrary $R_1<R$ there exists M such that we have

$$\int_C |f(z) - \pi_n(z)|^p n(z) |dz| \leqq \frac{M}{R_1^{np}}. \tag{13}$$

A consequence of this inequality (compare the proof of Theorem II) is that for an arbitrary $R_2<R_1<R$ there exists M' such that we have

$$|f(z) - \pi_n(z)| \leqq M'\left(\frac{R_2}{R_1}\right)^n, \qquad z \text{ on or within } C_{R_2}. \tag{14}$$

Inequality (14) asserts that the sequence $\{\pi_n(z)\}$ converges on or within C_{R_2} like a convergent geometric series. It follows from a very general theorem due to Ostrowski* that in any closed region interior to the region of uniform convergence of the given sequence we also have convergence like a convergent geometric series. If Theorem XI is not true, the sequence $\pi_n(z)$ converges in some region D'' which contains points interior to C_R—and on any closed point set interior to C_R inequality (14) holds for proper choice of R_1 and R_2—and such that D'' contains also some closed region D' which lies exterior to C_R. Let us take the convergence in D' in the form

$$|f(z) - \pi_n(z)| \leqq \frac{N}{S^n}, \qquad S > 1, \quad z \text{ in } D'. \tag{15}$$

* Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität, vol. 1 (1922), pp. 327–350; p. 329.

It is no loss of generality here to take S less than R. We set $R_2=R/S$, so we have $1<R_2<R_1<R$, with $R/R_2=S>R_1/R_2=S_1$. Then by (14) and (15) we can write

$$(16)\qquad |f(z)-\pi_n(z)| \leqq \frac{N'}{S_1^n},\quad z \text{ on or within } C_{R_2}, \text{ or } z \text{ in } D';$$

it will be noted that C_{R_2} and D' lie exterior to each other.

We are now in a position to apply a generalization of the second part of Theorem I, the present hypothesis being that an inequality of type (16) is given not merely in a single region but in two disconnected regions. Let $u(x, y)$ be Green's function for the region E, the exterior of C_{R_2} and D': the function $u(x, y)$ is harmonic in E, continuous in the corresponding closed region except at infinity, zero on the boundary of E, and can be written at infinity as $\log (x^2+y^2)^{1/2}$ plus a function harmonic at infinity. Then it follows* that the sequence $\{\pi_n(z)\}$ converges in the interior I_{S_1} of the locus $u(x, y)=\log S_1$, and the sequence converges uniformly on any closed point set interior to the locus. This locus may consist of a single curve or of two curves, and the set I_{S_1} is the interior of the one or of both, but in the latter case we shall be particularly concerned with I_{S_1}', that one of the two curves which contains C_{R_2} in its interior. Consider also the locus $I_{S_1}'':u_1(x, y)=\log S_1$, where $u_1(x, y)$ is harmonic exterior to C_{R_2}, continuous in the corresponding closed region except at infinity, zero on C_{R_2}, and can be expressed at infinity as $\log (x^2+y^2)^{1/2}$ plus a function harmonic at infinity. *The curve* I_{S_1}'' *is interior to the curve* I_{S_1}'. Indeed on the boundary of E we have

$$u(x, y) - u_1(x, y) \leqq 0;$$

the left-hand member is harmonic in E even at infinity, so interior to E the strong inequality obtains:

$$u(x, y) - u_1(x, y) < 0.$$

That is, on $I_{S_1}'':u_1(x, y)=\log S_1$ we have $u(x, y)<\log S_1$, so the curve I_{S_1}'' lies interior to I_{S_1}'.

The curve I_{S_1}'' is of course a curve of the family C_ρ; in fact if the function $w=\phi(z)$ maps the exterior of C onto the exterior of γ, the function $w=\phi(z)/R_2$ maps the exterior of C_{R_2} onto the exterior of γ, and I_{S_1}'' is defined as the curve $|\phi(z)|/R_2=S_1$. The curve C_R is the curve of the family I_{S_1}'' corresponding to $|\phi(z)|/R_2=R/R_2$, that is, the curve C_R is the curve I_S'', and lies interior to the curve I_S'. Allow now R_1 to approach R, R_2 being fixed, so that S_1 ap-

* See p. 229 of the reference given for Theorem I.

proaches S and the point sets I_{S_1}, I_{S_1}', I_{S_1}'' are variable, approaching uniformly* the respective point sets I_S, I_S', I_S''. The sequence $\{\pi_n(z)\}$ always converges uniformly on an arbitrary closed point set interior to I_{S_1}, hence converges uniformly on an arbitrary closed point set interior to I_S. The curve C_R lies interior to I_S, so the sequence $\{\pi_n(z)\}$ converges uniformly on a closed point set containing C_R in its interior, yet the limit of the sequence is $f(z)$ in C, hence also in I_S, and $f(z)$ has a singularity on C_R. This contradiction completes the proof.

Theorem XI has already been established by various writers and by other methods, and even more specific results, for certain special types of approximation (1)–(6). References to the literature have already been given for these cases.

Many consequences (e.g. Lückensätze) other than Theorem XI follow from the results of Ostrowski (loc. cit.) as applied to the present sequence $\{\pi_n(z)\}$ in the cases (1)–(6). We mention the theorem analogous to that of Jentzsch, that each point of C_R is a limit point of the zeros of polynomials of the sequence $\{\pi_n(z)\}$.

Theorem XI has obvious application to results on overconvergence recently proved† concerning approximation to rational functions, or to arbitrary analytic functions by polynomials satisfying prescribed auxiliary conditions. In each case (loc. cit., Theorems A, B, 4, 5, 6, 7, 9, 12, 15; §12.2) the sequence of approximating polynomials converges to its limit with the same degree of approximation as does the sequence of polynomials of best approximation. Hence in each case there is no region of uniform convergence containing in its interior any arc of the curve C_R, provided that we have uniform convergence interior to C_{R_1} whenever R_1 is less than R, and that the limit function has a singularity on C_R.

It is not without interest to notice that under the hypothesis of Theorem XI, *divergence of the sequence of polynomials* $\{\pi_n(z)\}$ *at all points exterior to* C_R *cannot be proved.* Indeed, it can be shown (see the reference just given, §12.2) that if the function $f(z)$ satisfies the hypothesis of Theorem XI, and if there are assigned arbitrary points β_i, finite in number and exterior to C_R, then there exist polynomials $\pi_n(z)$ which approximate $f(z)$ on C with the same degree of approximation as the sequence of polynomials of best approximation‡ and which for n sufficiently large satisfy the auxiliary conditions

* See for instance Lebesgue, Palermo Rendiconti, vol. 24 (1907), pp. 371–402.

† Walsh, these Transactions, vol. 32 (1930), pp. 335–390.

‡ This fact is there proved merely for approximation measured in the sense of Tchebycheff, but that implies the corresponding fact for any of the methods (1)–(6) of measuring approximation.

$\pi_n(\beta_i)=0$. This sequence $\{\pi_n(z)\}$ naturally converges not merely interior to C_R but also at the points β_i exterior to C_R.

The proof of Theorem XI that we have given contains incidentally a proof of the following theorem: *If $f(z)$ is analytic interior to C_R but has a singularity on C_R, then neither the sequence of polynomials $\{\pi_n(z)\}$ of best approximation to $f(z)$ on C (measured in any one of the ways considered) nor any other sequence of polynomials which converges on C like the sequence of polynomials of best approximation converges like a geometric series in any region or on any Jordan arc exterior to C_R.*

The result analogous to Theorem XI holds also for the sequences $\{\pi_n(x, y)\}$ of harmonic polynomials of best approximation to harmonic functions, where best approximation is taken in any of the senses (1), (2), (3), (4).* If we assume that this result is false, the sequence $\{\pi_n(x, y)\}$ converges in some region D'' which contains in its interior an arc of C_R and hence D'' contains some simply connected closed region D' which lies exterior to C_R. The sequence of polynomials $p_n(z)$ which vanish at a particular point of C and of which the polynomials $\pi_n(x, y)$ are the respective real parts, converges like a convergent geometric series on any closed point set interior to C_R,† hence converges like a convergent geometric series also in the region D'. Detailed study of the ratios of these geometric series, as in Theorem XI, yields directly a contradiction.

10. **Addendum.‡ Discussion of case $0<p<1$.** In the study of approximation by polynomials to an arbitrary analytic function of a complex variable where approximation is in the sense of least pth powers and is measured by an integral, we have consistently set aside the case $0<p<1$, for inequality (3) is not valid in this case. The case $p>1$ is naturally the most interesting, for here a polynomial of degree n of *best* approximation always exists and is unique, and $p>1$ includes also the case $p=2$ for which the approximating polynomials have a particularly simple form; the approximating functions may be found by orthogonalization and normalization of the sequence $\{[n(z)]^{1/2}z^n\}$. Nevertheless the case $0<p<1$ possesses some interest; in fact Professor Dunham Jackson has recently published§ some results on overconvergence, which are broadly speaking less general than those of the present writer, but which nevertheless treat the cases $0<p<1$, $p=1$, and $p>1$ without distinction by a single method. We shall therefore now show how the

* The point sets C are restricted as in our previous discussion (loc. cit.) of these measures of approximation for harmonic functions by harmonic polynomials.

† Compare Walsh, Bulletin of the American Mathematical Society, vol. 33 (1927), pp. 591–598.

‡ Received by the editors, February 3, 1931.

§ Bulletin of the American Mathematical Society, vol. 36 (1930), pp. 851–857.

methods of the present paper and those previously used by the present writer can be modified so as to include the case $0<p<1$ as well as the case $p\geqq 1$. We shall not take up in detail each of the cases (1)–(6) mentioned in the preceding section, for the treatments in all of these cases are similar; we do take up (2) and (4) respectively in detail.

LEMMA. *If each of the functions $f_n(z)$, $n=1, 2, \cdots$, is analytic and uniformly bounded* interior to the rectifiable Jordan curve C, and if we set*

$$\int_C |f_n(z)|^p \cdot |dz| = \epsilon_n, \qquad p > 0, \tag{17}$$

then $\epsilon_n \to 0$ implies $f_n(z)\to 0$ for z interior to C, uniformly on any closed point set C' interior to C. Moreover, we have

$$|f_n(z)| \leqq Q\epsilon_n^{1/p}, \qquad z \text{ on } C', \tag{18}$$

where Q depends on C' and on p but not on $f_n(z)$.

The boundary value of $f_n(z)$ exists almost everywhere on C, for approach to an arbitrary point of C along the normal, and it is this boundary value that is used in (17).

Let the zeros of $f_n(z)$, if any, interior to C be $\alpha_1, \alpha_2, \cdots$. We omit from the given sequence $f_n(z)$ any functions which vanish identically interior to C; it is sufficient to prove the Lemma for the remaining sequence. Consider the function

$$F_n(z) = \frac{|f_n(z)\prod|\phi(\alpha_i)|}{\prod \dfrac{\phi(z)-\phi(\alpha_i)}{\phi(z)-1/\bar{\phi}(\alpha_i)}}, \tag{19}$$

where $w=\phi(z)$ is a function which maps the interior of C conformally onto the interior of the unit circle $|w|=1$. There may be an infinity of points α_i, but if so the infinite products in the right-hand member of (19) converge, by Blaschke's theorem. We assume $\phi(\alpha_i)\neq 0$, which involves no loss of generality. The function $F_n(z)$ is analytic and has no zeros interior to C, and has the same modulus as $f_n(z)$ on C. The function $(F_n(z))^p$ is likewise analytic and uniformly bounded interior to C, if we consider an arbitrary determination of the pth power at a point interior to C and its analytic extension, so we have Cauchy's integral

* The requirement of uniform boundedness is inserted here for simplicity. It can, in fact, be replaced by the mere existence of the integrals in (17), where the boundary values of $f_n(z)$ on C are the limit values obtained almost everywhere on C by normal approach, together with the validity of Cauchy's integral formula.

$$[F_n(z)]^p = \frac{1}{2\pi i}\int_C [F_n(t)]^p \frac{dt}{t-z};$$

Cauchy's integral is naturally valid here, for the boundary values of $(F_n(z))^p$ for normal approach to C exist almost everywhere on C. If l denotes the length of C and δ denotes the minimum distance from C' to C, we have

$$(20) \qquad |F_n(z)|^p \leqq \frac{l}{2\pi\delta}\int_C |F_n(t)|^p |dt| = \frac{l\epsilon_n}{2\pi\delta}, \qquad z \text{ on } C'.$$

Each function

$$\frac{|\phi(\alpha_i)|}{\dfrac{\phi(z)-\phi(\alpha_i)}{\phi(z)-1/\bar{\phi}(\alpha_i)}}$$

is of absolute value greater than unity for z interior to C, so we have from (19)

$$|f_n(z)|^p \leqq |F_n(z)|^p, \qquad z \text{ interior to } C.$$

The entire lemma, including (18), now follows from (20).

The lemma is of precisely the form for application in the proof of overconvergence in case (2). The lemma also yields the proof of ordinary convergence (not overconvergence) for approximation by polynomials to a function analytic interior to C, continuous in the corresponding closed region, in the sense of least weighted pth powers (p positive, but greater than, equal to, or less than unity) as measured by a line integral over C. The proof of this ordinary convergence in case (3) and of overconvergence in cases (3), (5), (6) follows now with only obvious modifications of the methods already given. We need to consider the case (4) more in detail.

Lemma. *Let C be an arbitrary region. If each of the functions $f_n(z)$, $n=1, 2, \cdots$, is analytic interior to C, and if we set*

$$(21) \qquad \iint_C |f_n(z)|^p dS = \epsilon_n, \quad p > 0,$$

then $\epsilon_n \to 0$ implies $f_n(z) \to 0$ for z interior to C, uniformly on any closed point set C' interior to C. Moreover, we have

$$(22) \qquad |f_n(z)| \leqq Q\epsilon_n^{1/p}, \quad z \text{ on } C',$$

where Q depends on C' and on p but not on $f_n(z)$.

The integral

$$\frac{1}{2\pi}\int_0^{2\pi} | f(z_0 + re^{i\theta}) |^p d\theta, \quad p > 0,$$

is well known to be a non-decreasing function of r, in an arbitrary circle K which together with its interior lies interior to C. Here (r, θ) are polar coordinates with pole at the point z_0. The limit of this integral as r approaches zero is obviously $|f(z_0)|^p$, from which follows the inequality

$$| f(z_0) |^p \leqq \frac{1}{2\pi}\int_0^{2\pi} | f(z_0 + re^{i\theta}) |^p d\theta.$$

We multiply both sides of this inequality by $r\,dr$ and integrate from zero to κ, the radius of K. The resulting inequality is

$$\frac{\kappa^2}{2} | f(z_0) |^p \leqq \frac{1}{2\pi}\iint_K | f(z) |^p dS,$$

so we may write

$$| f(z_0) |^p \leqq \frac{1}{\pi\kappa^2}\iint_K | f(z) |^p dS \leqq \frac{\epsilon_n}{\pi\kappa^2} \cdot$$

This inequality holds for every point z_0 interior to C provided merely that the distance from z_0 to the nearest point of the boundary of C is not less than κ. The inequality therefore holds for proper choice of κ for z_0 on the boundary of an arbitrary closed point set C' interior to C and implies (22) immediately for z on the boundary of C'. Such an inequality, holding on the boundary of a closed point set C', holds at every point of the set, and the lemma is completely established.

This lemma yields the proof of overconvergence in case (4) and also the proof of ordinary convergence for approximation by polynomials to a function analytic interior to a simply connected region C, continuous in the corresponding closed region, in the sense of least weighted pth powers $(p>0)$ as measured by a surface integral over C, provided that the function can be uniformly approximated as closely as desired in the closed region by polynomials. There are only obvious modifications to be made in the proofs already given.

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MATRICES OF INTEGERS ORDERING DERIVATIVES*

BY

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1. **Introduction.** Riquier† in his treatise on partial differential equations has employed matrices of integers, which he calls cotes, to establish order relations among the derivatives of the unknown functions. The matrix effecting a given ordering of the derivatives is not uniquely determined. Certain simple transformations which preserve order relations have been employed by Riquier and Janet. The object of the present paper is to study systematically the matrices in question with special attention to equivalence. The principal result is a method of reducing any matrix to a canonical form which characterizes all matrices establishing the same order relations as the given one.

Some of the transformations are applicable only to a restricted class of matrices. The totality of transformations described has the property that any transformation preserving order can be expressed as a product of them.

Except when the contrary is expressly stated, the results obtained are valid whatever the first cotes of the independent variables may be.

It is expected to follow this paper with another which will treat the existence of a matrix establishing given order relations.

2. **Definitions and completeness of ordering.** Consider a rectangular array of integers, the term integer including zero. Let there be $n+r$ rows, the first n rows corresponding to independent variables x and the last r to unknown functions u. The number in the qth column will be called the *qth cote* of the corresponding variable. We shall use the ordinary matrix notation for the cotes: c_q^p will be the qth cote of the pth independent variable and γ_q^α the qth cote of the αth unknown.

The qth cote of the derivative

$$D_i u_\alpha = \frac{\partial^{i_1+i_2+\cdots+i_n} u_\alpha}{\partial x_1^{i_1} \partial x_2^{i_2} \cdots \partial x_n^{i_n}}$$

* Presented to the Society, December 31, 1930; received by the editors in November, 1930.

† C. Riquier, *Les Systèmes d'Équations aux Dérivées Partielles*, Paris, 1910, p. 195. For an exposition of the application of cotes to the proof of existence theorems one may consult the following papers also:

M. Janet, *Sur les systèmes d'équations aux dérivées partielles*, Journal de Mathématiques Pures et Appliquées, (8), vol. 3 (1920), p. 65.

J. M. Thomas, *Riquier's existence theorems*, Annals of Mathematics, (2), vol. 30 (1929), p. 285.

is defined as

$$C_q = \sum_{p=1}^{n} c_q^p i_p + \gamma_q^\alpha . \tag{2.1}$$

By definition also the derivative $D_i u_\alpha$ precedes or follows $D_j u_\beta$, whose cotes we denote by C_q', according as the first non-zero difference of the set

$$C_1 - C_1', C_2 - C_2', \cdots \tag{2.2}$$

is negative or positive.

If all the differences (2.2) are zero, the given matrix will not establish an order relation between the two derivatives. If the matrix is augmented by columns of arbitrarily chosen integers, the order relations established by the original matrix are not disturbed because the additional cotes will only play a rôle when the original cotes give no answer. Moreover, if the new columns are properly chosen, additional order relations are established by the augmented matrix.

In particular, if the last column is made $0, 0, \cdots, 0, 1, \cdots, r$, any two u's whose relative order is not established by the cotes before the last will have the order relation of their subscripts. Consequently, the augmented matrix completely orders the unknowns, and the equations

$$\gamma_q^\alpha - \gamma_q^\beta = 0 \qquad (q = 1, 2, \cdots, s), \tag{2.3}$$

where s represents the total number of cotes, imply $\alpha=\beta$, that is, if the differences (2.2) formed for two unknowns are all zero, the unknowns are the same.

Likewise if the matrix is further augmented by the n columns

$$\begin{array}{cccc} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \\ \hline 0 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 0 \end{array},$$

and if all the differences (2.2) for the derivatives $D_i u_\alpha$ and $D_j u_\beta$ are zero, the vanishing of the last n differences gives $i_p = j_p$. The vanishing of the other differences then shows that (2.3) hold, that is, u_α and u_β are the same. Consequently the vanishing of the differences (2.2) implies that the derivatives are identical.

THEOREM 1. *For any matrix of integers there exists an augmented matrix whose ordering is consistent with that of the original and is complete.*

In the future, unless the contrary is expressly stipulated, we shall deal only with matrices whose ordering is complete so that equality of all the corresponding cotes of two derivatives implies identity of the derivatives.

3. **Simplest transformations preserving order.** Certain transformations which can be performed upon the elements of a matrix without disturbing the order relations are rather obvious. It is clear that the qth cotes of all the unknowns can be increased by the same integer μ_q without altering the differences (2.2), that is, an admissible transformation is

$$\bar{\gamma}_q^\alpha = \gamma_q^\alpha + \mu_q \qquad (\alpha = 1, 2, \cdots, r). \tag{3.1}$$

Clearly a second is

$$\bar{c}_q^p = \lambda c_q^p, \quad \bar{\gamma}_q^\alpha = \lambda \gamma_q^\alpha, \quad \lambda > 0 \qquad (p = 1, 2, \cdots, n; \alpha = 1, 2, \cdots, r). \tag{3.2}$$

In these formulas, λ may be fractional provided its denominator is a divisor of all the elements of the qth column. Thus the highest common factor of the elements of any column can be removed.

If we put

$$\bar{c}_q^p = c_q^p + \sum \lambda_q^\sigma c_\sigma^p, \quad \bar{\gamma}_q^\alpha = \gamma_q^\alpha + \sum \lambda_q^\sigma \gamma_\sigma^\alpha,$$

we have

$$\overline{C}_q = C_q + \sum \lambda_q^\sigma C_\sigma,$$

whence for the differences of the sequence (2.2) the transformation

$$\overline{C}_q - \overline{C}_q' = C_q - C_q' + \sum \lambda_q^\sigma (C_\sigma - C_\sigma').$$

Since the qth cote plays a rôle in determining order (i.e. the vanishing or sign of $C_q - C_q'$ is of significance) only when the first $q-1$ cotes are equal, if we fix the range of the index σ as follows:

$$\bar{c}_q^p = c_q^p + \sum_{\sigma=1}^{q-1} \lambda_q^\sigma c_\sigma^p, \quad \bar{\gamma}_q^\alpha = \gamma_q^\alpha + \sum_{\sigma=1}^{q-1} \lambda_q^\sigma \gamma_\sigma^\alpha, \tag{3.3}$$

we have

$$\overline{C}_q - \overline{C}_q' = C_q - C_q'$$

whenever either $\overline{C}_q - \overline{C}_q'$ or $C_q - C_q'$ has significance. Hence (3.3) is a transformation preserving order.

If we like, we may combine (3.1), (3.2), (3.3) into the single transformation

$$\bar{c}_q^p = \sum_{\sigma=1}^{q} \lambda_q^\sigma c_\sigma^p, \quad \bar{\gamma}_q^\alpha = \sum_{\sigma=1}^{q} \lambda_q^\sigma \gamma_\sigma^\alpha + \mu_q, \quad \lambda_q^q > 0 \tag{3.4}$$

$$(p = 1, 2, \cdots, n; \alpha = 1, 2, \cdots, r).$$

The λ's in these formulas are not necessarily integers. The only restriction is that the result of applying (3.4) be a matrix of integers. Thus the inverse of a transformation (3.4) in general has fractional coefficients, and the set of matrices to which it can be applied is restricted.

We may summarize the transformation (3.3) in

THEOREM 2. *A matrix of integers formed by increasing or diminishing the elements of a column in a given matrix by any equimultiples of the corresponding elements in any column which precedes it establishes the same order relations as the original.*

It is evident that by use of (3.1) the cotes of all the unknowns can be made positive. If all the first cotes of the independent variables are positive,* we can accomplish the same result for the whole matrix by using the subsequent transformation

$$\bar{c}_q^p = c_q^p + (a + 1)c_1^p, \quad \bar{\gamma}_q^\alpha = \gamma_q^\alpha + (a + 1)\gamma_1^\alpha, \tag{3.5}$$

where a is the numerical value of the numerically greatest negative cote. A particular result of this is

THEOREM 3. *If the first cote of every independent variable is unity, the cotes of both independent variables and unknowns can be made positive without altering the order.*†

Consider two derivatives D_i and D_j of the same unknown, the ordering being complete. From the fact that these derivatives are identical if the differences (2.2) are zero, we know that the system

$$\sum_{p=1}^{n} c_q^p (i_p - j_p) = 0 \qquad (q = 1, 2, \cdots, s) \tag{3.6}$$

has only the trivial solution $i_p - j_p = 0$. Hence

THEOREM 4. *If the ordering is complete, the matrix of the cotes of the independent variables is of rank equal to the number of variables n.*

* It will be a result of Theorem 12, which is to follow, that the signs of the first cotes of the independent variables cannot be changed without changing the ordering.

† This result is given by Janet, who uses (3.1) and (3.5) for $c_1^p = 1$ to obtain it. Thus Janet employs a special transformation (3.4).

Since, conversely, (3.6) has only the trivial solution if $\|c\|$ is of rank n, we have also

THEOREM 5. *If there is only one unknown, a necessary and sufficient condition for complete ordering is that the matrix of cotes of the independent variables be of rank equal to the number of variables n.*

If a column of the whole matrix is linearly dependent on those to the left of it, its elements can be made zero by a transformation (3.4). The column can then be suppressed, for the difference (2.2) corresponding to it is always zero. Hence we may assume that *the rank of the matrix is equal to the number of its columns*. The rank is an integer between n and $n+r$. That it may be as high as $n+r$ follows from the existence of a matrix of integers of rank $n+r$.

If the rank of the whole matrix is $n+r$, the ordering is complete because the $n+1$ additional columns adjoined in §2 to insure completeness, being linearly dependent on those already there, are superfluous.

THEOREM 6. *A sufficient condition for complete ordering is that the matrix of cotes be of rank $n+r$.*

By transformations (3.1) the row of cotes corresponding to any unknown can be made zero. Consequently, the condition in Theorem 6 is not necessary. Likewise we have

THEOREM 7. *The rank of a matrix of cotes is not invariant under transformations preserving order.*

That the transformations considered in this section are not the only ones admissible can be seen from the following example:

$$(3.7)\qquad \begin{array}{cccc||ccc} x & 1 & 2 & 1 & 2 & 1 & 1 \\ y & 1 & 2 & 0 & 2 & 1 & 0 \\ \hline u & 1 & 1 & 0 & 1 & 1 & 0 \\ v & 2 & 2 & 0 & 2 & 2 & 0 \end{array}.$$

It is easy to give a direct proof that the above two matrices bring about the same ordering. We shall not do this, but shall content ourselves with remarking that no transformation (3.4) will throw one of them into the other. This is obvious because their first columns are not proportional for any choice of μ_1 in (3.1). Later (§8) a proof that the two matrices (3.7) are equivalent will be given by means of another sort of transformation.

4. **Properties of forms with integral coefficients.** For convenience of reference we state in a form best adapted to our purpose several consequences of well known theories.

Consider the system

$$\sum_{p=1}^{n} c_t^p k_p = b_t \qquad (t = 1, 2, \cdots, q), \tag{4.1}$$

where the rank of $\|c\|$ is the same as the number of equations q. If every b is divisible by the product of all the determinant factors of $\|c\|$, then (4.1) has a solution in integers because any determinant in the augmented matrix but not in the original contains a column composed solely of b's and hence is divisible by any determinant factor of $\|c\|$.*

In particular, if certain of the b's are made positive, others negative and the rest zero, we have a result conveniently stated as

THEOREM 8. *A set of integers k can be found for which the distribution of signs and zero values is anything we wish among the q linearly independent forms*

$$\sum_{p=1}^{n} c_t^p k_p \qquad (t = 1, 2, \cdots, q). \tag{4.2}$$

By choosing the b's divisible by a sufficiently great power of the product of the determinant factors we see that the values assumed by the non-zero forms can be made numerically greater than any given number.

The above result for a single form will be applied to prove

THEOREM 9. *If the non-homogeneous equations*

$$\sum_{p=1}^{n} c_t^p k_p + \kappa_t = 0 \qquad (t = 1, 2, \cdots, q-1) \tag{4.3}$$

have a solution and if the matrices

$$\|c_t^p\|, \quad \|c_t^p c_q^p\| \qquad (t = 1, 2, \cdots, q-1) \tag{4.4}$$

are of rank $\rho-1$, ρ, respectively, there exist integral values of the k's which satisfy (4.3) *and for which*

$$\sum_{p=1}^{n} c_q^p k_p + \kappa_q \tag{4.5}$$

has arbitrary sign.

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* An exposition of the theory of systems (4.1) with integral coefficients will be found in §12 of a paper by O. Veblen and P. Franklin, *On matrices whose elements are integers*, Annals of Mathematics, (2), vol. 23 (1921), p. 1.

The general solution in integers of (4.3) is any particular solution plus the general solution of the corresponding homogeneous system*

$$\sum_{p=1}^{n} c_t^p k_p = 0 \qquad (t = 1, 2, \cdots, q-1). \tag{4.6}$$

If on substitution of the general solution of (4.3) in (4.5) the variables cancel out, every solution of (4.6) will satisfy

$$\sum_{p=1}^{n} c_q^p k_p = 0. \tag{4.7}$$

But the general solution of (4.6) depends on $n-\rho+1$ parameters, whereas that of the system composed of (4.6) and (4.7) involves only $n-\rho$. Hence when the general solution of (4.3) is substituted in it, (4.5) still contains a variable form, which by Theorem 8 and the remark immediately following can be made to assume either a positive or a negative value exceeding $|\kappa_q|$ numerically. The desired result is therefore established.

The following is geometrically obvious:

THEOREM 10. *The expressions*

$$\sum_{p=1}^{n} c_q^p k_p + \kappa_q, \quad \sum_{p=1}^{n} c_q^p k_p + \bar{\kappa}_q \tag{4.8}$$

do not have contradictory signs if and only if the form

$$\sum_{p=1}^{n} c_q^p k_p \tag{4.9}$$

does not assume a value on the segment† $(-\kappa_q, -\bar{\kappa}_q)$.

Coupling the above with the theorem on existence of solutions in integers‡ we get

THEOREM 11. *The expressions* (4.8) *do not have opposite signs for integers k satisfying* (4.3) *if and only if the invariant factors of*

$$\begin{Vmatrix} c_t^p & c_q^p \\ \kappa_t & \kappa \end{Vmatrix} \text{ and } \|c_t^p \ c_q^p\| \qquad (p = 1, 2, \cdots, n;\ t = 1, 2, \cdots, q-1) \tag{4.10}$$

are the same for no value of κ on the segment $(\kappa_q, \bar{\kappa}_q)$.

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* O. Veblen and P. Franklin, loc. cit.

† By the term "segment (a, b)" we mean the set of integers x satisfying $a<x<b$ or $b<x<a$, according as $a<b$ or $b<a$.

‡ O. Veblen and P. Franklin, loc. cit., p. 10.

5. **Equivalence in the case of a single unknown.** Two matrices of cotes will be called *equivalent* if they establish exactly the same order relations among the derivatives.

Suppose two matrices $\|c\|$ and $\|\bar{c}\|$ ordering the derivatives of a single unknown are equivalent. When the derivatives D_i, D_j are compared, the differences (2.2) computed in the two systems of cotes are

$$\sum_{p=1}^{n} c_q^p k_p \qquad (q = 1, 2, \cdots, n), \tag{5.1}$$

$$\sum_{p=1}^{n} \bar{c}_q^p k_p \qquad (q = 1, 2, \cdots, n), \tag{5.2}$$

where

$$k_p = i_p - j_p. \tag{5.3}$$

A necessary condition for equivalence is that the first non-vanishing expression (5.1) have the same sign as the first non-vanishing expression (5.2), whatever integral values be given to the k's.

If the matrix

$$\|c_1^p \ \bar{c}_1^p\| \qquad (p = 1, 2, \cdots, n)$$

is of rank two, by Theorem 8 the first pair of forms (5.1) and (5.2) can be made to have opposite signs for integral values of the k's. Hence

$$\bar{c}_1^p = \lambda_1^1 c_1^p.$$

To proceed by induction we assume

$$\bar{c}_t^p = \sum_{\sigma=1}^{t} \lambda_t^\sigma c_\sigma^p \qquad (p = 1, 2, \cdots, n; t = 1, 2, \cdots, q-1). \tag{5.4}$$

If the matrix

$$\| c_1^p \cdots c_q^p \ \bar{c}_q^p \| \tag{5.5}$$

is of rank $q+1$, Theorem 8 shows the existence of integral values of the k's for which the first $q-1$ of forms (5.1) vanish and for which the qth forms (5.1) and (5.2) have opposite signs. Because of (5.4) the first $q-1$ of forms (5.2) also vanish for these k's. Thus the equivalence of the matrices is contradicted and the rank of (5.5) must be q. Therefore formulas (5.4) hold also for $t=q$, and the induction is complete.

Now for values of the k's making the first $q-1$ forms (5.1) zero, the qth form (5.2) reduces to

$$\lambda_q^q \sum_{p=1}^{n} c_q^p k_p.$$

Since its sign must not be opposite to that of the qth form (5.1), we conclude $\lambda_q^q \geqq 0$. The fact that the ordering is complete, that is, the rank of $\|\bar{c}\|$ is n, excludes the value zero; for if λ_q^q were zero, any k's making the first $q-1$ forms (5.2) zero would also make the qth zero.

If both c and $\bar{c}$ are regarded as given in (5.4), the equations of that system corresponding to a fixed value of the index t form a system of n linear equations in the t unknowns

$$\lambda_t^1, \lambda_t^2, \cdots, \lambda_t^t.$$

The matrix of these equations consists of the first t columns of $\|c\|$ and consequently is of rank t. The λ's in question can therefore be determined from t of the equations by Cramer's rule as rational functions of the integers c, $\bar{c}$. The λ's are therefore all rational. We cannot conclude, however, that they are integers.

A necessary condition for equivalence is accordingly that formulas (3.4) hold, in so far as they apply to the independent variables. The condition was previously known to be sufficient. Hence

THEOREM 12. *When there is only one unknown, two matrices of cotes are equivalent if and only if their elements are related by the formulas*

$$\bar{c}_q^p = \sum_{\sigma=1}^{q} \lambda_q^\sigma c_\sigma^p, \quad \lambda_q^q > 0 \qquad (p, q = 1, 2, \cdots, n). \tag{5.6}$$

6. **Canonical form for one unknown.** Consider the matrix $\|c\|$ for a single unknown. At least one of the elements on the first column is different from zero. Select the highest non-zero element, say c_1^a. Multiply the elements of the second column by $|c_1^a|$. Replace the second column by itself plus or minus c_2^a times the first, the sign being chosen so that in the new equivalent matrix $c_2^a = 0$. In the same way $c_3^a, \cdots, c_n^a$ can all be made zero. Suppose this done.

In the modified matrix there is a non-zero element on the second column. Suppose the highest one is c_2^b. As above, we make $c_3^b = \cdots = c_n^b = 0$. The zeros already obtained on the ath row persist under this operation.

The process is repeated until the nth column contains a single non-zero element, the $(n-1)$th at most two, and so on.

Finally, any positive factor common to all the elements of a column is removed.

The resulting matrix is called the *canonical form* of the original. Suppose we have two equivalent matrices in canonical form. By Theorem 12 the elements of their first columns must be related by the formulas $\bar{c}_1^p = \lambda_1^1 c_1^p$, where λ_1^1 is known to be rational. Suppose it is in its lowest terms. Its nu-

merator will have to be a divisor of all the integers $\bar{c}_1^p$. Since these numbers are relatively prime, the numerator must be unity. In the same way the denominator is a factor of the relatively prime integers c_1^p and is consequently unity. Therefore $\lambda_1^1=1$ and the first columns are identical.

A consequence of this is that exactly the same row will have had $n-1$ of its elements reduced to zero in the two matrices. For the purposes of the present proof there is no loss of generality in supposing this common row is the first, a situation which can be realized by a change in notation. Hence we assume

$$c_a{}^1 = \bar{c}_a{}^1 = 0 \qquad (a = 2, 3, \cdots, n).$$

From the equivalence and Theorem 12 we have

$$\bar{c}_2{}^1 = \lambda_2{}^1 c_1{}^1 + \lambda_2{}^2 c_2{}^1,$$

whence by substitution and use of the fact that $c_1^1 \neq 0$ we get $\lambda_2^1 = 0$. By use of (5.6) we find

$$\bar{c}_2{}^p = \lambda_2{}^2 c_2{}^p,$$

and as before we conclude the identity of the columns. Since the first non-zero elements on the second columns occupy the same position in the two matrices, we may asume that $c_a^2 = \bar{c}_a^2 = 0$ for $a=3, 4, \cdots, n$.

To complete the proof by induction, assume that

$$\bar{c}_t{}^p = c_t{}^p \qquad (p = 1, 2, \cdots, n; t = 1, 2, \cdots, q-1), \tag{6.1}$$

relations equivalent to

$$\lambda_t{}^t = 1, \ \lambda_t{}^p = 0 \qquad (t = 1, 2, \cdots, q-1; p < t), \tag{6.2}$$

and that

$$\bar{c}_a{}^t = c_a{}^t = 0 \qquad (t = 1, 2, \cdots, q-1; t < a). \tag{6.3}$$

In accordance with (6.1) we may assume that the highest non-zero element on the tth column is c_t^t. From (5.6) we have

$$\bar{c}_q^p = \sum_{\sigma=1}^{q} \lambda_q^\sigma c_\sigma^p . \tag{6.4}$$

Make in these relations $p=1, 2, \cdots, q-1$ successively, and use (6.2), (6.3). There results

$$\lambda_q^a = 0, \qquad a < q.$$

These values introduced in (6.4) give

$$\bar{c}_q^p = \lambda_q^q c_q^p,$$

whence as before we conclude the identity of the two columns. Moreover, we may consistently assume that c_q^q is the highest non-zero element on the qth column, so that (6.3) hold for $t=q$. The induction is therefore complete.

THEOREM 13. *Two matrices of cotes ordering the derivatives of a single function are equivalent if and only if they are identical when reduced to canonical form.*

As an example, consider the case of three independent variables, choosing the first cotes all equal to unity, as is customary in applications.* The reduction to canonical form gives either

$$\begin{array}{cccc} x & 1 & 0 & 0 \\ y & 1 & p & 0 \\ z & 1 & q & \pm 1 \end{array} \tag{6.5}$$

or

$$\begin{array}{cccc} x & 1 & 0 & 0 \\ y & 1 & 0 & \pm 1 \\ z & 1 & \pm 1 & 0, \end{array} \tag{6.6}$$

where p and q are relatively prime. The orderings as p and q take on all possible values are all distinct.

Of course (6.6) can be obtained from (6.5) by putting $p=\pm 1$, $q=0$ and *changing the order of the variables*, but (6.5) and (6.6) are not equivalent for any values of p, q.

7. **Canonical form for matrix ordering unknowns alone.** Suppose the unknowns arranged in a definite order. If we attribute to the first unknown the cote 0, to the second the cote 1, $\cdots$, to the rth the cote $r-1$, the unknowns are arranged in the given order by any matrix with its first column composed of the integers $0, 1, \cdots, r-1$ written in the appropriate order, and the other columns anything we wish. We shall call the single column whose formation is described above the *canonical form* for any matrix ordering the unknowns in the given manner.

If the matrix does not completely order all the unknowns, we may still give it a canonical form by assigning 0 cote to all the unknowns whose relative order is indeterminate but which precede all the rest, etc. We mean in this case by equivalence that the orderings are not only consistent but equally complete. Obviously we have

* Riquier, loc. cit., p. 207, footnote 2.

THEOREM 14. *Two matrices ordering unknowns alone are equivalent if and only if identical when reduced to canonical form.*

Consider now in a complete matrix a column in which the cotes of the independent variables are all zero. The corresponding difference (2.2) is the same for u_α, u_β as for $D_i u_\alpha$, $D_j u_\beta$, that is, only the relative order of the unknowns is of any consequence. Hence the column can be put in the canonical form for unknowns.

If several adjacent columns contain nothing but zeros in the places corresponding to the independent variables, they likewise can be replaced by a single column in the canonical form for a matrix of cotes alone. We may therefore replace a given matrix by an equivalent matrix in which no two adjacent columns have zeros in all places corresponding to the independent variables.

8. **Two additional transformations preserving order.** Let c be a positive integer. We write

$$\gamma_q^\alpha = c[\gamma_q^\alpha/c] + g_q^\alpha,$$

where [ ] denotes the "greatest integer in." The numbers g_q^α are non-negative and less than c. We introduce here the abbreviations

$$\kappa_q = \gamma_q^\alpha - \gamma_q^\beta, \quad \bar{\kappa}_q = \bar{\gamma}_q^\alpha - \bar{\gamma}_q^\beta, \tag{8.1}$$

which will be useful throughout the rest of the paper. For present purposes we in addition put

$$\bar{\gamma}_q^\alpha = c[\gamma_q^\alpha/c].$$

The difference $\kappa_q - \bar{\kappa}_q$, being equal to the difference of two non-negative numbers both less than c, is numerically less than c. Since $\bar{\kappa}_q$ is divisible by c, there is no multiple of c on the segment $(-\kappa_q, -\bar{\kappa}_q)$.

Suppose c is a factor of all the c_q^p on the qth column. Since any value assumed by the form (4.9) is divisible by c, Theorem 10 shows that the expressions (4.8) never have contradictory signs. Hence the qth cotes of the matrices

$$\left\| \begin{matrix} \cdots & c_q^p & c_{q+1}^p & \cdots \\ \cdots & \gamma_q^\alpha & \gamma_{q+1}^\alpha & \cdots \end{matrix} \right\| \tag{8.2}$$

and

$$\left\| \begin{matrix} \cdots & c_q^p & c_q^p & c_{q+1}^p & \cdots \\ \cdots & c[\gamma_q^\alpha/c] & c[\gamma_q^\alpha/c] + g_q^\alpha & \gamma_{q+1}^\alpha & \cdots \end{matrix} \right\| \tag{8.3}$$

are such that the expressions (4.8) have the same sign, if they are both different from zero.

If the first expression in (4.8) is zero, the number c must be a divisor of κ_q, since it is a divisor of c_q^p. It must therefore be a divisor of

$$\kappa_q - \bar{\kappa}_q = g_q^\alpha - g_q^\beta .$$

The difference on the right, being numerically less than c, must be zero. Hence $\kappa_q = \bar{\kappa}_q$, and the second expression (4.8) is also zero. The burden of the decision is thus thrown upon the $(q+1)$th and succeeding cotes in (8.2) and the $(q+2)$th and succeeding cotes in (8.3). As these series of cotes are identical, the decision they render is the same.

If the first expression in (4.8) is not zero and the second is zero, the decision is made by the qth column of (8.2) and the $(q+1)$th of (8.3). Since these columns are identical, the decision is the same.

Hence the two matrices are equivalent.

By two rather obvious transformations (8.3) becomes

$$\left\| \begin{matrix} \cdots & c_q^p/c & 0 & c_{q+1}^p & \cdots \\ \cdots & [\gamma_q^\alpha/c] & g_q^\alpha & \gamma_{q+1}^\alpha & \cdots \end{matrix} \right\| .$$

We accordingly have

THEOREM 15. *If all the qth cotes of the independent variables have a positive factor c in common, without disturbing order relations the qth column can be replaced by two columns the first of which has for elements c_q^p/c and the greatest integers in γ_q^α/c, and the second, zeros in the places corresponding to the independent variables and the non-negative remainders from the divisions γ_q^α/c in the others.**

Theorem 15 can be applied to show the equivalence of matrices (3.7). Reducing the first cotes of the independent variables in the second matrix by the above principle gives

$$\begin{array}{cccc} 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array} .$$

Multiply the first column by 2 and add to the second:

$$\begin{array}{cccc} 1 & 2 & 1 & 1 \\ 1 & 2 & 1 & 0 \\ \hline 0 & 1 & 1 & 0 \\ 1 & 2 & 2 & 0 \end{array} .$$

* Riquier, loc. cit., p. 202, gives essentially this result for the first column.

When the first cotes of the unknowns are increased by unity, the third column, being identical with the first, can be omitted. The first matrix of (3.7) results.

A converse to Theorem 15 can also be proved.

THEOREM 16. *If all the* $(q+1)$*th cotes of the independent variables are zero, by modifying the qth column appropriately the* $(q+1)$*th can be suppressed without disturbing order relations.*

Let the matrix be

$$\overline{M} = \left\| \begin{matrix} \cdots & \bar{c}_q^p & 0 & \cdots \\ \cdots & \bar{\gamma}_q^\alpha & \bar{\gamma}_{q+1}^\alpha & \cdots \end{matrix} \right\|.$$

It is only essential that the elements on the $(q+1)$th column be made non-negative. For definiteness, however, we suppose that column in canonical form (§7). Let c be a positive integer satisfying

$$\bar{\gamma}_{q+1}^\alpha < c \qquad (\alpha = 1, 2, \cdots, r).$$

Again we make the unessential restriction that c be the least integer satisfying the condition. Consider the matrix with the single column

$$M = \left\| \begin{matrix} \cdots & c\bar{c}_q^p & \cdots \\ \cdots & c\bar{\gamma}_q^\alpha + \bar{\gamma}_{q+1}^\alpha & \cdots \end{matrix} \right\|$$

in place of the two in $\overline{M}$. Theorem 15 shows that M is equivalent to $\overline{M}$ and the theorem is proved.

9. **Reduced form for the general matrix.** If the rank of the matrix

$$\|c_t^p\| \qquad (p = 1, 2, \cdots, n;\ t = 1, 2, \cdots, q-1)$$

is the same as that of the matrix with one additional column $t=q$, rational multipliers λ which satisfy

$$\sum_{\sigma=1}^{q} \lambda_q^\sigma c_\sigma^p = 0 \qquad (p = 1, 2, \cdots, n)$$

can be found, the last λ_q^q being different from zero. Because of the homogeneity of this relation, the λ's can always be rendered integral and λ_q^q positive. By a transformation (3.4) the elements in the qth column of $\|c\|$ can therefore be replaced by zeros. We suppose this done wherever possible. The matrix $\|c\|$ may then have r columns of zeros.

By transformations (3.4) and the transformations of Theorem 15 the square matrix obtained from $\|c\|$ by disregarding the columns of zeros can be put in the canonical form for a single unknown.

If equations (4.3) have no solution in integers whatever distinct values be given α, β in the definition of κ_t (8.1), the qth cotes of the unknowns play no rôle. They can be replaced by zeros. We assume this has been done wherever possible.

Let the columns whose elements in $\|c\|$ are all zero be treated as described at the end of §7 with the result that no two of them are consecutive and that each is in the canonical form described in §7. In addition, if for such a column, say the qth, there is no pair of distinct values α, β for which (4.3) have a solution and $\kappa_q \neq 0$ simultaneously, the qth cotes of the unknowns are useless and can be replaced by zeros. We assume this has been carried out wherever possible.

Finally any column containing only zeros is to be omitted.

The resulting matrix will be called a *reduced form* of the original.

Consider two equivalent matrices M and $\overline{M}$ in reduced form and such that the first $q-1$ columns of $\|c\|$ and $\|\bar{c}\|$ are identical.

If there is a non-zero element on the qth columns of both $\|c\|$ and $\|\bar{c}\|$, those columns, being corresponding columns in the canonical form of two equivalent matrices for a single unknown, are identical by Theorem 13.

Let $\bar{c}_q^p = 0$ for all values of p. For any k's satisfying (4.3) the first q differences (2.2) formed for $\overline{M}$ are

$$\bar{\kappa}_1 - \kappa_1, \cdots, \bar{\kappa}_{q-1} - \kappa_{q-1}, \bar{\kappa}_q.$$

The expression (4.5) must not have sign contradictory to the first of these which is not zero. The last is known to be different from zero for some choice of α, β. With α, β so chosen, the sign of (4.5) is fixed. The cotes c_q^p are therefore linearly dependent on $c_1^p, \cdots, c_{q-1}^p$: if they were not, Theorem 9 would say that (4.5) could be made to change sign for integral k's satisfying (4.3). Since the columns in $\|c\|$ linearly dependent on those preceding have been replaced by zeros, we have $c_q^p = 0$ for all values of p. The same argument also shows that $\bar{c}_q^p = 0$ if $c_q^p = 0$.

In any case, therefore, the qth columns of $\|c\|$ and $\|\bar{c}\|$ are identical, and by induction we have proved

THEOREM 17. *A necessary condition for the equivalence of two matrices of cotes is that the portions of them corresponding to the independent variables be identical when they are put in reduced form.*

10. **Transformations of the reduced form.** Let V be any transformation which converts a matrix M into an equivalent matrix $\overline{M}$:

$$V(M) = \overline{M}. \tag{10.1}$$

Suppose

$$U(M) = R, \quad S(\overline{M}) = \overline{R}, \tag{10.2}$$

where R and $\overline{R}$ are in reduced form. By Theorem 17 R and $\overline{R}$ have their corresponding matrices $\|c\|$ identical. Hence there exists a transformation T which affects the cotes of the unknowns alone and which sends R into $\overline{R}$:

$$T(R) = \overline{R}.$$

By substitution from (10.2) we get

$$TU(M) = S(\overline{M}),$$

or

$$S^{-1}TU(M) = \overline{M}.$$

Comparison with (10.1) gives

$$V = S^{-1}TU.$$

Hence to complete the determination of all transformations preserving order it will suffice to consider the transformations of the cotes of the unknowns alone, the matrix being assumed in reduced form.

Let R and $\overline{R}$ be two equivalent matrices in reduced form. Consider any distinct pair α, β in the definition (8.1). Suppose that equations (4.3) have integral solutions, but that none of these makes (4.5) vanish. For such values of the k's the first q differences (2.2) formed for $\overline{R}$ can be written

$$\bar{\kappa}_1 - \kappa_1,\ \bar{\kappa}_2 - \kappa_2,\ \cdots,\ \kappa_{q-1} - \bar{\kappa}_{q-1},\ \sum_{p=1}^{n} c_q^p k_p + \bar{\kappa}_q. \tag{10.3}$$

The sign of (4.5) must not be opposite to that of the first non-vanishing difference in the sequence (10.3).

First case, $c_q^p \neq 0$ for some p. The quantities c_q^p are linearly independent of $c_1^p, \cdots, c_{q-1}^p$, and by Theorem 9 expression (4.5) can be made to change sign for values of the k's satisfying (4.3). Hence we have

$$\bar{\kappa}_1 - \kappa_1 = \bar{\kappa}_2 - \kappa_2 = \cdots = \bar{\kappa}_{q-1} - \kappa_{q-1} = 0. \tag{10.4}$$

A further necessary condition on $\bar{\kappa}_q$ is furnished by Theorem 11.

If the value of $\bar{\kappa}_q$ is such that values of the k's satisfying (4.3) make the last expression (10.3) vanish, expression (4.5), which becomes $\kappa_q - \bar{\kappa}_q$ for the values in question, must not have opposite sign to

$$\sum_{p=1}^{n} c_{q+1}^{p} k_p + \bar{\kappa}_{q+1}. \tag{10.5}$$

Theorem 9 forces us to conclude $c_{q+1}^{p}=0$ for all values of p. If $\bar{\kappa}_{q+1}$ is also zero, then

$$\sum_{p=1}^{n} c_{q+2}^{p} k_p + \bar{\kappa}_{q+2} \tag{10.6}$$

must not have opposite sign to $\kappa_q - \bar{\kappa}_q$, which is not zero because (4.5) does not vanish. Since adjacent columns of $\|c\|$ cannot consist solely of zeros, $c_{q+2}^{p} \neq 0$ for some p. Theorem 9 shows that (10.6) can be made to change sign for values of k making the first $q+1$ differences (2.2) zero. It is therefore impossible for $\bar{\kappa}_{q+1}$ to be zero, and the sign of $\bar{\kappa}_{q+1}$ is the same as that of $\kappa_q - \bar{\kappa}_q$.

To summarize, the conditions are (10.4) and that stated in Theorem 11. The invariant factors of the two matrices (4.9) can become the same for $\kappa = \bar{\kappa}_q$ only if $c_{q+1}^{p}=0$ for all values of p, and in such a case there is the further condition that $\bar{\kappa}_{q+1}^{p}$ is not zero and has the same sign as $\kappa_q - \bar{\kappa}_q$. These conditions are also sufficient for equivalence, so far as comparison of derivatives of the unknowns u_α, u_β is concerned, because they assure that the first non-vanishing difference in the sequence (2.2) has the same sign for the two matrices. The quantities $\bar{\kappa}_{q+2}, \cdots$ are unrestricted, as is also κ_{q+1} except in the special case noted above

Second case, $c_q^p = 0$ *for all values of* p. In this case, $c_{q-1}^{p} \neq 0$ for some p. By Theorem 9 we can find integral k's satisfying the first $q-2$ of equations (4.3) and for which

$$\sum_{p=1}^{n} c_{q-1}^{p} k_p + \kappa_{q-1} \tag{10.7}$$

has arbitrary sign. Since (10.7) must not have opposite sign to the first non-zero expression in the sequence

$$\bar{\kappa}_1 - \kappa_1, \bar{\kappa}_2 - \kappa_2, \cdots, \bar{\kappa}_{q-2} - \kappa_{q-2}, \sum_{p=1}^{n} c_{q-1}^{p} k_p + \bar{\kappa}_{q-1}, \tag{10.8}$$

we conclude as before that

$$\bar{\kappa}_1 - \kappa_1 = \bar{\kappa}_2 - \kappa_2 = \cdots = \bar{\kappa}_{q-2} - \kappa_{q-2} = 0, \tag{10.9}$$

and that the condition of Theorem 11 with q replaced by $q-1$ holds.

Moreover, by considering values of the k's satisfying all of (4.3) we find that $\bar{\kappa}_{q-1} - \kappa_{q-1}$ cannot have sign opposite to κ_q, which cannot be zero from the definition of q.

If there are integers k satisfying the first $q-2$ of equations (4.3) and making the last expression (10.8) vanish, the first of the following expressions

$$\bar{\kappa}_q,\quad \sum_{p=1}^{n} c_{q+1}^{p} k_p + \bar{\kappa}_{q+1}$$

which does not vanish for these values must not have sign opposite to the first of the following,

$$\kappa_{q-1} - \bar{\kappa}_{q-1}.\ \kappa_q,$$

which does not vanish. Theorem 9 and the facts that $\kappa_q \neq 0$ and $c_{q+1}^{p} \neq 0$ show that $\bar{\kappa}_q \neq 0$ in this case. Hence $\bar{\kappa}_q$ has the sign of $\kappa_{q-1} - \bar{\kappa}_{q-1}$, unless the latter expression vanishes. In the exceptional case the hypothesis under which we are working (i.e., that the last expression (10.8) vanishes) is surely fulfilled and $\bar{\kappa}_q$ has the same sign as κ_q.

To summarize, the conditions are (10.9) and Theorem 11 with q replaced by $q-1$. Further, if $\bar{\kappa}_{q-1} - \kappa_{q-1}$ is zero, $\bar{\kappa}_q$ has the sign of κ_q; the expression $\bar{\kappa}_{q-1} - \kappa_{q-1}$, if not zero, has the same sign as κ_q, and, whenever $\bar{\kappa}_q$ is of significance, opposite sign to $\bar{\kappa}_q$. These necessary conditions are readily seen to be sufficient as far as comparison of derivatives of u_α, u_β is concerned. The $\bar{\kappa}$'s not mentioned are subjected to no restriction.

Any transformation satisfying the restrictions given above for all pairs α, β will preserve order.

The number q defined above, being a function of α and β, might be written $q(\alpha, \beta)$. The same is true of κ.

As an example, consider the matrix

$$\begin{array}{ccccc} 1 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 1 & 3 & 1 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 & 0 \end{array}, \tag{10.10}$$

which is in reduced form. For it,

$$q(2, 3) = q(3, 1) = q(1, 2) = 4.$$

The matrix

$$
(10.11)\qquad
\begin{array}{ccccc}
1 & 0 & 0 & 0 & 0 \\
1 & 2 & 0 & 0 & 0 \\
1 & 3 & 1 & 0 & 0 \\
1 & 2 & 2 & 0 & 1 \\
\hline
0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 1 & 2 & 0
\end{array}
$$

is also in reduced form, and can be obtained from (10.10) by the process described under case two. For it, however,

$$q(2, 3) = 4, \quad q(3, 1) = q(1, 2) = 3.$$

This illustrates the fact that $q(\alpha, \beta)$ may decrease by unity under case two, and is therefore not an invariant.

We may, of course, pass from (10.11) to (10.10) by the process of case one, and $q(3, 1)$, $q(1, 2)$ are seen to increase by unity.

We saw above, under case one, that $\bar{\kappa}_{q+1}$ can not be zero when all the expressions

$$\sum_{p=1}^{n} c_t^p k_p + \bar{\kappa}_t \qquad (t = 1, 2, \cdots, q)$$

vanish. Hence $q(\alpha, \beta)$ cannot increase by more than unity. It cannot decrease by more than unity, because if it did, it would increase by more than unity under the inverse transformation.

11. **Canonical form for the general matrix.** Consider a matrix in reduced form. By a transformation of the preceding section make the difference $\gamma_1^1 - \gamma_1^2$ numerically as small as possible. With this value fixed, make $\gamma_1^1 - \gamma_1^3$ numerically as small as possible. And so on, for each of the differences

$$\gamma_1^1 - \gamma_1^2, \gamma_1^1 - \gamma_1^3, \cdots, \gamma_1^1 - \gamma_1^r$$

in turn. Then treat the differences

$$\gamma_2^1 - \gamma_2^2, \gamma_2^1 - \gamma_2^3, \cdots, \gamma_2^1 - \gamma_2^r$$

successively in like manner. And so on.

The differences of the cotes of the unknowns are thus determined, provided we agree to choose the positive value when a difference can be made numerically least with either sign. We render the cotes themselves determinate by specifying that the algebraically least on each column be made zero by a transformation (3.1).

Any column containing only zeros is to be omitted.

The matrix finally obtained will be called the *canonical form* of the original. Since in reduction to canonical form we single out a matrix among all those equivalent to the given one by means of a certain minimum property, we evidently have

THEOREM 18. *Two matrices of cotes are equivalent if and only if identical when reduced to canonical form.*

When a matrix of cotes has been put in the above canonical form, the submatrix ordering the derivatives of any single unknown is of course in the canonical form of §6. Moreover, we can prove that any column whose elements in $\|c\|$ are all zero is in the canonical form of §7. To do this, suppose the ath column of the $\|c\|$ of a matrix in canonical form consists solely of zeros. Let that column be reduced by the process of §7. Suppose the differences

$$\gamma_a^1 - \gamma_a^2, \cdots, \gamma_a^1 - \gamma_a^b \tag{11.1}$$

are unchanged in the process, and that $\gamma_a^1 - \gamma_a^{b+1}$ is changed into $\bar{\gamma}_a^1 - \bar{\gamma}_a^{b+1}$. Now the reduction of §7 has the property of making all the differences which it changes numerically smaller. Hence

$$\left| \bar{\gamma}_a^1 - \bar{\gamma}_a^{b+1} \right| < \left| \gamma_a^1 - \gamma_a^{b+1} \right| .$$

In reducing the matrix as a whole to canonical form, however, when the differences (11.1) and those preceding them have been fixed at their final values, of all equivalent matrices we pick one for which $|\gamma_a^1 - \gamma_a^{b+1}|$ is least. Hence there is a contradiction, and all the differences of the ath cotes

$$\gamma_a^1 - \gamma_a^2, \cdots, \gamma_a^1 - \gamma_a^r$$

are the same in the two matrices. Since the algebraically least cote in both cases is zero, the ath cotes are identical, and the desired result is established.

As an example, consider the following reduced form in two unknowns and two independent variables:

$$\begin{array}{cccc} x & 1 & 0 & 0 \\ y & 1 & 1 & 0 \\ \hline u & a & b & 1 \\ v & 0 & 0 & 0 \end{array} . \tag{11.2}$$

There is a single distinct pair α, β. The differences (2.2) are

$$k_1 + k_2 + a, \; k_2 + b, \; 1.$$

Hence $q=3$, and since $c_3^{\rho}=0$, we are under the second case of §10. κ_2 can be changed. Since the invariant factors of $\|c\|$ are both unity, the maximum change in κ_2 is unity, and the new $\bar{\kappa}_3$ is necessarily of significance. Consequently there are two possibilities: $\bar{\kappa}_2-\kappa_2$ has same sign as $+1$ and opposite sign to $\bar{\kappa}_3$; or $\bar{\kappa}_2-\kappa_2=0$ and $\bar{\kappa}_3$ has same sign as $+1$. If b is negative, the canonical form is therefore

$$\begin{array}{ccc} 1 & 0 & 0 \\ 1 & 1 & 0 \\ \hline a & b+1 & 0 \\ 0 & 0 & 1 \end{array};$$

if b is positive, the canonical form is (11.2).

12. **Normal form.** If the transformation described in the discussion of Theorem 16 be applied to the canonical form, the columns containing nothing but zeros in $\|c\|$, other than the first, can be suppressed. The resulting matrix will be called the *normal form* of the original. The reason for making the process in Theorem 16 uniquely defined becomes apparent, and we have

THEOREM 19. *Two matrices of cotes are equivalent if and only if identical when reduced to normal form.*

It is to be noted that when the whole matrix is in normal form, the matrix $\|c\|$ has the same properties as for the canonical form *except that the cotes of the same column are not necessarily relatively prime.*

The normal form contains $n+1$ or n columns according as the first column of $\|c\|$ consists solely of zeros or not. Hence we have

THEOREM 20. *Any ordering effected by a matrix of cotes with the first cotes of the independent variables all equal to unity can be accomplished by a matrix with columns equal in number to the independent variables.*

Thus the normal form of

$$\begin{array}{ccccc} 1 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & 0 & 0 & \cdots & 1 \\ \hline 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & r-1 & 0 & \cdots & 0 \end{array}$$

is

$$\begin{array}{cccc} r & 0 & \cdots & 0 \\ r & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ r & 0 & \cdots & 1 \\ \hline 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ r-1 & 0 & \cdots & 0 \end{array}.$$

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THE EFFECTS OF GENERAL REGULAR TRANSFORMATIONS ON OSCILLATIONS OF SEQUENCES OF FUNCTIONS*

BY

RALPH PALMER AGNEW

1. Introduction

Recently the writer† has considered the behavior of continuous oscillation, continuous convergence, uniform oscillation, and uniform convergence of complex and real sequences of functions under complex and real regular transformations with triangular matrices; oscillation and convergence being in each case considered (1) over a set as a whole, (2) at a single point of a set, and (3) at all points of a set. It is the object of the present paper to outline an extension of that investigation, considering regular transformations of a general form which includes practically all of the transformations used in the theory of summability.‡

2. Transformations

Let T and A be sets of metric spaces, let T have a limit point t_0 not belonging to T, and let functions $a_k(t)$, $k=1, 2, 3, \cdots$, be defined over T. If a sequence $\{s_n(x)\}$, defined over A, is such that

$$(G) \qquad \sigma(t, x) = \sum_{k=1}^{\infty} a_k(t) s_k(x)$$

converges for all t in T and x in A and if

$$\lim_{t \to t_0} \sigma(t, x) = \sigma(x),§$$

then (G) is said to assign the value $\sigma(x)$ to the sequence $\{s_n(x)\}$.

It is convenient to regard (G) as being a transformation which carries a *given sequence* $\{s_n(x)\}$ into a *transformed function* $\sigma(t, x)$. The transformation (G) is said to be real if $a_k(t)$ is real for all k and for all t in T; otherwise it is

* Presented to the Society, September 11, 1930; received by the editors August 2, 1930.

† These Transactions, vol. 32 (1930), pp. 669–708. This paper will be referred to hereafter as Trans.

‡ See Carmichael, Bulletin of the American Mathematical Society, vol. 25 (1918–19), p.118; J. Schur, Journal für Mathematik, vol. 151 (1920), p. 82; and W. A. Hurwitz, Bulletin of the American Mathematical Society, vol. 28 (1922), p. 18.

§ Here, as elsewhere in this paper, t is restricted to approaching t_0 over the set T.

complex. Except in cases where a specific statement to the contrary is made, the transformations and sequences considered in this paper may be complex. The following conditions* are listed together for convenience:

C_1: $\sum_{k=1}^{\infty} |a_k(t)|$ is bounded for all t in T;

C_2: for each k, $\lim_{t\to t_0} a_k(t) = 0$;

C_3: $\lim_{t\to t_0} \sum_{k=1}^{\infty} a_k(t) = 1$;

C_4: $\lim_{t\to t_0} \sum_{k=1}^{\infty} | a_k(t) | = 1$;

C_5: for each k, $a_k(t) = 0$ for all sufficiently advanced t;†

C_6: $\sum_{k=1}^{\infty} a_k(t) = 1$ for all sufficiently advanced t.

Since we are considering only *regular* transformations‡ we shall use the symbol (G) to represent a regular transformation. It is well known that C_1, C_2, and C_3 are necessary and sufficient for the regularity of a complex transformation when applied to complex sequences, and for the regularity of a real transformation when applied to real sequences. *Hence* (G), *complex or real, satisfies* C_1, C_2, *and* C_3.

3. Oscillations

Let a sequence $\{s_n(x)\}$ be defined over a set A. For continuous oscillations of a sequence we have the two following definitions. The continuous oscillation of $\{s_n(x)\}$ over the set A (which we shall call the *Ω-oscillation of* $\{s_n(x)\}$ *over* A) is denoted by $\Omega(\{s_n\}, A)$ and is defined as follows: for each sequence $\{x_i\}$ of points of A, form

$$\limsup_{m\to\infty,\, n\to\infty,\, i\to\infty,\, j\to\infty} | s_m(x_i) - s_n(x_j) | = v;$$

the least upper bound of all such v is $\Omega(\{s_n\}, A)$. Similarly the continuous

* These conditions are analogous to the corresponding conditions of Trans. To see this, we may specialize (G) as in §9 of this paper, and then impose the further condition $a_{nk}=0$, $k>n$. Then (G) assumes the form $\sigma_n(x)=\sum_{k=1}^{n} a_{nk}s_k(x)$, of a transformation with a triangular matrix; and the conditions $C_1, \cdots, C_6$ become the corresponding conditions of Trans. Owing to these circumstances, the lemmas and theorems of §§4–7 include corresponding results of Trans. to which the reader will be referred by footnotes.

† I.e. for all points t of T which lie in a sufficiently small neighborhood of t_0 in T.

‡ A transformation is said to be regular when it assigns to each convergent sequence the value to which it converges.

oscillation of $\{s_n(x)\}$ at a point x_0 of A^0* over the set A (which we shall call the *Ω-oscillation of* $\{s_n(x)\}$ *at* x_0 *over* A) is denoted by $\Omega(x_0; \{s_n\}, A)$ and is defined as follows: for each sequence $\{x_i\}$ of points of A with the limit x_0, form

$$\limsup_{m\to\infty, n\to\infty, i\to\infty, j\to\infty} |s_m(x_i) - s_n(x_j)| = v;$$

the least upper bound of all such v is $\Omega(x_0; \{s_n\}, A)$.

For uniform oscillations of sequences, we have the two following definitions. The uniform oscillation of $\{s_n(x)\}$ over the set A (which we shall call the *O-oscillation of* $\{s_n(x)\}$ *over* A) is denoted by $O(\{s_n\}, A)$ and is defined as follows: for each sequence $\{x_i\}$ of A, form

$$\limsup_{m\to\infty, n\to\infty, i\to\infty} |s_m(x_i) - s_n(x_i)| = v;$$

the least upper bound of all such v is $O(\{s_n\}, A)$. Similarly the uniform oscillation of $\{s_n(x)\}$ at a point x_0 of A^0 over the set A (which we shall call the *O-oscillation of* $\{s_n(x)\}$ *at* x_0 *over* A) is denoted by $O(\{s_n\}, A)$ and is defined as follows: for each sequence $\{x_i\}$ of points of A with the limit x_0, form

$$\limsup_{m\to\infty, n\to\infty, i\to\infty} |s_m(x_i) - s_n(x_i)| = v;$$

the least upper bound of all such v is $O(x_0; \{s_n\}, A)$.

For the corresponding Ω-oscillations of transformed functions, we have the two following definitions. Let t and u be points of T, and for each sequence $\{x_i\}$ of points of A, form

$$\limsup_{t\to t_0, u\to t_0, i\to\infty, j\to\infty} |\sigma(t, x_i) - \sigma(u, x_j)| = v;$$

the least upper bound of all such v is the *Ω-oscillation of* $\sigma(t, x)$ *over* A and will be denoted by $\Omega(\sigma, A)$. Similarly, for each sequence $\{x_i\}$ of A with the limit x_0, form

$$\limsup_{t\to t_0, u\to t_0, i\to\infty, j\to\infty} |\sigma(t, x_i) - \sigma(u, x_j)| = v;$$

the least upper bound of all such v is the *Ω-oscillation of* $\sigma(t, x)$ *at* x_0 *over* A and will be denoted by $\Omega(x_0; \sigma, A)$.

For the *O-oscillations* of transformed functions, we have the two following definitions. For each sequence $\{x_i\}$ of points of A, form

$$\limsup_{t\to t_0, u\to t_0, i\to\infty} |\sigma(t, x_i) - \sigma(u, x_i)| = v;$$

* A^0 is used to denote the set consisting of A and its limit points.

the least upper bound of all such v is the *O-oscillation of* $\sigma(t, x)$ *over* A and will be denoted by $O(\sigma, A)$. Similarly, for each sequence $\{x_i\}$ of points of A with the limit x_0, form

$$\limsup_{t\to t_0,\, u\to t_0,\, i\to\infty} |\sigma(t, x_i) - \sigma(u, x_i)| = v;$$

the least upper bound of all such v is the *O-oscillation of* $\sigma(t, x)$ *at* x_0 *over* A and will be denoted by $O(x_0; \sigma, A)$.

4. Some fundamental lemmas

Lemma 4.1. *If* (G) *fails to satisfy* C_4, *then there is a bounded sequence* $\{s_n\}$ *of constants such that*

$$\limsup_{t\to t_0,\, u\to t_0} |\sigma(t) - \sigma(u)| > \limsup_{m\to\infty,\, n\to\infty} |s_m - s_n|.$$

If (G) *is real,* $\{s_n\}$ *may be taken real.**

From C_3 and a denial of C_4, it follows that there is a number θ for which

$$\limsup_{t\to t_0} \sum_{k=1}^{\infty} |a_k(t)| > \theta > 1.$$

Hence there is a sequence $\{t_n\}$ with the limit t_0 such that

$$\sum_{k=1}^{\infty} |a_k(t_n)| > \theta \qquad (n = 1, 2, 3, \cdots). \tag{4.11}$$

Let n_1 be any positive integer and choose $N_1 > n_1$ such that

$$\sum_{k=N_1+1}^{\infty} |a_k(t_{n_1})| < 1.$$

Using C_2, choose $n_2 > N_1$ such that

$$\sum_{k=1}^{N_1} |a_k(t_{n_2})| < \tfrac{1}{2};$$

then choose $N_2 > n_2$ such that

$$\sum_{k=N_2+1}^{\infty} |a_k(t_{n_2})| < \tfrac{1}{2}.$$

Proceeding in this manner, we may define a sequence $N_0 = 0 < n_1 < N_1 < n_2 < N_2 < n_3 < \cdots$ such that for $p = 1, 2, 3, \cdots$

* Compare Trans. Lemma 4.01 of which the proof was given by W. A. Hurwitz, American Journal of Mathematics, vol. 52 (1930), pp. 611–616.

$$\sum_{k=1}^{N_{p-1}} | a_k(t_{n_p}) | < \frac{1}{p} \quad \text{and} \quad \sum_{k=N_p+1}^{\infty} | a_k(t_{n_p}) | < \frac{1}{p} \cdot \tag{4.12}$$

From these inequalities and (4.11) we obtain

$$\sum_{k=N_{p-1}+1}^{N_p} | a_k(t_{n_p}) | > \theta - \frac{2}{p} \cdot \tag{4.13}$$

Now define* for $p=1, 2, 3, \cdots$

$$s_k = (-1)^{p+1} \operatorname{sgn} a_k(t_{n_p}), \quad N_{p-1} < k \leqq N_p. \tag{4.14}$$

Then s_k is real for all k if (G) is real; and $|s_k| \leqq 1$ for all k so that

$$\limsup_{m\to\infty, n\to\infty} | s_m - s_n | \leqq 2.$$

But

$$\sigma(t_{n_{p+1}}) - \sigma(t_{n_p}) = \sum_{k=1}^{\infty} a_k(t_{n_{p+1}}) s_k - \sum_{k=1}^{\infty} a_k(t_{n_p}) s_k,$$

and using (4.14) we obtain

$$\begin{aligned}\sigma(t_{n_{p+1}}) - \sigma(t_{n_p}) = {} & \sum_{k=1}^{N_p} a_k(t_{n_{p+1}}) s_k - \sum_{k=1}^{N_{p-1}} a_k(t_{n_p}) s_k + \sum_{k=N_{p+1}+1}^{\infty} a_k(t_{n_{p+1}}) s_k \\ & - \sum_{k=N_p+1}^{\infty} a_k(t_{n_p}) s_k + (-1)^p \sum_{k=N_p+1}^{N_{p+1}} | a_k(t_{n_{p+1}}) | + (-1)^p \sum_{k=N_{p-1}+1}^{N_p} | a_k(t_{n_p}) | \,.\end{aligned}$$

Using (4.12) and the fact that $|s_k| \leqq 1$ for all k, we see that the sum of the absolute values of the first four terms of the right member of the last expression is less than

$$\frac{1}{p+1} + \frac{1}{p} + \frac{1}{p+1} + \frac{1}{p} < \frac{4}{p};$$

and using (4.13) we see that the absolute value of the sum of the last two terms (which are real and of like sign) is greater than

$$\left(\theta - \frac{2}{p+1}\right) + \left(\theta - \frac{2}{p}\right) > 2\theta - \frac{4}{p} \cdot$$

Hence

$$| \sigma(t_{n_{p+1}}) - \sigma(t_{n_p}) | > 2\theta - \frac{8}{p} \cdot$$

Thus

---

* For complex z, sgn $z = |z|/z$ when $z \neq 0$ and $=0$ when $z=0$.

$$\limsup_{t\to t_0,\, u\to t_0} |\sigma(t)-\sigma(u)| \geqq \limsup_{p\to\infty} |\sigma(t_{n_{p+1}})-\sigma(t_{n_p})| \geqq \lim_{p\to\infty}\left(2\theta-\frac{8}{p}\right)$$
$$= 2\theta > 2 \geqq \limsup_{m\to\infty,\, n\to\infty} |s_m - s_n|,$$

and the lemma is proved.

The four following lemmas may be proved together.

LEMMAS 4.2, 4.3. *Let A be an infinite set and let $\{x_\alpha\}$ be a sequence of distinct points of A. In case A has a limit point x_0, $\{x_\alpha\}$ may be a sequence with the limit x_0. If (G), real or complex, fails to satisfy C_5, then there is a real sequence $\{s_n(x)\}$ bounded above (below) over A, such that*

$$\limsup_{t\to t_0,\, u\to t_0,\, i\to\infty,\, j\to\infty} |\sigma(t, x_i)-\sigma(u, x_j)| > \limsup_{m\to\infty,\, n\to\infty,\, i\to\infty,\, j\to\infty} |s_m(x_i')-s_n(x_j')| = 0$$

*where $\{x_\alpha'\}$ is any sequence of points of A.**

LEMMAS 4.4, 4.5. *Under the hypotheses of Lemmas 4.2, 4.3, there is a real sequence $\{s_n(x)\}$, bounded above (below) over A such that*

$$\limsup_{t\to t_0,\, u\to t_0,\, i\to\infty} |\sigma(t, x_i)-\sigma(u, x_i)| > \limsup_{m\to\infty,\, n\to\infty,\, i\to\infty} |s_m(x_i')-s_n(x_i')| = 0,$$

where $\{x_\alpha'\}$ is any sequence of points of A.†

From a denial of C_5 it follows that there is a value of k, say λ, and a sequence $\{t_\alpha\}$ with the limit t_0 such that

$$a_\lambda(t_\alpha) \neq 0 \qquad (\alpha = 1, 2, 3, \cdots).$$

Define the sequence $\{s_n(x)\}$ over A as follows: $s_n(x)=0$ over A for $n\neq\lambda$; $s_\lambda(x)=0$, $x\neq x_1, x_2, \cdots$; and $s_\lambda(x_\alpha)=(-1)^h/|a_\lambda(t_\alpha)|$ where $h=1(h=2)$. Evidently $s_n(x)$ is bounded above or below over A according as h is 1 or 2, and since $s_n(x)=0$ over A for $n>\lambda$,

$$\limsup_{m\to\infty,\, n\to\infty,\, i\to\infty,\, j\to\infty} |s_m(x_i')-s_n(x_j')| = \limsup_{m\to\infty,\, n\to\infty,\, i\to\infty} |s_m(x_i')-s_n(x_i')| = 0$$

where $\{x_\alpha'\}$ is any sequence of points of A. But $\sigma(t, x)=a_\lambda(t)s_\lambda(x)$ so that $|\sigma(t_\alpha, x_\alpha)|=1$ and $|\sigma(t_\beta, x_\alpha)|=|a_\lambda(t_\beta)|/|a_\lambda(t_\alpha)|$; hence

$$\limsup_{t\to t_0,\, u\to t_0,\, i\to\infty,\, j\to\infty} \Big|\sigma(t, x_i)-\sigma(u, x_j)\Big| \geqq \limsup_{t\to t_0,\, u\to t_0,\, i\to\infty} \Big|\sigma(t, x_i)-\sigma(u, x_i)\Big|$$
$$\geqq \limsup_{\alpha\to\infty,\, \beta\to\infty} \Big|\sigma(t_\alpha, x_\alpha)-\sigma(t_\beta, x_\alpha)\Big| \geqq \limsup_{\alpha\to\infty,\, \beta\to\infty} \Big||\sigma(t_\alpha, x_\alpha)|-|\sigma(t_\beta, x_\alpha)|\Big|$$
$$\geqq \limsup_{\alpha\to\infty,\, \beta\to\infty} \Big|1-|a_\lambda(t_\beta)|/|a_\lambda(t_\alpha)|\Big| = +\infty,$$

* Compare Trans. Lemmas 4.02, 4.03.

† Compare Trans. Lemmas 4.04, 4.05.

and the lemmas are proved.

LEMMAS 4.6, 4.7. *Let A be an infinite set and let $\{x_\alpha\}$ be a sequence of distinct points of A. In case A has a limit point x_0, $\{x_\alpha\}$ may be a sequence with the limit x_0. If (G), real or complex, fails to satisfy C_6, then there is a real sequence $\{s_n(x)\}$, bounded above (below) over A, such that*

$$\limsup_{t\to t_0,\, u\to t_0,\, i\to\infty} |\sigma(t, x_i) - \sigma(u, x_i)| > \limsup_{m\to\infty,\, n\to\infty,\, i\to\infty} | s_m(x_i') - s_n(x_i')| = 0$$

*where $\{x_i'\}$ is any sequence of points of A.**

From a denial of C_6 it follows that there is a sequence $\{t_\alpha\}$ with the limit t_0 such that

$$\sum_{k=1}^{\infty} a_k(t_\alpha) \neq 1 \qquad (\alpha = 1, 2, 3, \cdots).$$

Define a real function $s(x)$ over A, bounded above (below) over A, such that

$$s(x_\alpha) = (-1)^h \Big/ \left| 1 - \sum_{k=1}^{\infty} a_k(t_\alpha) \right| ;$$

and let $s_n(x) = s(x)$, $n = 1, 2, 3, \cdots$. Then $s_m(x) - s_n(x) = 0$ over A so that

$$\limsup_{m\to\infty,\, n\to\infty,\, i\to\infty} | s_m(x_i') - s_n(x_i')| = 0$$

where $\{x_i'\}$ is any sequence of points of A. But

$$\limsup_{t\to t_0,\, u\to t_0,\, i\to\infty} |\sigma(t, x_i) - \sigma(u, x_i)| = \limsup_{t\to t_0,\, u\to t_0,\, i\to\infty} |s(x_i)| \left| \sum_{k=1}^{\infty} a_k(t) - \sum_{k=1}^{\infty} a_k(u) \right|$$

$$\geqq \limsup_{u\to t_0,\, i\to\infty} |s(x_i)| \left| 1 - \sum_{k=1}^{\infty} a_k(u) \right| \geqq \limsup_{\alpha\to\infty} |s(x_\alpha)| \left| 1 - \sum_{k=1}^{\infty} a_k(t_\alpha) \right| = 1$$

and the lemmas are proved.

5. PROOFS OF TYPICAL THEOREMS INVOLVING Ω-OSCILLATIONS OVER A SET

THEOREM 5.11. *In order that (G) may be such that*

$$\Omega(\sigma, A) \leqq \Omega(\{s_n\}, A)$$

for every sequence $\{s_n(x)\}$, defined over an arbitrary set A and bounded over A for all n, C_4 is necessary and sufficient.†

The necessity of C_4 follows from Lemma 4.1 since, for the sequence of constants there defined, $\Omega(\sigma, A) > \Omega(\{s_n\}, A)$. To establish sufficiency of C_4,

* Compare Trans. Lemmas 7.01, 7.02.

† Compare Trans. Theorem 4.111.

choose M such that $|s_n(x)|<M$ over A for all n, B such that $\sum_{k=1}^\infty|a_k(t)|<B$ for all t in T and let $\sum_{k=1}^\infty|a_k(t)|=B(t)$. Let $\{x_i\}$ be any sequence of points of A, and let q be any number greater than $\Omega(\{s_n\}, A)$; then there is an index p such that

$$(5.111)\qquad |s_\mu(x_i)-s_\nu(x_j)|<q \text{ for } \mu\geqq p,\ \nu\geqq p,\ i\geqq p,\ j\geqq p.$$

We readily obtain the identity

$$(5.112)\qquad \begin{aligned}\sigma(t,x_i)-\sigma(u,x_j) &= \sum_{k=1}^{p}a_k(t)s_k(x_i)-\sum_{k=1}^{p}a_k(u)s_k(x_j)\\ &+\Big(\sum_{\mu=p+1}^\infty a_\mu(t)s_\mu(x_i)\Big)\Big(1-\sum_{\nu=p+1}^\infty a_\nu(u)\Big)-\Big(\sum_{\nu=p+1}^\infty a_\nu(u)s_\nu(x_j)\Big)\Big(1-\sum_{\mu=p+1}^\infty a_\mu(t)\Big)\\ &+\sum_{\mu=p+1}^\infty\sum_{\nu=p+1}^\infty a_\mu(t)a_\nu(u)[s_\mu(x_i)-s_\nu(x_j)].\end{aligned}$$

The absolute values of the first four terms of the right member of (5.112) are respectively less than or equal to

$$M\sum_{k=1}^p|a_k(t)|,\quad M\sum_{k=1}^p|a_k(u)|,\quad MB\Big|1-\sum_{\nu=p+1}^\infty a_\nu(u)\Big|,\quad\text{and}\quad MB\Big|1-\sum_{\mu=p+1}^\infty a_\mu(t)\Big|,$$

each of which, by C_2 and C_3, approaches 0 as t and u approach t_0. Hence

$$\begin{aligned}&\limsup_{t\to t_0,\,u\to t_0,\,i\to\infty,\,j\to\infty}|\sigma(t,x_i)-\sigma(u,x_j)|\\ &=\limsup_{t\to t_0,\,u\to t_0,\,i\to\infty,\,j\to\infty}\Big|\sum_{\mu=p+1}^\infty\sum_{\nu=p+1}^\infty a_\mu(t)a_\nu(u)[s_\mu(x_i)-s_\nu(x_j)]\Big|.\end{aligned}$$

But by (5.111)

$$\begin{aligned}\Big|\sum_{\mu=p+1}^\infty\sum_{\nu=p+1}^\infty a_\mu(t)a_\nu(u)[s_\mu(x_i)-s_\nu(x_j)]\Big| &\leqq q\sum_{\mu=p+1}^\infty\sum_{\nu=p+1}^\infty|a_\mu(t)|\,|a_\nu(u)|\\ &\leqq qB(t)B(u)\end{aligned}$$

so that

$$(5.113)\qquad \limsup_{t\to t_0,\,u\to t_0,\,i\to\infty,\,j\to\infty}|\sigma(t,x_i)-\sigma(u,x_j)|\leqq\limsup_{t\to t_0,\,u\to t_0}[qB(t)B(u)];$$

and using C_4 we have

$$(5.114)\qquad \limsup_{t\to t_0,\,u\to t_0,\,i\to\infty,\,j\to\infty}|\sigma(t,x_i)-\sigma(u,x_j)|\leqq q.$$

Since $\{x_i\}$ is any sequence of points of A, it follows from (5.114) that

$\Omega(\sigma, A) \leqq q$; and since q is any number greater than $\Omega(\{s_n\}, A)$, $\Omega(\sigma, A) \leqq \Omega(\{s_n\}, A)$ and the theorem is proved.

On noting that the proof of the preceding theorem is undisturbed by supposing (G) and the $\{s_n(x)\}$ sequences to be real, we obtain

THEOREM 5.12. *In order that a real* (G) *may be such that*

$$\Omega(\sigma, A) \leqq \Omega(\{s_n\}, A)$$

for every real sequence $\{s_n(x)\}$ *defined over an arbitrary set* A, *and bounded over* A *for all* n, C_4 *is necessary and sufficient.**

THEOREM 5.13. *In order that* (G) *may be such that*

$$\Omega(\sigma, A) \leqq \Omega(\{s_n\}, A)$$

for every sequence $\{s_n(x)\}$, *defined over an infinite set* A, C_4 *and* C_5 *are necessary and sufficient.†*

Necessity of C_4 follows from Lemma 4.1; and that of C_5 from Lemma 4.2 (or 4.3) since for the $\{s_n(x)\}$ sequences there defined, $\Omega(\{s_n\}, A) = 0$ while $\Omega(\sigma, A) > 0$. No proof of sufficiency is required if $\Omega(\{s_n\}, A) = +\infty$. If $\Omega(\{s_n\}, A)$ is finite, let q be any greater number, let $\{x_i\}$ be any sequence of points of A, and choose an index p such that (5.111) is satisfied. Then $|s_n(x_i) - s_p(x_p)| < q$ for $n \geqq p$, $i \geqq p$; hence there is a constant M such that $|s_n(x_i)| < M$ for $n \geqq p$, $i \geqq p$. Using C_5, choose a neighborhood Δ of t_0 in T such that $a_k(t) = 0$, $k = 1, 2, 3, \cdots, p$ for t in Δ. Then, referring to the identity (5.112), we see that the first two terms of the right member vanish for t and u in Δ and that, for $i \geqq p, j \geqq p$, the second and third terms approach zero as t and u approach t_0. Therefore we may write (5.113) and sufficiency follows as in Theorem 5.11. The same proof establishes the following two theorems.

THEOREM 5.14. *In order that a real* (G) *may be such that*

$$\Omega(\sigma, A) \leqq \Omega(\{s_n\}, A)$$

for every real sequence $\{s_n(x)\}$, *defined over an infinite set* A, C_4 *and* C_5 *are necessary and sufficient.‡*

THEOREM 5.15. *In order that a real* (G) *may be such that*

$$\Omega(\sigma, A) \leqq \Omega(\{s_n\}, A)$$

for every real sequence $\{s_n(x)\}$, *defined over an infinite set A and bounded above (below) over* A *for all* n, C_4 *and* C_5 *are necessary and sufficient.§*

* Compare Trans. Theorem 4.112.

† Compare Trans. Theorem 4.131.

‡ Compare Trans. Theorem 4.132.

§ Compare Trans. Theorem 4.133.

THEOREM 5.21. *In order that (G) may be such that $\Omega(\sigma, A)=0$ for every sequence $\{s_n(x)\}$, defined over an arbitrary set A and bounded over A for all n, such that $\Omega(\{s_n\}, A)=0$, no further conditions need be imposed upon a_{nk}.**

Letting $\{x_i\}$ be any sequence of points of A, and q be an arbitrarily small positive number, we can choose an index p for which (5.111) holds; and using (5.112) obtain (5.113) precisely as in Theorem 5.11. But

$$\limsup_{t\to t_0,\, u\to t_0} qB(t)B(u) \leqq qB^2;$$

hence $\Omega(\sigma, A)\leqq qB^2$. Since qB^2 is arbitrarily small, $\Omega(\sigma, A)=0$ and the theorem is proved.

THEOREM 5.22. *In order that (G) may be such that $\Omega(\sigma, A)=0$ for every sequence $\{s_n(x)\}$, defined over an infinite set A, such that $\Omega(\{s_n\}, A)=0$,* C_5 *is necessary and sufficient.*†

Necessity follows from Lemma 4.2 (or 4.3). The sufficiency proof is a modification of that of Theorem 5.13 in the same sense that the proof of Theorem 5.21 is a modification of the sufficiency proof of Theorem 5.11. The same proof establishes the two following theorems.

THEOREM 5.23. *In order that a real (G) may be such that $\Omega(\sigma, A)=0$ for every real sequence $\{s_n(x)\}$, defined over an infinite set A, such that $\Omega(\{s_n\}, A)=0$,* C_5 *is necessary and sufficient.*‡

THEOREM 5.24. *In order that a real (G) may be such that $\Omega(\sigma, A)=0$ for every real sequence $\{s_n(x)\}$, defined over an infinite set A and bounded above (below) over A for all n, such that $\Omega(\{s_n\}, A)=0$,* C_5 *is necessary and sufficient.*§

6. PROOF OF A TYPICAL THEOREM INVOLVING Ω-OSCILLATIONS AT A POINT OVER A SET

THEOREM 6.1. *In order that (G) may be such that*

$$\Omega(x_0; \sigma, A) \leqq \Omega(x_0; \{s_n\}, A)$$

for every sequence $\{s_n(x)\}$, defined over a set A such that x_0 is in A^0 and bounded over a neighborhood D of x_0 in A for all n, C_4 *is necessary and sufficient.*||

Necessity follows from Lemma 4.1. To establish sufficiency, choose M such that $|s_n(x)| < M$ over D for all n, let $\{x_i\}$ be any sequence of points of

* Compare Trans. Theorem 4.22.
† Compare Trans. Theorem 4.231.
‡ Compare Trans. Theorem 4.232.
§ Compare Trans. Theorem 4.233.
|| Compare Trans. Theorem 5.111.

A with the limit x_0, and let q be any number greater than $\Omega(x_0;\ \{s_n\},\ A)$. Then there is an index p for which (5.111) holds; let p be increased if necessary so that x_i is in D for $i \geqq p$. Using (5.112), we obtain (5.114) precisely as in Theorem 5.11; therefore $\Omega(x_0, \sigma, A) \leqq q$ and sufficiency follows.

7. Proof of a typical theorem involving O-oscillations over a set

THEOREM 7.1. *In order that* (G) *may be such that*

$$O(\sigma, A) \leqq O(\{s_n\}, A)$$

for every sequence $\{s_n(x)\}$, *defined over an infinite set* A, C_4, C_5, *and* C_6 *are necessary and sufficient.**

Necessity of C_4 follows from Lemma 4.1; and that of C_5 and C_6 from Lemmas 4.4 (or 4.5) and 4.6 (or 4.7) respectively, since, for the $\{s_n(x)\}$ sequences there defined, $O(\{s_n\},\ A)=0$ while $O(\sigma,\ A)>0$. If $O(\{s_n\},\ A) = +\infty$, no proof of sufficiency is required. If $O(\{s_n\}, A)$ is finite, let q be any greater number and let $\{x_i\}$ be any sequence of points of A; then there is an index p such that $|s_\mu(x_i)-s_\nu(x_i)| <q$ for $\mu \geqq p$, $\nu \geqq p$, $i \geqq p$. Using C_5 and C_6, choose a neighborhood Δ of t_0 in T such that $a_k(t)=0$, $k=1, 2, 3, \cdots, p$, for t in Δ and also $\sum_{k=p+1}^{\infty} a_k(t)=1$ for t in Δ. Then, considering the identity (5.112) with j replaced by i, we see that the first four terms of the right member vanish for t and u in Δ; hence we obtain (5.113) and, using C_4, (5.114) with j replaced by i. Therefore $O(\sigma, A) \leqq q$, $O(\sigma, A) \leqq O(\{s_n\}, A)$, and the theorem is proved.

8. A catalogue of theorems

A comparison of the proofs which have been given in §§5–7 with those of the corresponding theorems of Trans. will suggest to the reader all of the modifications of the proofs of the theorems of Chapters II and III of Trans. which are necessary to obtain new theorems involving the general regular transformation (G). We shall, to save space, not give formal statements of the new theorems but shall specify the changes which must be made in the theorems of Trans. to produce the new theorems.

A group of theorems, giving necessary and sufficient conditions that (G) may be such that the *Ω-oscillation over a set of a transformed function shall not exceed the Ω-oscillation over the set of a given sequence*, is obtained by replacing (T) by (G) and $\Omega(\{\sigma_n\}, A)$ by $\Omega(\sigma, A)$ in Trans. Theorems 4.111–4.133.

A group of theorems, giving necessary and sufficient conditions that (G) may be such that the *Ω-oscillation over a set of a transformed function shall be zero whenever the Ω-oscillation over the set of a given sequence is zero*, is obtained

* Compare Trans. Theorem 7.131.

by replacing (T) by (G) and $\Omega(\{\sigma_n\}, A)$ by $\Omega(\sigma, A)$ in Trans. Theorems 4.21–4.233.

A group of theorems, giving necessary and sufficient conditions that (G) may be such that *at a single point or limit point of a set, the Ω-oscillation over the set of a transformed function shall not exceed the Ω-oscillation over the set of a given sequence*, is obtained by replacing (T) by (G) and $\Omega(x_0; \{\sigma_n\}, A)$ by $\Omega(x_0, \sigma, A)$ in Trans. Theorems 5.111–5.133.

A group of theorems, giving necessary and sufficient conditions that (G) may be such that *at a single point or limit point of a set, the Ω-oscillation over the set of a transformed function shall be zero whenever the Ω-oscillation over the set of a given sequence is zero*, is obtained by replacing (T) by (G) and $\Omega(x_0; \{\sigma_n\}, A)$ by $\Omega(x_0, \sigma, A)$ in Trans. Theorems 5.21–5.233.

A group of theorems, giving necessary and sufficient conditions that (G) may be such that *at each point and limit point of a set, the Ω-oscillation over the set of a transformed function shall not exceed the Ω-oscillation over the set of a given sequence*, is obtained by replacing (T) by (G) and $\Omega(x; \{\sigma_n\}, A)$ by $\Omega(x, \sigma, A)$ in Trans. Theorems 6.111–6.133.

A group of theorems, giving necessary and sufficient conditions that (G) may be such that *the Ω-oscillation of a transformed function shall be zero at each point and limit point of a set whenever the Ω-oscillation of a given sequence is zero at each point and limit point of the set*, is obtained by replacing (T) by (G) and $\Omega(x; \{\sigma_n\}, A)$ by $\Omega(x, \sigma, A)$ in Trans. Theorems 6.21–6.233.

A group of theorems, giving necessary and sufficient conditions that (G) may be such that the *O-oscillation over a set of a transformed function shall not exceed the O-oscillation over the set of a given sequence*, is obtained by replacing (T) by (G) and $O(\{\sigma_n\}, A)$ by $O(\sigma, A)$ in Trans. Theorems 7.111–7.133.

A group of theorems, giving necessary and sufficient conditions that (G) may be such that the *O-oscillation over a set of a transformed function shall be zero whenever the O-oscillation over the set of a given sequence is zero*, is obtained by replacing (T) by (G) and $O(\{\sigma_n\}, A)$ by $O(\sigma, A)$ in Trans. Theorems 7.21–7.233.

A group of theorems, giving necessary and sufficient conditions that (G) may be such that *at a single point or limit point of a set, the O-oscillation over the set of a transformed function shall not exceed the O-oscillation over the set of a given sequence*, is obtained by replacing (T) by (G) and $O(x_0; \{\sigma_n\}, A)$ by $O(x_0, \sigma, A)$ in Trans. Theorems 8.111–8.133.

A group of theorems, giving necessary and sufficient conditions that (G) may be such that *at a single point or limit point of a set, the O-oscillation over the set of a transformed function shall be zero whenever the O-oscillation over the

set of a given sequence is zero, is obtained by replacing (T) by (G) and $O(x_0; \{\sigma_n\}, A)$ by $O(x_0, \sigma, A)$ in Trans. Theorems 8.21–8.233.

A group of theorems, giving necessary and sufficient conditions that (G) may be such that *at each point and limit point of a set, the O-oscillation over the set of a transformed function shall not exceed the O-oscillation over the set of a given sequence,* is obtained by replacing (T) by (G) and $O(x; \{\sigma_n\}, A)$ by $O(x, \sigma, A)$ in Trans. Theorems 9.111–9.133.

A group of theorems, giving necessary and sufficient conditions that (G) may be such that the *O-oscillation of a transformed function shall be zero at each point and limit point of a set whenever the O-oscillation of a given sequence is zero at each point and limit point of the set,* is obtained by replacing (T) by (G) and $O(x; \{\sigma_n\}, A)$ by $O(x, \sigma, A)$ in Trans. Theorems 9.21–9.233.

9. Application to transformations with square matrices

A well known family of regular transformations of the form (G) is obtained by taking T to be the set of positive integers and t_0 to be the symbolic limit point $+\infty$. Then $a_k(n)$ and $\sigma(n, x)$ may be written a_{nk} and $\sigma_n(x)$, and (G) becomes a transformation of the form

$$(S) \qquad \sigma_n(x) = \sum_{k=1}^{\infty} a_{nk} s_k(x)$$

which assigns to the sequence $\{s_n(x)\}$ the value $\lim_{n\to\infty}\sigma_n(x) = \sigma(x)$ when the limit exists. In this case what we have called the *transformed function* becomes a *transformed sequence* and we see on referring to the definitions of §3 that the oscillations $\Omega(\sigma, A)$, $\Omega(x_0, \sigma, A)$, $O(\sigma, A)$, and $O(x_0, \sigma, A)$ become respectively $\Omega(\{\sigma_n\}, A)$, $\Omega(x_0; \{\sigma_n\}, A)$, $O(\{\sigma_n\}, A)$, and $O(x_0; \{\sigma_n\}, A)$. Hence for regular transformations (S), the statements of the theorems of §8 become practically identical with the statements of the corresponding theorems of Chapters II and III of Trans. In fact, we may obtain, from each theorem of Chapters II and III of Trans., a theorem involving (S) by replacing (T) by (S) and interpreting C_4, C_5, and C_6 to be the conditions obtained by replacing t by n, $a_k(t)$ by a_{nk}, and $\lim_{t\to t_0}$ by $\lim_{n\to\infty}$ in the conditions of §2.

10. Application of the Euler-Abel power series method

The Euler-Abel transformation assigns to a series $u_1+u_2+\cdots$ the value

$$\lim_{t\to 1} \sigma(t) = \lim_{t\to 1} (u_1 + u_2 t + u_3 t^2 + \cdots)$$

when the limit exists, and to a sequence $\{s_n\}$ the value $\lim_{t\to 1}\sigma(t)$, where

$$(E) \qquad \sigma(t) = \sum_{k=1}^{\infty} t^{k-1}(1-t)s_k,$$

when the limit exists. That the transformation (E) is regular when the set T over which t approaches 1 is the real set $-1<t<1$ was first shown by Abel. It has been pointed out by Hurwitz* that a necessary and sufficient condition that (E) be regular is that the set T be a set of a region R interior to the circle $|t|=1$ and between some pair of chords through $t=1$.

For (E) we have, in the notation of (G), $a_k(t)=t^{k-1}(1-t)$; and we see at once that *(E) fails to satisfy* C_5 *when T is any set of R, and that (E) satisfies* C_6 *when T is any set of R.*

We find further: *In order that a regular (E) may satisfy* C_4, *it is necessary and sufficient that T be such that for each sequence $t_n=\xi_n+i\eta_n$ of points of T with the limit* 1, *we have* $\lim_{n\to\infty}\eta_n/(1-\xi_n)=0$. In particular, *if T is the set* $-1<t<1$, *then (E) satisfies* C_4.

11. Application of the Borel-Sannia transformations

For each integer r (positive, zero, and negative), the Borel-Sannia transformation of order r† is given by

$$(B_r) \qquad \sigma^{(r)}(t) = \sum_{k=1}^{\infty} e^{-t}\frac{t^{k-r}}{(k-r)!}s_k,\ddagger$$

and it assigns to a sequence $\{s_n\}$ the value $\lim_{R(t)\to+\infty}\sigma^{(r)}(t)$ when this limit exists. A necessary and sufficient condition that (B_r) be regular is that for all points $t=\xi+i\eta$ of T with sufficiently great positive abscissas, η^2/ξ shall be bounded.§

For (B_r), $a_k^{(r)}(t)=e^{-t}t^{k-r}/(k-r)!$ and we see that *any regular (B_r) fails to satisfy* C_5. Considering C_6, we find that *any regular (B_r) of order $r\geqq 1$ satisfies* C_6 *and any regular (B_r) of order $r\leqq 0$ fails to satisfy* C_6.

For each integer r we find the following: *In order that a regular (B_r) may satisfy* C_4, *it is necessary and sufficient that T be such that for each sequence $t_n=\xi_n+i\eta_n$ of points of T such that $\xi_n\to+\infty$, we have* $\lim \eta_n^2/\xi_n=0$. In particular, *if T is the set of positive real numbers, then (B_r) satisfies* C_4.

* Bulletin of the American Mathematical Society, vol. 28 (1922), p. 24.

† G. Sannia, Rendiconti del Circolo Matematico di Palermo, vol. 42 (1917), pp. 303–322. Note that B_1 is the Borel mean or the Borel exponential transformation, and that B_0 is the Borel integral transformation.

‡ Here $1/(k-r)!=0$ where $k-r<0$. This convention is justified by the behavior of the reciprocal of the gamma function.

§ W. A. Hurwitz, Bulletin of the American Mathematical Society, loc. cit., p. 25.

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THE DISCRIMINANT MATRIX OF A SEMI-SIMPLE ALGEBRA*

BY

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1. **Introduction.** Let $\mathfrak{A}$ be a linear associative algebra with basis $e_1, e_2, \cdots, e_n$ over a field $\mathfrak{F}$ in which n has a reciprocal, and let the constants of multiplication be denoted by c_{ijk}. Let $t_1(x)$ and $t_2(x)$ denote, respectively, the first and second traces of x. In a recent paper† the writer called the symmetric matrices

$$T_1 = \|t_1(e_re_s)\| = \|\sum_{i,j} c_{rij}c_{sji}\|, \quad T_2 = \|t_2(e_re_s)\| = \|\sum_{i,j} c_{irj}c_{jsi}\|$$

the *first* and *second discriminant matrices* of $\mathfrak{A}$ relative to the given basis. The relation of these matrices to the discriminant of an algebraic field was shown, so that the names were justified. It was also shown that under a linear transformation of basis,

$$e_i' = \sum a_{ij}e_j, \quad A = (a_{rs}), \quad |A| = a \neq 0,$$

the matrices are transformed cograediently, i.e.,

$$T_1' = AT_1\bar{A}, \quad T_2' = AT_2\bar{A} \tag{1}$$

where $\bar{A}$ denotes the transpose of A. Thus the ranks of T_1 and T_2 are invariant, and if $\mathfrak{F}$ is a real field, so are the signatures. Other elementary properties of these matrices were discussed and their occurrence in the literature noted.

In the first part of the present paper the behavior of T_1 under transformation of basis is used to establish the existence for every algebra with a principal unit of a normal basis of simple form. This normal basis has a cyclic property generalizing that of the familiar basis $1, i, j, k$ for quaternions. By means of this normal basis several new theorems in the theory of semi-simple algebras are obtained, e.g., the fact that T_1 and T_2 are identical, and that the first and second characteristic functions are identical.

In the second part of the paper (§4 et seq.) the discriminant matrices of a direct sum, direct product and complete matric algebra are investigated.

2. **The normal basis.** Let us now assume that e_1 is a principal unit so that

* Presented to the Society, April 18, 1930; received by the editors in November, 1930.

† Annals of Mathematics, (2), vol. 32, p. 60.

$$c_{i1j} = c_{1ij} = \delta_{ij}.$$

We denote by τ_{rs} the elements of T_1. By means of the associativity conditions we may write τ_{rs} in the alternative forms*

$$\tau_{rs} = \sum_{i,k} c_{rik}c_{ski} = \sum_{h,k} c_{srh}c_{hkk}.$$

Then if we set $\sum_k c_{hkk} = d_h$, we may write $\tau_{rs} = \sum_h c_{srh}d_h$. In particular

$$\tau_{r1} = \sum_{i,k} c_{rik}c_{1ki} = \sum_{i,k} c_{rik}\delta_{ki} = \sum_i c_{rii} = d_r,$$

$$\tau_{11} = \sum_i c_{1ii} = n.$$

Thus T_1 is of the form

$$\begin{Vmatrix} n & d_2 & d_3 & \cdots & d_n \\ d_2 & \tau_{22} & \tau_{23} & \cdots & \tau_{2n} \\ d_3 & \tau_{32} & \tau_{33} & \cdots & \tau_{3n} \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ d_n & \tau_{n2} & \tau_{n3} & \cdots & \tau_{nn} \end{Vmatrix}.$$

By means of a transformation of matrix

$$A = \begin{Vmatrix} 1 & 0 & 0 & \cdots & 0 \\ -d_2/n & 1 & 0 & \cdots & 0 \\ -d_3/n & 0 & 1 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ -d_n/n & 0 & 0 & \cdots & 1 \end{Vmatrix}$$

we find that

$$T_1' = AT_1\bar{A} = \begin{Vmatrix} n & 0 & 0 & \cdots & 0 \\ 0 & \tau_{22}' & \tau_{23}' & \cdots & \tau_{2n}' \\ 0 & \tau_{32}' & \tau_{33}' & \cdots & \tau_{3n}' \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & \tau_{n2}' & \tau_{n3}' & \cdots & \tau_{nn}' \end{Vmatrix}.$$

By the above transformation $e_1' = e_1$ is the principal unit. It is now obvious that the symmetric matrix T_1' can be reduced to a diagonal matrix by transformations in $\mathfrak{F}$ which leave the principal unit e_1 invariant. We have

* MacDuffee, loc. cit., p. 62 (2).

THEOREM 1. *If $\mathfrak{A}$ has a principal unit, a basis can be so chosen that the principal unit is e_1 and*

$$T_1 = \left\| \begin{matrix} g_1 & 0 & 0 \cdots 0 \\ 0 & g_2 & 0 \cdots 0 \\ 0 & 0 & g_3 \cdots 0 \\ \cdots & \cdots & \cdots \\ 0 & 0 & 0 \cdots g_n \end{matrix} \right\| \tag{2}$$

where the g's are in $\mathfrak{F}$ and $g_1 = n$.

Such a basis will be called a *normal basis* for the algebra.

We have seen that when e_1 is a principal unit, $\tau_{r1} = d_r$, so for a normal basis

$$d_1 = n, \; d_2 = d_3 = \cdots = d_n = 0.$$

Hence* $\tau_{rs} = nc_{sr1}$ so that

$$c_{sr1} = \frac{g_r}{n}\delta_{rs}.$$

In the associativity conditions

$$\sum_h c_{rih}c_{phs} = \sum_h c_{prh}c_{his}$$

set $s=1$ and use the above relation. We obtain

$$\frac{1}{n} \sum_h c_{rih}\delta_{ph}g_p = \frac{1}{n} \sum_h c_{prh}\delta_{hi}g_i,$$

$$g_p c_{rip} = g_i c_{pri} \qquad (i, p, r = 1, 2, \cdots, n). \tag{3}$$

THEOREM 2. *When a normal basis is taken for an algebra $\mathfrak{A}$ with a principal unit, the constants of multiplication are in the relation* (3), *the g's being given by* (2).

It is interesting to note that for quaternions with the familiar basis 1, i, j, k where $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$, the basis is in normal form with $g_1 = 4$, $g_2 = g_3 = g_4 = -4$. The cyclic property of quaternion multiplication is generalized in the cyclic advance of the subscripts in (3).

3. **Some applications.** We shall now suppose that $\mathfrak{A}$ is semi-simple with a normal basis $e_1, e_2, \cdots, e_n$. Then each g_i in (2) is different from zero.

* Cf. these Transactions, vol. 31 (1929), p. 81, Lemma 7.

Theorem 3. *In every semi-simple algebra, for every basis, the discriminant matrices T_1 and T_2 are identical.*

By definition

$$T_1 = \left\| \sum_{h,k} c_{rkh} c_{shk} \right\|.$$

By means of (3),

$$T_1 = \left\| \sum_{h,k} \frac{g_k}{g_h} c_{hrk} \frac{g_h}{g_k} c_{ksh} \right\|$$
$$= \left\| \sum_{h,k} c_{hrk} c_{ksh} \right\| = T_2.$$

Thus for a normal basis T_1 and T_2 are identical. Since they are transformed cogrediently (1), they are identical for all bases.

Henceforth we shall speak of *the* discriminant matrix of a semi-simple algebra, and denote it by T.

Theorem 4. *For every semi-simple algebra and for all bases,*

$$S(x) = TR(x)T^{-1}, \tag{4}$$

where $R(x)$ and $S(x)$ are, respectively, the first and second matrices of x, and T is the discriminant matrix.

If $\mathfrak{A}$ has a normal basis, (3) gives

$$c_{ris} g_s = g_i c_{sri} = g_r c_{isr}.$$

Hence

$$S(e_i)T = TR(e_i).$$

Multiplying by x_i and summing for i gives

$$S(x)T = TR(x)$$

where

$$x = x_1 e_1 + x_2 e_2 + \cdots + x_n e_n$$

is the general number of the algebra. Thus (4) is established for a normal basis.

Under transformation of basis

$$S(x) = A^{-1} S'(x') A, \quad R(x) = \bar{A} R'(x') \bar{A}^{-1},^*$$

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* Dickson, *Algebren und ihre Zahlentheorie*, p. 38. The $S(x)$ is the transpose of Dickson's S_x.

and by (1), $T=A^{-1}T'\bar{A}^{-1}$. Hence from (4)

$$A^{-1}S'(x')A = A^{-1}T'\bar{A}^{-1}\bar{A}R'(x')\bar{A}^{-1}\bar{A}T'^{-1}A = A^{-1}T'R'(x')T'^{-1}A,$$

so that

$$S'(x') = T'R'(x')T'^{-1}.$$

Thus (4) holds for every basis.

THEOREM 5. *For every semi-simple algebra, and for all bases, the first and second characteristic functions are identical.*

By definition* the first and second characteristic functions of the general number are

$$C_1(\omega) = | R(x) - \omega I |,$$
$$C_2(\omega) = | S(x) - \omega I |.$$

The theorem follows immediately from (4).

4. **The discriminant matrix of a direct sum.** An algebra $\mathfrak{A}$ is the direct sum $\mathfrak{B}\oplus\mathfrak{C}$ of two algebras $\mathfrak{B}$ and $\mathfrak{C}$ if $\mathfrak{A}=\mathfrak{B}+\mathfrak{C}$, $\mathfrak{B}\mathfrak{C}=\mathfrak{C}\mathfrak{B}=0$, $\mathfrak{B}\wedge\mathfrak{C}=0$. It is known that every semi-simple algebra is the direct sum of simple algebras, and the components are unique except for order.

The direct sum of two matrices M_1 and M_2 is the matrix

$$\left\| \begin{matrix} M_1 & O \\ O & M_2 \end{matrix} \right\|$$

where the O's stand for rectangular blocks of 0's. Let $T(\mathfrak{A})$ denote the discriminant matrix of $\mathfrak{A}$.

THEOREM 6. *If $\mathfrak{A}=\mathfrak{B}\oplus\mathfrak{C}$, a basis for $\mathfrak{A}$ may be so chosen that*

$$T(\mathfrak{A}) = T(\mathfrak{B}) \oplus T(\mathfrak{C}).$$

We choose the basis numbers $e_1, e_2, \cdots, e_n$ of $\mathfrak{A}$ so that $e_1, e_2, \cdots, e_h$ form a basis for $\mathfrak{B}$ and $e_{h+1}, \cdots, e_n$ a basis for $\mathfrak{C}$. Then $e_re_s=0$ unless $r\leqq h$, $s\leqq h$ or $r>h$, $s>h$. Since $T(\mathfrak{A})=\|t(e_re_s)\|$, obviously

$$T(\mathfrak{A}) = \left\| \begin{matrix} T(\mathfrak{B}) & 0 \\ 0 & T(\mathfrak{C}) \end{matrix} \right\|.$$

COROLLARY 6. *If $\mathfrak{F}$ is a real field, and $\mathfrak{A}=\mathfrak{B}\oplus\mathfrak{C}$, then the signature of $T(\mathfrak{A})$ is the sum of the signatures of $T(\mathfrak{B})$ and $T(\mathfrak{C})$.*

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* Dickson, ibid., p. 37.

5. **The discriminant matrix of a direct product.** An algebra $\mathfrak{A}$ is the direct product $\mathfrak{B}\times\mathfrak{C}$ of two algebras $\mathfrak{B}$ and $\mathfrak{C}$ if $\mathfrak{A}=\mathfrak{B}\mathfrak{C}$, the order of $\mathfrak{A}$ is the product of the orders of $\mathfrak{B}$ and $\mathfrak{C}$, and if every number of $\mathfrak{B}$ is commutative with every number of $\mathfrak{C}$.

If $P=(p_{rs})$ and $Q=(q_{rs})$ are two square matrices, the direct product* $P\times Q$ is defined to be the matrix

$$\begin{Vmatrix} Pq_{11} & Pq_{12} \cdots & Pq_{1n} \\ Pq_{21} & Pq_{22} \cdots & Pq_{2n} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ Pq_{n1} & Pq_{n2} \cdots & Pq_{nn} \end{Vmatrix}$$

where Pq_{ij} stands for the block of elements

$$\begin{matrix} p_{11}q_{ij} & p_{12}q_{ij} \cdots & p_{1n}q_{ij} \\ p_{21}q_{ij} & p_{22}q_{ij} \cdots & p_{2n}q_{ij} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ p_{n1}q_{ij} & p_{n2}q_{ij} \cdots & p_{nn}q_{ij}. \end{matrix}$$

Theorem 7. *If $\mathfrak{A}=\mathfrak{B}\times\mathfrak{C}$, a basis for $\mathfrak{A}$ can be so chosen that*

$$T(\mathfrak{A}) = T(\mathfrak{B}) \times T(\mathfrak{C}).$$

Suppose that $\mathfrak{B}$ has the basis $e_1, e_2, \cdots, e_g$ and the constants of multiplication b_{ijk}, while $\mathfrak{C}$ has the basis $f_1, f_2, \cdots, f_h$ and the constants of multiplication c_{ijk}. Then the numbers $e_{i_1}f_{i_2}$ form a basis for $\mathfrak{A}$, and

$$\begin{aligned} e_{i_1}f_{i_2}e_{j_1}f_{j_2} &= e_{i_1}e_{j_1}f_{i_2}f_{j_2} \\ &= \sum_{\substack{k_1=1,\cdots,g \\ k_2=1,\cdots,h}} b_{i_1j_1k_1}c_{i_2j_2k_2}e_{k_1}f_{k_2}. \end{aligned} \tag{5}$$

Let us denote the basis numbers $e_{i_1}f_{i_2}$ of $\mathfrak{A}$ by the symbols E_i and order them so that

$$i - 1 = g(i_2 - 1) + i_1 - 1, \quad 0 \leqq i_1 - 1 < g. \tag{6}$$

Evidently i_1 and i_2 determine i uniquely and conversely. Then (5) may be written

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* A. Hurwitz, Mathematische Annalen, vol. 45, p. 381.
C. Stephanos, Journal de Mathématiques, (5), vol. 6, p. 73.

$$E_iE_j = \sum_{k=1}^{n} D_{ijk}E_k$$

where $D_{ijk} = b_{i_1j_1k_1}c_{i_2j_2k_2}$, the j_1, j_2, k_1, k_2 being determined from j and k by relations similar to (6).

It now follows that

$$\begin{aligned} T(\mathfrak{A}) &= \left\| \sum_{p,q=1}^{gh} D_{rpq}D_{sqp} \right\| \\ &= \left\| \sum_{p_1,q_1,p_2,q_2} b_{r_1p_1q_1}c_{r_2p_2q_2}b_{s_1q_1p_1}c_{s_2q_2p_2} \right\| \\ &= \left\| \sum_{p_1,q_1} b_{r_1p_1q_1}b_{s_1q_1p_1} \sum_{p_2,q_2} c_{r_2p_2q_2}c_{s_2q_2p_2} \right\| \\ &= T(\mathfrak{B}) \times T(\mathfrak{C}). \end{aligned}$$

COROLLARY 7. *If $\mathfrak{F}$ is a real field, and $\mathfrak{A} = \mathfrak{B} \times \mathfrak{C}$, the signature of $T(\mathfrak{A})$ is equal to the product of the signatures of $T(\mathfrak{B})$ and $T(\mathfrak{C})$.*

As in Theorem 1 bases for $\mathfrak{B}$ and $\mathfrak{C}$ may be so chosen that $T(\mathfrak{B})$ and $T(\mathfrak{C})$ are diagonal matrices. It is evident that $T(\mathfrak{A})$ is now a diagonal matrix whose diagonal elements are the products of the diagonal elements of $T(\mathfrak{B})$ by the diagonal elements of $T(\mathfrak{C})$. Let p_1, p_2 and p denote, respectively, the number of positive terms in the main diagonals of $T(\mathfrak{B})$, $T(\mathfrak{C})$and $T(\mathfrak{A})$. Similarly let n_1, n_2 and n denote the number of negative terms. Then evidently

$$p = p_1p_2 + n_1n_2, \quad n = p_1n_2 + p_2n_1,$$

$$p - n = (p_1 - n_1)(p_2 - n_2).$$

6. **The discriminant matrix of a complete matric algebra.** If $\mathfrak{A}$ is a complete matric algebra of order n^2, basal numbers e_{ij} can be so chosen that

$$e_{ij}e_{kl} = \delta_{jk}e_{il}$$

where δ_{jk} is Kronecker's delta. We shall arrange the e_{ij} in the order e_{11}, e_{22}, $\cdots$, e_{nn} followed by e_{ij}, e_{ji}, $j > i$; $i = 1, 2, \cdots$; $j = i+1, \cdots$. If we denote $T(\mathfrak{A})$ by (τ_{rs}), then

$$\begin{aligned} \tau_{rs} &= t(e_{r_1r_2}e_{s_1s_2}) = t(\delta_{r_2s_1}e_{r_1s_2}) = n\delta_{r_2s_1}t(e_{r_1s_2}) \\ &= n\delta_{r_2s_1}\delta_{r_1s_2}. \end{aligned}$$

Thus $T(\mathfrak{A})$ is in the form

$$(7) \qquad \left\| \begin{array}{cccccccc} n & 0 \cdots 0 & & & & & & \\ 0 & n \cdots 0 & & & & & & \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot & & & & & \\ 0 & 0 \cdots n & & & & & & \\ & & & 0 & n & & & \\ & & & n & 0 & & & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & & & & 0 & n \\ & & & & & & n & 0 \end{array} \right\| .$$

Theorem 8. *If $\mathfrak{A}$ is a complete matric algebra, basal numbers may be so chosen that $T(\mathfrak{A})$ is of the form* (7).

Corollary 8. *If $\mathfrak{F}$ is a real field and $\mathfrak{A}$ is a complete matric algebra of order n^2, the signature of $T(\mathfrak{A})$ is n.*

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THE INTEGRABILITY OF A SEQUENCE OF FUNCTIONS*

BY

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1. **Introduction.** Let $f=f_1, f_2, \cdots$ be a sequence of functions summable on the measurable set E, and convergent on E to the summable function F. If

$$\lim_{n\to\infty} \int_E f_n dx = \int_E F dx$$

the sequence is said to be *integrable*. If the above equality holds when the set E is replaced by any measurable part of E the sequence is said to be *completely integrable*. These questions of integrability and complete integrability have received considerable attention from various writers.† It has been shown by Vitali‡ that the equi-convergence§ of the sequence of integrals is both necessary and sufficient for complete integrability. It would then follow that this condition is sufficient for integrability. One can, however, easily construct examples which show that it is not necessary. The chief aim of the present paper is to determine conditions which are both necessary and sufficient for the integrability of the sequence f. This is accomplished by methods which yield, as special cases, some of the results already obtained by Vitali and de la Vallée Poussin.

2. **Definitions and preliminary results.** In what follows, without again making mention of it, we shall use e to denote any measurable sub-set of E. We further define $S(l, \eta)$, $\eta>0$ and arbitrary, to be the part of E for which $|F-f_n|<\eta$ $(n\geqq l)$, and $C(l, \eta)$ its complement on E; $e(l, n)_+$ and $e(l, n)_-$ the parts of $C(l, \eta)$ for which $f_n\geqq 0, f_n<0$ respectively. It is easily verified that these sets are measurable, and that $mC(l, \eta)$ tends to zero as l becomes infinite. Finally we use $g=g_1, g_2, \cdots$ to denote a sub-sequence $f_{n_1}, f_{n_2}, \cdots$ of f.

We shall have occasion to use

* Presented to the Society, December 27, 1929; received by the editors April 26, 1930.

† Osgood, American Journal of Mathematics, vol. 19, p. 182; Hobson, Proceedings of the London Mathematical Society, (1), vol. 35, p. 254; W. H. Young, Proceedings of the London Mathematical Society, (2), vol. 1, p. 89; Arzelà, Memorie, Reale Accademia delle Scienze, Bologna, (5), vol. 8, p. 703; Vitali, Rendiconti del Circolo Matematico di Palermo, vol. 23, p. 137; de la Vallée Poussin, these Transactions, vol. 16, p. 444 ff.

‡ Loc. cit.

§ Hobson, *Real Variable*, second edition, vol. II, p. 208.

A. *Corresponding to* $\epsilon>0$ *there exists* $\delta>0$ *such that if* $me<\delta$ *then*

$$\left|\int_{eS(l,\eta)} f_n dx\right| < \epsilon \qquad (n \geqq l).$$

This readily follows from the summability of F and the fact that, on $S(l, \eta)$, $f_n=F+t_n$, where $|t_n|<\eta$ for $n\geqq l$.

We now prove

I. *It is necessary and sufficient for the integrability of* f *that for every* $\eta>0$

$$\lim_{l\to\infty}\int_{C(l,\eta)} f_n dx = 0 \qquad (n \geqq l).$$

To show that this is sufficient we write

$$\left|\int_E f_n dx - \int_E F dx\right| \leqq \int_{S(l,\eta)} |f_n - F|\, dx + \int_{C(l,\eta)} |f_n - F|\, dx.$$

We thus see that for l sufficiently large and for η sufficiently small the left-hand side of this inequality is arbitrarily small for all $n\geqq l$.

The condition is also necessary. Obviously

$$\int_{C(l,\eta)} f_n dx = \int_{C(l,\eta)} F dx + \int_E (f_n - F) dx - \int_{S(l,\eta)} (f_n - F) dx.$$

Then for any η and $n\geqq l$, we have

$$\left|\int_{C(l,\eta)} f_n dx\right| \leqq \left|\int_{C(l,\eta)} F dx\right| + \left|\int_E (f_n - F) dx\right| + \eta\cdot mE. \tag{1}$$

Now if $0<\eta'<\eta$, then $C(l, \eta')$ contains $C(l, \eta)$ and $C(l, \eta')=C(l, \eta)+C(l, \eta')\cdot S(l, \eta)$ so that

$$\begin{aligned}\left|\int_{C(l,\eta)} f_n dx\right| &\leqq \left|\int_{C(l,\eta')} f_n dx\right| + \left|\int_{C(l,\eta')S(l,\eta)} f_n dx\right| \\ &\leqq \left|\int_{C(l,\eta')} f_n dx\right| + \eta\cdot mC(l,\eta') + \int_{C(l,\eta')} |F|\, dx \\ &\leqq 2\int_{C(l,\eta')} |F|\, dx + \left|\int_E (f_n - F) dx\right| + \eta'\cdot mE + \eta\cdot mC(l,\eta'),\end{aligned}$$

the last inequality following from (1) by replacing η by η'. The necessity of the condition follows at once from this inequality, since it is valid for all $\eta'<\eta$ and since $mC(l, \eta')$ tends to zero as l becomes infinite.

3. The integral of f_n over e bounded in e and n. We first prove

B. *If the sequence of functions $f=f_1, f_2, \cdots$ is summable on the measurable set E, converges to the function F which is measurable on E, and is such that the integral of f_n over e is bounded in e and n, then F is summable on E.*

Let M be the least upper bound of

$$\int_e |f_n|\,dx$$

for all e of E and all n. Further let E_N be the part of E for which

$$-N \leqq F \leqq N.$$

Then F will be summable on E if

$$\int_{E_N} |F|\,dx$$

is bounded as to N. Now

$$\begin{aligned}\int_{E_N} |F|\,dx &= \int_{E_N S(l,\eta)} |F|\,dx + \int_{E_N C(l,\eta)} |F|\,dx \\ &\leqq \int_{E_N S(l,\eta)} (|F-f_n| + |f_n|)dx + \int_{E_N C(l,\eta)} |F|\,dx \\ &\leqq \eta\cdot mE + M + N\cdot mC(l,\eta).\end{aligned}$$

Since $mC(l, \eta)$ approaches zero as l approaches infinity independently of N, we have at once the desired boundedness.

If the sequence of integrals is equi-convergent then, obviously, the integral of f_n over e is bounded in e and n. We thus get

B′*. *If the sequence of summable functions f converges on E to a measurable function F, and if the sequence of integrals is equi-convergent, then F is summable.*

If the sequence of integrals is not equi-convergent, by making use of A it is easy to show that there exists $\lambda > 0$ and a sequence of values $n_1, n_2, \cdots$ of n such that either

$$\int_{e(l_i,n_i)+} f_{n_i}dx > \lambda \text{ or } \int_{e(l_i,n_i)-} f_{n_i}dx < -\lambda \quad (l_i = n_i,\ i = 1, 2, \cdots).$$

Assuming that the first holds we can select from the sequence n_i a subsequence n_j such that for ϵ arbitrary

* de la Vallée Poussin, loc. cit., Theorem I.

$$\int_{e(l_j, n_j)+} f_{n_j} dx > \lambda \text{ and } \int_{C(l_{j+1}, \eta)} |f_{n_j}|\, dx < \epsilon.$$

Set $e_j = \{C(l_j, \eta) - C(l_{j+1}, \eta)\} e(l_j, n_j)_+$. Then

$$\int_{e_j} f_{n_j} dx > \lambda - \epsilon.$$

Also $e_1 + e_2 + \cdots + e_{j-1}$ is contained in $S(l_j, \eta)$ and in $C(l_1, \eta)$, and $e_{j+1} + \cdots$ is contained in $C(l_{j+1}, \eta)$. Hence if l_1 is taken sufficiently large and $e = \sum e_j$, we have both

$$\int_e |F|\, dx < \epsilon, \quad \text{and} \int_e f_{n_j} dx > \lambda - 2\epsilon \qquad (j = 1, 2, \cdots),$$

which shows that the sequence is not completely integrable. In case f is such that the sequence of integrals is equi-convergent, then, if E' is any measurable part of E, complete integrability readily follows from

$$\int_{E'} |F - f_n|\, dx = \int_{E'S(l,\eta)} |F - f_n|\, dx + \int_{E'C(l,\eta)} |F - f_n|\, dx \qquad (n \geqq l).$$

We thus have

C*. *If a sequence of summable functions converges on E to a summable function F, it is then necessary and sufficient for complete integrability that the sequence of integrals be equi-convergent.*

Let $g = g_1, g_2, \cdots$ be any sub-sequence of f, and δ any positive number. Let $U(g, n, \delta)$ and $L(g, n, \delta)$ be the least upper bound and greatest lower bound respectively of

$$\int_e g_i dx \qquad (i = 1, 2, \cdots, n)$$

for every e with $me < \delta$. It is evident that, for a given δ, $U(g, n, \delta)$ does not decrease as n increases, and consequently, since the integral of f_n over e is bounded in e and n, this function converges to a limit $U(g, \delta)$. But $U(g, \delta) \geqq 0$, and obviously does not increase as δ decreases. Hence $U(g, \delta)$ converges to a limit $U(g)$. Dealing in a similar manner with the function $L(g, n, \delta)$ we arrive at the corresponding limit $L(g)$. It is now possible to prove

II. *If the sequence of summable functions f converges on E to a measurable function F, and if the integral of f_n over e is bounded in n and e, it is then neces-*

* See also de la Vallée Poussin, loc. cit., Theorem IV.

sary and sufficient for the integrability of the sequence that $U(g)+L(g)=0$ *for every* g.

That F is summable follows from B. To show that the conditions are necessary, let f be integrable and assume that $U(g)+L(g)\neq 0$ for at least one g. For the sake of definiteness let

$$U(g) + L(g) = \lambda > 0. \tag{1}$$

Making use of A it is possible to fix δ so that both

$$U(g, \delta) + L(g, \delta) > \frac{\lambda}{2}, \tag{2}$$

and if $me<\delta$,

$$\left| \int_{eS(l,\eta)} g_n dx \right| < \frac{\lambda}{4} \qquad (n \geqq l). \tag{3}$$

Again, since the sequence is integrable, we can use I to fix $l=l'$ for which both $mC(l, \eta)<\delta(l\geqq l')$, and

$$\left| \int_{C(l,\eta)} g_n dx \right| < \frac{\lambda}{4} \qquad (n \geqq l \geqq l'). \tag{4}$$

But for any $l\geqq l'$ it is evident that there exist e and $n\geqq l$ such that we have

$$\begin{aligned} \int_{e(l,n)+} g_n dx \leqq U(g, \delta) &\leqq \int_e g_n\, dx + \frac{\lambda}{8} \\ &\leqq \int_{ee(l,n)+} g_n dx + \int_{ee(l,n)-} g_n dx + \int_{eS(l,\eta)} g_n dx + \frac{\lambda}{8} \\ &\leqq \int_{e(l,n)-} g_n dx + \frac{\lambda}{4} \qquad (n \geqq l \geqq l'). \end{aligned} \tag{5}$$

Similarly we arrive at

$$\int_{e(l,n)-} g_n dx - \frac{\lambda}{4} \leqq L(g, \delta) \leqq \int_{e(l,n)-} g_n dx \qquad (n \geqq l \geqq l'). \tag{6}$$

But (5) and (6) combine to contradict (4). We conclude, therefore, that the conditions of (2) are necessary.

The conditions are also sufficient. Suppose that, for every g, $U(g)+L(g)=0$. If f is not integrable then I is denied, and we can assume the existence of a number $\lambda>0$ and two sequences $l_1, l_2, \cdots$ and $n_1, n_2, \cdots$ where $n_i\geqq l_i$ and such that either

$$\int_{C(l_i,\eta)} f_{n_i}dx > \lambda \qquad (i = 1, 2, \cdots), \tag{7}$$

or the left-hand side of (7) is less than $-\lambda$. For definiteness suppose that (7) holds. Let $g_i = f_{n_i}$. Fix δ so that we have

$$U(g, \delta) + L(g, \delta) < \frac{\lambda}{2}, \tag{8}$$

and so that for $me < \delta$ we also have

$$\left| \int_{eS(l_i,\eta)} g_i dx \right| < \frac{\lambda}{4}. \tag{9}$$

Fix $i = i'$ such that for $i \geqq i'$, $mC(l_i, \eta) < \delta$. Then, obviously, we have

$$U(g, \delta) \geqq \int_{e(l_i,n_i)+} g_i dx \qquad (i \geqq i'). \tag{10}$$

By methods similar to those used in obtaining (5) it is possible to show that

$$L(g, \delta) \geqq \int_{e(l_i,n_i)-} g_i dx - \frac{\lambda}{2} \qquad (i \geqq i'). \tag{11}$$

But evidently (8), (10), and (11) can now be used to contradict (7). We conclude, therefore, that the conditions are sufficient.

If the sequence f is such that f_n is uniformly bounded below, then $L(g) = 0$ for every g. Furthermore, if the sequence is integrable, it follows from II that for every g, $U(g) = 0$. But if, for every g, $U(g) = L(g) = 0$, it is evident that the sequence of integrals is equi-convergent. We thus get

III. *If the sequence f converges on E to a measurable function F, and if f_n is uniformly bounded below, or above, it is then necessary and sufficient for the integrability of f that the sequence of integrals be equi-convergent.**

We have seen that the equi-convergence of the sequence of integrals is both necessary and sufficient for complete integrability. This, with III, gives

III′. *If the sequence f is integrable, and is such that f_n is uniformly bounded below, or uniformly bounded above, then f is completely integrable.*

One might hope that the existence of the double limit $\lim_{\substack{\delta\to 0\\ n\to\infty}} \{U(g, n, \delta) + L(g, n, \delta)\} = 0$ would prove to be both necessary and sufficient for integrability. On account of II, it is evidently sufficient. An example shows that it is not necessary.

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* Cf. de la Vallée Poussin, loc. cit., p. 448.

Let e_n be a sequence of distinct measurable sets on E with a single limit point not belonging to any set of the sequence, and such that me_n tends to zero monotonically. Let e_n' be a similar sequence distinct from e_n with a single limit point not belonging to any set of either sequence, and with $me_n' = 2me_n$. Let λ be any positive real number, and let $f_n = \lambda/me_n$ on e_n, $f_n = -\lambda/(2me_n)$ on e_n', and $f_n = 0$ elsewhere on E. We see that f_n converges to zero, and for each n the integral of f_n over E is zero. But for every $\delta > 0$ and every n it is possible to find $\delta' < \delta$ and $n' \geqq n$ such that

(1) $$U(f, n', \delta') + L(f, n', \delta') = \frac{\lambda}{2}\cdot$$

Let δ and n be given. Take $n' \geqq n$ and such that $me_{n'} < \delta$, and fix $\delta' = me_{n'}$. Then $U(f, n', \delta') = \lambda$, and $L(f, n', \delta') = -\delta'\lambda/me_{n'}' = -\lambda/2$, and these give (1).

4. **The integral of f_n over e not bounded in n and e.** We prove

IV. *If the sequence of summable functions f converges on E to the summable function F, it is necessary and sufficient for the integrability of the sequence that corresponding to every g of f and every $\epsilon > 0$ there is a number $\delta_{g\epsilon} > 0$ such that for $\delta < \delta_{g\epsilon}$ it is possible to find $n = n_\delta$ for which* $|U(g, n, \delta) + L(g, n, \delta)| < \epsilon$ $(n \geqq n_\delta)$.

Let f be integrable. Suppose that for some g the conditions of IV do not hold. If for this g it so happens that for every δ only one of the functions $U(g, n, \delta)$, $L(g, n, \delta)$ is unbounded in n, it is very easy to show that I is contradicted. If for every δ both these functions are bounded, then the methods of II can be used to contradict I. We thus have to consider only the case for which both these functions are, for every δ, unbounded in n. In such a case there exists a number $\lambda > 0$ and independent of δ, and for each δ a sequence of values $n_1, n_2, \cdots$ of n for which either

(1) $$U(g, n_i, \delta) + L(g, n_i, \delta) > \lambda \qquad (i = 1, 2, \cdots),$$

or the left-hand side of (1) is less than $-\lambda$. For the sake of definiteness let (1) hold. Since λ is independent of δ we can consider only such values of δ that for $me < \delta$ we have

(2) $$\left| \int_{eS(l,\eta)} g_n dx \right| < \frac{\lambda}{4} \qquad (n \geqq l).$$

Since for any δ both the functions involved in (1) become monotonically infinite with i, it is possible to choose from n_i a sub-sequence n_j such that

(3) $$U(g, n_j, \delta) > U(g, n_{j-1}, \delta) + \frac{\lambda}{4}$$

and

$$L(g, n_j, \delta) < L(g, n_{j-1}, \delta) - \frac{\lambda}{4} \tag{4}$$

both hold. It then follows from (3) that there exists some n_j' where $n_{j-1} \leqq n_j' \leqq n_j$, and some e_j with $me < \delta$, for which

$$\int_{e_j} g_{n_j'} dx > U(g, n_j, \delta) - \frac{\lambda}{4} \cdot$$

Then, by making use of (2) and the methods used in obtaining (5) and (6) in the proof of II, we arrive at

$$\int_{e(l, n')_+} g_{n_j'} dx > U(g, n_j, \delta) - \frac{\lambda}{2} \cdot \tag{5}$$

But for j sufficiently large, $me(l, n_j')_- < \delta(l = n_j')$, and then we evidently have

$$\int_{e(l, n_j')_-} g_{n_j'} dx \geqq L(g, n_j, \delta). \tag{6}$$

But (5), (6), and (1) can now be used to contradict I. We conclude, therefore, that the conditions of IV are necessary.

In order to show that the conditions are sufficient, let IV hold and suppose that f is not integrable. Then I is denied. Consequently, there exists a number $\lambda > 0$ and two sequences $l_1, l_2, \cdots$ and $n_1, n_2, \cdots$ where $n_i \geqq l_i$ and such that either

$$\int_{C(l_i, \eta)} f_{n_i} dx > \lambda \qquad (i = 1, 2, \cdots) \tag{1}$$

holds, or the left side of (1) is less than $-\lambda$. For the sake of definiteness assume that (1) holds, and set $g_i = f_{n_i}$. It is then possible, by methods similar to those we have used above, to show that the hypotheses of IV are contradicted. This leads to the conclusion that the conditions of IV are sufficient.

The example given above can easily be modified to show that when the integral of f_n over e is unbounded the double limit $\lim_{\substack{\delta \to 0 \\ n \to \infty}} \{U(g, n, \delta) + L(g, n, \delta)\} = 0$ is not necessary for integrability.

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ON THE INTERCHANGE OF LIMIT AND LEBESGUE INTEGRAL FOR A SEQUENCE OF FUNCTIONS*

BY

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R. L. Jeffery in a paper on *The integrability of a sequence of functions*† has given a number of necessary and sufficient conditions for $\lim_n \int_E f_n = \int_E F$, where f_n and F are summable on the measurable set E and $\lim_n f_n = F$ on E. The object of this note is to give an additional condition for the validity of this interchange, embodied in the

THEOREM. *If $U(n, \delta)$ is the least upper bound, and $L(n, \delta)$ is the greatest lower bound of $\int_e f_n$ for all measurable subsets e of E for which $me \leqq \delta$, then a necessary and sufficient condition that* $\lim_n \int f_n = \int F$ *on E, is that*

$$\lim_{\delta \to 0} \lim_{n \to \infty} [U(n, \delta) + L(n, \delta)] = 0.$$

We are using the notation $\lim_\delta \lim_n$ in the standard sense of the common value of $\lim_\delta \underline{\lim}_n$ and $\lim_\delta \overline{\lim}_n$. In (ϵ, δ) form the condition of the theorem is equivalent to the following:‡ For every $\epsilon > 0$ there exists a δ_ϵ such that for every $\delta \leqq \delta_\epsilon$, there exists an $n_{\delta\epsilon}$ such that if $n \geqq n_{\delta\epsilon}$, then $|U(n, \delta) + L(n, \delta)| \leqq \epsilon$.

We establish the equivalence of our condition with the necessary and sufficient condition I of Jeffery, viz.

$$\lim_{l \to \infty} \int_{C(l, \eta)} f_n = 0 \qquad (n \geqq l)$$

where $C(l, \eta)$ is the complement relative to E of the set for which

$$|f_n - f| \leqq \eta$$

for every $n \geqq l$.

By the definition of $U(n, \delta)$, for every ϵ and δ, there exists a subset e of E of measure less than δ such that

$$U(n, \delta) - \epsilon \leqq \int_e f_n \leqq U(n, \delta).$$

Since $U(n, \delta) \geqq 0$, we can obviously assume that $f_n \geqq 0$ on e. Now for any l and η

* Presented to the Society, December 31, 1930; received by the editors December 1, 1930.

† Cf. the present number of these Transactions.

‡ Cf., for instance, Hildebrandt, *Note on interchange of limits*, Bulletin of the American Mathematical Society, vol. 34 (1928), p. 80.

$$\left|\int_e f_n - \int_{eC(l,\eta)} f_n\right| \leqq \int_{e-eC(l,\eta)} |f_n - F| + \int_{e-eC(l,\eta)} |F|.$$

As a consequence if we take δ so that $\delta\eta \leqq \epsilon/2$ and so that, for $me \leqq \delta$, it is true that $\int_e |F| < \epsilon/2$, then provided $n \geqq l$

$$\left|\int_e f_n - \int_{eC(l,\eta)} f_n\right| \leqq \epsilon.$$

Let C^+ be the subset of $C(l, \eta)$ for which $f_n \geqq 0$, and C^- the set for which $f_n < 0$. If then l be chosen so that $mC(l, \eta) \leqq \delta$, and $n \geqq l$, then

$$U(n, \delta) - 2\epsilon \leqq \int_{eC(l,\eta)} f_n = \int_{eC^+} f_n \leqq \int_{C^+} f_n \leqq U(n, \delta).$$

In a similar way we show that

$$L(n, \delta) \leqq \int_{C^-} f_n \leqq L(n, \delta) + 2\epsilon.$$

Hence if $\delta \leqq \delta_\epsilon$, and $l \geqq l_{\delta\epsilon}$, with $n \geqq l$, we have

$$\left|\int_{C(l,\eta)} f_n - [U(n, \delta) + L(n, \delta)]\right| \leqq 2\epsilon.$$

From this statement follows the equivalence of the conditions

$$\lim_{l\to\infty} \int_{C(l,\eta)} f_n = 0 \ (n \geqq l) \text{ and } \lim_{\delta\to 0} \lim_{n\to\infty} [U(n, \delta) + L(n, \delta)] = 0.$$

In so far as

$$\lim_{\delta\to 0} [U(n, \delta) + L(n, \delta)] = 0$$

for every n, our condition is equivalent to the equality of the iterated limits $\lim_\delta \lim_n$ and $\lim_n \lim_\delta$ of the function $U(n, \delta) + L(n, \delta)$. Since the existence of the double limit $\lim_{(n,\delta)\to(\infty,0)}$ is sufficient for this interchange it follows that

$$\lim_{(n,\delta)\to(\infty,0)} [U(n, \delta) + L(n, \delta)] = 0$$

is also a sufficient condition for $\lim_n \int_E f_n = \int_E F$. That it is not necessary follows from the example given by Jeffery in §3.

If the integrals $\int f_n$ are equicontinuous on E, i.e., if

$$\lim_{me\to 0} \int_e f_n = 0$$

uniformly in n, then obviously $\lim_\delta[|U(n, \delta)| + |L(n, \delta)|] = 0$, uniformly in n, U and L being even taken with respect to any subset e of E. Applying the standard theorem on interchange of iterated limits, we get at once the Vitali theorem for all subsets e of E. On the other hand, if all the functions f_n are positive on E, then $U(n, \delta)$ converges to zero monotonically in δ, while $L(n, \delta) = 0$. By applying the following generalization of Dini's Theorem:*

If $\lim_n \lim_\delta U(n, \delta) = \lim_\delta \lim_n U(n, \delta)$, *and* $U(n, \delta)$ *is monotone in* δ, *then* $\lim_\delta U(n, \delta)$ *exists uniformly in* n,

we find the well known result that when $f_n \geqq 0$, equicontinuity of $\int f_n$ is a necessary condition for $\lim_n \int f_n = \int F$ on E.

The theorem of this note is still valid if the convergence of f_n to f on E is convergence in a measure,† i.e., if $D(n,\eta)$ is the set of points of E for which $|f_n - f| > \eta$ then $\lim_n mD(n,\eta) = 0$ for each η. Jeffery's Theorem must be altered so that the condition becomes

$$\lim_n \int_{D(n,\eta)} f_n = 0$$

for every η. The proofs for this more general case require only slight changes from those given.

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* Cf. Bulletin of the American Mathematical Society, vol. 21 (1914), p. 113.

† The possibility of this extension was suggested by L. M. Graves.

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NON-SEPARATED CUTTINGS OF CONNECTED POINT SETS*

BY

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1. We shall consider a connected, metric and separable space which we denote by M. A subset X of M is called a cutting of M provided that the complement $M-X$ of X is not connected and hence is the sum of two mutually separated sets $M_1(X)$ and $M_2(X)$; X is said to separate two points or point sets A and B in M when the sets $M_1(X)$ and $M_2(X)$ can be so chosen that $M_1(X)\supset A$ and $M_2(X)\supset B$, and is said to separate a single set N in M when $M_1(X)$ and $M_2(X)$ can be chosen so that $N\cdot M_1(X)\neq 0\neq N\cdot M_2(X)$.

A collection G of subsets of M will be called *non-separated* provided that the elements of G are mutually exclusive and no element of G separates any other element of G in M.

A subset P of M is said to have the potential order α in M relative to a given collection G of subsets of M provided that α is the least cardinal number such that there exists a monotonic decreasing sequence $[U_i]$ of neighborhoods of P such that $P=\prod_1^\infty \overline{U}_i$ and such that for each i, the boundary $F(U_i)$ of U_i is a subset of the sum of α of the sets of the collection G.

In this paper we shall show, first, that if G is any uncountable non-separated collection of cuttings of M then *all save a countable number of the elements of G have the potential order* 2 *in M relative to G.* Now obviously if the elements of any collection G of mutually exclusive cuttings of M are connected or if they reduce to single points, then the collection G is non-separated. And since for the case where M is compact, the potential order of a point of M is the same as its order in the Menger-Urysohn sense, our theorem yields as corollaries many important known results concerning the cut points and connected cuttings of connected sets and of continua; for example: (1) the theorem of Wazewski-Menger‡ that the ramification points of any acyclic continuous curve are countable, (2) the theorem of Kuratowski and Zarankiewicz§ that the set of all points of any connected set M whose complement in M is neither connected nor the sum of two connected point

* Presented to the Society, February 22, 1930; received by the editors July 5, 1930.

† Fellow, John Simon Guggenheim Memorial Foundation.

‡ See Wazewski, Annales de la Société Polonaise de Mathématique, vol. 2 (1923), p. 49; and Menger, Fundamenta Mathematicae, vol. 10 (1927), p. 108.

§ Bulletin of the American Mathematical Society, vol. 33 (1927), p. 571.

sets is countable; (3) the theorem of the author* that all save a countable number of the cut points of any continuum are points of order 2 of M in the Menger-Urysohn sense; and (4) other results concerning cuttings due to Zarankiewicz† and to the author.‡

Second, with the aid of this theorem we shall show that if the space M contains an uncountable non-separated collection G of cuttings, then there exists an upper semi-continuous collection S of elements such that all save a countable number of the sets of G are elements of S and such that every two elements of S may be separated in M by some third element. In case M is compact, the decomposition space S is an acyclic continuous curve.

Finally, we shall prove an existence theorem to the effect that *every locally connected space M contains an uncountable non-separated collection of cuttings.* Therefore, the above mentioned decomposition is always realisable for locally connected sets M, and notably for the case where M is a continuous curve, this decomposition gives rise to a decomposition space which is a non-degenerate acyclic continuous curve.

2. **Preliminary lemmas.** Let X and Y be any two cuttings of M and set

$$M - X = M_1(X) + M_2(X), \tag{i}$$

$$M - Y = M_1(Y) + M_2(Y), \tag{ii}$$

representing decompositions of $M-X$ and $M-Y$ respectively into mutually separated sets. Then if i, j, r, and s are positive integers such that $i+j=3=r+s$, it follows immediately that the following equation is valid:

$$M = M_i(X) + M_r(Y) + M_j(X)\cdot M_s(Y) + X + Y. \tag{2.1}$$

With the aid of this equation, we deduce at once the result

(2.2) *If neither of the sets X and Y separates the other, we may choose the indices i and r such that*

(a) $$X \subset M_r(Y) \textit{ and } Y \subset M_i(X);$$

and these relations imply also the relations

(b) $$M_j(X)\cdot M_s(Y) = 0,\ M_j(X) + X \subset M_r(Y), \textit{ and } M_s(Y) + Y \subset M_i(X).$$

Clearly this is the case, because by virtue of the relations (a) we may omit the last two terms in equation (2.1); and since M is connected, the term $M_j(X)\cdot M_s(Y)$ must vanish. This fact gives at once the remaining two relations (b).

* These Transactions, vol. 30 (1928), p. 606.

† See Fundamenta Mathematicae, vol. 12 (1928), pp. 119–125.

‡ See Bulletin of the American Mathematical Society, vol. 35 (1929), pp. 87–104.

Now let G be any non-separated collection of cuttings of M and let $E(a, b)$ be the collection of all those elements of G which separate two given points a and b in M. Let X and Y be any two elements of $E(a, b)$ and let the indices in (i) and (ii) be chosen so that

(iii) $$M_1(X)\cdot M_1(Y) \supset a \text{ and } M_2(X)\cdot M_2(Y) \supset b.$$

The element X of $E(a, b)$ will be said to precede the element Y, and this fact is indicated by the notation $X<Y$, provided that for at least one set of decompositions satisfying (i), (ii) and (iii) it is true that $X\subset M_1(Y)$. We shall now show that this definition gives a natural order to the elements of $E(a, b)$.

First, for any two elements X and Y of $E(a, b)$, at least one of the relations $X<Y$ and $Y<X$ must be valid. For if X does not precede Y, then by (2.2), (a), $r=2$ and hence $s=1$. By (b) and (iii) it follows that $j=2$ and hence $i=1$. Therefore by (a), $Y\subset M_1(X)$, which means $Y<X$.

Second, only one of the relations $X<Y$ and $Y<X$ can be valid. For if $X<Y$, then [for any set of decompositions whatever satisfying (i), (ii), (iii)], in (2.2), $r=1$ and hence $s=2$. By (b) and (iii) it follows that $j=1$ and hence $i=2$. Therefore by (a), $Y\subset M_2(X)$, which is incompatible with $Y<X$.

Finally, for any three elements Z, X and Y of $E(a, b)$, the relations $Z<X$, $X<Y$ imply that $Z<Y$. For then $Z\subset M_1(X)$ and $X\subset M_1(Y)$. Hence in (2.2), $r=1$ and $s=2$. By (b) and (iii) it follows that $j=1$. Therefore by the second relation in (b), $Z\subset M_1(X)+X\subset M_1(Y)$, which gives $Z<Y$.

Thus we have proved the following result:

(2.3) *If each element of the non-separated collection $E(a, b)$ of subsets of M separates the two points a and b in M, then the collection $E(a, b)$ possesses a natural order.*

For convenience we give here a lemma concerning ordered sets due to Zarankiewicz* which will be used below.

Lemma (Zarankiewicz). *If K is any ordered subset of M, then the set H of all points p of K such that p is not at the same time a limit point of the set P_p of all points of K preceding p and also of the set F_p of all points of K following p is countable.*

The space M being separable and metric, it therefore contains a countable sequence $R_1, R_2, R_3, \cdots$ of open sets which is equivalent to the set of all open subsets of M. Now let H_1 be the set of all points of K which are not limit points of their predecessors, and let $H_2=H-H_1$. For each point p of H_1 let $n(p)$ be the least positive integer such that $R_{n(p)}$ contains p but contains no point of K which precedes p. Then if p and q are distinct points of H_1 and

* See Fundamenta Mathematicae, vol. 12 (1928), p. 119.

$p<q$, then since $R_{n(q)}$ does not contain p, it follows that $n(p)\neq n(q)$, and therefore H_1 is countable. A similar argument proves H_2 countable; and hence H is countable.

3. THEOREM. *If G is any uncountable non-separated collection of cuttings of a connected, metric, and separable space M, then all save possibly a countable number of the elements of G have the potential order* 2 *in M relative to G.*

Suppose, on the contrary, that G contains an uncountable subcollection G_1 no element of which has the potential order 2 in M relative to G. Now there exist two points a and b of M such that the collection $E(a, b)$ of all those elements of G_1 which separate a and b in M is uncountable; for M being separable, there exists a countable subset D of M such that $\overline{D}=M$; and since every element of G_1 which contains no point of D must separate some pair of points of D in M, and since the set of all pairs of points of D is countable, it follows that for at least one pair of points a, b of D, the set $E(a, b)$ is uncountable.

By §2 the elements of the collection $E(a, b)$ possess a natural order; and if K is a point set which contains exactly one point x of each element X of $E(a, b)$ and contains no other points, then K is an ordered point set. Indeed for each pair x, y of points of K, set $x<y$ provided that $X<Y$. By the Zarankiewicz lemma, the set H of all points p of K which are not at the same time a limit point both of their predecessors and of their successors is countable. Let $H(a, b)$ be the collection of all those sets X of $E(a, b)$ such that the corresponding point x in K belongs to $K-H$. Then $H(a, b)$ is uncountable and each element X of $H(a, b)$ contains a point x which is a limit point of the sum of the predecessors of X and also of the sum of the successors of X.

Now for each element X of $H(a, b)$, there exist mutually separated sets $M_1(X)$ and $M_2(X)$ such that

$$M - X = M_1(X) + M_2(X),\ M_1(X) \supset a \text{ and } M_2(X) \supset b.$$

And with the aid of what has just been shown it follows immediately that there exist two infinite sequences of elements $X_1, X_2, X_3, \cdots$ and $Y_1, Y_2, Y_3, \cdots$ of $H(a, b)$ such that, for each n,

$$X_n < X_{n+1} < X < Y_{n+1} < Y_n, \tag{1}$$

and such that X contains a point which is a limit point both of $\sum X_n$ and $\sum Y_n$.

Since by supposition no element of $H(a, b)$ can have the potential order 2 in M relative to G, it follows that if for each element X of $H(a,b)$, $V_n(X)$ denotes the set of points $M-[M_1(X_n)+M_2(Y_n)]$, then there exists at least one point p_x belonging to the point set

$$\prod_1^\infty V_n(X) - X,$$

for if this were not the case, then by virtue of (1) and equation (2.1) in which substitute X_n for X, Y_n for Y, 1 for i and 2 for r, it follows that $V_n(X) \supset M_2(X_n)\cdot M_1(Y_n) \supset X$; and if for each point p of $M_2(X_n)\cdot M_1(Y_n)$ we take a neighborhood N_p of p of diameter less than 1/4 the distance from p to the set of points $\overline{M_1(X_n)}+\overline{M_2Y(_n)}$, and call $U_n(X)$ the sum of all the neighborhoods N_p, then it follows readily that

$$X \subset M_2(X_n)\cdot M_1(Y_n) \subset U_n(X) \subset \overline{U_n(X)} \subset V_n(X);$$

and hence $F[U_n(X)] \subset X_n + Y_n$, $U_n(X) \subset U_{n-1}(X)$ and $X = \prod_1^\infty \overline{U_n(X)}$; but then X has the potential order 2 in M relative to G, contrary to supposition.

Now if X and Y are any two elements of $H(a, b)$, $X \neq Y$, it follows that $p_x \neq p_y$. For suppose $X < Y$. Then since X contains a limit point of the sum of its successors in $E(a, b)$ but contains no limit point of $M_2(Y)$, it follows that there exist two elements Y_k and Y_m in the "Y-sequence" in (1) for the element X such that

$$X < Y_k < Y_m < Y;$$

and since Y contains a limit point of the sum of its predecessors in $E(a, b)$ but contains no limit point of $M_1(Y_m)$, it follows that there exists an element X_n of the "X-sequence" for Y in (1) such that

$$X < Y_k < Y_m < X_n < Y.$$

Consequently it follows with the aid of (2.2) that

$$p_x \subset M_1(Y_k) + Y_k \subset M_1(Y_m)$$

and

$$p_y \subset M_2(X_n) + X_n \subset M_2(Y_m),$$

and hence $p_x \neq p_y$.

Now let L denote the set of all points $[p_x]$ for all elements X of $H(a, b)$. Then L is uncountable and is an ordered set; indeed, it is only necessary to set $p_x < p_y$ when $X < Y$. Therefore by the Zarankiewicz lemma, there exists a point p_x of L which is a limit point both of its predecessors and of its followers, and hence both of $\sum X_n$ and of $\sum Y_n$, where the sequences $[X_n]$ and $[Y_n]$ satisfy (1). But $\sum X_n \subset M_1(X)$ and $\sum Y_n \subset M_2(X)$; and p_x must then belong either to $M_1(X)$ or to $M_2(X)$ and be a limit point of the other, contrary to the fact that these two sets are mutually separated. Thus the supposition that our theorem is false leads to a contradiction.

4. **Consequences of §3.** Let G be any uncountable non-separated collection of cuttings of M. Then since the product of any family $[\overline{U}_n]$ of closed sets is closed, §3 yields at once the result

(α) *All save a countable number of the elements of G are closed point sets.*

Now if X is any element of G such that $M-X$ is not the sum of two connected point sets, X cannot have a potential order 2 in M relative to G. For $M-X=M_1(X)+M_2(X)+M_3(X)$, where the sets $M_1(X)$, $M_2(X)$, and $M_3(X)$ are mutually separated and contain points a_1, a_2 and a_3 respectively; and if X had the potential order 2 relative to G, there would exist two elements X_1 and X_2 of G and a neighborhood U of X such that $F(U)\subset X_1+X_2$, $X_1\subset M_1(X)$, $X_2\subset M_2(X)$ and $\overline{U}\cdot(a_1+a_2+a_3)=0$; but then it would readily follow that the point set $M_3(X)\cdot(M-\overline{U})$ is non-vacuous and is both open and closed, contrary to the fact that M is connected. Thus in consequence of the theorem in §3 we have

(β) *The complement of each element of G, with the exception of a countable number of such elements, consists of exactly two components.*

Let us denote by ρ the property of any subset N of M not to be separated in M by any single element of G. Clearly each element X of G has the property ρ. We shall now show that

(γ) *All save a countable number of the elements of G are saturated in M relative to the property ρ.*

If, on the contrary, G contains an uncountable subcollection G_1 no element of which is saturated relative to the property ρ, then for each element Z of G_1 there exists at least one point p_z which is not separated from Z in M by any single element of G. Under these conditions it follows by the theorem and proof in §3 that there exist two points a and b of M and three elements Z, X and Y of $E(a, b)$ (the collection of all those elements of G_1 which separate a and b) such that $X<Z<Y$, and $M_2(X)\cdot M_1(Y)$ contains Z but does not contain the point p_z and also such that $X+Y$ does not contain p_z. But then by equation (2.1) we have either $p_z\subset M_1(X)$ or $p_z\subset M_2(Y)$. This is impossible because in the first case X separates p_z and Z in M and in the second case Y separates p_z and Z in M.

A cutting X of M is said to be an irreducible cutting of M provided that no proper subset of X is a cutting of M.

(δ) *All save a countable number of the elements of G are irreducible cuttings of M.*

If this is not so, there exists an uncountable collection G^0 of cuttings of M such that for each element X^0 of G^0 there exists an element X of G and a point p_z of X such that $X^0\subset X-p_z$. Since G is non-separated, it follows at once that G^0 is non-separated. Therefore by (γ) there exists an element X^0

of G^0 which is saturated relative to the property ρ defined by the collection G^0. Consequently there exists an element Y^0 of G^0 which separates X^0 and p_x in M, and one has $M-Y^0=M_1(Y^0)+M_2(Y^0)$, where $M_1(Y^0)\supset X^0$ and $M_2(Y^0)\supset p_x$. But then $M-Y=M_1(Y^0)\cdot(M-Y)+M_2(Y^0)\cdot(M-Y)$, and thus Y separates X in M (for $Y\cdot(X^0+p_x)\subset Y\cdot X=0$), which contradicts the non-separatedness of G.

We prove now the following general theorem:

THEOREM. *Every uncountable non-separated collection G of cuttings of a connected, metric, and separable space M contains a subcollection Q which contains all save possibly a countable number of the elements of G and such that each element X of Q has the following properties*: (a) *X is closed*; (b) *$M-X$ is the sum of two mutually separated connected point sets*; (c) *X is saturated in M relative to the property ρ defined by the collection Q, i.e., for every point p of $M-X$, there exists an element Y of Q which separates X and p in M*; (d) *X is an irreducible cutting of M*; *and* (e) *X has the potential order 2 in M relative to Q.*

To obtain the collection Q, let D be a countable subset of M which is dense in M and let us omit from G: (1) every element which does not possess each of the properties (a), (b), and (d); (2) every element which separates in M some pair of points a, b of D which are separated by only a countable number of elements of G; (3) every element which separates some pair a, b of points of D and contains no point p having the property that every neighborhood of p contains points of uncountably many distinct elements of G which separate a and b. Let G_1 denote the collection of the elements of G remaining after these omissions. Then by virtue of (α), (β) and (δ), together with the facts that there are only a countable number of pairs of points of D and that in the space M every uncountable set of points contains a point of condensation of itself, it follows that G_1 contains all save possibly a countable number of the elements of G.

Now let us omit from G_1 every element which is not saturated in M relative to the property ρ defined by the collection G_1 and also every element which does not have the potential order 2 in M relative to G_1. Let Q be the collection of elements of G_1 remaining after these omissions. Then Q contains all save a countable number of the elements of G_1 and hence also of G, and every element X of Q has the desired properties (a)-(e). Clearly X has properties (a), (b) and (d), for every element of G_1 has these properties. It remains to show that X has properties (c) and (e).

To show that X has property (c), let p be any point of $M-X$. There exists an element Y of G_1 which separates X and p, because every element of Q is saturated in M relative to the property ρ defined by G_1. Hence $M-Y$

$=M_1(Y)+M_2(Y)$, where $M_1(Y)\supset X$ and $M_2(Y)\supset p$. Also $M-X=M_1(X)+M_2(X)$, where $M_2(X)\supset Y$. Thus if a and b are points of $M_1(X)$ and $M_2(Y)$ respectively belonging to D, both X and Y separate a and b in M, and we have $X<Y$ in the order from a to b. Now there exists also an element Z of G_1 which separates X and Y in M, and it follows from §2 that Z also separates a and b in M, and we have the order $X<Z<Y$. Thus $Z\subset M_2(X)\cdot M_1(Y)$. Since X and Y are closed, $M_2(X)\cdot M_1(Y)$ is a neighborhood of Z, and hence it contains points of (and therefore contains all of) uncountably many elements of G which separate a and b in M. Therefore there exists at least one of these elements, say W, which belongs to Q, for all but a countable number of the elements of G belong to Q. Thus we have the order $X<W<Y$; and since $p\subset M_2(Y)$, it follows that W separates X and p in M. Consequently X has property (c).

Since X has the potential order 2 in M relative to G_1, there exist, as shown in §3, two points a and b of M such that X belongs to the collection $E(a, b)$ of all those elements of G_1 which separate a and b in M and such that there exist two sequences $X_1, X_2, \cdots$ and $Y_1, Y_2, \cdots$ of elements of $E(a, b)$ so that

$$X_n < X_{n+1} < X < Y_{n+1} < Y_n$$

and such that if $U_n=M_2(X_n)\cdot M_1(Y_n)$, then $X=\prod_1^\infty \overline{U}_n$. Now for each n there exist, by virtue of property (c), two elements X_n' and Y_n' of Q belonging to $E(a, b)$ such that $X_n<X_n'<X<Y_n'<Y_n$. Hence if U_n' denotes the point set $M_2(X_n')\cdot M_1(Y_n')$, one has $U_n'\subset U_n$. Hence $X=\prod_1^\infty \overline{U}_n'$, and since $F(U_n')\subset X_n'+Y_n'$ and since clearly the sequence $[U_n]$ contains an infinite subsequence $[U_{n_i}]$ such that $U_{n_{i+1}}\subset U_{n_i}$, it follows that X has the potential order 2 in M relative to Q. This completes the proof.

5. **Decomposition of M by means of a non-separated collection G every element of which is saturated relative to property ρ.** Let G be any non-separated collection of subsets of M each of which is saturated in M relative to the property ρ defined by G. For each point e of M which belongs to no element of G, let E denote the point set consisting of e together with all points p of M which are not separated in M from e by any single element of G. Let S denote the collection whose elements are the elements of G together with all such point sets E thus defined. Clearly each element of S is closed and every point of M belongs to exactly one element of S. We shall show next that the collection S is non-separated.

Suppose, on the contrary, that some element X of S separates some pair of points p and q belonging to an element Y of S. Then $M-X=M_1(X)+M_2(X)$, where $M_1(X)\supset p$ and $M_2(X)\supset q$. Now by virtue of the definition of the collections G and S, it follows that there exists an element Z of G which

separates X and p in M. Hence $M-Z=M_1(Z)+M_2(Z)$, where $M_1(Z)\supset X$ and $M_2(Z)\supset p$. Since Z belongs to G, it cannot separate Y in M; and therefore $p+q\subset Y\subset M_2(Z)$. But then

$$M-Z=[M_1(Z)+M_1(X)\cdot M_2(Z)]+M_2(X)\cdot M_2(Z),$$

and we have a separation of $M-Z$ into two mutually separated sets containing the points p and q respectively of Y, contrary to the fact that since Z belongs to G it cannot separate Y in M. Therefore S is non-separated.

Now clearly every element of S is saturated in M relative to the property ρ defined by the collection S. Consequently every two elements X and Y of S are separated in M by some third element of S. With the aid of this property it follows immediately that the collection S is upper semi-continuous,* i.e., there does not exist a sequence $X_1, X_2, X_3, \cdots$ of elements of S and two sequences $[p_i]$ and $[q_i]$ of points such that $p_i+q_i\subset X_i$ and which have sequential limit points p and q respectively belonging to two different elements P and Q respectively of S. For there exists an element X of S such that $M-X=M_1(X)+M_2(X)$ where $M_1(X)\supset P$ and $M_2(X)\supset Q$; and since for each i, X_i is a subset either of $M_1(X)$ or of $M_2(X)$, either $M_1(X)$ or $M_2(X)$ contains X_i for infinitely many i's; but this is impossible, for both p and q are limit points of every infinite subsequence of $X_1, X_2, X_3, \cdots$.

Now in case the space M is compact, the elements of S are closed and compact, and if for each pair of elements X and Y of S we define the distance $\rho(X, Y)$ between X and Y as the upper limit of the distances $\rho(x, y)$, where x and y are points of X and Y respectively, it readily follows that the space S' so obtained is compact, separable, metric and connected; and since it readily follows that every two "points" of S' are separated in S' by some third "point" of S', therefore† S' is an acyclic continuous curve.

6. Existence Theorem. *If the space M is connected im kleinen, there exists an uncountable non-separated collection of cuttings of M.*

Let a and b be any two points of M, and for each positive number r which is less than the distance from a to b, let $S(a, r)$ denote the set of all points of M whose distance from a is equal to r and let $I(a, r)$ denote the set of all points at a distance $<r$ from a. Then for each r, $S(a, r)$ separates a and b in M. Let $R(a, r)$ denote the component of $M-S(a, r)$ containing a, let $R(b, r)$ denote the component of $M-\overline{R(a, r)}$ containing b, and let X_r denote the point set $\overline{R(a, r)}\cdot\overline{R(b, r)}$. Then clearly X_r separates a and b in M and we have

(i) $$X_r\subset F[R(a, r)]\subset S(a, r), \text{ and } X_r=F[R(b, r)];$$

(ii) $$R(a, r)\subset I(a, r).$$

* See R. L. Moore, these Transactions, vol. 27 (1925), pp. 416–428.

† See R. L. Moore, Fundamenta Mathematicae, vol. 7 (1925), pp. 302–307.

Obviously the collection of cuttings $[X_r]$ is uncountable. It remains to show that it is non-separated. Let X_{r_1} and X_{r_2} be any two elements of this collection and suppose $r_1 < r_2$. By (ii) it follows that $\overline{R(a, r_1)} \subset R(a, r_2)$. Thus $X_{r_1} \subset R(a, r_2)$, and therefore X_{r_2} does not separate X_r in M. From the inclusion $\overline{R(a, r_1)} \subset R(a, r_2)$ and (i) it follows that $\overline{R(b, r_2)} = R(b, r_2) + X_{r_2} \subset R(b, r_1)$, and consequently X_r does not separate X_{r_2} in M. Thus the collection $[X_r]$ is non-separated, and the theorem is proved.

Now since a may be any point whatever of M and since every neighborhood of a contains uncountably many of the sets $[X_r]$, it follows by §4, (δ), that every such neighborhood contains at least one set X_r which is an irreducible cutting of M. Thus we have the following

Corollary. *Every open subset of a connected and connected im kleinen point set M lying in a separable metric space contains an irreducible cutting I of M.*

This corollary answers a question raised by the author.*

As a result of this existence theorem it follows that the decomposition treated in §5 is always realisable in case M is locally connected; and in case M is a continuous curve, M may be decomposed upper semi-continuously into a collection S of the type attained in §5, and the decomposition space S' is a non-degenerate acyclic continuous curve.

7. **Concluding remarks.** Although it is easily seen with the aid of a very simple example that two cuttings X and Y of M may have the property that neither of them separates the other in M and yet the set $M_2(X)+X$ not be connected, where $M_2(X) \supset Y$, nevertheless the following lemma is true.

Lemma. *If a and b are two points of M and $X_1, X_2, X_3, \cdots$ is any infinite sequence of distinct mutually exclusive sets each of which separates a and b in M and no one of which separates any other one, and we have*

$$X_1 < X_2 < X_3 < \cdots,$$

then the set of points $\sum_1^\infty M_1(X_i)$ is connected.

For if on the contrary this set of points is the sum of two mutually separated sets N_1 and N_2, then since a belongs to all of the sets $M_1(X_i)$, there exists an integer n such that $N_1 \cdot M_1(X_n) \neq 0 \neq N_2 \cdot M_1(X_n)$. Since by (2.2), (b), it follows that $M_1(X_n) \subset M_1(X_{n+1})$, therefore $N_1 \cdot M_1(X_{n+1}) \neq 0 \neq N_2 \cdot M_1(X_{n+1})$. Since these two sets are mutually separated, one of them, say $N_1 \cdot M_1(X_{n+1})$,

* See Fundamenta Mathematicae, vol. 13 (1929), p. 50, where the question is raised for *continuous curves M*. A solution of this problem for the case where M is a plane continuous curve has been given by J. H. Roberts; see these Transactions, vol. 32 (1930), p. 19.

contains X_n. But then it is easily seen that the sets $N_2 \cdot M_1(X_n)$ and $M - N_2 \cdot M_1(X_n)$ are mutually separated, contrary to the fact that M is connected.

With the aid of this lemma it can be shown without great difficulty that if X is any element of a non-separated collection G of subsets of M which is saturated in M relative to the property ρ defined by the collection G, then

(1) each component of $M - X$ is open in M;

(2) the components of $M - X$ are countable;

(3) X is a potentially regular element of G in M relative to G, i.e., a monotone decreasing sequence of neighborhoods $[U_i]$ of X exists such that $F(U_i)$ is a subset of a finite number of the elements of G and $X = \prod_1^\infty \overline{U}_i$;

(4) the potential order of X in M relative to G is equal to the number of components of $M - X$ when this number is finite, and is equal to ω (i.e., X is of increasing order) when and only when this number is infinite.

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EXPANSION THEORY ASSOCIATED WITH LINEAR DIFFERENTIAL EQUATIONS AND THEIR REGULAR SINGULAR POINTS*

BY

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Introduction. Birkhoff† has developed an expansion theory associated with linear differential equations on an interval of the real axis containing no singular points of the equation. The purpose of this paper is to develop a similar theory for the domain of complex variables in a region containing a regular singular point of the equation. To take the place of the two-point boundary conditions of the real variable case, we have corresponding relations between the values of the solution and its derivatives at a given point, and the values obtained by analytic continuation about the regular singular point back to the given point. The boundary conditions, n in number, are assumed to be linear, homogeneous and linearly independent, the order of the equation being n.

An expansion theory associated with the linear equation of the second order with polynomial coefficients has been developed by O. Volk.‡ The method used in this paper is different from that employed by Volk and the coefficients are not assumed to be polynomials. The results of this paper and those of Volk overlap each other for the second-order equation, but neither includes the whole of the other. The Neumann-Gegenbauer expansions in Bessel's functions are obtained as special cases. The Legendre and hypergeometric differential equations give rise to expansion theories if infinity is considered as the regular singular point and the parameter is suitably chosen as a function of the arbitrary constants of the respective equations. The expansion theory for the hypergeometric equation as developed by Reinsch§ is not included as a special case, since the parameter does not enter in the same way. In another paper it is hoped to extend the present theory so as to include the latter as a special case of a more general theory. The Laurent expansion is included as a special case arising from differential equations of every order.

A solution of a differential equation with a regular singular point is, in

* Presented to the Society, April 4, 1931; received by the editors October 30, 1930.

† These Transactions, vol. 9 (1908), p. 373.

‡ Mathematische Annalen, vol. 86 (1922), p. 296.

§ American Journal of Mathematics, vol. 47 (1925), p. 45.

general, multiple-valued in the neighborhood of the singular point. An infinite series of such solutions in which the type of multiple-valuedness changes from term to term of the series will in general not converge to a function with a simple type of multiple-valuedness. For this and other reasons the boundary conditions will be limited as set forth later.

The method of the paper is to obtain an expansion of $(t-x)^{-1}$ by means of the solutions of the original equation and its adjoint. The expansion of an arbitrary analytic function of one variable is then obtained by use of the Cauchy integral theorem. Absolute and uniform convergence of the expansion is proved by obtaining asymptotic properties of the functions involved. The results are generalized by developing an expansion theory for functions of any finite number of complex variables.

1. **Boundary conditions.** There is no generality lost by assuming that the regular singular point is at $x=0$. In order that the formulas be as simple as possible we consider the equation† to be

$$
\begin{aligned}
(1)\quad L(y) + \lambda x^{-1} y \equiv x^{n-1} y^{(n)} &+ x^{n-2} \alpha y^{(n-1)} \\
&+ x^{n-3} p_2(x) y^{(n-2)} + \cdots + p_n(x) x^{-1} y + \lambda x^{-1} y = 0,
\end{aligned}
$$

where $p_2(x), \cdots, p_n(x)$ are analytic and single-valued within a circle of radius Γ about $x=0$ and α is a constant. The equation adjoint to (1) may be written

$$
\begin{aligned}
(2)\quad M(v) + \lambda x^{-1} v \equiv (-1)^n x^{n-1} v^{(n)} &+ x^{n-2} \beta v^{(n-1)} \\
&+ x^{n-3} q_2(x) v^{(n-2)} + \cdots + q_n(x) x^{-1} v + \lambda x^{-1} v = 0,
\end{aligned}
$$

where $q_2(x), \cdots, q_n(x), \beta (q_n(x) \equiv p_n(x))$ have the same properties as $p_2(x), \cdots, p_n(x), \alpha$. We have the Lagrange identity

$$
(3)\qquad vL(y) - yM(v) \equiv \frac{d}{dx} R(y, v),
$$

where $R(y, v)$ is the bilinear concomitant. Let

$$
\begin{aligned}
(4)\quad -F(\rho) = \rho(\rho-1)\cdots(\rho-n+1) &+ \alpha\rho(\rho-1)\cdots(\rho-n+2) \\
&+ \sum_{\mu=2}^{n} p_\mu(0)\rho(\rho-1)\cdots(\rho-n+\mu+1).
\end{aligned}
$$

---

† The equation

$$
z^{n-1} u^{(n)} + z^{n-2} p_1(z) u^{(n-1)} + \cdots + p_n(z) z^{-1} u + R(z) \lambda z^{-1} u = 0
$$

is not essentially more general if $R(0) \neq 0$, since substituting

$$
u = \exp\left[-\frac{1}{n} \int_a^z \frac{p_1(t) - p_1(0)}{t} dt \right], \qquad z = \exp\left[\int_b^x \{R(t)\}^{1/n} t^{-1} dt \right],
$$

$a \neq 0$, $b \neq 0$, in the order given transforms it into the type (1).

Then the indicial equation of (1) for $x=0$ is

$$\lambda - F(\rho) = 0. \tag{5}$$

If $\rho_1, \rho_2, \cdots, \rho_n$ are the exponents belonging to the point $x=0$ of the original equation (1), then, since we have taken the equation in this form, $-\rho_1, -\rho_2, \cdots, -\rho_n$ are the corresponding exponents for (2). This fact may be easily inferred from a result in Forsyth's *Theory of Differential Equations*, Part III, volume IV, pages 256–257.

Let $y^{(j)}(a)$ be the value at $x=a$ of the jth derivative of $y(x)$, a solution of the differential equation, and let $Y^{(j)}(a)$ be the value obtained for $y^{(j)}(x)$ at $x=a$ after analytic continuation about $x=0$. The $2n$ quantities $y^{(j)}(a)$, $Y^{(j)}(a)$, $j=0, 1, \cdots, n-1$, are the quantities used in the boundary conditions. We assume a set of boundary conditions which have, in addition to the properties described in the introduction, the following properties: they determine an enumerable infinitude of values of λ, called characteristic values, for each of which the equation has a solution (not identically zero) satisfying the boundary conditions; for each characteristic value the solution for the present is assumed unique. The case where the solution is not unique is discussed separately for the boundary conditions which are used. The boundary conditions to be associated with the adjoint equation are to have the same properties. We shall later put on still further restrictions and arrive at the boundary conditions

$$Y^{(j)}(a) = Ky^{(j)}(a) \qquad (j = 0, 1, \cdots, n-1),$$

for the original equation; for the adjoint equations we shall then have the conditions

$$V^{(j)}(a) = K^{-1}v^{(j)}(a) \qquad (j = 0, 1, \cdots, n-1).$$

Let λ^* and λ^{**} be two different characteristic values for a given set of boundary conditions, and let $y^*(x)$, $y^{**}(x)$ be solutions of (1) for $\lambda=\lambda^*$ and $\lambda=\lambda^{**}$ respectively, and let $v^*(x)$ and $v^{**}(x)$ be the corresponding solutions of the adjoint equation. Suppose $\rho_1, \rho_2, \cdots, \rho_n$ to be the exponents corresponding to $\lambda=\lambda^{**}$ and $\sigma_1, \sigma_2, \cdots, \sigma_n$ be those corresponding to $\lambda=\lambda^*$. Since a differential equation having two exponents equal or differing by an integer usually has in its general solution logarithms, we shall assume, except for a finite number at most of characteristic values, that the exponents are not equal and no two differ by an integer. Consider the n^2 differences $\rho_i-\sigma_j$, $i, j=1, 2, \cdots, n$. We shall make the additional assumption that not more than one of these differences is equal to another difference plus an integer. The reason for this will appear later.

The solutions in question for the original equation and its adjoint $\lambda=\lambda^*$ may be written

$$(6)\qquad \begin{aligned} y^*(x) &= c_1 y_{\sigma_1}(x) + \cdots + c_n y_{\sigma_n}(x), \\ v^*(x) &= b_1 v_{-\sigma_1}(x) + \cdots + b_n v_{-\sigma_n}(x), \end{aligned}$$

where the functions in the right members have the formulas

$$(7)\qquad y_{\sigma_i}(x) = x^{\sigma_i}\left\{1 + \sum_{m=1}^{\infty} a_{\sigma_i,m} x^m\right\} = x^{\sigma_i}\phi_{\sigma_i}(x),$$

$$(8)\qquad v_{-\sigma_i}(x) = x^{-\sigma_i}\left\{1 + \sum_{m=1}^{\infty} b_{-\sigma_i,m} x^m\right\} = x^{-\sigma_i}\psi_{-\sigma_i}(x),$$

and similarly for $\lambda=\lambda^{**}$. The series (7) and (8) converge for $|x| < \Gamma$.

We have from (3)

$$v^*L(y^{**}) - y^{**}M(v^*) + (\lambda^{**} - \lambda^*)v^*y^{**}x^{-1}$$
$$\equiv \frac{d}{dx}\{R(y^{**}, v^*)\} + (\lambda^{**} - \lambda^*)v^*y^{**}x^{-1} = 0.$$

Multiply the above by dx and integrate once about the origin along the contour C_a which passes through the point $x=a$ and encloses only the singularity $x=0$ of the equation. The first part becomes $R(y^{**}(a), v^*(a)) - R(Y^{**}(a), V^*(a))$. This difference is a non-singular† bilinear form in the $2n$ quantities involved. Hence if the original conditions were linear, homogeneous, linearly independent, and n in number there exist unique boundary conditions for the adjoint such that if v^* is a solution satisfying these conditions the foregoing expression has the value zero. Letting v^* be such a solution, we have, since $\lambda^*\neq\lambda^{**}$,

$$(9)\qquad \int_{C_a} v^*y^{**}x^{-1}dx = 0.$$

This condition together with the condition

$$\int_{C_a} v^*y^*x^{-1}dx \neq 0$$

would enable us to determine formally the coefficients in the expansion

$$f(x) = \sum_{m=0}^{\infty} a_m y_m(x),$$

† This follows from the formal properties of the bilinear concomitant as in the real variable case. Compare Carmichael, American Journal of Mathematics, vol. 43 (1921), p. 234.

where $y_m(x)$ is the solution, supposed for the present to be unique, satisfying the original equation and boundary conditions for $\lambda=\lambda_m$, a characteristic value. However, in order to do this using only the foregoing, the path of integration would have to pass through the point $x=a$. We would then be unable to obtain the expansions of functions whose region of existence did not extend out to $x=a$. We shall then limit ourselves to boundary conditions for which (9) is true for all paths which enclose only the singularity $x=0$.

At any point sufficiently near $x=0$ the function $v^*y^{**}x^{-1}$ is single-valued and analytic. There exists then a function $\psi(x)$ such that its derivative is $v^*y^{**}x^{-1}$. If we continue analytically about $x=0$ the analytic continuation of $\psi(x)$ retains this property. From (9) for all paths we conclude that $\psi(x)$ is single-valued near $x=0$. Its derivative, and hence v^*y^{**}, has this property.

We have the following†

LEMMA. *If $S_1(x)$, $S_2(x)$, $\cdots$, $S_k(x)$, $T(x)$ are single-valued and analytic functions near $x=0$ and α_1, $\cdots$, α_k are complex numbers different from integers and such that no difference $\alpha_i-\alpha_j$, $i, j=1, 2, \cdots, k$; $i\neq j$, is an integer, and we have*

$$x^{\alpha_1}S_1(x) + x^{\alpha_2}S_2(x) + \cdots + x^{\alpha_k}S_k(x) \equiv T(x),$$

it follows that $S_1(x)\equiv S_2(x)\equiv \cdots \equiv S_k(x)\equiv T(x)\equiv 0$.

For the conditions named we shall show that the boundary conditions must lead to solutions which are constant multiples of the fundamental solutions corresponding to one exponent for each characteristic value. Suppose on the contrary that y^{**} and v^* are both combinations of more than one solution corresponding to the exponents. The product $y^{**}v^*$ is of the form of the left member of the equation of the lemma with the α's replaced by $\rho_s-\sigma_p$. One of the differences must be an integer. If this were not true the product would be identically zero by the lemma, and hence either v^* or y^{**} would be identically zero, contrary to hypothesis. We may then take $\rho_1-\sigma_1=m_1$, m_1 an integer and $c_1\neq 0$ and $b_1\neq 0$. If this were the only such difference equal to an integer, then by the lemma the product would be equal to $c_1b_1x^{\rho_1-\sigma_1}\phi_{\rho_1}(x) \cdot\psi_{-\sigma_1}(x)$. We would then have

$$b_1c_2x^{\rho_2-\sigma_1}\phi_{\rho_2}(x)\psi_{-\sigma_1}(x) + b_2c_1x^{\rho_1-\sigma_2}\phi_{\rho_1}(x)\psi_{-\sigma_2}(x) + \cdots + \cdots \equiv 0.$$

Now $\phi_{\rho_i}(x)\not\equiv 0$ and $\psi_{-\sigma_i}(x)\not\equiv 0$, $i=1, 2, \cdots, n$. We shall first suppose that the notation can be so chosen that $c_2\neq 0$, $b_2\neq 0$. Since the above equation is satisfied and none of the members written is identically zero, the lemma is

† For proof see for instance Schlesinger, *Handbuch der Theorie der linearen Differentialgleichungen*, vol. I, p. 176.

contradicted unless the differences $\rho_1-\sigma_2$, $\rho_2-\sigma_1$, $\rho_2-\sigma_2$ are equal to similar differences plus an integer. We have at least two such and this is contrary to hypothesis. Hence at least either v^* or y^{**} is of the character we wished to establish for them. Next suppose $c_2=c_3=\cdots=c_n=0$ but $b_2\neq 0$. Then we would have to have either $\rho_1-\sigma_2=m_2$ or $\rho_1-\sigma_2=\rho_1-\sigma_j+m_3, j>2$, m_2 and m_3 being integers. In both cases using the relation $\rho_1-\sigma_1=m_1$ we would have $\sigma_i-\sigma_j=$ integer, $i\neq j$, contrary to hypothesis. Hence $b_2=b_3=\cdots=b_n=0$. Similarly if we chose $c_2\neq 0$ and $b_2=b_3=\cdots=b_n=0$ we would get a contradiction. The same result would be obtained if it were assumed that another difference of exponents were an integer.

If ν is the exponent to which the solution satisfying the boundary conditions corresponds for $\lambda=\lambda_0$, then the exponents for the other characteristic values are $\nu+m$, m a positive or negative integer. From (5) we have

$$\lambda_m = F(\nu + m) \qquad (m = 0, \pm 1, \pm 2, \cdots). \tag{10}$$

The solutions of the original equation evidently satisfy the boundary conditions

$$\begin{aligned} Y(a) &= Ky(a), \\ Y^{(s)}(a) &= Ky^{(s)}(a) \qquad (s = 1, \cdots, n-1); \end{aligned} \tag{11}$$

and the adjoint solutions satisfy the conditions

$$\begin{aligned} V(a) &= K^{-1}v(a), \\ V^{(s)}(a) &= K^{-1}v^{(s)}(a) \qquad (s = 1, \cdots, n-1), \end{aligned} \tag{12}$$

where $K=e^{2\pi i\nu}$, $i=(-1)^{1/2}$ and a is any point different from zero and within the circle of radius Γ about $x=0$. The adjoint conditions have the required character that the bilinear concomitant is single-valued near $x=0$. Any other set having this character is made up of linear combinations of these and hence equivalent.

These boundary conditions can be seen to lead to the characteristic values λ_m and to the solutions we have obtained. We have the orthogonality conditions

$$\int_C v_{-\nu-s}(x)y_{\nu+m}(x)x^{-1}dx = 0,\ s \neq m, \tag{13}$$

where C is a circle of radius $R<\Gamma$ about $x=0$. The arbitrary multipliers may be taken equal to unity since they can be absorbed in the coefficients of the expansion. It is evident from (7) and (8) that

$$\int_C v_{-\nu-m}(x)y_{\nu+m}(x)x^{-1}dx = 2\pi i. \tag{14}$$

The set $\lambda_m = F(\nu+m)$ will furnish an enumerably infinite number of characteristic values as required for the expansion. Unless $F(\nu+m)=F(\nu+p)$, $m\neq p$, the equation has only one solution satisfying the boundary conditions. The case $F(\nu+m)=F(\nu+p)$ we shall treat separately.

2. **Formal expansion of $x^\nu/(t-x)$.** For a given value of $K\neq 0$ of the boundary conditions we have for some ν, $e^{2\pi i\nu}=K$. Let ν be a determination such that either (a) a solution of the form (7) exists† for each exponent $\nu+m$, $m=0, 1, 2, \cdots$, even if $F(\nu+m)=F(\nu+p)$, $m\neq p$, or (b) the members of the set $F(\nu+m)$, $m=0, 1, 2, \cdots$, are all different. We shall see presently that we can do this. The set of characteristic values of λ are $\lambda_m=F(\nu+m)$ for m a positive or negative integer or zero. For our expansion theory we will need only those arising from $m\geqq 0$.

If for a given characteristic value λ_m the exponents are such that two of them differ by an integer or are equal but the exponent $\nu+m$ is not involved, the orthogonality conditions still hold. However, if the two exponents whose difference is an integer include as one of them the exponent $\nu+m$ we would have

$$\lambda_m = F(\nu+m) = F(\nu+p) = \lambda_p,\ p \neq m.$$

That is, there are two notations for the same characteristic value. If p is negative there would be the question as to the existence of the solution of the adjoint corresponding to the exponent $-\nu-m$, since $-\nu-p$ is also an exponent. If this solution exists in the form (8) there is no change in the theory.

For a given ν we have $\lambda_m=F(\nu+m)=G(m)$, where

$$\begin{aligned} G(m) &= -m^n + a_1m^{n-1} + \cdots + a_n \\ &= -m^n + \alpha_1 m^{n-1} + \cdots + \alpha_n + i\{\beta_1 m^{n-1} + \cdots + \beta_n\}, \end{aligned}$$

$a_s=\alpha_s+i\beta_s$, α_s and β_s being real. Then $F(\nu+m)=F_1(m)+iF_2(m)$ where $F_1(m)$ and $F_2(m)$ are real-valued polynomials in the real variable m. Now $F_1(m)$ and $F_2(m)$ as functions of the continuous variable m have a finite number of maxima and minima, and hence for m sufficiently large $F_1(m)$ is different for different values m, and likewise for $F_2(m)$. From this it follows that at most only a finite number of $F(\nu+m)$, $m=0, 1, \cdots$, are equal.

We shall now compute formally the coefficients in the expansion

$$x^\nu/(t-x) \sim \sum_{m=0}^{\infty} a_m(t)y_{\nu+m}(x). \tag{15}$$

† For the condition that this solution exists see Ince, *Ordinary Differential Equations*, Chapter XVI, pp. 404–6.

We shall first consider the case when the λ's with the positive subscripts are all distinct from those with negative subscripts. Let λ_m be a characteristic value which does not have two exponents $\nu+m$ and $\nu+p$, $p\neq m$, corresponding to it. Multiply both sides of (15) through by $v_{-\nu-m}(x)x^{-1}dx$ and assume for the moment that the series on the right permits of term by term integration. Integrating on the contour C about the origin and using (13) and (14) we get

$$a_m(t) = \frac{1}{2\pi i}\int_C \frac{x^\nu v_{-\nu-m}(x)dx}{x(t-x)}.$$

Substituting series (8) for $v_{-\nu-m}(x)$ and expanding $1/(t-x)$ in powers of x and simplifying we get

$$a_m(t) = t^{-1}P_{-\nu-m}(t), \tag{16}$$

where $P_{-\nu-m}(t)$ is the sum of the first $m+1$ terms of the power series expansion of $t^\nu v_{-\nu-m}(t)$.

Next suppose $F(\nu+r_1)=F(\nu+r_2)=\cdots=F(\nu+r_s)$, where $0\leqq r_1<r_2<\cdots<r_s$, and that solutions exist in the form (7) for the original equation and adjoint for the exponents $\nu+r_1$, $\nu+r_2$, $\cdots$, $\nu+r_s$ and $-\nu-r_1$, $-\nu-r_2$, $\cdots$, $-\nu-r_s$ respectively. Let $u_{\nu+r_1}(x)$, $\cdots$, $u_{\nu+r_s}(x)$ be a particular set of solutions of the original equation and $v_{-\nu-r_1}(x)$, $\cdots$, $v_{-\nu-r_s}$ be a particular set for the adjoint. These sets may be chosen, for example, by putting all the arbitrary elements equal to unity for each exponent. We have

$$\begin{aligned}\int_C v_{-\nu-r_q}(x)u_{\nu+r_q}(x)x^{-1}dx &= 0,\ q<p,\\ &= 2\pi i,\ q=p,\\ &= d_{q,p}\cdot 2\pi i,\ q>p,\end{aligned}$$

where $d_{q,p}$ is not necessarily zero. Now the solution

$$y_{\nu+r_p}(x) = u_{\nu+r_p}(x) + \alpha_{p,p+1}u_{\nu+r_{p+1}}(x) + \cdots + \alpha_{p,s}u_{\nu+r_s}(x)$$

is in the form of (7). Next form

$$\int_C v_{-\nu-r_q}(x)y_{\nu+r_p}(x)x^{-1}dx = l_{q,p}2\pi i,$$

for $q=p+1$, $\cdots$, s. We then have $s-p$ equations

$$\begin{aligned} d_{p+1,p} + \alpha_{p,p+1} &= l_{p+1,p},\\ d_{p+2,p} + \alpha_{p,p+1}d_{p+2,p+1} + \alpha_{p,p+2} &= l_{p+2,p},\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ d_{s,p} + \alpha_{p,p+1}d_{s,p+1} + \cdots + \alpha_{p,s} &= l_{s,p}.\end{aligned}$$

It is evident that we can determine uniquely $\alpha_{p,q}$, $q=p+1, \cdots, s$, so that $l_{q,p}=0$, $q=p+1, \cdots, s$. Let this be done for $p=1, 2, \cdots, s$. We then have

$$\int_C v_{-\nu-r_q}(x) y_{\nu+r_p}(x) x^{-1} dx = 0,\ q \neq p,$$
$$= 2\pi i,\ q = p,$$

and the solutions have the same orthogonality properties as do the remaining solutions if this is done for every such case. The determination of the coefficients in the expansion is then the same as before. We have the formal expansion

$$x^\nu/(t-x) \sim \sum_{m=0}^{\infty} t^{-1} P_{-\nu-m}(t) y_{\nu+m}(x). \tag{17}$$

We now take up the case when we have, for some λ_m, $\lambda_m = \lambda_{-p}$, m and p both positive. If a solution of the adjoint exists for the exponent $-\nu-m$ there is no change in the theory. However even if this solution does not exist we sometimes can obtain a formal expansion. In $P_{-\nu-m}(t)$ we have only the first $(m+1)$ terms of the power series expansion of $t^\nu v_{-\nu-m}(t)$. If we try by formal methods to find the solution of the adjoint corresponding to the exponent $-\nu-m$ we are able to compute the coefficients up to and including $b_{-\nu-m,\, m+p-1}$, being unable to compute $b_{-\nu-m,\, m+p}$ on account of a zero multiplier. Since $p>0$ we can compute the coefficients up to $b_{-\nu-m,\, m}$ which are all that are used in $P_{-\nu-m}(t)$. Whenever this case arises we shall use the coefficients for $P_{-\nu-m}(t)$ computed in this way. It is to be noted that these coefficients satisfy the same general formulas as do those when the solution exists.

3. **Asymptotic properties of $y_{\nu+m}(x)$ and $P_{-\nu-m}(t)$ with respect to m.** We now show that we have an actual expansion of $x^\nu/(t-x)$ by first showing that the series in (17) may be integrated term by term and next that it converges to $x^\nu/(t-x)$. This will be done by first proving the uniform convergence of the series in (17) with respect to both x and t in a suitable range.

The functions $y_{\nu+m}(x)$ and $P_{-\nu-m}(t)$ of (17) may be written

$$\begin{aligned} y_{\nu+m}(x) &= \sum_{s=0}^{\infty} a_{\nu+m,s} x^{s+\nu}, \quad a_{\nu+m,0} = 1, \\ P_{-\nu-m}(t) &= \sum_{s=0}^{m} b_{-\nu-m,s} t^s, \quad b_{-\nu-m,0} = 1. \end{aligned} \tag{18}$$

We have†

† For formulas (19) to (24) see Frobenius, Journal für Mathematik, vol. 76 (1873), p. 214.

$$
(19)\quad
\begin{aligned}
a_{\nu+m,0}f_0(\nu + m) &= 0,\\
a_{\nu+m,1}f_0(\nu + m + 1) + a_{\nu+m,0}f_1(\nu + m) &= 0,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &\\
a_{\nu+m,s+1}f_0(\nu + m + s + 1) + \cdots + a_{\nu+m,0}f_{s+1}(\nu + m) &= 0,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &,
\end{aligned}
$$

and

$$
(20)\quad
\begin{aligned}
b_{-\nu-m,0}\phi_0(- \nu - m) &= 0,\\
b_{-\nu-m,1}\phi_0(- \nu - m + 1) + b_{-\nu-m,0}\phi_1(- \nu - m) &= 0,\\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots &\\
b_{-\nu-m,m}\phi_0(- \nu) + \cdots + b_{-\nu-m,0}\phi_m(- \nu - m) &= 0,
\end{aligned}
$$

where

$$
\begin{aligned}
f(x, \nu + m + s) &= [\nu + m + s]_n + \alpha[\nu + m + s]_{n-1} + p_2(x)[\nu + m + s]_{n-2}\\
&\quad + \cdots + p_n(x) + \lambda_m\\
&= \sum_{\mu=0}^{\infty} f_\mu(\nu + m + s)x^\mu,
\end{aligned}
$$

$$
(21)\quad
\begin{aligned}
\phi(x, - \nu - m + s) &= (- 1)^n[- \nu - m + s]_n + \beta[- \nu - m + s]_{n-1}\\
&\quad + q_2(x)[- \nu - m + s]_{n-2} + \cdots + q_n(x) + \lambda_m\\
&= \sum_{\mu=0}^{\infty} \phi_\mu(- \nu - m + s)x^\mu,
\end{aligned}
$$

$$
[\nu + m + s]_r = (\nu + m + s)(\nu + m + s - 1) \cdots (\nu + m + s - r + 1).
$$

The first conditions in (19) and (20) are

$$
f_0(\nu + m) = \lambda_m - F(\nu + m) = 0 \text{ and } \phi_0(- \nu - m) = \lambda_m - F(\nu + m) = 0;
$$

both give rise to the indicial equation. If

$$
(22)\quad C_{\nu+m,s+1} = \frac{\{ |a_{\nu+m,s}| M(\nu+m+s) + |a_{\nu+m,s-1}| M(\nu+m+s-1)R^{-1} + \cdots + M(\nu+m)R^{-s} \}}{| f_0(\nu + m + s + 1) |},
$$

where

$$
M(\nu + m + s) = | [\nu + m + s]_{n-2} | M_2 + \cdots + M_n,
$$

$M_2, \cdots, M_n$ being the maxima of $|p_2'(x)|$, etc., within and on a circle of radius $R<\Gamma$, then

$$
(23)\quad | a_{\nu+m,s+1} | < C_{\nu+m,s+1}.
$$

We also have

$$C_{\nu+m,s+1} < C_{\nu+m,s}\left\{\frac{M(\nu+m+s)+|f_0(\nu+m+s)|R^{-1}}{|f_0(\nu+m+s+1)|}\right\},$$

which becomes

$$C_{\nu+m,s+1} < C_{\nu+m,s}\left\{\frac{M(\nu+m+s)+|F(\nu+m)-F(\nu+m+s)|R^{-1}}{|F(\nu+m)-F(\nu+m+s+1)|}\right\}. \tag{24}$$

For m sufficiently large† the denominator of the right-hand side of (24) is always different from zero. We consider first

$$P_{\nu+m,s} = \frac{F(\nu+m)-F(\nu+m+s)}{F(\nu+m)-F(\nu+m+s+1)} = \frac{G(m+s)-G(m)}{G(m+s+1)-G(m)} \tag{25}$$

when s is large. Put

$$T_{m+s}(r) = \frac{(m+s)^{n-r}}{(m+s)^n-m^n}, \quad U_{m+s}(r) = \frac{m^{n-r}}{(m+s)^n-m^n}\cdot$$

After dividing the numerator and denominator of (24) by $(m+s)^n-m^n$ and expanding $(m+s+1)^{n-r}$ into powers of $m+s$ by the binomial theorem, we may write

$$P_{-\nu-m,s} = \frac{1+\sum_{r=1}^{n}\{B_rT_{m+s}(r)+C_rU_{m+s}(r)\}}{1+\sum_{r=1}^{n}\{D_rT_{m+s}(r)+E_rU_{m+s}(r)\}}, \tag{26}$$

where B_r, C_r, D_r, and E_r are uniquely determined. Now

$$\frac{(m+s)^{n-r}}{(m+s)^n-m^n} = \frac{1}{(m+s)^r-m^r\left\{\dfrac{m}{m+s}\right\}^{n-r}},$$

and $m/(m+s)<1$; hence

$$\frac{(m+s)^{n-r}}{(m+s)^n-m^n} < \frac{1}{(m+s)^r-m^r}\cdot$$

Now we have

$$(m+s)^r-m^r = s\left\{rm^{r-1}+\frac{r(r-1)}{2}m^{r-2}s+\cdots+s^{r-1}\right\} \geqq s,\ r\geqq 1;$$

† Note that the argument for $y_{\nu+m}(x)$ holds for m becoming infinite through positive real values.

hence

$$\frac{(m+s)^{n-r}}{(m+s)^n - m^n} < \frac{1}{s}, \quad \frac{m^{n-r}}{(m+s)^n - m^n} < \frac{(m+s)^{n-r}}{(m+s)^n - m^n} < \frac{1}{s}.$$

If s is sufficiently large the denominator is surely positive and we can write

$$| P_{\nu+m,s} | < \frac{1 + \sum_{r=1}^{n} \{ | B_r | T_{m+s}(r) + | C_r | U_{m+s}(r) \}}{1 - \sum_{r=1}^{n} \{ | D_r | T_{m+s}(r) + | E_r | U_{m+s}(r) \}}.$$

For a given δ, $1 > \delta > 0$, there exists an s_1 such that for $s > s_1$

$$| P_{\nu+m,s} | < \frac{1+\delta}{1-\delta}.$$

Now consider one term of

$$\frac{M(\nu + m + s)}{| F(\nu + m) - F(\nu + m + s + 1) |}$$
$$= \sum_{r=2}^{n} \frac{M_r | (\nu + m + s) \cdots (\nu + m + s - n + r - 1) |}{| G(m + s + 1) - G(m) |}$$

which may be written

$$Q_{\nu+m,s} = \frac{M_r | (\nu + m + s) \cdots (\nu + m + s - n + r - 1) |}{| G(m + s + 1) - G(m) |}$$

$$= \frac{M_r \left| \left(1 + \frac{\nu}{m+s}\right) \cdots \left(1 + \frac{\nu - n + r - 1}{m + s}\right) \right|}{\left\{ (m+s)^r - \left(\frac{m}{m-s}\right)^{n-r} m^r \right\} \frac{| G(m+s+1) - G(m) |}{(m+s)^n - m^n}}.$$

We have for $s > s_1$

$$Q_{\nu+m,s} < \frac{\overline{M}_r}{(m+s)^r - m^r} < \frac{\overline{M}_r}{s(1-\delta)},$$

where $\overline{M}_r$ is independent of s. For a given δ there exists an s_2 such that for $s > s_2 \geqq s_1$

$$Q_{\nu+m,s}(r) < \frac{\delta}{n-1} \qquad (r = 2, 3, \cdots, n).$$

We can now write

$$C_{\nu+m,s+1} < C_{\nu+m,s}\left\{\delta + \frac{1+\delta}{1-\delta}R^{-1}\right\} < \frac{C_{\nu+m,s}}{R-\delta_2},\quad s > s_2 \geqq s_1,$$

where δ_2 may be made arbitrarily small.

For $s=1, 2, \cdots, s_2$ the right-hand side of (24) approaches $s/((s+1)R) < 1/R$ as m becomes infinite. This follows from the fact that the degree of the numerator and of the denominator of $P_{\nu+m,s}$ is $n-1$ in m, and the coefficients of m^{n-1} are respectively s and $s+1$. The rest of the numerator of the right-hand side of (24) is of degree $n-2$ in m. There exists then an m_1 such that for all $m>m_1$ we have for all s

$$C_{\nu+m,s+1} < \frac{C_{\nu+m,s}}{R-\delta_2}.$$

Using this bound for $C_{\nu+m,s}, \cdots, C_{\nu+m,2}$ we get

$$|a_{\nu+m,s+1}| < C_{\nu+m,s+1} < \left[\frac{1}{R-\delta_2}\right]^s C_{\nu+m,1},$$

where

$$C_{\nu+m,1} = \frac{M_2|[\nu+m]_{n-2}| + \cdots + M_n}{|F(\nu+m) - F(\nu+m+1)|}.$$

We may write

$$|x^{-\nu-m}y_{\nu+m}(x) - 1| < C_{\nu+m,1}\sum_{s=1}^{\infty}\left[\frac{1}{R-\delta_2}\right]^{s-1}|x|^s$$

$$= C_{\nu+m,1}|x|\left[\frac{1}{1-\dfrac{|x|}{R-\delta_2}}\right],\quad |x| < R-\delta_2.$$

Now for $|x| \leqq R' < R-\delta_2 < \Gamma$ we have for M independent of x and m

$$|x^{-\nu-m}y_{\nu+m}(x) - 1| < C_{\nu+m,1}M.$$

The numerator of $C_{\nu+m,1}$ is of degree $n-2$ in m and the denominator of degree $n-1$. Hence for a given arbitrary small positive number ϵ there exists an m_2 such that for $m>m_2$ we have

$$C_{\nu+m,1} < \frac{\epsilon}{M}.$$

Then for $m>m_3$, the larger of m_1 and m_2, we have

$$|x^{-\nu-m}y_{\nu+m}(x) - 1| < \epsilon,\quad |x| \leqq R' < R-\delta_2 < \Gamma,$$

or

$$y_{\nu+m}(x) = x^{\nu+m}\{1 + \eta_{\nu+m}(x)\}, \tag{27}$$

where

$$\mathop{\mathrm{L}}_{m\to\infty} \eta_{\nu+m}(x) = 0.$$

We have proved the following

THEOREM I. *Let $y_{\nu+m}(x)$ be the solution of the differential equation with $\lambda = \lambda_m$ formed with the exponent $\nu+m$; then if we write*

$$y_{\nu+m}(x) = x^{\nu+m}\{1 + \eta_{\nu+m}(x)\},$$

we have

$$\mathop{\mathrm{L}}_{m\to\infty} \eta_{\nu+m}(x) = 0,$$

provided that $|x| \leqq R' < \Gamma$.

The coefficients in the function $P_{-\nu-m}(t)$ for all the cases we have considered satisfy equation (19). Since $\phi_0(-\nu-m+s+1) = \lambda_m - F(\nu+m+s-1)$ the argument in this case goes much like the preceding when we observe that $m-s$ is always positive. It is necessary that m is always as large as s is taken. We shall not give the proof but state the corresponding

THEOREM II. *Let $P_{-\nu-m}(t)$ be the function previously defined; then if we write*

$$P_{-\nu-m}(t) = t^{-m}\{1 + r_{-\nu-m}(t)\}, \tag{28}$$

we have

$$\mathop{\mathrm{L}}_{m\to\infty} r_{-\nu-m}(t) = 0$$

provided that $|t| \leqq R'' < \Gamma$.

The following corollary is evident.

COROLLARY. *Let $y_{\nu+m}(x)$ and $P_{-\nu-m}(t)$ be the functions of Theorems* I *and* II; *then $x^{-\nu}y_{\nu+m}(x)$ and $P_{-\nu-m}(t)$ are dominated by $M_1|x|^m$ and $M_2|t|^{-m}$ respectively, where M_1 and M_2 are independent of x and t, provided $|x| \leqq R' < \Gamma$ and $|t| \leqq R'' < \Gamma$.*

4. **Convergence of the expansion for $x^\nu/(t-x)$.** Consider the series

$$\sum_{m=0}^{\infty} P_{-\nu-m}(t)t^{-1}x^{-\nu}y_{\nu+m}(x), \tag{29}$$

where $P_{-\nu-m}(t)$ and $y_{\nu+m}(x)$ are the functions previously defined. By the

corollary of the preceding section the general term is in absolute value dominated by $M_1M_2|x|^m|t|^{-m-1}$. This is a term of a convergent series if $|x|<|t|<\Gamma$. Hence for $|x|<|t|$ the series converges absolutely. If t is restrained to the circle $|t|=R'$ and $|x|\leqq R''<R'$, then the series is dominated term by term by a converging series of constants and hence by the Weierstrass M-test converges absolutely and uniformly with respect to both t and x. Next we must show that the series actually represents $1/(t-x)$.

We prove the following

LEMMA. *If $y_{\nu+m}(x)$ and $P_{-\nu-p}(x)$ are the functions defined above, then*

$$\int_C P_{-\nu-p}(x)x^{-\nu}x^{-1}y_{\nu+m}(x)dx = 2\pi i,\quad p = m, \tag{30}$$
$$= 0,\quad p \neq m,$$

and

$$\frac{1}{2\pi i}\int_C \frac{P_{-\nu-p}(x)dx}{x(t-x)} = t^{-1}P_{-\nu-p}(t).$$

The first and last results may be verified easily by using the expansions of the functions involved. We now prove the second result for the cases where the functions $v_{-\nu-p}(x)$ exist for each $p\geqq 0$. From the orthogonality conditions we have

$$\int_C v_{-\nu-p}(x)y_{\nu+m}(x)x^{-1}dx = \int_C P_{-\nu-p}(x)x^{-\nu-1}y_{\nu+m}(x)dx$$
$$+\int_C R_{-\nu-p}(x)x^{-\nu}y_{\nu+m}(x)dx = 0,$$

where $R_{-\nu-p}(x)$ is analytic at $x=0$. But

$$\int_C R_{-\nu-p}(x)x^{-\nu}y_{\nu+m}(x)dx = 0;$$

hence

$$\int_C P_{-\nu-p}(x)x^{-\nu-1}y_{\nu+m}(x)dx = 0.$$

Evaluating the foregoing integral for $m<p$ we get

$$\sum_{q=0}^{p-m} b_{-\nu-p,q}\cdot a_{\nu+m,p-m-q} = 0. \tag{31}$$

Now let $P_{-\nu-p}(x)$ be the function defined when the solution does not exist. We evidently have

$$\int_C P_{-\nu-p}(x)x^{-\nu-1}y_{\nu+m}(x)dx = 0,\ m > p,$$

(32) $$= 2\pi i \sum_{q=0}^{p-m} b_{-\nu-p,q}a_{\nu+m,p-m-q},\ m < p,$$

$$= 2\pi i,\ m = p.$$

Consider now the points δ for which $F(\nu+\delta+m)=F(\nu+\delta+r)$ for every pair r and m, $r\neq m$. For each pair there are at most n. The totality for all such pairs forms an enumerable set. The complementary set, then, is non-enumerable. There exists then a set $\alpha, \beta, \cdots, \tau, \cdots$ which has the limit zero and such that $F(\nu+\tau+m)\neq F(\nu+\tau+r)$, $r, m=0, \pm 1, \pm 2, \cdots$; $r\neq m$. Consider now the functions $y_{\nu+\tau+m}(x)$ and $P_{-\nu-\tau-p}(x)$ which are formed for $\lambda=\lambda_m'=F(\nu+\tau+m)$ and $\lambda=\lambda_p'=F(\nu+\tau+p)$ respectively. This is equivalent to replacing the K of the boundary conditions by $K_\tau'=e^{2\pi i\tau}\cdot K$. These latter functions come under the previous case and we can write from (31)

$$\sum_{q=0}^{p-m} b_{-\nu-\tau-p,q}a_{\nu+\tau+m,p-m-q} = 0.$$

We have

$$\mathop{\mathrm{L}}_{\tau\to 0} b_{-\nu-\tau-p,q} = b_{-\nu-p,q}, \quad \mathop{\mathrm{L}}_{\tau\to 0} a_{\nu+\tau+m,p-m-q} = a_{\nu+m,p-m-q}.$$

This follows from the continuity of the function involved in the equations satisfied by the a's and b's. Since (31) is only a finite sum we may write

$$\mathop{\mathrm{L}}_{\tau\to 0} \sum_{q=0}^{p-m} b_{-\nu-\tau-p,q}a_{\nu+\tau+m,p-m-q} = \sum_{q=0}^{p-m} b_{-\nu-p,q}a_{\nu+m,p-m-q} = 0.$$

Comparing with the second equation of (32) we get

$$\int P_{-\nu-p}(x)x^{-1}y_{\nu+m}(x)dx = 0,$$

which proves the lemma.

The series (29) is a series of analytic functions of x and t and converges uniformly if $|x|\leqq R<|t|$; hence it represents an analytic function of x and t. The difference

$$(t-x)^{-1} - \sum_{m=0}^{\infty} P_{-\nu-m}(t)t^{-1}x^{-\nu}y_{\nu+m}(x) = \sum_{m=0}^{\infty} A_m(t)x^m$$

is an analytic function of x and has an ascending power series expansion in x. To determine the coefficients $A_m(t)$ let t be any fixed point such that $0<|t|<\Gamma$. Multiply through by $x^{-1}P_{-\nu-m}(x)dx$ and integrate along the contour

C_1 defined by $|x| = R'' < |t| < \Gamma$. This is legitimate since the series converges uniformly along this circle. From $(t-x)^{-1}$ we obtain $t^{-1}P_{-\nu-m}(t)$ and from the series we get also $t^{-1}P_{-\nu-m}(t)$ and hence the right hand side gives zero. We have for $m=0, 1, 2, \cdots$,

$$\sum_{r=0}^{\infty} \int_{C_1} A_r(t) x^{r-1} P_{-\nu-m}(x) dx = 0.$$

Now

$$P_{-\nu-m}(x) = x^{-m}\left\{1 + \sum_{s=1}^{m} b_{-\nu-m,s} x^s\right\}.$$

Substituting in the above summation we see that if $r-m>0$ the integrals are zero, otherwise they are equal to $b_{-\nu-m,m-r}\cdot 2\pi i$; that is,

$$2\pi i \sum_{r=0}^{m} A_r(t) b_{-\nu-m,m-r} = 0.$$

The coefficient of $A_m(t)$ is $b_{-\nu-m,0}$ or unity, and $A_m(t)$ is expressed as a linear combination of $A_{m-1}(t)$, $A_{m-2}(t)$, etc. But for $m=0$ we have

$$A_0(t) b_{-\nu-m,0} = 0 \text{ or } A_0(t) = 0.$$

Hence by induction $A_m(t)=0$ for every m. This proves

THEOREM III. *Let x and t be two complex variables such that $|x| \leqq R' < \Gamma$ and $\Gamma > |t| \geqq R > R'$; then the expansion*

$$(33) \qquad (t - x)^{-1} = \sum_{m=0}^{\infty} t^{-1} P_{-\nu-m}(t) x^{-\nu} y_{\nu+m}(x)$$

is valid. If t is a variable point on the contour C defined by the circle $|t| = R$ and x a variable point within a smaller circle about $x=0$, then the expansion converges absolutely and uniformly with respect to t and x.

5. **Expansion of an arbitrary function.** Since the expansion (33) converges uniformly with respect to t on the circle C, then if $|x| < R$ we may write

$$\begin{aligned} f(x) &= \frac{1}{2\pi i}\int_C \frac{f(t)dt}{t - x} \\ &= \sum_{m=0}^{\infty} x^{-\nu} y_{\nu+m}(x) \frac{1}{2\pi i}\int_C f(t) P_{-\nu-m}(t) t^{-1} dt, \end{aligned}$$

where $f(x)$ is analytic within and on the circle of radius R. We may state this as the following

THEOREM IV. *Let $f(x)$ be a function of x which is analytic and single-valued inside and upon the contour C formed by the circle $|x| = R < \Gamma$; then the expansion*

$$x^\nu f(x) = \sum_{m=0}^{\infty} a_m y_{\nu+m}(x), \tag{34}$$

where

$$a_m = \frac{1}{2\pi i}\int_C f(t) t^{-1} P_{-\nu-m}(t)dt,$$

is valid in the circle $|x| = R'$, where $R' < R$. The expansion divided by x^ν converges absolutely and uniformly with respect to x provided that $|x| \leqq R'$.

The absolute and uniform convergence follows from bounding the coefficients a_m and the function $x^{-\nu}y_{\nu+m}(x)$ by use of the corollary of §3.

Let the Taylor's series for $f(t)$ be

$$f(t) = \sum_{s=0}^{\infty} \alpha_s t^s;$$

then

$$\begin{aligned} a_m &= \frac{1}{2\pi i}\int_C f(t)t^{-1}P_{-\nu-m}(t)dt \\ &= \frac{1}{2\pi i}\int_C t^{-1}\{\alpha_0 + \alpha_1 t + \cdots\}\{t^{-m} + b_{-\nu-m,1}t^{-m-1} + \cdots + b_{-\nu-m,m}\}dt. \end{aligned}$$

The coefficient of $1/t$ is $\sum_{n=0}^{m} b_{-\nu-m,m-n}\alpha_n$; hence

$$a_m = \sum_{n=0}^{m} b_{-\nu-m,m-n}\alpha_n \qquad (m = 0, 1, \cdots). \tag{35}$$

Let $f(x)$ be a single-valued analytic function in the annular region defined by $0 < r \leqq |x| \leqq R' < \Gamma$. Then by Cauchy's theorem

$$f(x) = \frac{1}{2\pi i}\int_C \frac{f(t)dt}{t - x} + \frac{1}{2\pi i}\int_c \frac{f(t)dt}{x - t},$$

where C and c are the contours $|t| = R' < \Gamma$ and $0 < |t| = r'$, $r < R'$, respectively. We use the expansion (33) for $1/(t-x)$ in the first integral and (33) with x and t interchanged for the second integral. Taking into account that these expansions converge uniformly we evidently have

THEOREM V. *If $f(x)$ is analytic and single-valued in the annular region defined by*

$$0 < r \leqq |x| \leqq R < \Gamma,$$

the expansion

$$f(x) = \sum_{m=0}^{\infty} a_m x^{-\nu} y_{\nu+m}(x) + \sum_{m=0}^{\infty} a_m' x^{-1} P_{-\nu-m}(x), \tag{36}$$

where

$$a_m = \frac{1}{2\pi i}\int_C f(t) t^{-1} P_{-\nu-m}(t) dt, \quad a_m' = \int_c f(t) t^{-\nu} y_{\nu+m}(t) dt,$$

is valid and converges absolutely and uniformly with respect to x when $r' \leqq |x| \leqq R'$ where $r < r'$ and $R' < R$.

If the Laurent expansion of $f(x)$ is

$$f(x) = \sum_{m=0}^{\infty} \alpha_m x^{-m} + \sum_{m=1}^{\infty} \alpha_m' x^{-m}$$

we can write

$$a_m = \sum_{n=0}^{m} b_{-\nu-n,m-n}\alpha_n, \quad a_m' = \sum_{n=0}^{\infty} a_{\nu+m+1,n}\alpha'_{m+n+1}. \tag{37}$$

6. **Extension to functions of several variables.** Let $x_1, t_1, x_2, t_2, \cdots, x_s, t_s$ be complex variables and let $f(x_1, x_2, \cdots, x_s)$ be a single-valued analytic function when $|x_1| < R_1 \leqq \Gamma$, $|x_2| < R_2 \leqq \Gamma, \cdots, |x_s| < R_s \leqq \Gamma$. The expansion (33) for $(t_k - x_k)^{-1}$ converges absolutely and uniformly with respect to x_k if t_k is restricted to the circle $|t_k| = R_k' < R_k$ and $|x_k| < R_k'$. A product of such absolutely convergent expansions for $k = 1, 2, \cdots, s$ will have the corresponding property for $x_1, \cdots, x_s$ and $t_1, \cdots, t_s$. We also have

$$f(x_1, x_2, \cdots, x_s) = \left(\frac{1}{2\pi i}\right)^s \int_{C_1}\int_{C_2}\cdots\int_{C_s} \frac{f(t_1, \cdots, t_s) dt_s \cdots dt_1}{(t_1 - x_1)\cdots(t_s - x_s)},$$

where $C_1, C_2, \cdots, C_s$ are contours defined by the circles

$$|t_1| = R_1', \ |t_2| = R_2', \cdots, |t_s| = R_s'.$$

The following extension is evident.

THEOREM VI. *Let $f(x_1, x_2, \cdots, x_s)$ be a function of $x_1, x_2, \cdots, x_s$ which is single-valued and analytic when $x_1, x_2, \cdots, x_s$ respectively satisfy the conditions $|x_1| < R_1$, $|x_2| < R_2, \cdots, |x_s| < R_s$; then the expansion*

$$f(x_1, x_2, \cdots, x_s) = \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \cdots$$

$$\sum_{m_s=0}^{\infty} A_{m_1,m_2,\cdots m_s} x_1^{-\nu_1} x_2^{-\nu_2} \cdots x_s^{-\nu_s} y_{\nu_1+m_1}(x_1) \cdots y_{\nu_s+m_s}(x_s), \tag{38}$$

where

$$A_{m_1,\cdots,m_s} = \left(\frac{1}{2\pi i}\right)^s \int_{C_1} \cdots$$

$$\int_{C_s} (t_1 t_2 \cdots t_s)^{-1} f(t_1, \cdots, t_s) P_{-\nu_1-m_1}(t_1) \cdots P_{-\nu_s-m_s}(t_s) dt_s \cdots dt_1,$$

is valid and converges absolutely and uniformly when $x_1, \cdots, x_s$ are respectively inside the circles $|x_1| = r_1, \cdots, |x_s| = r_s$, where $r_1 < R_1, \cdots, r_s < R_s$.

It is to be noted that it is not necessary that the expansions of the various $(t_k - x_k)$ used in proving the above theorem arise out of a single differential system. The various $y_{\nu_k+m_k}(x_k)$ may be replaced by a solution of a different differential system of the type considered here.

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EXPANSION PROBLEMS ASSOCIATED WITH A SYSTEM OF INTEGRAL EQUATIONS*

BY

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1. **Introduction.** Fredholm‡ has shown how a system of linear integral equations of the form

$$(1) \qquad y^{(i)}(x) = \lambda \int_a^b \sum_{\alpha=1}^n K_{i\alpha}(x;s) y^{(\alpha)}(s) ds + f^{(i)}(x) \quad (i = 1, 2, \cdots, n)$$

may be reduced to a single linear integral equation whose kernel is defined on $a \leqq x \leqq a+n(b-a)$, $a \leqq s \leqq a+n(b-a)$. Greggi§ considered a system of the form (1) and by use of the transformation introduced by Fredholm showed the form of the resolvent matrix for the system; for the symmetric system where $K_{ij}(x;s) = K_{ji}(s;x)$ $(i, j = 1, 2, \cdots, n)$ he also stated theorems analogous to those proved by Schmidt|| for a single integral equation with symmetric kernel. System (1) also comes under the class of systems treated by Platrier.¶

Weatherburn** has treated the system (1) without using the transformation introduced by Fredholm, but by vector method throughout. He states all results for the case $n = 3$, but his method of procedure is equally applicable to the general case.

In the present paper a special system of the form (1), to which is applied the term "definitely self-adjoint," is considered and the existence of a countable infinity of real characteristic numbers is established, together with expansion theorems in terms of the characteristic solutions of the system of integral equations. A definitely self-adjoint system of integral equations includes as a special case the symmetric system with closed matrix kernel. It also includes the system of integral equations to which a boundary value problem for a system of ordinary linear differential equations of the first order which

* Presented to the Society, April 18, 1930; received by the editors July 21, 1930.

† National Research Fellow in Mathematics.

‡ I. Fredholm, Acta Mathematica, vol. 27 (1903), pp. 365–390.

§ G. Greggi, Atti del Reale Istituto Veneto di Scienze, Lettere ed Arti, vol. 71 (1912), pp. 541–551.

|| E. Schmidt, Dissertation, Göttingen, 1905.

¶ Ch. Platrier, Journal de Mathématiques, (6), vol. 9 (1913), pp. 233–304; in particular, pp. 257–266.

** C. E. Weatherburn, Transactions of the Cambridge Philosophical Society, vol. 22 (1912–1923), pp. 133–158.

is definitely self-adjoint in the sense defined by Bliss* may be reduced by the introduction of the Green's matrix of the differential system. It is also shown that the idea of definite self-adjointness as defined by Bliss may be extended to a somewhat more general differential system, and that the system of integral equations to which the more general boundary value problem may be reduced by the introduction of the Green's matrix is also of the definitely self-adjoint type considered in this paper.

Matrix notation is used throughout this paper. All matrices used are denoted by capital letters and are supposed to have n rows and n columns, the element in the ith row and the jth column being denoted by the same letter with the subscript ij. Likewise, all vectors are supposed to have n components, and if v is a vector then $v^{(i)}$ is used to denote the ith component of v. If M is a matrix and v is a vector, then Mv denotes the vector whose ith component is $M_{i\alpha}v^{(\alpha)}$, where it is understood that α is an umbral label, i.e., its repetition in an expression indicates summation with respect to that label over the values $1, 2, \cdots, n$. Similarly, vM denotes the vector whose ith component is $v^{(\alpha)}M_{\alpha i}$. If M and N are matrices, then $M+N$ is the matrix $\|M_{ij}+N_{ij}\|$ and MN denotes the product matrix $\|M_{i\alpha}N_{\alpha j}\|$; also, if u and v are vectors, then uv denotes the scalar quantity $u^{(\alpha)}v^{(\alpha)}$.

2. **The Fredholm determinant and resolvent matrix.** We may write system (1) in vector form as

$$y(x) = \lambda \int_a^b K(x; s)y(s)ds + f(x), \tag{2}$$

where $y(x) \equiv (y^{(\alpha)}(x))$ and $f(x) \equiv (f^{(\alpha)}(x))$ are vectors each of whose components are continuous on $X: a \leqq x \leqq b$, and $K(x; s) \equiv \|K_{ij}(x; s)\|$ is a matrix. We will suppose that the discontinuities of each element $K_{ij}(x; s)$ $(i, j=1, 2, \cdots, n)$ are regularly distributed† in the square $W: a \leqq x \leqq b$, $a \leqq s \leqq b$, and that $|K_{ij}(x; s)|$ is bounded by a finite constant on W; finally, that $K_{ii}(x; x)$ is Riemann integrable on X.

As for a single integral equation we may prove for the vector equation (2) the existence of the Fredholm determinant $D(\lambda)$, which is a permanently converging series in λ, and the existence of the resolvent matrix $D(x; s|\lambda) \equiv \|D_{ij}(x; s|\lambda)\|$, each element of which converges absolutely and uniformly in W for all values of λ.

* G. A. Bliss, these Transactions, vol. 28 (1926), pp. 561–584.

† The discontinuities of a function of (x, s) are said to be regularly distributed in W if they all lie on a finite number of curves which have continuous tangents and no one of which is met by a line parallel to the axis of x or to the axis of s in more than a finite number of points. See Bôcher, *An Introduction to the Study of Integral Equations*, London, 1909, p. 3.

The matrix relations

$$D(x;s\,|\,\lambda) - D(\lambda)K(x;s) = \lambda\int_a^b K(x;t)D(t;s\,|\,\lambda)dt, \tag{3}$$

$$= \lambda\int_a^b D(x;t\,|\,\lambda)K(t;s)dt, \tag{4}$$

and the scalar relation

$$\frac{d^p}{d\lambda^p}D(\lambda) = -\int_a^b \frac{\partial^{p-1}}{\partial\lambda^{p-1}}\Big[\sum_{\alpha=1}^{n} D_{\alpha\alpha}(t;t\,|\,\lambda)\Big]dt \qquad (p = 1, 2, \cdots) \tag{5}$$

may be established by the same method that is used to prove the analogous relations for the single integral equation. The following theorem may also be proved, as for a single integral equation.*

THEOREM A. *If $\lambda = \lambda_0$ is not a root of $D(\lambda) = 0$, then* (2) *has a unique solution $y(x, \lambda_0)$ for $\lambda = \lambda_0$, and*

$$y(x, \lambda_0) = f(x) + \lambda_0\int_a^b \frac{D(x;s\,|\,\lambda_0)}{D(\lambda_0)}f(s)ds. \tag{6}$$

If λ_0 is a root of $D(\lambda) = 0$ of multiplicity m, then the equation

$$y(x) = \lambda\int_a^b K(x;s)y(s)ds \tag{7}$$

has $k(0 < k \leqq m)$ linearly independent solutions for $\lambda = \lambda_0$; furthermore, the associated vector integral equation

$$z(x) = \lambda\int_a^b z(s)K(s;x)ds \tag{8}$$

also has k linearly independent solutions for $\lambda = \lambda_0$. If $D(\lambda_0) = 0$, then a necessary and sufficient condition that (2) *have a solution for $\lambda = \lambda_0$ is that for every solution $z(x)$ of* (8) *for $\lambda = \lambda_0$, we have*

$$\int_a^b z(x)f(x)dx = 0. \tag{9}$$

3. **The definitely self-adjoint vector integral equation.** Throughout this section $f(x)$ and $g(x)$ are used to denote arbitrarily selected vectors whose components are continuous on X. Equation (7) is said to be definitely self-adjoint when the following conditions are satisfied:

* See Weatherburn, loc. cit., pp. 139–152. For a discussion of the Fredholm determinant and resolvent for a single integral equation, together with the proof of the Fredholm theorems, see Fredholm, loc. cit., Bôcher, loc. cit., pp. 38–46, or Goursat, *Cours d'Analyse*, Paris, vol. 3, pp. 368–380.

(H1) $K(x;\, s) \equiv H(x;\, s)S(s)$, where $S(x)$ is a symmetric matrix each of whose elements is real and continuous on X, and the discontinuities of the real elements $H_{ij}(x;\, s)$ are regularly distributed in the square W, $|H_{ij}(x;\, s)|$ is bounded by a finite constant on W, and $H_{ii}(x;\, x)$ is Riemann integrable on X;

(H2) $S(x)K(x;\, s) = K^*(s;\, x)S(s)$, where $K^*(x;\, s)$ is the transposed matrix of $K(x;\, s)$, i.e., $K^*_{ij}(x;\, s) = K_{ji}(x;\, s)$;

(H3) the bilinear form $\bar{v}S(x)v$, formed by the vectors $v \equiv (v^{(\alpha)})$ and $\bar{v} \equiv (\bar{v}^{(\alpha)})$, where $\bar{v}^{(\alpha)}$ is the conjugate imaginary of $v^{(\alpha)} (\alpha = 1, 2, \cdots, n)$, is non-negative on X and vanishes identically for a vector $f(x)$ of the form

$$f(x) = \int_a^b K(x;\, s)g(s)ds$$

only when $f(x) \equiv 0$;

(H4) if $\int_a^b K(x;\, s)f(s)ds \equiv 0$, then $S(x)f(x) \equiv 0$ on X.

Now consider an equation (7) which is definitely self-adjoint according to the above definition. If $y(x)$ is a solution of (7) for a characteristic number λ, we have in view of (H2) that $y(x)S(x)$ is a solution of the associated equation (8). Therefore, if $y(x)$ and $y_0(x)$ are solutions of (7) corresponding to characteristic values λ and λ_0, then

$$[\lambda - \lambda_0] \int_a^b y_0(x)S(x)y(x)dx = 0. \tag{10}$$

It then follows in view of (H3) that there exist no imaginary characteristic numbers for (7), and therefore

THEOREM 1. *For a definitely self-adjoint vector integral equation* (7) *all the zeros of the Fredholm determinant* $D(\lambda)$ *are real and the linearly independent characteristic solutions corresponding to each zero may be chosen real.*

Also, in view of (H3), we have the following theorem.

THEOREM 2. *If* $y_1(x), \cdots, y_k(x)$ *are linearly independent solutions of a definitely self-adjoint vector integral equation* (7) *for a characteristic number* λ, *then* $y_1(x)S(x), \cdots, y_k(x)S(x)$ *are linearly independent solutions of* (8) *for the same characteristic number.*

THEOREM 3. *If* λ_0 *is a root of* $D(\lambda) = 0$ *and* $f(x)$ *is such that*

$$y(x) = \lambda_0 \int_a^b K(x;\, s)y(s)ds + f(x) \tag{11}$$

has a solution, then there is a solution of the vector equation

$$y(x) = \lambda \int_a^b K(x; s)y(s)ds + f(x) \tag{12}$$

which is analytic in λ *at* $\lambda=\lambda_0$.

For since the zeros of $D(\lambda)$ are isolated, there is a neighborhood of $\lambda=\lambda_0$ in which there is no characteristic number of (7) distinct from λ_0. For λ in this neighborhood and $\lambda\neq\lambda_0$, the unique solution of (12) is given by

$$y(x, \lambda) = f(x) + \lambda \int_a^b \frac{D(x; s \mid \lambda)}{D(\lambda)} f(s)ds. \tag{13}$$

We will now show that near λ_0 the vector $y(x, \lambda)$ is still well-defined and analytic in λ. Let m be the multiplicity of λ_0 as a zero of $D(\lambda)$. Then $D(\lambda_0)=0 =d^pD(\lambda_0)/d\lambda^p(p=1, 2, \cdots, m-1)$, $d^mD(\lambda_0)/d\lambda^m\neq 0$, and therefore from (5) it follows that $\partial^{m-1}D(x; s|\lambda_0)/\partial\lambda^{m-1}\not\equiv 0$ on W. Suppose $\partial^rD(x; s|\lambda_0)/\partial\lambda^r$ is the first partial derivative of $D(x; s|\lambda)$ which is not identically zero on W for $\lambda=\lambda_0$. By (3), we have

$$\frac{\partial^r}{\partial\lambda^r}D(x; s \mid \lambda_0) = \lambda_0 \int_a^b K(x; t)\frac{\partial^r}{\partial\lambda^r} D(t; s \mid \lambda_0)dt. \tag{14}$$

If $r<m-1$, it would then follow that

$$\begin{aligned} \frac{\partial^{r+1}}{\partial\lambda^{r+1}} D(x; s \mid \lambda_0) &= \lambda_0 \int_a^b K(x; t) \frac{\partial^{r+1}}{\partial\lambda^{r+1}} D(t; s \mid \lambda_0)dt \\ &\quad + \int_a^b K(x; t) \frac{\partial^r}{\partial\lambda^r} D(t; s \mid \lambda_0)dt \\ &= \lambda_0 \int_a^b K(x; t) \frac{\partial^{r+1}}{\partial\lambda^{r+1}} D(t; s \mid \lambda_0)dt \\ &\quad + (1/\lambda_0) \frac{\partial^r}{\partial\lambda^r} D(x; s \mid \lambda_0). \end{aligned} \tag{15}$$

As each column of $\partial^rD(x; s|\lambda_0)/\partial\lambda^r$ is a solution of (7) for $\lambda=\lambda_0$, then each row of $[\partial^rD^*(x; s|\lambda_0)/\partial\lambda^r]S(x)$ is a solution of (8) for $\lambda=\lambda_0$, and therefore, since (15) has a solution, we have from Theorem A that

$$\int_a^b \frac{\partial^r}{\partial\lambda^r} D^*(x; s \mid \lambda_0)S(x) \frac{\partial^r}{\partial\lambda^r} D(x; s \mid \lambda_0)dx = 0,$$

and, in view of (H3), that $\partial^rD(x; s|\lambda_0)/\partial\lambda^r\equiv 0$ on X, which is a contradiction. Hence $r=m-1$. In view of (4) it follows that for each fixed x on X the rows

of $\partial^{m-1}D(x;s\,|\,\lambda_0)/\partial\lambda^{m-1}$ are solutions of (8) for $\lambda=\lambda_0$. Since, by hypothesis, there exists a solution of (11), from Theorem A we have

$$\int_a^b \frac{\partial^{m-1}}{\partial\lambda^{m-1}} D(x;s\,|\,\lambda_0)f(s)ds \equiv 0.$$

Hence (13) is well-defined near λ_0 and there exists a solution of (12) which is analytic in λ at $\lambda=\lambda_0$.

THEOREM 4. *If $f(x)$ is a vector which satisfies the relation*

$$\int_a^b y(x)S(x)f(x)dx = 0 \tag{16}$$

with every characteristic solution $y(x)$ of the definitely self-adjoint equation (7), *then $S(x)f(x)\equiv 0$ on X.*

Since if $y(x)$ is a characteristic solution of (7) corresponding to a characteristic number λ, then $y(x)S(x)$ is a characteristic solution of (8) for the same characteristic number, we have in view of Theorem 2 that the relation (16) implies

$$\int_a^b z(x)f(x)dx = 0,$$

and therefore, since $\lambda=0$ is not a zero of $D(\lambda)$, that

$$\int_a^b z(s)\left[\int_a^b K(s;x)f(x)dx\right]ds = 0,$$

for every characteristic solution $z(x)$ of (8). Then the equation

$$y(x) = \lambda\int_a^b K(x;s)y(s)ds + \int_a^b K(x;s)f(s)ds \tag{17}$$

has a solution for every value of λ. By Theorem 3 the solution $y(x,\lambda)$ of (17) is representable by a permanently convergent power series

$$y(x,\lambda) = u_0(x) + u_1(x)\lambda + u_2(x)\lambda^2 + \cdots . \tag{18}$$

By substituting (18) in (17) and comparing coefficients, we have

$$u_r(x) = \int_a^b K(x;s)u_{r-1}(s)ds \qquad (r = 0, 1, 2, \cdots), \tag{19}$$

where u_{-1} is defined as $f(x)$. Let

$$W_r = \int_a^b u_0(x)S(x)u_r(x)dx \qquad (r = 0, 1, 2, \cdots). \tag{20}$$

In the manner used by Bliss† to prove the analogous theorem for definitely self-adjoint boundary value problems, one may show that $W_0 = 0$, and therefore we have in view of (H3) that $u_0(x) \equiv 0$. It then follows from (H4) that $S(x)f(x) \equiv 0$ on X.

COROLLARY 1. *If* $|S(x)| \neq 0$, *then* $f(x) \equiv 0$ *is the only continuous vector satisfying* (16) *with all the characteristic solutions of* (7).

COROLLARY 2. *The vector* $f(x) \equiv 0$ *is the only vector of the form*

$$f(x) = \int_a^b K(x; s)g(s)ds$$

which satisfies (16) *with all the characteristic solutions of* (7).

THEOREM 5. *The totality of characteristic numbers and characteristic solutions of the definitely self-adjoint equation* (7) *is denumerably infinite and may be represented by* λ_i, $y_i(x)$ $(i = 1, 2, \cdots)$. *Furthermore, these characteristic solutions may be chosen normed and orthogonal in the sense that*

$$\int_a^b y_i(x)S(x)y_j(x)dx = E_{ij} \qquad (E_{ij} = 0 \text{ if } i \neq j,\ E_{ii} = 1). \tag{21}$$

For suppose there were only a finite number, $m-1$, of characteristic solutions of (7). Denote these by $y_1(x), \cdots, y_{m-1}(x)$. Then m continuous vectors $g_1(x), \cdots, g_m(x)$ may be chosen so that $S(x)g_1(x), \cdots, S(x)g_m(x)$ are vectors which are linearly independent on X. Let $f_i(x) = \int_a^b K(x; s)g_i(s)ds$ $(i = 1, 2, \cdots, m)$. There would then exist constants c_i not all zero and such that $\sum_{\alpha=1}^m f_\alpha(x)c_\alpha$ satisfies the relation (16) with all the solutions of (7). By the above Corollary 2 it then follows that $\int_a^b K(x; s)\,[\sum_{\alpha=1}^m g_\alpha(s)c_\alpha]ds \equiv 0$. But this, by (H2), implies that $\sum_{\alpha=1}^m S(x)g_\alpha(x)c_\alpha \equiv 0$, which is impossible unless $c_i = 0 (i = 1, 2, \cdots, m)$. Hence there is an infinity of characteristic numbers, and the number is denumerable since the zeros of $D(\lambda)$ are denumerable. The characteristic solutions of (7) may then be chosen to satisfy (21).‡

For a set of characteristic solutions $y_i(x)$ of (7) which are chosen to satisfy the relations (21), we have, in view of (H1) and (H4), the following lemmas which we state without proof.§

LEMMA 1. *If* $g(x)$ *is a vector each of whose components is continuous on* X, *then the series* $\sum_{\alpha=1}^\infty [\int_a^b g(x)S(x)y_\alpha(x)dx]^2$ *converges and is not greater in value than* $\int_b^a g(x)S(x)g(x)dx$.

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† Bliss, loc. cit., pp. 573–574.

‡ See Bliss, loc. cit., p. 575; also Schmidt, loc. cit., p. 4.

§ See Bliss, loc. cit., pp. 582, 583.

LEMMA 2. *If $h(x; s)$ is a vector each of whose components is bounded on W and for each fixed x on X is integrable in s on $a \leqq s \leqq b$, and $g(x)$ is a vector whose components are continuous on X, then the series $\sum_{\alpha=1}^{\infty} [\int_a^b g(s)S(s)y_\alpha(s)ds] \cdot [\int_a^b h(x; s)S(s)y_\alpha(s)ds]$ converges uniformly on X.*

THEOREM 6. *If $f(x)$ is a vector of the form*

$$f(x) = \int_a^b K(x; s)g(s)ds, \tag{22}$$

where $g(x)$ is a vector whose components are continuous on X, then

$$\phi(x) = \sum_{\alpha=1}^{\infty} \left[\int_a^b f(s)S(s)y_\alpha(s)ds \right] y_\alpha(x) \tag{23}$$

converges uniformly on X and $S(x)[f(x)-\phi(x)] \equiv 0$ on X.

For since

$$\begin{aligned} S(x)y_i(x) &= \lambda_i \int_a^b S(x)K(x; s)y_i(s)ds \\ &= \lambda_i \int_a^b y_i(s)S(s)K(s; x)ds \qquad (i = 1, 2, \cdots), \end{aligned}$$

in view of (H2), we have that

$$\int_a^b f(s)S(s)y_i(s)ds = (1/\lambda_i) \int_a^b g(s)S(s)y_i(s)ds. \tag{24}$$

Hence

$$\begin{aligned} \phi(x) &= \sum_{\alpha=1}^{\infty} (1/\lambda_\alpha) y_\alpha(x) \left[\int_a^b g(s)S(s)y_\alpha(s)ds \right] \\ &= \sum_{\alpha=1}^{\infty} \left[\int_a^b H(x; s)S(s)y_\alpha(s)ds \right] \left[\int_a^b g(s)S(s)y_\alpha(s)ds \right]. \end{aligned}$$

In view of the above Lemma 2 we then have that each component of $\phi(x)$ converges uniformly on X. We may then integrate $\int_a^b y_i(x)S(x)[f(x)-\phi(x)]dx$ term by term and obtain that this integral is zero for each $y_i(x)$. It then follows from Theorem 4 that $S(x)[f(x)-\phi(x)] \equiv 0$ on X.

COROLLARY. *If $|S(x)| \neq 0$ on X, then for every vector $f(x)$ of the form* (22) *the series $\phi(x)$ converges uniformly and represents $f(x)$ on X.*

THEOREM 7. *If $f(x)$ is of the form* (22) *and $g(x)$ is of the form*

$$g(x) = \int_a^b K(x; s)h(s)ds, \tag{25}$$

where $h(x)$ is a vector each of whose components is continuous on X, then the series $\phi(x)$ converges uniformly on X to the vector $f(x)$.

For by Theorem 6 we have that the series

$$\sum_{\alpha=1}^{\infty} y_\alpha(x)\left[\int_a^b y_\alpha(s)S(s)g(s)ds\right]$$

converges uniformly on X. From (24) it then follows that

$$f(x) - \phi(x) = \int_a^b K(x; s)\left[g(s) - \sum_{\alpha=1}^{\infty} y_\alpha(s)\int_a^b y_\alpha(t)S(t)g(t)dt\right]ds,$$

and since $S(x)[f(x)-\phi(x)]\equiv 0$ on X, we have from (H3) that $\phi(x)=f(x)$ on X.

4. **Remarks.** If for a definitely self-adjoint vector integral equation the matrix $S(x)$ is the identity matrix E, then the matrix kernel of the vector integral equation is symmetric and the vector integral equation corresponds to a single integral equation with a closed symmetric kernel.

Bliss† has treated a system of ordinary linear differential equations of the first order which may be written in vector form as

$$y' = [A(x) + \lambda B(x)]y, \tag{26}$$

where $A(x)$ and $B(x)$ are matrices of n rows and columns whose elements are continuous on the interval X and y is a vector with n components, each of which is continuous and has a continuous first derivative on X satisfying (26). With (26) are associated boundary conditions

$$My(a) + Ny(b) = 0, \tag{27}$$

where M and N are constant matrices such that the matrix $\|M, N\|$ is of rank n. According to Bliss the boundary value problem (26), (27) is said to be *self-adjoint* if the differential equations and also the boundary conditions of the adjoint system are equivalent to its own for all values of λ by means of a transformation $z=T(x)y$, where the elements of $T(x)$ are real, single-valued, and have continuous first derivatives on X, and such that $|T(x)|\neq 0$ on X.

The boundary value problem (26), (27) is said to be *definitely self-adjoint* if the matrix $S(x)\equiv T^*(x)B(x)$ is symmetric, the bilinear form $\bar{f}S(x)f$ is non-

† Bliss, loc. cit.

negative on X, and this form vanishes identically for a vector $f(x)$ which is a solution of an equation of the type

$$f'(x) = A(x)f(x) + B(x)g(x), \tag{28}$$

where $g(x)$ is an arbitrarily chosen vector whose components are continuous, only when $f(x) \equiv 0$ on X.

We may assume without loss of generality that the system

$$y' = A(x)y, \tag{29}$$

together with the boundary conditions (27), is incompatible. Then (26), (27) is equivalent to the vector integral equation

$$y(x) = \lambda \int_a^b G(x; s)B(s)y(s)ds, \tag{30}$$

where $G(x; s)$ is the Green's matrix for (29), (27). The matrix kernel of (30) is of the form $H(x; s)S(s)$, where $H(x; s) = G(x; s)T^{*-1}(s)$ and $S(x) = T^*(x) \cdot B(x)$. By using the properties of the Green's matrix for a definitely self-adjoint boundary value problem as established by Bliss,† it may be shown that (30) is a definitely self-adjoint vector integral equation, according to the definition of this paper. It is also to be noted that condition (H3) used in defining a definitely self-adjoint vector integral equation is somewhat weaker than the corresponding condition given by Bliss in defining a definitely self-adjoint boundary value problem.

If now the elements of $A(x)$ and $B(x)$ in (26) are not continuous, but merely Lebesgue summable, we define as a solution of (26) a vector y whose components are absolutely continuous and which satisfies (26) "almost everywhere" on X. The system (26), (27) will then be said to be self-adjoint if the differential equations and also the boundary conditions of the adjoint system are equivalent to its own for all values of λ by means of a transformation $z = T(x)y$, where now the elements of $T(x)$ are real, single-valued, such that $|T(x)| \neq 0$ on X, and merely absolutely continuous. By analogy with the definition of Bliss, the system (26), (27), when the elements of $A(x)$ and $B(x)$ are merely Lebesgue summable, is said to be definitely self-adjoint when $S(x) \equiv T^*(x)B(x)$ is symmetric, $\bar{f}S(x)f \geqq 0$, and if $\bar{f}(x)S(x)f(x) = 0$ "almost everywhere" on X for a solution of (28), where $g(x)$ is an arbitrary summable vector, then $f(x) \equiv 0$. System (26), (27) is then equivalent to the vector integral equation (30), where now it is understood that the integral is taken in the sense of Lebesgue. If, however, the elements of $B(x)$ are continuous, while

† See Bliss, loc. cit., pp. 577–581.

the elements of $A(x)$ are summable, then the matrix kernel of (30) satisfies the continuity condition of (H1), and (30) is again a definitely self-adjoint vector integral equation, according to the definition of this paper.

In defining a definitely self-adjoint vector integral equation we have imposed on the matrices $S(x)$ and $H(x; s)$ the continuity condition of (H1). Clearly a related treatment for a vector integral equation in which the matrix kernel satisfies weaker continuity conditions may be carried out, just as the treatment of a single scalar integral equation has been carried through under weaker conditions.

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ON THE DERIVATIVES OF HARMONIC FUNCTIONS ON THE BOUNDARY*

BY

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Abstract. Let U be harmonic in a closed region R, whose boundary contains a regular surface element E, with a representation $z=\phi(x, y)$. If E has bounded curvatures, and if $\phi(x, y)$ and the boundary values of U on E have continuous derivatives of order n which satisfy a Dini condition, then the partial derivatives of U of order n exist, as limits, on E, and are continuous in R at any interior point of E. Hölder conditions on the boundary values of U, or on their derivatives of order n, imply Hölder conditions on U, or the corresponding derivatives, in R, in the neighborhood of the interior points of E.

1. **Introduction.** A large number of articles contain studies of the existence and behavior of the limits of the derivatives, on the boundary, of harmonic functions, when these are given as the potentials of various spreads of attracting matter. On the other hand, studies of the derivatives of harmonic functions defined directly by their boundary values are surprisingly few, particularly in space of three dimensions. In a paper of my own,† the problem for the logarithmic potential has been investigated. In space, there are few actual results on derivatives of order higher than the first, and the conditions imposed on the boundary values are much heavier than need be.

The method used in previous work has been to express the given harmonic function as the potential of a double distribution, through a Neumann series. While this method has not yet yielded the results of which it is capable, it contains an element of indirectness, in that the conditions on the boundary values must first be translated into conditions on the moment of the double distribution, and from these, the behavior of the derivatives of the harmonic function must then be inferred. The method here used is based on Poisson's integral, applied to a sphere internally tangent to the boundary. In the case of the derivatives of the first order, this method requires more than is necessary for the theorems, for in order to apply it, we must assume that spheres, internally tangent to the boundary, and containing no exterior

* Presented to the Society, February 28, 1931; received by the editors January 23, 1931.

† *Harmonic functions and Green's integral*, these Transactions, vol. 13 (1912), pp. 109–132. References to the literature are given there, and in my two previous papers, ibid., vol. 9 (1908), pp. 39–66.

points, exist. For derivatives of higher order, however, this requirement ceases to be extraneous. As the Neumann method is comparatively simple for derivatives of the first order, the two procedures appear to complement each other nicely.

The results, in their generality, for derivatives of higher order are new. Those for derivatives of the first order are in every respect more general than any at hand, with the exception of Liapounoff's,* who requires less of the boundary surface, but more of the boundary values. The results here obtained with respect to Hölder conditions appear to be new for $n>1$, and those for U itself are more general than those at hand.† As an incidental result, a simple proof is given of the analytic character of harmonic functions.‡

2. **The derivatives of first order of Poisson's integral.** Let U be harmonic in a sphere of radius a. We consider first its derivative in the direction of its polar axis, $\theta=0$, at a point of that axis. Writing Poisson's integral in the form

$$U(\rho) = \frac{a(a^2-\rho^2)}{2}\int_0^\pi \frac{f(\theta)\sin\theta}{r^3}\,d\theta,\quad f(\theta) = \frac{1}{2\pi}\int_0^{2\pi} U(a,\phi,\theta)\,d\phi,$$

$$r^2 = a^2+\rho^2-2a\rho\cos\theta,$$

we find, for $\rho<a$,

$$\frac{\partial U}{\partial \rho} = -\,a\rho J_1 - \frac{3a(a+\rho)}{2}J_2,$$

where

$$J_1 = \int_0^\pi \frac{f(\theta)\sin\theta d\theta}{r^3},\quad J_2 = (a-\rho)\int_0^\pi \frac{f(\theta)(\rho-a\cos\theta)\sin\theta d\theta}{r^5}.$$

Our task is to show that these integrals approach limits as $\rho\to a$, under suitable conditions on $f(\theta)$, and to observe something as to the rate of approach. Assuming the existence of $f'(\theta)$ near $\theta=0$, it can be shown, by an integration

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* *Sur certaines questions qui se rattachent au problème de Dirichlet*, Journal de Mathématiques, (5), vol. 4 (1898), p. 241. To the literature cited in my papers referred to above, should be added Korn, Mathematische Annalen, vol. 53 (1900), pp. 593–608; P. Lévy, *Sur l'allure des fonctions de Green et de Neumann dans le voisinage du contour*, Acta Mathematica, vol 42 (1920), pp. 207–267.

† See Korn, *Sur les équations de l'élasticité*, Annales de l'Ecole Normale, (3), vol. 24 (1907), pp. 23, 25.

Since the writing of this paper, I have learned of one by Schauder, *Potentialtheoretische Untersuchungen, Erste Abhandlung*, about to appear in the Mathematische Zeitschrift. The contacts of the two papers are confined to results on Hölder conditions on U and its derivatives of the first order. Those for U itself are essentially the same; for the derivatives of first order, Schauder's are more general than mine, in that bounded curvatures of the bounding surface are not required.

‡ I wish to acknowledge my indebtedness to my colleague, Dr. Gergen, for his careful examination of the manuscript.

by parts, that the normal derivative of U approaches a limit provided $f'(\theta)$ satisfies a condition of the type used by Dini,* namely that the integral

$$\int_0^\eta \frac{|f'(\theta)|}{\theta}\, d\theta$$

is convergent. For our purposes, however, a somewhat different condition will be used. We shall show, namely, that the normal derivative of U has a limit provided $f(\theta)$ is integrable and bounded, and such that the integral

$$\int_0^\eta \frac{|f(\theta) - f(0)|}{\theta^2}\, d\theta$$

is convergent.

It is legitimate to assume $f(0) = 0$, since the subtraction of a constant from U affects neither its derivatives nor the validity of the hypotheses. With a number η, $0 < \eta \leqq \pi/2$, we break up the integrals J_1 and J_2 each into two,

$$J_1 = J_{11} + J_{12}, \quad J_2 = J_{21} + J_{22},$$

J_{11} and J_{21} being extended over the interval $(0, \eta)$, and J_{12} and J_{22}, over the interval (η, π). Then, for any fixed η, the functions J_{12} and J_{22} are analytic in ρ at $\rho = a$, and hence have limits from which they differ arbitrarily little for all ρ sufficiently near a. Hence, if it can be shown that η can be so restricted that J_{11} and J_{21} are arbitrarily small in absolute value, independently of ρ, the existence of a limit for the derivative of U will be established.

But this is immediate. From the equations

$$r^2 = (a - \rho)^2 + 4a\rho \sin^2 \frac{\theta}{2} = (a - \rho \cos\theta)^2 + \rho^2 \sin^2\theta$$
$$= (a\cos\theta - \rho)^2 + a^2 \sin^2\theta,$$

we derive the inequalities

$$\frac{a - \rho}{r} \leqq 1, \quad \frac{|\rho - a\cos\theta|}{r} \leqq 1, \quad \frac{\theta}{r} \leqq \frac{\pi \sin\theta}{2r} \leqq \frac{\pi}{2a},$$

the last holding for $0 \leqq \theta \leqq \eta$, since $\eta \leqq \pi/2$. Using them, we find

$$|J_{11}| \leqq \left(\frac{\pi}{2a}\right)^3 \int_0^\eta \frac{|f(\theta)|}{\theta^2}\, d\theta, \quad |J_{21}| \leqq \left(\frac{\pi}{2a}\right)^3 \int_0^\eta \frac{|f(\theta)|}{\theta^2}\, d\theta.$$

The integrals are convergent, by hypothesis, and so approach 0 with η. As they are independent of ρ, the existence of the limits of J_1 and J_2, and so of the normal derivative of U, is established. We note, moreover, that for fixed

* Acta Mathematica, vol. 25 (1902), p. 224.

η, the derivatives of J_{12} and J_{22} with respect to ρ are bounded in absolute value by a number depending only on the bound for $|f(\theta)|$, so that we may enunciate the results as follows:

THEOREM I. *Let U be harmonic in the sphere of radius a, and be given by Poisson's integral with the bounded integrable boundary values $U(a, \phi, \theta)$. Let the average of these values on parallel circles,*

$$f(\theta) = \frac{1}{2\pi}\int_0^{2\pi} U(a, \phi, \theta)d\phi,$$

be subject to the requirement that the integral

$$\int_0^{\eta} \frac{|f(\theta) - f(0)|}{\theta^2} d\theta, \quad 0 < \eta \leqq \frac{\pi}{2}, \tag{1}$$

be convergent. Then the derivative of U in the direction of the polar axis $\theta = 0$ approaches a limit at the surface of the sphere for approach along the polar axis. Moreover, the approach to the limit is uniform for any class of boundary functions which are uniformly bounded in absolute value, and for which the integral (1) *approaches* 0 *uniformly with η.*

Tangential derivatives. A similar theorem exists for the tangential derivatives. The derivative of U in the direction of increasing θ in the meridian half-plane $\phi = \phi_0$, at a point of the polar axis, is given by

$$\frac{1}{a}\frac{\partial U}{\partial \theta} = \frac{3a\rho(a^2 - \rho^2)}{2}\int_0^{\pi} \frac{F(\theta)\sin^2\theta}{r^5} d\theta,$$

$$F(\theta) = \frac{1}{2\pi}\int_0^{2\pi} U(a, \phi, \theta)\cos(\phi - \phi_0)d\phi.$$

The same reasoning as that just employed then leads to

THEOREM II. *Theorem* I *holds also for the tangential derivatives of U, provided the function $F(\theta)$ satisfies the conditions there imposed on $f(\theta)$.*

Remark. Even if $f(\theta)$ and $F(\theta)$ have continuous derivatives of the first two orders, the conditions of Theorems I and II will not be fulfilled, unless these functions, and their first derivatives, vanish at $\theta = 0$. This difficulty, however, may at once be met by the subtraction from U of a linear function, tangent to U at $\theta = 0$, $\rho = a$. The theorems are therefore more general than at first appears.

Limiting values. Under the hypotheses imposed on $f(\theta)$ and $F(\theta)$, it will be seen that the limiting values of the normal and tangential derivatives of U are given by the convergent integral

$$(2)\qquad \frac{\partial U}{\partial \rho} = -\frac{1}{4a}\int_0^\pi \frac{[f(\theta) - f(0)]\cos\frac{\theta}{2}}{\sin^2\frac{\theta}{2}}\,d\theta,$$

and by the derivatives in the same direction of the boundary values of U, respectively.

3. **Formulation of conditions which insure the existence and continuity of derivatives on the boundary.** An advantage of the present method of study is its local character. If we wish to consider the behavior of the derivatives of a harmonic function U only on a portion of the surface S bounding the region in which it is given, we need make special hypotheses on S and the boundary values of U only on this portion. Accordingly, we shall deal with a region whose boundary contains a *regular surface element* E, that is, a set of points, which is given, for a suitable orientation of the coördinate axes, by an equation

$$z = \phi(x, y),$$

$\phi(x, y)$ being one-valued, and having continuous derivatives of the first order, for (x, y) in a closed regular region of the (x, y)-plane. A regular region of the plane is one bounded by a regular curve, without double points.*

We shall employ the following conditions.

CONDITION A_n. *R is a bounded open continuum, whose boundary S contains a regular surface element E, with the following properties:*

(a) *it has definite radii of curvature at each point, which are uniformly bounded;*

(b) *with coördinate axes tangent and normal to E at any point p, it admits a representation $z=\phi(x, y)$, where z is one-valued and has continuous partial derivatives, of order n, with respect to x and y, which are such that if q is any second point of E, and $D^n\phi$ any definite one of these derivatives,*

$$| D^n\phi(q) - D^n\phi(p) | \leqq D(t),$$

where t is the projection of pq on the tangent plane at p, and where $D(t)$ is a never decreasing function, independent of p and of the direction of pq, such that

$$\int_0^\eta \frac{D(t)}{t}\,dt \quad (0 < \eta)$$

is convergent;

* For full details on these definitions, see Kellogg, *Foundations of Potential Theory*, Berlin, 1929, particularly p. 105.

(c) *to every regular surface element E' contained in the interior of E, there corresponds a positive number a_1, such that the sphere of radius a_1, about any point of E', contains no points of S other than those of E.*

CONDITION B_n. *The function $U=U(x, y, z)$ is one-valued and continuous in R, and harmonic in the interior of R. Its values $U(x, y, \phi(x, y))$ on E, the axes being tangent and normal to E at any point p, are subject to the condition imposed on $\phi(x, y)$ in condition A_n*(b).

Remarks. The requirement of a representation $z=\phi(x, y)$, where z is single-valued and has continuous derivatives, for a single tangent-normal position of the axes, does not, of itself, assure such a representation for all such positions of the axes, even though E be arbitrarily flat.* It does so, however, if, in addition to the requirement that every pair of normals make an acute angle with each other, we demand that the projection of E on one of its tangent planes be convex. It is convenience of application which has dictated the expression of the condition A_n(b) in this form. The condition A_n(c) excludes multiple boundary points on E. Otherwise S is unrestricted except that it must be bounded. A_n(a) is a consequence of A_n(b) for $n \geqq 2$.

We may now formulate the main theorem of the paper.

THEOREM III. *Let R satisfy Condition A_n and U Condition B_n. Let P be a point of R on the normal to E' at any of its points p. Then any given derivative of U, of order n, with respect to x, y and z, approaches a limit as P approaches p along the normal. If it is defined at p as equal to this limit, it is then a continuous function on E', for unrestricted approach.*

As the method of proof of this theorem is different, for $n=1$, from that for $n>1$, we consider the cases separately.

4. **Existence and continuity of the derivatives of the first order.** In order to infer properties of the boundary values of U on a sphere, internally tangent to E, from known properties of the boundary values on E, we shall need preliminary information on the rapidity of approach of U to its boundary values on E. This we shall obtain by means of a harmonic dominant function. It is the need of this function which largely accounts for the difference in the treatments of the derivatives of the first, and of higher orders.

First, however, we shall have need of a surface element E'', intermediate between E and E'. Let E'' denote the portion of E whose points are distant not more than $a_1/2$ from E'. Let a_2 be the lower bound of the radii of curvature of E, and let a be the less of the two positive numbers $a_1/8$ and $a_2/2$. Then a will have the properties

* See *Foundations of Potential Theory*, loc. cit., p. 107, also Theorem VII, p. 108.

(a) *any sphere of radius* $4a$ *about a point of* E'' *contains no points of* S *except those of* E,

(b) *any sphere of radius* $4a$ *about a point of* E' *contains no points of* S *except those of* E'',

(c) *the sphere* σ_i, *of radius* a, *internally tangent to* E *at any point* p *of* E', *will lie in the interior of* R *except at* p, *and the sphere* σ_e, *of radius* a, *externally tangent to* E *at any point* p *of* E', *will be exterior to* R *except at* p.

Let p be a point of E''. With axes in the tangent-normal position at p, we form the linear function

$$G_p = Ax + By,$$

A and B being the derivatives at p of the boundary values of U, with respect to x and y, respectively. Then $U_p = U - G_p$ is harmonic in R, and has boundary values on E which vanish, together with their derivatives of first order, at p. Moreover, G_p, and any of its derivatives, are uniformly bounded in R. As a consequence U_p is bounded in absolute value in R, by a constant M, independent of p. The law of the mean, and condition B_1, now yields, for any point q of E, not distant more than $4a$ from E'',

$$|U_p(q)| = |U_p'(\bar{q})|\, t = |U_p'(\bar{q}) - U_p'(p)|\, t \leqq tD(\bar{t}) \leqq tD(t), \tag{3}$$

the bars indicating appropriate mean points or values.

We now take up the harmonic dominant function. It is

$$W = \rho^\lambda P_\lambda(\cos\theta) \qquad (0 < \lambda < 1),$$

where $P_\lambda(u)$, $u = \cos\theta$, is that solution of Legendre's differential equation,

$$\frac{d}{du}(1 - u^2)\frac{dP_\lambda}{du} + \lambda(\lambda + 1)P_\lambda = 0,$$

which is regular at $u = 1$, there assuming the value 1. The greatest root of this function in the interval $(-1, +1)$, if it has any, is negative. Under any circumstances, there is a positive number α, which we may take less than $\pi/2$, such that for $\cos(\pi/2+\alpha) \leqq u \leqq 1$, $P_\lambda(u)$ is positive, and in this interval, $P_\lambda(u)$ is increasing. The last statement may be verified by forming the power series for $P_\lambda(u)$ in $z = 1 - u$, which converges for $|z| < 2$, and has all its coefficients, after the constant term, negative.

We have, then, the following properties for W. It is continuous in the region

$$0 \leqq \rho, \quad 0 \leqq \theta \leqq \pi/2 + \alpha, \tag{4}$$

and harmonic in the interior. Its value at any point (ρ, ϕ, θ) lies between ρ^λ and its boundary values

$$W = C\rho^\lambda,\quad C = P_\lambda\left(\cos\left(\frac{\pi}{2}+\alpha\right)\right) = P_\lambda(-\sin\alpha) > 0.$$

Let us take, as origin of the spherical coördinates in terms of which W is expressed, a point p of E'', the axis from which θ is measured being the inward normal. By the property (c) of the number a, all points of R within a distance $2a\sin\alpha$ of p will lie in the region (4). The derivatives of first order of the boundary values of U, being continuous in a closed region, are uniformly bounded. The same is true for G_p. Hence by the first equation (3), the boundary values of U_p do not exceed, in absolute value, a uniform constant times W, in the sphere σ of radius $2a\sin\alpha$ about p. On the portion of σ in R, $|U_p|\leqq M$, and as W has here a positive lower bound, there is a uniform constant, A, such that on the whole boundary of the portion of R in σ, $|U_p|\leqq AW$. As U_p and W are harmonic in this region, the inequality also holds in its interior. This leads to the inequality

$$(5)\qquad |U_p(Q)| \leqq A\overline{pQ}^\lambda,$$

valid, first, for any point Q of R in σ. But since $|U_p|$ is bounded throughout R, and ρ^λ is an increasing function, the number A can be so chosen that the inequality holds throughout R.

Finally, since the derivatives of G_p are uniformly bounded in R, we have, for suitable B',

$$(6)\qquad |G_p(Q) - G_p(p)| \leqq B'\overline{pQ},$$

and hence, combining (5) and (6),

$$(7)\qquad |U(Q) - U(p)| \leqq B''\overline{pQ}^\lambda,$$

first, for $\overline{pQ}\leqq 1$, and then for any Q in R, B'' being a constant independent of p. But (6) holds, if, without changing the linear function G_p, we substitute for the argument point p, any other point q of E'', and the same substitution may be made in (7). Hence we have, on combining the inequalities (6) and (7), thus altered,

$$(8)\qquad |U_p(Q) - U_p(q)| \leqq B\overline{qQ}^\lambda,$$

where q is any point of E'', Q any point of R, and B is a constant, independent of p.

This result is valid for any λ in the open interval (0, 1), the constant B depending, in general, on λ. For immediate purposes, we shall assign to λ a value greater than 1/2.

5. **Completion of the proof of the theorem for $n=1$.** Let p now denote any point of E', σ_i the sphere of radius a, internally tangent to E' at p, and Q

a point of the lower half of the surface of σ. We wish to know that Q is on a normal to E at a point of E''; for although the inequality (8) could be used without this knowledge, it will be useful later. By the properties (b) and (c) of the number a, we know that E lies between σ_i and the sphere σ_e externally tangent to E' at p, until it passes out of the sphere of radius $4a$ about p. Hence E must cut the sphere Σ, through Q, and tangent to σ_e at the extremity of the radius which points toward Q. There are therefore points of E within a distance d of Q, where d is the diameter of Σ. We may find an appraisal for d by the cosine law of trigonometry. If θ is the angle between the radii of σ_i to p and Q,

$$(a + d)^2 = a^2 + (2a)^2 - 2(2a)a\cos\theta, \text{ or } d(2a + d) = 4a^2(1 - \cos\theta),$$

so that

$$d = \frac{8a^2 \sin^2 \dfrac{\theta}{2}}{2a + d} \leqq 4a \sin^2 \frac{\theta}{2} \cdot$$

Since Q is on the lower half of σ_i, $\theta \leqq \pi/2$, and $d \leqq 2a$. If q is the point* of E nearest Q, its distance from Q cannot exceed d, since, as we have seen, there are points of E within Σ. Hence $\overline{pq} \leqq \overline{pQ} + d \leqq 2^{1/2}a + 2a < 4a$, and by the property (b) of a, q therefore lies on E''.

We now use the inequality (8). Since $\overline{qQ} \leqq d \leqq a\theta^2$, this yields

$$| U_p(Q) - U_p(q) | \leqq Ba^\lambda \theta^{2\lambda}. \tag{9}$$

On the other hand, since $t = a \sin\theta \leqq a\theta$, (3) yields

$$| U_p(q) | \leqq a\theta D(a\theta). \tag{10}$$

Combining the inequalities (9) and (10), we see that the values $U_p(Q)$ on the surface of σ_i are subject to the inequality

$$| U_p(a, \phi, \theta) | \leqq \theta(Ba^\lambda \theta^{2\lambda - 1} + aD(a\theta)) \qquad (\theta \leqq \pi/2).$$

It follows that the hypotheses of Theorems I and II are in force, since $2\lambda - 1 > 0$. Accordingly, the derivative of U_p, in any fixed direction, approaches a limit at p along the normal, and this, uniformly as to p. As the derivatives of G_p are bounded, uniformly as to p, the derivative of U itself approaches limits on E' along normals, uniformly. As the derivative is continuous in the interior of R, we infer that the same limits are approached for unrestricted approach of the argument point to the boundary. The assignment of these limiting values to the derivative, as values on E', therefore

* Or any of them, in case there are more than one. A similar comment applies at several points in the sequel.

makes the derivative continuous at the points of E'. Theorem III is thus proved for $n=1$.

Remarks on Condition B_1. It is known* that continuity of the boundary values, say on a circle, of the real part of a function of a complex variable, analytic in the circle, is not sufficient for the continuity of the boundary values of the conjugate function. We may conclude, by an integration, and by noting that a harmonic function of x and y may also be considered a harmonic function of x, y and z, that something stronger than mere continuity must be required of the derivatives of the boundary values of U if we are to have continuous normal derivatives. The condition selected, although somewhat conditioned by the proof, is a fairly liberal one. It is clearly less restrictive than a Hölder condition on the derivatives:

$$| U'(q) - U'(p) | \leqq At^{\lambda} \qquad (0 < \lambda < 1).$$

In fact, if merely

$$| U'(q) - U'(p) | \leqq A/[\log^{\alpha} (k/t)] \qquad (\alpha > 1),$$

where k exceeds the maximum value t assumes, the function on the right will be seen to have the properties required of $D(t)$ in Condition B_1.

6. **The derivatives of harmonic functions at interior points. Analytic character.** We shall need bounds for the derivatives of U at interior points of R. We may obtain these by applying a familiar inequality. Let V be harmonic in the sphere of radius c about P, and have there the upper and lower bounds M and m. Then if DV denote the derivative of V in any given direction, its value at P is subject to the inequality†

$$(11) \qquad | DV | \leqq \frac{3}{4c}(M - m),$$

or, in terms of the upper bound M of the absolute value of V on the sphere,

$$(12) \qquad |DV| \leqq \frac{3M}{2c} \cdot$$

If V is defined in a region R, and M is the maximum of $|V|$ in R, c may be understood as the distance from P to the nearest boundary point of R.

We next seek a bound for the absolute value of the derivative D^2V of DV in any given direction, by applying (12) to a sphere of radius uc about P, $0<u<1$. We have

* See, for instance, Kellogg, *Potential functions on the boundary of their regions of definition*, these Transactions, vol. 9 (1908), p. 39, footnote ‡.

† See, for instance, *Foundations of Potential Theory*, loc. cit., p. 227.

$$| D^2V | \leqq \frac{3}{2uc} \frac{3M}{2(1-u)c},$$

$(1-u)c$ being the distance from the sphere to the nearest point of the boundary of R. The result holds for any u in the given interval, and is closest when $u=1/2$. It then gives

$$| D^2V | \leqq \left(\frac{3}{2}\right)^2 M \frac{2^2}{c^2},$$

c being again the distance from P to the nearest boundary point. Continuing in this way, we find for any derivative of V, of order n,

$$(13) \qquad | D^nV | \leqq \left(\frac{3}{2}\right)^n M \frac{n^n}{c^n},$$

as we proceed to verify, by induction.

Assuming the formula (13), let us find, by means of (12), a bound for the value at P of the derivative in any given direction, of the harmonic function D^nV. On the sphere of radius uc about P, the absolute value of D^nV does not exceed

$$\left(\frac{3}{2}\right)^n M \frac{n^n}{(1-u)^n c^n}.$$

Using this bound in (12), and replacing c by uc in that inequality, we find

$$| D^{n+1}V | \leqq \left(\frac{3}{2}\right)^{n+1} M \frac{n^n}{c^{n+1}} \frac{1}{u(1-u)^n}.$$

We choose u so that the last factor will take its least value,

$$u = \frac{1}{n+1}, \quad \frac{1}{u(1-u)^n} = \frac{(n+1)^{n+1}}{n^n}.$$

The inequality for D^nV thus obtained coincides with that given by the formula (13) when n is there replaced by $n+1$. As it is valid for $n=1$, (13) therefore holds generally.

As $n^n \leqq n!e^n$, this inequality may be given the form

$$(14) \qquad | D^nV | \leqq \left(\frac{3e}{2c}\right)^n Mn!.$$

Suppose V be developed in a Taylor series about the interior point P of R, with remainder. It will be found, by means of this bound for the derivatives of V, that for points whose distance from P is less than $c/4$, the remain-

der after the terms of degree n approaches 0 as n becomes infinite. The infinite series therefore converges to V in this neighborhood of P. We thus have a simple proof that V is analytic at any interior point of R.

However, the purpose for which the inequality (14) was derived was the study of the derivatives of higher order of Poisson's integral. The factor of the integrand which concerns us is

$$g = g(x, y, z) = \frac{a^2 - \rho^2}{r^3},$$

where r is measured from the point $Q(\xi, \eta, \zeta)$ of the surface of the sphere of radius a about the origin of coördinates O to the point $P(x, y, z)$ in the sphere, and where ρ is the distance OP. As it stands, bounds for g, which is harmonic throughout space, except at Q, are not evident, at least not in a form adapted to our needs. However, if we write ψ for the angle PQO, we have $\rho^2 = a^2 + r^2 - 2ar \cos \psi$, so that g becomes

$$g = \frac{2a \cos \psi}{r^2} - \frac{1}{r},$$

the terms on the right being harmonic except at Q, and being bounded in absolute value, at a distance r from Q, by $2a/r^2$, and by $1/r$, respectively.

If, now, we replace $D^2 V$ by $(2a \cos \psi)/r^2$, and, correspondingly, M by $2a/3^2$, (14), with n replaced by $n+2$, becomes

$$\left| D^n \frac{2a \cos \psi}{r^2} \right| \leqq \left(\frac{3e}{2r}\right)^{n+2} \frac{2a}{3^2}(n+2)!.$$

Similarly, if we replace DV by $1/r$, and M by $2/3$, (14), with n replaced by $n+1$, becomes

$$\left| D^n \frac{1}{r} \right| \leqq \left(\frac{3e}{2r}\right)^{n+1} \frac{2}{3}(n+1)!.$$

Combining these results, we have, for all points in the sphere, since there $r \leqq 2a$,

$$|D^n g| \leqq 10a \left(\frac{3e}{2}\right)^n \frac{(n+2)!}{r^{n+2}}. \tag{15}$$

7. **The derivatives of order n of Poisson's integral.** We write Poisson's integral in the form

$$U = \frac{a}{4\pi} \int_0^{\pi} \int_0^{2\pi} U(a, \phi, \theta) g \sin \theta d\phi d\theta,$$

so that for $\rho<a$, any of the partial derivatives of order n with respect to x, y and z, may be written

$$D^nU = \frac{a}{4\pi}\int_0^\pi\int_0^{2\pi} U(a, \phi, \theta)D^ng \sin\theta d\phi d\theta. \tag{16}$$

We regard this derivative as reckoned at a point of the polar axis $\theta=0$, and divide the integral with respect to θ into two parts, one from 0 to η, and one from η to π, where $0<\eta\leqq\pi/2$. Thus

$$D^nU = J_1 + J_2,$$

where J_2, for fixed η, is analytic in ρ at $\rho=a$, and where

$$\begin{aligned} |J_1| &\leqq \frac{a}{4\pi}\int_0^\eta\int_0^{2\pi} |U(a, \phi, \theta)|\,|D^ng| \sin\theta d\phi d\theta \\ &\leqq C_n\int_0^\eta \frac{f(\theta)d\theta}{\theta^{n+1}}. \end{aligned} \tag{17}$$

We have here used the fact, that for $\theta\leqq\pi/2$, $\theta/r\leqq\pi/(2a)$, and have employed the abbreviations

$$C_n = \frac{5\pi^2}{4}\left(\frac{3\pi e}{4a}\right)^n(n+2)!,\quad f(\theta) = \frac{1}{2\pi}\int_0^{2\pi}|U(a, \phi, \theta)|\,d\theta.$$

The reasoning used to establish Theorem I now yields

THEOREM IV. *Let U be harmonic in the sphere of radius a, and be given by Poisson's integral with the bounded integrable boundary values $U(a, \phi, \theta)$. Let the average of the absolute value of this boundary function on parallel circles,*

$$f(\theta) = \frac{1}{2\pi}\int_0^{2\pi}|U(a, \phi, \theta)|\,d\phi,$$

be subject to the requirement that

$$\int_0^\eta \frac{f(\theta)d\theta}{\theta^{n+1}} \qquad (0<\eta) \tag{18}$$

be convergent. Then any of the partial derivatives of order n of U, with respect to x, y and z, at the point P of the polar axis, approaches a limit as P approaches the surface of the sphere along this axis. Moreover, the approach is uniform for any class of boundary functions which are uniformly bounded in absolute value, and for which the integral (18) *approaches* 0 *uniformly with* η.

As remarked in connection with Theorem I, this result is broader than is at first apparent. For, provided that merely the derivatives of the boundary

values, of order n, have differences at neighboring points which approach 0 sufficiently rapidly with the distance between the points, the condition (18) may be brought to fulfillment by the subtraction from U of a suitable harmonic polynomial. We shall revert to this point in the next section.

8. **A lemma on osculating harmonic polynomials.** We now consider the existence of the polynomials, mentioned at the close of the last section, which broaden the scope of Theorem IV.

LEMMA. *Let the region R be subject to condition A_n, and the function U to condition B_n. We assume, moreover, that the derivatives of U of order $n-1$ exist as limits on E, and are continuous there. Then, corresponding to each point p of E, there exists a harmonic polynomial, G_p, of degree n, such that*

$$U_p = U - G_p$$

vanishes at p, together with all its derivatives of orders $1, 2, \cdots, n-1$, and further, such that the derivatives of order n of its boundary values on E vanish at p. The values of G_p, and of its derivatives, are bounded in R, uniformly as to p.

Taking the axes in the tangent-normal position at p, let $G_{n-1,h}$ denote the sum of the terms of degree less than n in the development of U in spherical harmonics about the point $P(0, 0, h)$ in the interior of R. As $h \to 0$, this harmonic polynomial approaches a limit G_{n-1}, since its coefficients, which are binomial coefficients times the derivatives of U of order $n-1$ and lower, are continuous at the points of E. As these coefficients are subject to the equations which make $G_{n-1,h}$ harmonic, these equations are satisfied in the limit, and so G_{n-1} is also harmonic. Thus $U - G_{n-1}$ is harmonic in R, and vanishes, together with its derivatives of order $n-1$ and lower, at p.

The derivatives of the boundary values of $U - G_{n-1}$ of the same orders also vanish at p, while those of order n are the same as those of U. We form a homogeneous harmonic polynomial of order n as follows. We start with

$$f(x, y) = \frac{1}{n!}\left[\left\{x\frac{\partial}{\partial \xi} + y\frac{\partial}{\partial \eta}\right\}^n U(\xi, \eta, \phi(\xi, \eta))\right]_{\xi=\eta=0},$$

whose derivatives of order n coincide at p with those of U; from it, we form the homogeneous harmonic polynomial

$$H_n = f(x, y) - \frac{\nabla^2 f}{2!} z^2 + \frac{\nabla^2\nabla^2 f}{4!} z^4 - \cdots,$$

$\nabla^2 f$ denoting, as usual, the Laplacian of f. Because of the special position of the axes, z and its partial derivatives with respect to x and y vanish at p, so that the derivatives of the boundary values of H_n of order n reduce to those

of the first term at p, and thus to those of U. Because of the continuity of the derivatives of U and of ϕ on the closed set E, the coefficients of G_{n-1} and H_n are bounded, uniformly as to p, and hence so are its values and those of its derivatives in R. Thus

$$G_p = G_{n-1} + H_n$$

has the properties required in the lemma.

It may be noted that G_p, although uniquely determined by the procedure for setting it up, is not uniquely determined by the properties enunciated in the lemma. Thus, if $\psi(x, y)$ is any homogeneous polynomial of degree $n-1$, the harmonic polynomial

$$\psi(x, y)z - \frac{\nabla^2\psi}{3!}z^3 + \frac{\nabla^2\nabla^2\psi}{5!}z^5 - \cdots$$

may be added to G_p without impairing the requisite properties.

9. **Proof of the theorem for the derivatives of order $n>1$.** The proof of Theorem III, for $n>1$, is essentially a proof by induction, although, as we shall see, the case $n=2$ occupies a somewhat special position. We therefore begin by noting that the conditions A_n and B_n imply the conditions A_{n-1} and B_{n-1}. Thus, since the derivatives of ϕ of order n are continuous functions of the coördinates x and y and of the position of p in a closed region of these variables, they are uniformly bounded in absolute value. This means that the difference quotients of the derivatives of order $n-1$ are bounded, and accordingly the function $D(t) = \text{const.}\times t$ will serve as the required dominant function for them. The situation is the same with the boundary values of U.

Let n denote an integer, $n \geqq 2$. We assume that Theorem III has been proved for all smaller values of n. That is, we assume that the conditions A_n and B_n are in force, and that all the partial derivatives of U of orders 1, 2, $\cdots$, $n-1$ exist as limits on a regular surface element E, and are continuous there. We shall identify this regular surface element with the E of the theorem so as not to multiply notations. A later remark will make clear that this is legitimate.

We consider the function $U_p = U - G_p$ of the lemma, and take the axes in the usual tangent-normal position at p, a point of E'. We construct the sphere σ_i, internally tangent to E' at p of radius a. Let Q be a point of the lower half of the surface of σ_i and q the foot of a normal to E through Q (see §5). Then

$$U_p(Q) = U_p(q) + \left.\frac{\partial U_p}{\partial \nu}\right|_q \overline{qQ} + \frac{1}{2}\left.\frac{\partial^2 U_p}{\partial \nu^2}\right|_q \overline{qQ}^2 + \cdots \tag{19}$$
$$+ \frac{1}{(n-1)!}\left.\frac{\partial^{n-1} U_p}{\partial \nu^{n-1}}\right|_{q_1} \overline{qQ}^{n-1},$$

where q_1 is an interior point of the segment $\overline{qQ}$. This equation is legitimate, since the derivatives of U_p of order $n-1$ exist.

To avoid a whole series of different notations, let us agree to denote by K any function which has a bound for its absolute value which may depend on R, E', U, and n, but not on p, and similarly, let us denote by $D(t)$ any function dominated by a function satisfying the requirements of condition A_n on $D(t)$. These notations may therefore mean different functions from time to time, or even in the same equation, but no difficulty will arise if this fact be kept in mind. Our object is to establish an equation

$$|U_p(Q)| = t^n D(t). \tag{20}$$

To do this, we develop the coefficients in (19), by Taylor's series with remainders, about the point p. For the first, since the derivatives of lower order vanish at p, we have

$$U_p(q) = \frac{1}{n!}\left.\frac{\partial^n U_p}{\partial t^n}\right|_{p_1} t^n,$$

where p_1 is an appropriate mean point on E. But, by Condition B_n,

$$\left|\left.\frac{\partial^n U_p}{\partial t^n}\right|_{p_1}\right| = \left|\left.\frac{\partial^n U_p}{\partial t^n}\right|_{p_1} - \left.\frac{\partial^n U_p}{\partial t^n}\right|_{p}\right| \leqq D(t).$$

Hence

$$|U_p(q)| = t^n D(t). \tag{21}$$

Passing to the second term in the development (19), we have

$$\begin{aligned} \left.\frac{\partial U_p}{\partial \nu}\right|_q &= \left.\frac{\partial U_p}{\partial \nu}\right|_p + \left.\frac{\partial^2 U_p}{\partial t \partial \nu}\right|_p t + \cdots + \frac{1}{(n-2)!}\left.\frac{\partial^{n-1} U_p}{\partial^{n-2} t \partial \nu}\right|_{p_1} t^{n-2}, \\ &= K t^{n-2}, \end{aligned} \tag{22}$$

since the derivatives of U_p of order $n-1$ are uniformly bounded on E and vanish at p. Since $\overline{qQ} = Kt^2$, as we say at the beginning of §5, we have for this second term in (19),

$$\left.\frac{\partial U_p}{\partial \nu}\right|_q \overline{qQ} = Kt^n.$$

Similar considerations show that the later terms in (19) are bounded functions times t^{n+1}, the order with respect to t increasing, at each step, by unity. We thus have the preliminary result

$$U_p(Q) = Kt^n. \tag{23}$$

While this is not as sharp an appraisal as the needed one (20), it will serve us in attaining that goal. The difficulty is obviously with the second term in (19). If $n=2$, the argument in the derivative must be a mean point q_1, since the second is then the final term. It is in this sense that the case $n=2$ is special. We see, then, that if we show that

$$(24)\qquad \left|\left|\frac{\partial U_p}{\partial \nu}\right|_q\right| = t^{n-2}D(t) \text{ for } n > 2, \text{ and } \left|\left|\frac{\partial U_p}{\partial \nu}\right|_{q_1}\right| = D(t) \text{ for } n = 2,$$

the desired equation (20) will be assured.

The case $n>2$. We shall specialize the axes still further by taking the x-axis along the projection of $\overline{pq}$, so that $x=t$. We have, then, as we see by (22), to prove that

$$(25)\qquad \left|\left|\frac{\partial U_p}{\partial \nu}\right|_q \Big/ t^{n-2}\right| = \frac{1}{(n-2)!}\left|\left|\frac{\partial^{n-1} U_p}{\partial x^{n-2}\partial \nu}\right|_{p_1}\right| = D(t).$$

We may, however, replace the mean argument point p_1 by q, since the projection on the tangent plane of pp_1 is not greater than that of pq, or t. In (25)

$$\frac{\partial U_p}{\partial \nu} = \left[-\frac{\phi_x}{w}\frac{\partial U_p}{\partial x} - \frac{\phi_y}{w}\frac{\partial U_p}{\partial y} + \frac{1}{w}\frac{\partial U_p}{\partial z}\right]_{z=\phi(x,y)}, \quad w = (1+\phi_x{}^2+\phi_y{}^2)^{1/2}.$$

By Leibnitz' rule for products,

$$\frac{\partial^{n-2}}{\partial x^{n-2}}\left[-\frac{\phi_x}{w}\frac{\partial U_p}{\partial x}\right]_{z=\phi(x,y)} = \sum_0^{n-2}\binom{n-2}{i}\frac{\partial^{n-2-i}}{\partial x^{n-2-i}}\left(-\frac{\phi_x}{w}\right)\frac{\partial^i}{\partial x^i}\left[\frac{\partial U_p}{\partial x}\right]_{z=\phi(x,y)}.$$

For all values of i less than $n-2$, the second factor in each term vanishes at p, and has bounded derivatives of the first order with respect to x and y, while the first factor is bounded. These terms are therefore of the form Kt. For $i=n-2$, the second factor is bounded, while the first one vanishes at p, and has bounded derivatives, and so is also of the form Kt, because of the special position of the axes. The same is true of

$$\frac{\partial^{n-2}}{\partial x^{n-2}}\left[-\frac{\phi_y}{w}\frac{\partial U_p}{\partial y}\right]_{z=\phi(x,y)}.$$

Finally, we see in a similar way that

$$\begin{aligned}\frac{\partial^{n-2}}{\partial x^{n-2}}\left[\frac{1}{w}\frac{\partial U_p}{\partial z}\right]_{z=\phi(x,y)} &= Kt + \frac{1}{w}\frac{\partial^{n-2}}{\partial x^{n-2}}\left[\frac{\partial U_p}{\partial z}\right]_{z=\phi(x,y)}\\ &= Kt + \frac{1}{w}\left[\frac{\partial^{n-1}U_p}{\partial x^{n-2}\partial z}\right]_{z=\phi(x,y)}\end{aligned}$$

since the difference of the derivatives on the right is a sum of terms each of which is a bounded function times a derivative of U_p of lower order, or times a power of ϕ_x.

As Kt is a function $D(t)$, and as the sum of two such functions belongs to the same class, the establishment of the equation (25) is thus reduced to proving that

$$|V(q)| = |D^{n-1}U_p|_q| = \left|\frac{\partial^{n-1}U_p}{\partial x^{n-2}\partial z}\bigg|_q\right| = D(t). \tag{26}$$

Since $V(p)=0$, the problem is to determine how rapidly the function $V(q)$ approaches its value at p as $q\to p$. We may proceed as follows. Let P and Q be points of R on the normals to E at p and q, respectively, with $\overline{pP}=\overline{qQ}=\delta>0$. We compare $V(P)$ with $V(p)$, $V(Q)$ with $V(q)$, and then $V(P)$ with $V(Q)$.

For the first, we apply Poisson's integral to U_p, using the sphere σ_i tangent to E at p. By (16), we have

$$V(P) = \frac{a}{4\pi}\int_0^\pi\int_0^{2\pi} U_p(a,\phi,\theta)D^{n-1}g\sin\theta d\phi d\theta,$$

and if p' is a point between p and P, distant ρ' from the center of σ_i,

$$V(p') - V(P) = \frac{a}{4\pi}\int_\rho^{\rho'}\int_0^\pi\int_0^{2\pi} U_p(P')\frac{\partial}{\partial\rho}D^{n-1}g\sin\theta d\phi d\theta d\rho,$$

the integrand being continuous. P' is the point (a,ϕ,θ). We break the integral with respect to θ into two parts, the first over the interval $(0,\pi/2)$, and the second over the interval $(\pi/2,\pi)$. In the first, $U_p(P')=Kt^n=K\theta^n$, by (23). In the second, $|U_p(P')|\leqq M$, and $r>a$, if $pP<a$, as we have already implicitly assumed. In both integrals,

$$\frac{\partial}{\partial\rho}D^{n-1}g = \frac{K}{r^{n+2}},$$

by (15). Accordingly, we may write

$$V(p') - V(P) = J_1 + J_2,$$

where

$$|J_2| \leqq \frac{a}{4\pi}\int_\rho^{\rho'}\int_{\pi/2}^{\pi}\int_0^{2\pi}\frac{MK}{a^{n+2}}d\phi d\theta d\rho = K(\rho'-\rho) = K(a-\rho),$$

and

$$|J_1| \leqq \frac{a}{4\pi}\int_\rho^{\rho'}\int_0^{\pi/2}\int_0^{2\pi} K\theta^n \frac{K}{r^{n+2}}\theta d\phi d\theta d\rho$$
$$= K\int_\rho^{\rho'}\int_0^{\pi/2}\frac{1}{r}d\theta d\rho,$$

where, in the last step, we have used the inequality $\theta/r \leqq \pi/(2a)$.

For the inner integral, we find

$$\int_0^{\pi/2}\frac{1}{r}d\theta \leqq 2^{1/2}\int_0^{\pi/2}\frac{\cos\frac{\theta}{2}d\theta}{\left[(a-\rho)^2+\left(2(a\rho)^{1/2}\sin\frac{\theta}{2}\right)^2\right]^{1/2}}$$
$$= \left(\frac{2}{a\rho}\right)^{1/2}\log\frac{(2a\rho)^{1/2}+(a^2+\rho^2)^{1/2}}{a-\rho} \leqq \frac{2}{a}\log\frac{2\cdot 2^{1/2}a}{a-\rho},$$

if $\rho \geqq a/2$. Hence

$$J_1 = K\int_\rho^{\rho'}\log\frac{3a}{a-\rho}d\rho = K\left[(a-\rho')\log\frac{a-\rho'}{3ae}-(a-\rho)\log\frac{a-\rho}{3ae}\right].$$

As $V(p')\rightarrow V(p)$ as $\rho'\rightarrow a$, we find, therefore,

$$V(p)-V(P) = K\left[(a-\rho)+(a-\rho)\log\frac{3ae}{a-\rho}\right] = K(a-\rho)\log\frac{a}{a-\rho}\cdot$$

This gives, in terms of $\delta = a-\rho$, for $\delta < a/2$,

$$V(p)-V(P) = K\delta\log\frac{a}{\delta}\cdot \tag{27}$$

When we consider $V(q)-V(Q)$, we must first make sure that q is in the region for which (23), with p replaced by q, and Q by a point on the lower half of the corresponding sphere σ_i, is valid. But this is true because q is on E'', and distant from its edge at least $4a-(2^{1/2}+2)a = (2-2^{1/2})a$, by §5. We have also to consider the effect of adding to U_p the harmonic polynomial $G_p - G_q$. Since the derivatives of G_p are all bounded in R, uniformly as to p, this addition affects $V(q)-V(Q)$ only by adding a term $K\overline{qQ} = K\delta$. Hence we infer also that

$$|V(q)-V(Q)| = K\delta\log\frac{a}{\delta}\cdot \tag{28}$$

When it comes to comparing $V(P)$ with $V(Q)$, we connect P and Q by a curve γ, never nearer than δ to the boundary of R. We have, then,

$$V(Q) - V(P) = \int_{s_P}^{s_Q} \frac{\partial V}{\partial s}\, ds.$$

Let us take for γ the locus of the centers of the spheres of radius δ, internally tangent to E at the points where E is cut by the (x, z)-plane, i.e., at the points of the curve $x=x$, $y=0$, $z=\phi(x, 0)$. Then γ is given by

$$\xi = x - \frac{\phi_x}{w}\delta, \quad \eta = -\frac{\phi_y}{w}\delta, \; \zeta = \phi + \frac{1}{w}\delta.$$

We find that the derivatives of ξ, η, and ζ with respect to x are uniformly bounded, and hence the length of γ is a bounded function times the x-coördinate of q, or t. Moreover, the harmonic function V is uniformly bounded in the portion of R swept out by the spheres of radius $4a$ about the points of E'. Let B be a bound for its absolute value in this region. Then at the points of γ, by (11),

$$\left|\frac{\partial V}{\partial s}\right| \leqq \frac{3}{2\delta} B.$$

Accordingly, we have

$$| V(Q) - V(P) | \leqq K\frac{t}{\delta} \cdot \tag{29}$$

We now combine the results (27), (28), (29), writing $\delta = t^{1/2}$. Then, for $t < a^2/4$,

$$V(q) = V(q) - V(p) = Kt^{1/2} \log \frac{a^2}{t} + Kt^{1/2} = Kt^\lambda,$$

if λ is any number between 0 and 1/2. There is no difficulty in extending such a relation to values of t greater than $a^2/4$, since $V(q)$ is bounded on E. As Kt^λ has the properties required of $D(t)$, the equation (26), and with it the first equation (24), is established.

The case $n=2$. We have to show that

$$| V(q_1) | = \left| \left.\frac{\partial U_p}{\partial \nu}\right|_{q_1} \right| = D(t).$$

If δ_1 is the distance from q to q_1, we find, as before,

$$| V(q_1) - V(q) | = K\delta_1 \log \frac{a}{\delta_1} = K\delta \log \frac{a}{\delta}.$$

But the preceding considerations have proved that $V(q) = V(q) - V(p) = D(t)$. Accordingly

$$|V(q_1)| = D(t),$$

and the second equation (24) is established. But this, it will be recalled, is sufficient for the equation (20), which we set out to establish.

The proof of Theorem III is now readily completed. The equation (20) leads at once to

$$|U_p(Q)| = \theta^n D(\theta),$$

so that the hypothesis of Theorem IV is fulfilled by the boundary values of U_p on σ_i, and accordingly, the derivatives of order n of U_p approach limits at p along the normal. Moreover, the approach is uniform as to p, and these derivatives, rightly defined on E', are then continuous in R at the points of E'.

It remains only to justify the assumption that the derivatives of order $n-1$ and lower were continuous at the points of E. We may interpolate a set of surface elements between E and E', each interior to the preceding. On the first, the derivatives of first order are continuous, on the second, those of second order are continuous, and so on. Letting the $(n-1)$th play the rôle of E in the above proof, we have established the existence and continuity of the derivatives of the nth order on E'. Theorem III is thus completely proved.

10. **Hölder conditions on U.** We shall consider, in this section, Hölder conditions on U itself, and in the next, Hölder conditions on the derivatives. We assume

CONDITION A_λ. *This is obtained from Condition A_1, with* (a) *omitted, and with $D(t)$ specialized so as to take the form At^λ, $0<\lambda<1$, so that*

$$|D\phi(q) - D\phi(p)| \leqq At^\lambda.$$

CONDITION B_λ. *U is continuous in R, and harmonic in the interior of R, and if p and q are any two points of E,*

$$|U(q) - U(p)| \leqq At^\lambda.$$

We then have the theorem

THEOREM V. *If R is subject to Condition A_λ, and U to Condition B_λ, there is a region R', containing all the points of R in a neighborhood of E', and a constant B, such that for any two points P and Q of R',*

$$|U(Q) - U(P)| \leqq Br^\lambda, \quad r = \overline{PQ}.$$

We may choose R' at once as those points of R whose distances from E' do not exceed a (see §4). Reverting to the dominant harmonic function $W = \rho^\lambda P_\lambda$ (cos θ) of §4, we take the origin of the system of spherical coör-

dinates at any point p of E'', with the axis of θ in the direction of the inward normal. Then a portion of E in the neighborhood of p lies in the region (4). For, with axes of cartesian coördinates in the usual tangent-normal position at p, we have, by Condition A_λ,

$$|z| = |\phi_x(q_1)x + \phi_y(q_1)y| \leqq 2^{1/2}At^{1+\lambda},$$

while the boundary of (4) is given by

$$z = -\tan \alpha t.$$

Hence all points of R in a sphere about p, of radius not greater than $[(\tan\alpha)/(2^{1/2}A)]^{1/\lambda}$, lie in the region (4).

We conclude, as in §4, that there is a constant B', independent of p, such that for any point p on E'', and any point Q in R,

$$|U(Q) - U(p)| \leqq B'\overline{pQ}^\lambda. \tag{30}$$

The problem is now to extend this inequality to points P in R'. If P is any point of R', its distance from E' is not more than a, while the distance from E' of any point of S not in E'' is at least $4a$. Hence any point P of R' is nearer to some point p of E'' than to any other boundary point of R. Let p be the nearest point of E'', distant c, say, from P. Let σ denote the sphere of radius $c/2$ about P. Then, by (30), the oscillation of U on σ does not exceed twice the maximum on σ_i of $B'\overline{pQ}^\lambda$. Accordingly, by (11), the derivative DU of U, in any direction, at P, is subject to the inequality

$$|DU| \leqq \frac{3B'(3c/2)^\lambda}{2(c/2)} = B''c^{\lambda-1}. \tag{31}$$

Now let P and Q be any two points of R'. We consider first the case in which $r = PQ$ is less than the distance of the segment PQ from E''. Here, integrating along the segment PQ, we have

$$|U(Q) - U(P)| = \left|\int_0^r \frac{\partial U}{\partial s} ds\right| \leqq B''c^{\lambda-1}r \leqq B''r^\lambda.$$

On the other hand, if the length r of the segment PQ is greater than or equal to its distance from E'', let s be the point of E'' nearest the segment. Then sQ and sP are not greater than $2r$, and (30) yields, when applied to the pairs of points s, Q, and s, P,

$$|U(Q) - U(P)| \leqq 2B'2^\lambda r^\lambda.$$

Hence if B denotes the larger of the two constants B'' and $2^{1+\lambda}B'$, we have, for any two points P and Q in R',

$$|U(Q) - U(P)| \leqq Br^{\lambda}.$$

Theorem V is thus proved.

Remark. The exponent λ has been confined to the open interval (0, 1). For $\lambda=1$, the Hölder condition becomes a Lipschitz condition, and such a condition on the boundary values of U does not imply a similar condition for neighboring interior points. This may be shown by an example. When the sphere to which Poisson's integral is applied becomes the infinite plane, we have the following representation of a function, harmonic to one side of this plane, and assuming the boundary values f:

$$U(P) = \frac{1}{2\pi}\int\int f d\Omega,$$

where $\Delta\Omega$ denotes the solid angle subtended at P by the element of surface ΔS of the plane, the integral being extended over the infinite plane. Using cylindrical coördinates (ρ, ϕ, z), with origin in the plane, and z-axis normal to it, we consider the function defined by the boundary values $f=\rho/(1+\rho^2)$, at points of the z-axis, $z>0$. The evaluation of the integral gives, for such points,

$$U = \frac{z}{(1-z^2)^{3/2}}\left[\log\frac{1+(1-z^2)^{1/2}}{z} - (1-z^2)^{1/2}\right],$$

and U therefore fails to have bounded difference quotients near the origin, although its boundary values do have.

11. **Hölder conditions on the derivatives of U.** The conditions which we here assume are $A_{n+\lambda}$ and $B_{n+\lambda}$; they are simply the conditions obtained from A_n and B_n by specializing the function $D(t)$ to be of the form $At^{\lambda}(0<\lambda<1)$. As the definition of $D(t)$ implies, A and λ are independent of p and of the direction of pq. We conclude by establishing

Theorem VI. *If R is subject to Condition $A_{n+\lambda}$ and U to Condition $B_{n+\lambda}$, then there is a region R', containing all points of R in a neighborhood of E', and a constant B, such that for any two points P and Q of R',*

$$|D^nU(Q) - D^nU(P)| \leqq Br^{\lambda}, \quad r = \overline{PQ}.$$

Here as before, D^nU means any one of the derivatives of U of order n with respect to x, y and z, the axes of these coördinates being fixed.

By Theorem III, we know that the derivatives of order n of U exist and are continuous at the points of any closed surface element interior to E. We may infer that these derivatives are bounded in the region R'' containing all points of R whose distances from E'' do not exceed $(2^{1/2}+2)a$, and no others.

Let p be any point of E'', and σ_i the sphere of radius a, internally tangent

to E'' at p. Let Q lie on the lower half of the surface of σ_i. Its distance from p is then not more than $2^{1/2}a$, and hence the nearest point of S to Q is in R'', and so on E. Call such a point q. Then (19) holds for the function U_p, defined in §8. We conclude, as in §9—except that the steps are much simplified by our present knowledge, by Theorem III, that the derivatives of U of order n are bounded in R''—that

$$U_p(Q) = Kt^{n+\lambda}, \tag{32}$$

where we are again adopting the convention that K means any function whose absolute value has a bound independent of p, and of any other argument points.

From this, we infer, using the method of §9, and applying Poisson's integral in the sphere σ_i, that if D^nU is any given derivative of U of order n,

$$D^nU_p(P) - D^nU_p(p) = K\delta^\lambda, \tag{33}$$

for any point P on the normal at p, distant δ from p, $\delta \leqq a$. This latter restriction may, however, be dropped, as we have seen, provided we remain in R''. This leads, as in the preceding section, to

$$D^{n+1}U_p(P) = Kc^{\lambda-1},$$

where c is the distance from P to E, and P is on a normal to E at a point of E'', and in R''. Since all derivatives of G_p are bounded, uniformly as to p, in R'', the last equation yields

$$D^{n+1}U(P) = Kc^{\lambda-1}. \tag{34}$$

For R', we take, as before, the set of all points of R whose distances from E' do not exceed a. Any point of R' is on a normal to E at some point of E''. Let P and Q be any two points of R'. As before, we have two cases to consider. If the distance $r = PQ$ is not greater than the distance between the segment PQ and E'', we argue, as before, that

$$|D^nU(Q) - D^nU(P)| \leqq Br^\lambda. \tag{35}$$

This is the desired result, established for this case.

If r is greater than the distance between the segment PQ and E'', new geometric considerations are needed. Because of the continuity of ϕ_x and ϕ_y, there corresponds to any positive angle β, a number b, such that if p and q are any two points of E, whose distance is not more than b, the normals to E at p and q make an angle not greater than β. We shall take β as the acute angle for which $\sin(\beta/2) = 1/8$. Let p and q be two points of E'', whose distance r does not exceed b, b being further restricted, if necessary, so as not

to exceed $a/4$. Let P be in R'', on the normal to E'' at p, and let Q be in R'' and on the normal to E'' at q, such that $pP = qQ = 4r$.

We find, then, that $PQ \leqq 2(4r) \sin (\beta/2) + r = 2r$. Since $r \leqq b \leqq a/4$, the nearest point of E to P is p, and its distance is $4r$. Thus the whole segment $\overline{PQ}$ is distant from E'' at least as much as the length $\overline{PQ}$, and (35) is applicable. It gives

$$D^n U(Q) - D^n U(P) = K(2r)^\lambda = Kr^\lambda.$$

But, by (33), we have also

$$D^n U(P) - D^n U(p) = K(4r)^\lambda = Kr^\lambda,$$

and, similarly,

$$D^n U(Q) - D^n U(q) = Kr^\lambda.$$

Combining the last three equations, we have

$$D^n U(q) - D^n U(p) = Kr^\lambda. \tag{36}$$

The preliminary restriction that $r \leqq b$ may now be removed by the usual argument.

Now let P and Q be any two points of R' whose distance exceeds the distance of the segment $\overline{PQ}$ from E''. There will be a point s of E'' whose distance from $\overline{PQ}$ is less than r, and therefore, whose distances from P and Q are less than $2r$. Hence the nearest points of E'' to P and Q, which we call p and q, respectively, will be distant from P and Q, respectively, less than $2r$. The distance $\overline{pq}$, accordingly, cannot exceed $5r$. Applying the equation (33) to the pairs of points P, p and Q, q and the equation (36) to the points p, q, we obtain the inequality (35) for the second case. Here P or Q or both may lie on E'', and so be any points in the closed region R'. Theorem VI is thus established.

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ON THE PADÉ APPROXIMANTS ASSOCIATED WITH A POSITIVE DEFINITE POWER SERIES*

BY

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1. Introduction. To every power series:

(1) $$\mathfrak{P}(z) = c_0 - c_1 z + c_2 z^2 - \cdots \qquad (c_0 \neq 0),$$

and to every pair of numbers m, n of the sequence $0, 1, 2, 3, \cdots$, there corresponds uniquely a rational function:

(2) $$[m, n] = N_{m,n}(z)/D_{m,n}(z),$$

in which the degrees of numerator and denominator do not exceed n and m, respectively, and such that the formal power series

$$\mathfrak{P}(z)D_{m,n}(z) - N_{m,n}(z)$$

shall begin with the $(m+n+1)$th or a higher power of z. The function (2) is called a Padé approximant† of $\mathfrak{P}(z)$.

Following Padé we shall form with the approximants the accompanying table of double entry:

(3)

$[0, 0] = c_0$	$[0, 1] = c_0 - c_1 z$	$[0, 2] = c_0 - c_1 z + c_2 z^2$	$\cdots$
$[1, 0]$	$[1, 1]$	$[1, 2]$	$\cdots$
$[2, 0]$	$[2, 1]$	$[2, 2]$	$\cdots$
.	.	.	.

Let S_i (S_{-i}), $i>0$, designate the ith diagonal file of approximants to the right of (below) and parallel to the principal diagonal, S_0, in (3). Then S_i is the infinite sequence

(4) $$S_i = [0, i],\ [1, i+1],\ [2, i+2], \cdots;$$

and

(5) $$S_{-i} = [i, 0],\ [i+1, 1],\ [i+2, 2], \cdots.$$

Both (4) and (5) give S_0 for $i=0$.

* Presented to the Society, March 29 and December 30, 1929; received by the editors December 30, 1929.

† For details see Perron, *Die Lehre von den Kettenbrüchen*, Leipzig and Berlin, 1913, Chapter X.

Let

$$\Delta_{m,n} = \begin{vmatrix} c_{n-m}, & c_{n-m+1}, & \cdots, & c_n \\ c_{n-m+1}, & c_{n-m+2}, & \cdots, & c_{n+1} \\ \cdot & \cdot & & \cdot \\ c_n, & c_{n+1}, & \cdots, & c_{n+m} \end{vmatrix} \qquad (m, n = 0, 1, 2, \cdots; c_i = 0 \text{ if } i < 0).$$

Then if

$$\Delta_{n,n} > 0 \qquad (n = 0, 1, 2, \cdots), \tag{6}$$

$\mathfrak{P}(z)$ is called positive definite. The object of the following article is to study the convergence of the diagonal files, S_i, in the Padé table associated with a positive definite series, and to investigate the relationship among, and the character of, the limits (when they exist) of these files.* A summary of the principal results is contained in §7.

2. **The polynomials A_n^k, B_n^k.** With each of the series†

$$\mathfrak{P}^k(z) = c_k - c_{k+1}z + c_{k+2}z^2 - \cdots$$

we shall associate four polynomials, namely

$$\begin{aligned} A_{2n}^k &= \alpha_0^{n,k} + \alpha_1^{n,k}z + \cdots + \alpha_{n-1}^{n,k}z^{n-1}, \quad B_{2n}^k = \beta_0^{n,k} + \beta_1^{n,k}z + \cdots + \beta_n^{n,k}z^n, \\ A_{2n+1}^k &= \gamma_0^{n,k} + \gamma_1^{n,k}z + \cdots + \gamma_n^{n,k}z^n, \quad B_{2n+1}^k = \delta_0^{n,k} + \delta_1^{n,k}z + \cdots + \delta_n^{n,k}z^n, \end{aligned} \tag{7}$$

obtained by requiring that the formal power series

$$\mathfrak{P}^k B_m^k - A_m^k \qquad (m = 2n \text{ or } 2n + 1)$$

shall begin with the mth power of z. It will be seen that this requirement yields the following systems of equations:

$$\begin{aligned} \sum_{i=0}^{p}(-1)^i c_{k+i}\beta_{p-i}^{n,k} &= \alpha_p^{n,k} \qquad (p = 0, 1, \cdots, n-1), \\ \sum_{i=0}^{p}(-1)^i c_{k+i}\delta_{p-i}^{n,k} &= \gamma_p^{n,k} \qquad (p = 0, 1, \cdots, n); \end{aligned} \tag{8}$$

$$\begin{aligned} \sum_{i=0}^{n}(-1)^i c_{k+p+i}\beta_{n-i}^{n,k} &= 0 \qquad (p = 0, 1, \cdots, n-1), \\ \sum_{i=0}^{n}(-1)^i c_{k+p+i}\delta_{n-i}^{n,k} &= 0 \qquad (p = 1, 2, \cdots, n). \end{aligned} \tag{9}$$

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* Cf. Wall, these Transactions, vol. 31 (1929), pp. 91–110, in which the same question is studied under the further restriction that (1) shall be a series of Stieltjes.

† In §§2, 3, 4 the series $\mathfrak{P}(z)$ is not restricted to be positive definite, but is an arbitrary series with constant term different from zero.

Since the homogeneous systems (9) both possess fewer equations (by one) than unknowns they may be satisfied in every case by sets of values of the $\beta_i^{n,k}$, $\delta_i^{n,k}$, respectively, each set having at least one element not zero. The $\alpha_i^{n,k}$, $\gamma_i^{n,k}$ are then determined by the $\beta_i^{n,k}$, $\delta_i^{n,k}$, respectively, in accordance with (8).

Though the solution of (8), (9) is not unique, the functions A_m^k/B_m^k are unique provided the $\beta_i^{n,k}$ or the $\delta_i^{n,k}$, as the case may be, are not all zero. In fact, if the power series

$$\mathfrak{P}^k B_m^k - A_m^k,$$

$$\mathfrak{P}^k B_{m,1}^k - A_{m,1}^k$$

both begin with the mth or a higher power of z, the polynomial

$$A_{m,1}^k B_m^k - A_m^k B_{m,1}^k,$$

which is identical with

$$(\mathfrak{P}^k B_m^k - A_m^k) B_{m,1}^k - (\mathfrak{P}^k B_{m,1}^k - A_{m,1}^k) B_m^k,$$

contains no power of z lower than the mth. But since it is of degree $m-1$ at most it must vanish identically, and therefore

$$A_{m,1}^k / B_{m,1}^k \equiv A_m^k / B_m^k.$$

We shall suppose in general that the polynomials (7) are determined by an arbitrary non-trivial solution of (9). Only when the determinant $\Delta_{n-1,n+k}$ is different from zero do we take a particular determination of the A_{2n}^k, B_{2n}, which is in fact unique apart from a common constant factor, namely

$$A_{2n}^k(-z) = \frac{(-1)^n}{\Delta_{n-1,n+k}} \begin{vmatrix} 0, & c_k z^{n-1}, & \cdots, & c_k + c_{k+1}z + \cdots + c_{k+n-1}z^{n-1} \\ c_k, & c_{k+1}, & \cdots, & c_{k+n} \\ \cdot & \cdot & \cdot & \cdot \\ c_{k+n-1}, & c_{k+n}, & \cdots, & c_{k+2r-1} \end{vmatrix},$$

(10)

$$B_{2n}^k(-z) = \frac{(-1)^n}{\Delta_{n-1,n+k}} \begin{vmatrix} z^n, & z^{n-1}, & \cdots, & 1 \\ c_k, & c_{k+1}, & \cdots, & c_{k+n} \\ \cdot & \cdot & & \cdot \\ c_{k+n-1}, & c_{k+n}, & \cdots, & c_{k+2n-1} \end{vmatrix}.$$

If $\Delta_{n,n+k} \neq 0$, we shall take

$$A^k_{2n+1}(-z) = \frac{(-1)^n}{\Delta_{n,n+k}} \begin{vmatrix} c_k z^n, & c_k z^{n-1} + c_{k+1} z^n, & \cdots, & c_k + c_{k+1} z + \cdots + c_{k+n} z^n \\ c_{k+1}, & c_{k+2}, & \cdots, & c_{k+n+1} \\ \cdot & \cdot & \cdot & \cdot \\ c_{k+n}, & c_{k+n+1}, & \cdots, & c_{k+2n} \end{vmatrix},$$

(11)

$$B^k_{2n+1}(-z) = \frac{(-1)^n}{\Delta_{n,n+k}} \begin{vmatrix} z^n, & z^{n-1}, & \cdots, & 1 \\ c_{k+1}, & c_{k+2}, & \cdots, & c_{k+n+1} \\ \cdot & \cdot & \cdot & \cdot \\ c_{k+n}, & c_{k+n+1}, & \cdots, & c_{k+2n} \end{vmatrix}.$$

These are likewise unique except for a common constant factor.

3. **Expressions for the A^k_n, B^k_n in terms of the A^{k-1}_n, B^{k-1}_n.** Let us assume that

(12) $$\Delta_{n-1,n+k-1},\ \Delta_{n,n+k-1} \neq 0.$$

Then we shall determine constants K_1, K_2 such that

(13) $$B^k_{2n-1} = K_1 B^{k-1}_{2n} - K_2 B^{k-1}_{2n+1},$$

(14) $$z \cdot A^k_{2n-1} = c_{k-1}(K_1 B^{k-1}_{2n} - K_2 B^{k-1}_{2n+1}) - (K_1 A^{k-1}_{2n} - K_2 A^{k-1}_{2n+1}).$$

The polynomials (7) involved in (13), (14) are given by (10), (11) by virtue of (12). If then we equate the coefficients of the highest and of the lowest powers of z in (13) we shall find that

$$K_2 = 1,\quad K_1 = \Delta_{n-1,n+k}/\Delta_{n,n+k-1}.$$

One may then verify that the right member of (14) is divisible by z, and that the power series for $\mathfrak{P}^k B^k_{2n-1} - A^k_{2n-1}$ (using the values of A^k_{2n-1}, B^k_{2n-1} from (13), (14)) begins with the $(2n-1)$th or a higher power of z.

In like manner, if

(15) $$\Delta_{n-1,n+k},\ \Delta_{n,n+k-1} \neq 0,$$

one will find that

$$B^k_{2n} = K B^{k-1}_{2n+1},\quad z A^k_{2n} = c_{k-1} K B^{k-1}_{2n+1} - K A^{k-1}_{2n+1},$$

where $K = 1/K_1$.

We have proved the following theorem.

THEOREM 1. *If* (12) *holds, then*

(16) $$B^k_{2n-1} = h^{k-1}_n B^{k-1}_{2n} - B^{k-1}_{2n+1},$$

(17) $$z\overset{k}{A}_{2n-1} = c_{k-1}[\overset{k-1}{h_n}\overset{k-1}{B}_{2n} - \overset{k-1}{B}_{2n+1}] - [\overset{k-1}{h_n}\overset{k-1}{A}_{2n} - \overset{k-1}{A}_{2n+1}],$$

where

(18) $$\overset{k-1}{h_n} = \Delta_{n-1,n+k}/\Delta_{n,n+k-1}.$$

If (15) *holds, then*

(19) $$\overset{k-1}{h_n}\overset{k}{B}_{2n} = \overset{k-1}{B}_{2n+1},$$

(20) $$z\overset{k-1}{h_n}\overset{k}{A}_{2n} = c_{k-1}\overset{k-1}{B}_{2n+1} - \overset{k-1}{A}_{2n+1}.$$

4. **The Padé approximants.** Let $\mathfrak{P}_k$ denote the sum of the first k terms of $\mathfrak{P}$, and consider the functions

(21) $$\overset{k}{F_n} = (\mathfrak{P}_k\overset{k}{B}_{2n-1} + (-z)^k\overset{k}{A}_{2n-1})/\overset{k}{B}_{2n-1},$$

(22) $$\overset{k-1}{F}_{n+1} = (\mathfrak{P}_k\overset{k}{B}_{2n} + (-z)^k\overset{k}{A}_{2n})/\overset{k}{B}_{2n} \qquad (n,\ k = 1, 2, 3, \cdots).$$

The degrees of numerator and denominator of (21) do not exceed $n+k-1$, $n-1$, respectively. Then since the formal power series

$$\mathfrak{P}\overset{k}{B}_{2n-1} - (\mathfrak{P}_k\overset{k}{B}_{2n-1} + (-z)^k\overset{k}{A}_{2n-1}) = (-z)^k(\overset{k}{\mathfrak{P}}\overset{k}{B}_{2n-1} - \overset{k}{A}_{2n-1})$$

begins with the $(k+2n-1)$th or a higher power of z, it follows (see Introduction) that F_n^k is the Padé approximant $[n-1, n+k-1]$ of $\mathfrak{P}$, i.e.,

(23) $$\overset{k}{F_n} = [n-1, n+k-1] \qquad (n,\ k = 1, 2, \cdots).$$

Similarly, (22) is the Padé approximant $[n, n+k-1]$ of $\mathfrak{P}$.

Set $A_m^0 \equiv A_m$, $B_m^0 \equiv B_m$. Then clearly

(24) $$F_n = A_{2n-1}/B_{2n-1},\quad \overset{-1}{F_n} = A_{2n}/B_{2n} \qquad (n = 1, 2, \cdots)$$

are the approximants $[n-1, n-1]$, $[n, n-1]$, respectively.

Let C_n^k, D_n^k denote the polynomials A_n^k, B_n^k, respectively, connected with the reciprocal of $\mathfrak{P}$, namely

$$\mathfrak{E}(z) = d_0 - d_1 z + d_2 z^2 - \cdots.$$

If $\mathfrak{E}_k$ is the sum of the first k terms, then

(25) $$\overset{-k}{F_n} = \overset{k}{D}_{2n-1}/(\mathfrak{E}_k\overset{k}{D}_{2n-1} + (-z)^k\overset{k}{C}_{2n-1}),$$

(26) $$\overset{-k+1}{F}_{n+1} = \overset{k}{D}_{2n}/(\mathfrak{E}_k\overset{k}{D}_{2n} + (-z)^k\overset{k}{C}_{2n}) \qquad (n,\ k = 1, 2, \cdots)$$

are the approximants $[n+k-1,\ n-1]$, $[n+k-1,\ n]$, respectively, of $\mathfrak{P}$. Taking (25), for example, we see that the degrees of numerator and denominator do not exceed $n-1$, $n+k-1$, respectively, while the formal power series

$$\mathfrak{P}(\mathfrak{C}_k \overset{k}{D}_{2n-1} + (-z)^k \overset{k}{C}_{2n-1}) - \overset{k}{D}_{2n-1} = -(-z)^k \mathfrak{P}(\overset{k}{\mathfrak{C}} \overset{k}{D}_{2n-1} - \overset{k}{C}_{2n-1})$$

begins with at least the $(2n+k-1)$th power of z.

Along with (25), (26) we have

$$F_n = D_{2n-1}/C_{2n-1},\quad F_n^1 = D_{2n}/C_{2n}. \tag{27}$$

5. **The convergence of the diagonal files S_k, $k \geqq -1$, for a positive definite series.** If $\mathfrak{P}(z)$ is positive definite, then the series $\mathfrak{P}^{2k}(z)$, $k=1, 2, 3, \cdots$, are all positive definite inasmuch* as the determinants $\Delta_{n,n+2k}$, $n, k=0, 1, 2, \cdots$, are all positive if they are positive when $k=0$.

It is seen from the relation

$$\Delta_{n,n+2k+1}\Delta_{n-2,n+2k+1} = \Delta_{n-1,n+2k}\Delta_{n-1,n+2k+2} - \Delta^2_{n-1,n+2k+1}$$

that for each $k=0, 1, 2, \cdots$, the determinants $\Delta_{n,n+2k+1}$, $n=0, 1, 2, \cdots$, cannot vanish for two consecutive values of n. Let (n') denote the set of all distinct indices n for which $\Delta_{n-1,n+2k} \neq 0$ (k being fixed). Then (n') is infinite.

If n_1 is not in (n') then the four approximants

$$\begin{matrix} [n_1-1,\ n_1+2k-1], & [n_1-1,\ n_1+2k], \\ [n_1,\ n_1+2k-1], & [n_1,\ n_1+2k], \end{matrix} \tag{28}$$

are identical.† No other approximants of the table are equal to these.

From these considerations we immediately deduce the following result (vide supra, (4), (23)).

THEOREM 2. *If $\mathfrak{P}(z)$ is positive definite and the limits*

$$\lim_{n=\infty} F_n^{2k-1} = S_{2k-1},\quad \lim_{n=\infty} F_n^{2k} = S_{2k},\quad \lim_{n=\infty} F_n^{2k+1} = S_{2k+1} \tag{29}$$

exist and two (or all) of them are distinct for $z=z_0$, then there exists a number N such that

$$\Delta_{n-1,n+2k} \neq 0 \text{ if } n \geqq N. \tag{30}$$

* Sylvester, Philosophical Magazine, (4), vol. 4 (1852), pp. 140–141.

† Perron, loc. cit., p. 426. Note that my $\Delta_{m,n}$ differs from that used by Perron by the factor ± 1. Van Vleck showed that all irregularities of the table must be of the first order. Cf. these Transactions, vol. 4 (1903), pp. 297–332; p. 330.

In order to connect with Hamburger's work* on positive definite power series, we now set

$$P^{k}_{2n}(z') = z'^{n-1}A^{k}_{2n}(z), \quad Q^{k}_{2n}(z') = z'^{n}B^{k}_{2n}(z), \tag{31}$$

$$P^{k}_{2n+1}(z') = z'^{n}A^{k}_{2n+1}(z), \quad Q^{k}_{2n+1}(z') = z'^{n+1}B^{k}_{2n+1}(z), \tag{32}$$

where $z'=1/z$. When (10), (11) hold, and k is even, one will readily identify these polynomials with Hamburger's polynomials $P_n(z')$, $Q_n(z')$ connected with the positive definite power series $\mathfrak{P}^k(z')/z'$ (k even). When $\Delta_{n-1,n+2k}=0$, $P^{2k}_{2n}(z')/Q^{2k}_{2n}(z')$ is to be identified with Hamburger's $U_n(z')/V_n(z')$ (for the series $\mathfrak{P}^{2k}(z')/z'$).

Now by (21), (22), (31), (32),

$$F^{2k}_{n} = \mathfrak{P}_{2k}(z) + z^{2k-1}P^{2k}_{2n-1}(z')/Q^{2k}_{2n-1}(z'), \tag{33}$$

$$F^{2k-1}_{n+1} = \mathfrak{P}_{2k}(z) + z^{2k-1}P^{2k}_{2n}(z')/Q^{2k}_{2n}(z'), \tag{34}$$

$k=0, 1, 2, \cdots$ ($\mathfrak{P}_0=0$, F^{-1}_{n+1} to be replaced by F^{-1}_{n}). Then by (4), (23), (24), (33), and Theorem XIX of Hamburger's memoir,† we have immediately

THEOREM 3. *If $\mathfrak{P}(z)$ is positive definite, then over every closed finite region of the z-plane which contains no part of the real axis,*

$$S_{2k} = \lim_{n=\infty} F^{2k}_{n} = \mathfrak{P}_{2k} + z^{2k-1}I^{2k}(z') \qquad (k = 0, 1, 2, \cdots),$$

where

$$I^{2k}(z') = \int_{-\infty}^{+\infty} \frac{d\psi^{2k}(u)}{z'+u},$$

a Stieltjes integral in which $\psi^{2k}(u)$ is a monotone non-decreasing real function $\phi(u)$ satisfying the equations

$$c_{2k+i} = \int_{-\infty}^{+\infty} u^i d\phi(u) \qquad (i = 0, 1, 2, \cdots). \tag{35}$$

The function‡ $\psi^{2k}(u)$ is determined by the condition that it shall have a greater saltus at $u=0$ than every§ other function $\phi(u)$ satisfying (35).

* Hans Hamburger, Mathematische Annalen, vol. 81 (1920), pp. 235–319, and vol. 82 (1921), pp. 120–187. This excellent memoir furnishes the basis for the present paper.

† Loc. cit., vol. 81, p. 310.

‡ The "Maximalbelegungsfunktion" of Hamburger: cf. loc. cit., vol. 81, pp. 298–299, Theorem XVI and Definition XII.

§ Two functions ϕ and ϕ_1 will be considered identical if they agree, with the possible exception of an additive constant, at all points of continuity.

We shall call the functions $\phi(u)$ satisfying (35) *moment functions* of $\mathfrak{P}^{2k}$. In particular the moment function $\psi^{2k}(u)$ defined in Theorem 3 is the *maximal* moment function of $\mathfrak{P}^{2k}$.

There are two cases to be considered. Either the maximal moment function is the only moment function of $\mathfrak{P}^{2k}$, or else $\mathfrak{P}^{2k}$ has an infinite number of different moment functions. These will be called the *determinate* and the *indeterminate* cases, respectively, and it will be convenient to term $\mathfrak{P}^{2k}$ determinate and indeterminate.

In the indeterminate case,* the saltus of $\psi^{2k}(u)$, the maximal moment function, is positive at $u=0$, i.e.,

$$\lim_{\epsilon=0} [\psi^{2k}(\epsilon) - \psi^{2k}(-\epsilon)] = \delta > 0, \ \epsilon > 0, \tag{36}$$

and in fact,

$$\frac{1}{\delta} \equiv h^{2k} = \lim_{n=\infty} h_n^{2k}, \tag{37}$$

where h_n^{2k} is given by (18).

For the determinate case, the work of Hamburger† and (34), (33) give at once the following theorem:

Theorem 4. *If $\mathfrak{P}(z)$ is positive definite and $\mathfrak{P}^{2k}$ determinate, then over every finite closed region containing no part of the real axis, S_{2k-1} converges uniformly and*

$$S_{2k-1} \equiv S_{2k}.$$

Let us set

$$g_{n'}^{2k} = P_{2n'}^{2k}(0)/Q_{2n'}^{2k}(0), \ n' \text{ in } (n') \qquad (k = 0, 1, 2, \cdots). \tag{38}$$

Then according to Hamburger,‡

$$\begin{aligned} P_{2n'}^{2k}(z') &= \Omega_{n'}^{2k}(z') + g_{n'}^{2k} P_{2n'-1}^{2k}(z'), \\ Q_{2n'}^{2k}(z') &= \Theta_{n'}^{2k}(z') + g_{n'}^{2k} Q_{2n'-1}^{2k}(z') \qquad (k = 0, 1, 2, \cdots), \end{aligned} \tag{39}$$

where $\Omega_{n'}^{2k}$, $\Theta_{n'}^{2k}$ are polynomials defined for all n. In the indeterminate case,§

$$\begin{aligned} &\lim_{n=\infty} P_{2n-1}^{2k}(z') = p_1^{2k}(z'), \quad \lim_{n=\infty} Q_{2n-1}^{2k}(z') = q_1^{2k}(z'), \\ &\lim_{n=\infty} \Omega_n^{2k}(z') = \omega^{2k}(z'), \quad \lim_{n=\infty} \Theta_{n'}^{2k}(z') = \theta^{2k}(z'), \end{aligned} \tag{40}$$

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* Hamburger, loc. cit., vol. 81, p. 295, Theorem XV, p. 263, Definition III, and formulas (57), (58), pp. 262–263.

† Loc. cit., vol. 82, p. 144, Theorem XXII; vol. 81, p. 292, Theorem XIV, p. 289, Definition XI, and p. 290.

‡ Loc. cit., vol. 82, p. 123, equations (10).

§ Loc. cit., vol. 82, pp. 139–141, Theorems XX and XXI.

uniformly over every finite closed region. Here p_1^{2k}, q_1^{2k}, ω^{2k}, θ^{2k} are entire transcendental functions of z' subject to the identity

$$(41)\qquad p_1^{2k}\theta^{2k} - q_1^{2k}\omega^{2k} = +1.$$

Let Z be a constant or a function of z and set

$$m^{2k}(z', Z) = \frac{\omega^{2k}(z') + Zp_1^{2k}(z')}{\theta^{2k}(z') + Zq_1^{2k}(z')},$$

$k=0, 1, 2, 3, \cdots$, $m^0(z', Z) \equiv m(z', Z)$. We define this function for $|Z| = \infty$ as follows:

$$m^{2k}(z', \infty) \equiv p_1^{2k}(z')/q_1^{2k}(z').$$

Theorem 5. (a) *If $\mathfrak{P}(z)$ is positive definite and $\mathfrak{P}^{2k}(z)$ is indeterminate, then*

$$S_{2k} = \mathfrak{P}_{2k} + z^{2k-1}m^{2k}(z', \infty) \qquad (k = 0, 1, 2, \cdots; \mathfrak{P}_0 = 0).$$

(b) *Under the same conditions, the files S_{2k-1}, S_{2k+1} converge if and only if*

$$(42)\qquad \lim_{n'=\infty} |g_{n'}^{2k}| = \infty,$$

or else

$$(43)\qquad \lim_{n'=\infty} g_{n'}^{2k} = g^{2k},\ g^{2k} \textit{ finite},$$

and (30) *hold. In the former case,*

$$(44)\qquad S_{2k-1} \equiv S_{2k} \equiv S_{2k+1},$$

while in the latter,

$$(45)\quad S_{2k-1} = \mathfrak{P}_{2k} + z^{2k-1}m^{2k}(z', g^{2k}), S_{2k+1} = \mathfrak{P}_{2k} + z^{2k-1}m^{2k}(z', g^{2k} - z/h^{2k})$$

(*where h^{2k} is the positive constant given by* (37)), *and*

$$(46)\qquad S_{2k-1} \not\equiv S_{2k} \not\equiv S_{2k+1}.$$

Part (a) is the direct consequence of (33), (40).

Now by (34), (39), (40),

$$(47)\quad \lim_{n'=\infty} F_{n'+1}^{2k-1} = \mathfrak{P}_{2k} + z^{2k-1}m^{2k}(z', \infty),\ \text{or}\ \mathfrak{P}_{2k} + z^{2k-1}m^{2k}(z', g^{2k}),$$

according as (42), (43), respectively, holds. It is easily seen with the aid of (41) that the latter is different from S_{2k}.

Again, by (21), (16), (17), the argument being z throughout,

$$F_{n'}^{2k+1} = \mathfrak{P}_{2k+1} - z^{2k+1} A_{2n'-1}^{2k+1}/B_{2n'-1}^{2k+1} = \mathfrak{P}_{2k} + z^{2k}\left[\frac{h_{n'}^{2k} A_{2n'}^{2k} - A_{2n'+1}^{2k}}{h_{n'}^{2k} B_{2n'}^{2k} - B_{2n'+1}^{2k}}\right],$$

which becomes, by (31), (32), (39),

$$F_{n'}^{2k+1} = \mathfrak{P}_{2k} + z^{2k-1}\left[\frac{\Omega_{n'}^{2k}(z') + (g_{n'}^{2k} P_{2n'-1}^{2k}(z') - zP_{2n'+1}^{2k}(z')/h_{n'}^{2k}}{\Theta_{n'}^{2k}(z') + (g_{n'}^{2k} Q_{2n'-1}^{2k}(z') - zQ_{2n'+1}^{2k}(z')/h_{n'}^{2k}}\right].$$

Hence, by (40),

$$\lim_{n'=\infty} F_{n'}^{2k+1} = \mathfrak{P}_{2k} + z^{2k-1} m^{2k}(z', \infty), \text{ or } \mathfrak{P}_{2k} + z^{2k-1} m^{2k}(z', g^{2k} - z/h^{2k}), \tag{48}$$

according as (42), (43), respectively, holds. As before the latter is different from S_{2k}.

Let (n_1) be the set of all indices n not in (n'). Then if (n_1) is finite, we may clearly replace n' by n in (47), (48) and hence in this case we have (44) or (45) according as (42), (43), respectively, holds.

If, on the other hand, (n_1) is infinite, it follows from the identity of the approximants in the squares (28) that

$$\lim_{n_1=\infty} F_{n_1+1}^{2k-1} = \lim_{n_1=\infty} F_{n_1}^{2k+1} = S_{2k},$$

and if the second limits in (47), (48) hold, S_{2k-1}, S_{2k+1} surely diverge inasmuch as they both contain an infinite subsequence with the different limit S_{2k}.

The completion of the proof of (46) is accomplished, again making use of (41), by showing that the difference of the functions (45) is not zero.

We prove next a theorem connecting the various files S_{2k}.

THEOREM 6. (a) *If $\mathfrak{P}^{2k}$ is determinate, then $\mathfrak{P}^{2k-2t}$, $t \leqq k$, is determinate and*

$$S_{2k} \equiv S_{2k-2t}. \tag{49}$$

(b) *If $\mathfrak{P}^{2k-2t}$, $t \leqq k$, is indeterminate, then $\mathfrak{P}^{2k}$ is indeterminate, and*

$$S_{2k} \not\equiv S_{2k-2t}. \tag{50}$$

To prove (a) we write, by Theorem 3,

$$S_{2k-2t} = \mathfrak{P}_{2k-2t} + z^{2k-2t-1} I^{2k-2t}(z'), \quad S_{2k} = \mathfrak{P}_{2k} + z^{2k-1} I^{2k}(z').$$

But clearly

$$I^{2k-2t}(z') = c_{2k-2t} z - c_{2k-2t+1} z^2 + \cdots + z^{2t} \int_{-\infty}^{+\infty} u^{2t} d\psi^{2k-2t}(u)/(z' + u).$$

Hence

$$(51)\qquad S_{2k} - S_{2k-2t} = z^{2k-1}\left[\int_{-\infty}^{+\infty}\frac{d\psi^{2k}(u)}{z'+u} - \int_{-\infty}^{+\infty}\frac{d\psi_1^{2k}(u)}{z'+u}\right],$$

where

$$d\psi_1^{2k}(u) = u^{2t}d\psi^{2k-2t}(u).$$

But $\psi_1^{2k}(u)$ is a moment function of the determinate series $\mathfrak{P}^{2k}$, and therefore

$$\psi_1^{2k}(u) = \int_{-\infty}^{u} u^{2t}d\psi^{2k-2t}(u) \text{ and } \psi^{2k}(u) = \int_{-\infty}^{u} d\psi^{2k}(u)$$

agree at all points of continuity and at the points $z=+\infty, -\infty$. It then follows from a known theorem* that the right member of (51) is identically zero. Thus we prove (49).

To prove that $\mathfrak{P}^{2k-2t}$ is determinate, suppose the contrary. Then $\mathfrak{P}^{2k-2t}$ has two different moment functions:

$$\psi_i^{2k-2t}(u),\ i = 0, 1.$$

Therefore $\mathfrak{P}^{2k}$ has the two different moment functions:

$$\psi_i^{2k}(u) = \int_{-\infty}^{u} u^{2t}d\psi_i^{2k-2t}(u),\quad i = 0, 1,$$

which is contrary to hypothesis.

To prove (b), assume that $\mathfrak{P}^{2k}$ is determinate. Then by (a), just proved, $\mathfrak{P}^{2k-2t}$ is determinate, contrary to hypothesis. Therefore $\mathfrak{P}^{2k}$ must be indeterminate.

Again, by Theorem 3,

$$S_{2k-2t} = \mathfrak{P}_{2k} + z^{2k-1}\int_{-\infty}^{+\infty} u^{2t}d\psi^{2k-2t}(u)/(z'+u),$$

$$S_{2k} = \mathfrak{P}_{2k} + z^{2k-1}I^{2k}(z'),$$

and if these were identical it would follow that

$$\psi^{2k}(u) = \int_{-\infty}^{u} u^{2t}d\psi^{2k-2t}(u).$$

But this is impossible since if it held, $\psi^{2k}(u)$ would have zero saltus at $u=0$. This contradicts (36).

* Perron, loc. cit., p. 367, Lemma 1.

THEOREM 7. (a) *If $\mathfrak{P}(z)$ is indeterminate, then the files S_{2k-1}, $k=0, 1, 2, 3, \cdots$, all converge or else all diverge.*

(b) *A necessary and sufficient condition for convergence is that*

$$\lim_{n'=\infty} | g_{n'} | = \infty ,$$

or else

$$\lim_{n'=\infty} g_{n'} = g, \quad g \text{ finite},$$

and $\Delta_{n-1,n}=0$ for but a finite number of values of n.

(c) *When the odd files are convergent, two successive files S_n, S_{n+1} have limits which are meromorphic functions of $z'=1/z$ expressible in the form $u_n(z)/v_n(z)$, $u_{n+1}(z)/v_{n+1}(z)$, where $u_n(z)$, $v_n(z)$, $\cdots$, are entire functions of $1/z$ among which there is, in general, the relation*

$$u_n(z)v_{n+1}(z) - u_{n+1}(z)v_n(z) = (-z)^n. \tag{52}$$

An exception arises whenever, for some k,

$$\lim_{n'=\infty} | \overset{2k}{g}_{n'} | = \infty ,$$

whereupon

$$S_{2k-1} \equiv S_{2k} \equiv S_{2k+1}. \tag{53}$$

The identity (53) cannot hold for two or more consecutive values of k. If $S_{2k-1} \not\equiv S_{2k}$ or $S_{2k} \not\equiv S_{2k+1}$, then $S_{2k-1} \not\equiv S_{2k} \not\equiv S_{2k+1}$.

(d) *A sufficient condition for divergence of the odd files is that $\Delta_{n-1,n+2k}=0$ for an infinite number of values of n when $k=k_1$ and $k=k_1+1 (k_1 \geqq 0)$.*

In fact, by Theorem 6, all the series $\mathfrak{P}^{2k}$, $k \geqq 0$, are indeterminate. Hence by Theorem 5 it is seen that if S_{2k-1} converges (diverges) then S_{2k+1} must converge (diverge), $k=0, 1, 2, \cdots$. Thus all the files S_{2k-1} converge or else they all diverge. The condition in (b) is obviously correct since it is a necessary and sufficient condition for the convergence of S_{-1}, by Theorem 5.

The first part of (c) follows from Theorem 5 if we take

$$v_n(z) = \overset{2k}{\theta}(z') + \overset{2k}{g}\,\overset{2k}{q_1}(z'), \quad \overset{2k}{q_1}(z'), \text{ or } \overset{2k}{\theta}(z') + (\overset{2k}{g} - z/\overset{2k}{h})\overset{2k}{q_1}(z'),$$

according as $n=2k-1$, $2k$, or $2k+1$, and then put

$$u_n(z) = v_n(z)\left[\mathfrak{P}_{2k} + z^{2k-1}m^{2k}(z', Z)\right], \text{ where } Z = g^{2k}, \ \infty, \text{ or } g^{2k} - z/h^{2k},$$

according as $n=2k-1$, $2k$, or $2k+1$. The relation (52) then follows from (41) after an easy calculation.

That (53) cannot hold for two consecutive values of k follows at once from (50). The last statement in (c) follows from Theorem 5.

Under the hypothesis of (d), it would follow from Theorem 2 that $S_{2k_1} \equiv S_{2k_1+2}$, again contradicting (50).

6. **The convergence of the files S_{-k}, $k \geqq 1$, for a positive definite series.** To investigate the convergence of the files S_{-k}, we turn to the reciprocal series $\mathfrak{E}(z)$. The following theorem is fundamental.

THEOREM 8. *If $\mathfrak{P}(z)$ is a determinate (indeterminate) positive definite power series, then* $-\mathfrak{E}^2(z)$ is a determinate (indeterminate) positive definite power series.*

On the supposition that $\mathfrak{P}$ has a corresponding continued fraction of the form

$$\frac{1}{a_1} + \frac{z}{a_2} + \frac{z}{a_3} + \cdots, \tag{54}$$

a condition equivalent to (6) is

$$a_{2i+1} > 0 \qquad (i = 0, 1, 2, \cdots). \tag{55}$$

Now $-\mathfrak{E}^1(z)$ will have a corresponding continued fraction†

$$\frac{1}{a_2} + \frac{z}{a_3} + \frac{z}{a_4} + \cdots,$$

and consequently if

$$g_n = a_2 + a_4 + \cdots + a_{2n} \neq 0 \qquad (n = 1, 2, 3, \cdots), \tag{56}$$

$-\mathfrak{E}^2(z)$ will have the corresponding continued fraction‡

$$\frac{1}{e_1} + \frac{z}{e_2} + \frac{z}{e_3} + \cdots,$$

where

$$e_{2i+1} = a_{2i+3}(g_{i+1})^2, \quad e_{2i} = a_{2i+2}/(g_i g_{i+1}). \tag{57}$$

Hence by (55), (56), $e_{2i+1} > 0$, and therefore $-\mathfrak{E}^2$ is positive definite in this restricted case.

Under the same restrictions, *if $\mathfrak{P}$ is indeterminate, then $-\mathfrak{E}^2$ is indeterminate, and conversely.* To verify this it is sufficient§ to prove the convergence of the two following series:

* Here we write $\mathfrak{E}^2$ for $\mathfrak{E}^{(2)}$.

† Wall, loc. cit., p. 99. Theorem 1.

‡ Wall, loc. cit., pp. 102–103, formulas (49), (50).

§ Hamburger, loc. cit., vol. 82, p. 148, Theorem XXIV.

$$(58)\qquad (A)\quad \sum e_{2i+1}, \qquad\qquad (B)\quad \sum e_{2i+1}(e_2+e_4+\cdots+e_{2i})^2,$$

on the hypothesis that the like series connected with $\mathfrak{P}$, namely

$$(59)\qquad (A)\quad \sum a_{2i+1}, \qquad\qquad (B)\quad \sum a_{2i+1}(a_2+a_4+\cdots+a_{2i}),$$

converge; and conversely.

Clearly (58) (A) converges, being by (57) the same as (59) (B), which we now suppose convergent. Now, by (57),

$$(60)\qquad \begin{aligned}\sum_{i=1}^{n} e_{2i+1}(e_2+e_4+\cdots+e_{2i})^2 = \frac{1}{(a_2)^2}\sum_{i=1}^{n} a_{2i+3}(g_{i+1})^2 \\ -\frac{2}{a_2}\sum_{i=1}^{n} a_{2i+3}g_{i+1}+\sum_{i=1}^{n} a_{2i+3},\end{aligned}$$

from which the convergence of (58) (B) follows on the hypothesis that (59) converge, since $\sum a_{2i+3}g_{i+1}$ converges then as Hamburger showed.*

Conversely, let (58) be convergent. Then as before (59) (B) converges. Now $\sum e_{2i+1}(e_2+e_4+\cdots+e_{2i})$ is convergent, and, by (57),

$$(61)\qquad \sum_{i=1}^{n} e_{2i+1}(e_2+e_4+\cdots+e_{2i}) = \frac{1}{a_2}\sum_{i=1}^{n} a_{2i+3}(g_{i+1})^2-\sum_{i=1}^{n} a_{2i+3}g_{i+1}.$$

Multiplying the members of (61) by $2/a_2$ and subtracting the resulting equation from (60), we get

$$(62)\qquad \begin{aligned}\sum_{i=1}^{n} a_{2i+3} = \sum_{i=1}^{n} e_{2i+1}(e_2+e_4+\cdots+e_{2i})^2 \\ -\frac{2}{a_2}\sum_{i=1}^{n} e_{2i+1}(e_2+e_4+\cdots+e_{2i})^2+\frac{1}{(a_2)^2}\sum_{i=1}^{n} a_{2i+3}(g_{i+1})^2.\end{aligned}$$

Since the sums on the right of (62) converge for $n=\infty$, the like is true of the sum on the left, and hence (59) (A) converges.

Let now $\overline{\mathfrak{P}}(z)$ be obtained from $\mathfrak{P}(z)$ in the following manner. Suppose t real, and write

$$\frac{c_0}{(z'+t)}-\frac{c_1}{(z'+t)^2}+\frac{c_2}{(z'+t)^3}-\cdots=\frac{\bar c_0}{z'}-\frac{\bar c_1}{z'^2}+\frac{\bar c_2}{z'^3}-\cdots,$$

where the series on the right is obtained by expanding in descending powers of z' each term of the series on the left and then collecting the like powers of $1/z'$. Then we set

$$\overline{\mathfrak{P}}(z) = \bar c_0-\bar c_1 z+\bar c_2 z^2-\cdots.$$

* Loc. cit., vol. 82, p. 139, formula (36).

LEMMA. *If $\mathfrak{P}(z)$ is an indeterminate (determinate) positive definite power series, then if t is real $\overline{\mathfrak{P}}(z)$ is an indeterminate (determinate) positive definite power series. There exists in every real interval an uncountable infinity of numbers t such that $\overline{\mathfrak{P}}(z)$ will have a corresponding continued fraction*

$$\frac{1}{\bar{a}_1} + \frac{z}{\bar{a}_2} + \frac{z}{\bar{a}_3} + \cdots, \tag{63}$$

in which

$$\bar{a}_2 + \bar{a}_4 + \cdots + \bar{a}_{2i} \neq 0 \qquad (i = 1, 2, \cdots). \tag{64}$$

The series $\mathfrak{C}^2$ connected with $\overline{\mathfrak{P}}$ is $\overline{\mathfrak{C}}^2$, i.e., bears the same relationship to $\mathfrak{C}^2$ that $\overline{\mathfrak{P}}$ bears to $\mathfrak{P}$.

With the aid of this lemma the completion of the proof of our theorem may be accomplished. In fact, if $\mathfrak{P}$ fails to have a corresponding continued fraction, or if, while having a corresponding continued fraction, (56) fails, then we may turn to $\overline{\mathfrak{P}}$ in which t has been chosen so that (63) exists and (64) holds. By what we have already proved, $-\overline{\mathfrak{C}}^2$ will then be positive definite and determinate (indeterminate) if $\overline{\mathfrak{P}}$ and hence $\mathfrak{P}$ are of the same character. The same will then be true of $-\mathfrak{C}^2$ since the latter is obtainable from $-\overline{\mathfrak{C}}^2$ on subjecting it to the above described transformation, using $-t$ as the parameter.

Now that part of the lemma up to and including (63) was proved by Hamburger.* He showed that the function $P_{2n}(z')/Q_{2n}(z')$ connected with $\overline{\mathfrak{P}}(z)$ is equal† to $P_{2n}(z'+t)/Q_{2n}(z'+t)$. It follows that the sum (64) is equal to $P_{2n}(t)/Q_{2n}(t)$, and this will be different from 0 for all n if we do not take t to be one of the countable number of zeros of the polynomials $P_{2n}(t)$, and this is clearly possible inasmuch as we have at our disposal an uncountable number of choices for t. Now since

$$1/\mathfrak{P}(z) = \mathfrak{C}(z),$$

it follows that

$$1\Big/\left[\frac{c_0}{(z'+t)} - \frac{c_1}{(z'+t)^2} + \cdots\right] = d_0 z' - [d_1 - d_0 t] + \frac{\bar{d}_2}{z'} - \frac{\bar{d}_3}{z'^2} + \cdots.$$

Thus

$$\overline{\mathfrak{C}}(z) = d_0 - (d_1 - d_0 t)z + \bar{d}_2 z^2 - \bar{d}_3 z^3 + \cdots,$$

and hence

$$\overline{\mathfrak{C}}^2(z) = \bar{d}_2 - \bar{d}_3 z + \bar{d}_4 z^2 - \cdots,$$

* Loc. cit., vol. 81, §4, and p. 300.

† Hamburger, loc. cit., vol. 81, §4, p. 265, formula (64).

which was to be proved.

By (25), (26) we may now write

$$F_n^{-2k} = 1/[\mathfrak{C}_{2k} + z^{2k-1}P_{2n-1}^{-2k}(z')/Q_{2n-1}^{-2k}(z')], \tag{65}$$

$$F_{n+1}^{-2k+1} = 1/[\mathfrak{C}_{2k} + z^{2k-1}P_{2n}^{-2k}(z')/Q_{2n}^{-2k}(z')], \quad k \geqq 1, \tag{66}$$

where P_n^{-2k}, Q_n^{-2k} are related to C_n^{2k}, D_n^{2k} by equations of the form (31), (32).

Theorems exactly analogous to the theorems of §4 can now be stated for the files S_{-k}, $k \geqq 1$, using (65), (66) and Theorem 8 as our point of departure. It is important to observe that when $\mathfrak{P}(z)$ is indeterminate, the convergence or divergence of the file S_{-1} implies the convergence or divergence, respectively, of all the other files S_k where k is odd.

7. **Summary.** From the preceding discussion one will readily see that there are just three cases, as follows.

Case I. When $\mathfrak{P}$ is indeterminate, all the even files of the table are convergent, and the limits are meromorphic functions of $z' = 1/z$, such that, for every $i \geqq 0$, $k \geqq 1$,

$$S_{2i} \not\equiv S_{2i+2k}, \quad S_{-2i-2} \not\equiv S_{-2i-2k-2}.$$

The odd files either all converge or else all diverge, and when convergent the limits of adjacent files of the table are, in general, unequal. In exceptional cases we may have

$$S_{2i-1} \equiv S_{2i} \equiv S_{2i+1} \qquad (i = 0, \pm 1, \pm 2, \cdots),$$

but if this hold for $i = i'$, i'', then $|i' - i''| > 1$.

Case II. The series $\mathfrak{P}^{2k}$ and $-\mathfrak{C}^{2k'}$ for $k = 0, 1, 2, \cdots, p$; $k' = 1, 2, 3, \cdots, p'$, are determinate, while for $k > p$, $k' > p'$, respectively, these series are indeterminate. In this case the files S_i, $i = 0, 1, \cdots, 2p$; $-1, -2, \cdots, -2p'$, all converge to a common limit. For the remaining files in the upper and lower halves of the table taken separately, the discussion is essentially the same as in Case I.*

Case III. The series $\mathfrak{P}$ and the associated series $\mathfrak{P}^{2k}$, $-\mathfrak{C}^{2k}$ are all determinate. Then all the files of the table have a common limit which is a function that is analytic over the entire plane except the whole or a part of the real axis.

In Case I the limits of the files S_i, $i \geqq -1$, are meromorphic functions of $1/z$ with poles on the real axis only. As for S_{-2}, S_{-3}, S_{-4}, $\cdots$, the limits are

* See §9 for further discussion of Case II.

of the form $1/[d_0-d_1z+z^2f(z)]$, where $f(z)$ is a meromorphic function of $1/z$ with poles on the real axis only, and therefore these limits are meromorphic functions of $1/z$ with zeros on the real axis only.

8. **On extended positive definite series.** In order to continue the discussion of the diagonal files we shall need to consider the question of "extending" positive definite series. We make the following definition: The positive definite power series $P(z)$ will be said to admit of an extension of order n if there exist $2n$ real numbers, $c_{-1}, c_{-2}, \cdots, c_{-2n}$, such that the power series

$$\mathfrak{P}^{(-2i)}(z) = c_{-2i} - c_{-2i+1}z + \cdots \qquad (i = 1, 2, 3, \cdots, n) \tag{67}$$

shall all be positive definite.

Two cases will be distinguished according as $\mathfrak{P}(z)$ is determinate or indeterminate. In the former case we have the following theorem:*

THEOREM 9. *Let $\mathfrak{P}(z)$ be a determinate positive definite power series with moment function $\psi(u)$ (cf. §5). Then a necessary and sufficient condition that $\mathfrak{P}(z)$ shall admit of an extension (67) of order n is that the integral*

$$\int_{-\infty}^{+\infty} \frac{d\psi(u)}{u^{2n}} \tag{68}$$

shall be convergent. The coefficients c_{-i} in an extension are as follows:

$$c_{-i} = \int_{-\infty}^{+\infty} d\psi(u)/u^i,\ i < 2n; \quad c_{-2n} \geqq \int_{-\infty}^{+\infty} d\psi(u)/u^{2n}. \tag{69}$$

No other extensions of order n are possible.

To prove that the convergence of (68) is sufficient for the existence of an extension of $\mathfrak{P}(z)$ of order n, we write

$$\psi_1(u) = \int_{-\infty}^{u} \frac{d\psi(u)}{u^{2n}},$$

and note that this is a non-decreasing real function of u such that all the moments

$$c_i' = \int_{-\infty}^{+\infty} u^i d\psi_1(u) \qquad (i = 0, 1, 2, \cdots)$$

exist. Consequently,† the series

$$c_0' - c_1'z + c_2'z^2 - \cdots \tag{70}$$

* Cf. the corresponding theorem for Stieltjes series given by the writer in these Transactions, vol. 31, pp. 771–781.

† Hamburger, loc. cit., vol. 81, §5, pp. 266–270.

is positive definite. But

$$c'_{2n+i} = c_i \qquad (i = 0, 1, 2, \cdots),$$

and therefore (70) is an extension of $\mathfrak{P}(z)$ of order n.

To prove the necessity of the condition, assume that an extension (67) of $\mathfrak{P}(z)$ exists. Clearly, since $\mathfrak{P}(z)$ is determinate, $\mathfrak{P}^{(-2n)}(z)$ is determinate by Theorem 6. Let $\psi^{(-2n)}(u)$ be the moment function of $\mathfrak{P}^{(-2n)}(z)$. Then $\psi(u)$ and $\psi_2(u) = \int_{-\infty}^{u} u^{2n} d\psi^{(-2n)}(u)$ are both moment functions of the determinate series $\mathfrak{P}(z)$, and are therefore equal, with the possible exception of an additive constant, at all points of continuity. Thus if $-a<0$, $b>0$, $i\geqq 0$, $-a$, b points of continuity of $\psi(u)$,

$$\int_{-\infty}^{-a} \frac{d\psi(u)}{u^{2n-i}} = \int_{-\infty}^{-a} u^i d\psi^{(-2n)}(u), \tag{71}$$

$$\int_{b}^{\infty} \frac{d\psi(u)}{u^{2n-i}} = \int_{b}^{\infty} u^i d\psi^{(-2n)}(u). \tag{72}$$

Now if a, b approach 0 over points of continuity of $\psi(u)$, the right members of (71), (72), and therefore the left members, will approach definite limits, L_i, L_i', respectively. Let $-a'$ be an arbitrary real number subject to the condition $0<a'<\delta$, and let $-a_1$, $-a_2$ be points of continuity of $\psi(u)$ such that $-\delta<-a_1<-a'<-a_2<0$. Then if $\epsilon>0$, and i is even,

$$L_i - \frac{\epsilon}{2} < \int_{-\infty}^{-a_1} \frac{d\psi(u)}{u^{2n-i}} \leqq \int_{-\infty}^{-a'} \frac{d\psi(u)}{u^{2n-i}} \leqq \int_{-\infty}^{-a_2} \frac{d\psi(u)}{u^{2n-i}} \leqq L_i, \tag{73}$$

if δ is sufficiently small; if i is odd, (73) is to be replaced by

$$L_i + \frac{\epsilon}{2} > \int_{-\infty}^{-a_1} \frac{d\psi(u)}{u^{2n-i}} \geqq \int_{-\infty}^{-a'} \frac{d\psi(u)}{u^{2n-i}} \geqq \int_{-\infty}^{-a_2} \frac{d\psi(u)}{u^{2n-i}} \geqq L_i.$$

Hence, whether i is even or odd,

$$\left| L_i - \int_{-\infty}^{-a'} \frac{d\psi(u)}{u^{2n-i}} \right| < \frac{\epsilon}{2} \qquad (0 < a' < \delta). \tag{74}$$

Similarly,

$$\left| L_i' - \int_{b'}^{+\infty} \frac{d\psi(u)}{u^{2n-i}} \right| < \frac{\epsilon}{2} \qquad (0 < b' < \delta'). \tag{75}$$

It follows, if we combine (74), (75), that

$$\int_{-\infty}^{+\infty} \frac{d\psi(u)}{u^{2n-i}}, \quad i \geqq 0,$$

converges, and its limit is

$$L_i + L_i' = c_{-i}, \quad 2n \geqq i \geqq 0.$$

When $i=2n$, $L_i=\psi^{(-2n)}(-0)$ while $L_i'=c_{-2n}-\psi^{(-2n)}(+0)$, and hence in this case

$$\int_{-\infty}^{+\infty} \frac{d\psi(u)}{u^{2n}} = c_{-2n} - \omega,$$

where $\omega \geqq 0$ is the saltus of $\psi^{(-2n)}(u)$ at $u=0$.

When $\mathfrak{P}(z)$ is indeterminate, we proceed as follows. Set

$$m(z';t) = \frac{\omega(z')+tp_1(z')}{\theta(z')+tq_1(z')}, \quad z'=1/z, \tag{76}$$

where t is a real parameter, and $\omega(z')$, $\theta(z')$, $\cdots$, are the functions introduced in (40). Hamburger* has shown that $m(z';t)$ is a meromorphic function of z'†, which may be expressed as a Stieltjes integral:

$$m(z';t) = \int_{-\infty}^{+\infty} \frac{d\psi(u;t)}{z'+u}, \tag{77}$$

the latter having the asymptotic development $\mathfrak{P}(1/z')/z'$.

For any given finite value of t, $\psi(u;t)$ is constant in the neighborhood of $u=0$, and hence all the moments

$$c_{-i} = \int_{-\infty}^{+\infty} d\psi(u;t)/u^i \qquad (i = 1, 2, \cdots) \tag{78}$$

exist. Thus if we set

$$\psi^{(-2n)}(u) = \int_{-\infty}^{u} d\psi(u;t)/u^{2n},$$

where n is an arbitrary positive integer, we see that $\psi^{(-2n)}(u)$ is a non-decreasing function of u such that all the moments

$$c_{-2n+i} = \int_{-\infty}^{+\infty} u^i d\psi^{(-2n)}(u) \qquad (i = 0, 1, 2, \cdots)$$

exist. Hence $\mathfrak{P}^{(-2n)}(z)=c_{-2n}-c_{-2n+1}z+\cdots$ is a positive definite power series and is an extension of $\mathfrak{P}(z)$ of order n.

Hamburger‡ showed that $\mathfrak{P}^{(-2)}(z)$ is determinate. It then follows from

* Loc. cit., vol. 82, §§18–19.

† The z' of the preceding work.

‡ Loc. cit. vol. 82, p. 179.

Theorem 6 that $\mathfrak{P}^{(-2n)}(z)$ is determinate for $n=1, 2, \cdots$. Also, $M+\mathfrak{P}^{(-2)}(z)$, $M>0$, is indeterminate, and it may in turn be extended, giving an extension $\mathfrak{P}_1^{(-4)}(z)$ of $\mathfrak{P}(z)$ of order 2 which may be determinate or indeterminate, and is in any case different from $\mathfrak{P}^{(-4)}(z)$. This new series, $\mathfrak{P}_1^{(-4)}(z)$, may then be extended, etc. We sum up these facts in the following theorem. The formulas (72) are given by Hamburger.

THEOREM 10. *Let $\mathfrak{P}(z)$ be an indeterminate positive definite power series, and let $m(z; t)$ be the meromorphic function of z defined by* (76), (77), *which gives rise to a moment function $\psi(u; t)$ of $\mathfrak{P}(z)$. Then $\mathfrak{P}(z)$ admits of a determinate extension of arbitrary order n corresponding to every finite value of the real parameter t. The coefficients in the extension are given by* (78).

*The first two coefficients, c_{-1}, c_{-2}, are polynomials in t as follows:**

$$\begin{aligned} c_{-1} &= t, \\ c_{-2} &= q_1'(0)t^2 + [\theta'(0) - p_1'(0)]t - \omega'(0). \end{aligned} \tag{79}$$

The series $(M+c_{-2})-c_{-1}z+c_0z^2- \cdots$, $M>0$, is positive definite and indeterminate. This series may then be extended to any desired order, forming thereby, at pleasure, extensions of $\mathfrak{P}(z)$ all of which are indeterminate, or else such that all from and after a certain order are determinate.

Let us suppose that there exists a sequence of indices

$$n_1 < n_2 < n_3 < \cdots$$

such that (cf. (38))

$$\lim_{p=\infty} g_{n_p}^{2i} = g^{2i},$$

where g^{2i} is finite. Then by (34), (39), (40), (23), the sequence of approximants in S_{2i-1},

$$[n_p, n_p + 2i - 1] \qquad (p = 1, 2, \cdots),$$

will have the limit $c_0-c_1z+ \cdots -c_{2i-1}z^{2i-1}+z^{2i-1}m^{2i}(z', g^{2i})$.

Let $\psi^{2i}(u, t)$ be the moment function of $\mathfrak{P}^{2i}$ occurring in the expression (76) for $m^{2i}(z', g^{2i})$, and set

$$\int_{-\infty}^{+\infty} \frac{d\psi^{2i}(u, g^{2i})}{u^k} = c_k^{(i)} \qquad (k = 1, 2, 3, \cdots). \tag{80}$$

Then we shall show that the series

$$\frac{c_1^{(i)}}{z} - \frac{c_2^{(i)}}{z^2} + \frac{c_3^{(i)}}{z^3} - \cdots \tag{81}$$

* The primes denote differentiation with respect to z.

converges outside a circle of radius R where R is the distance from the origin to the most remote pole of $m^{2i}(z', g^{2i})/z$ (and is necessarily finite).

In fact, $\psi^{2i}(u, g^{2i})$ is constant in the interval $(-R^{-1}, +R^{-1})$ and hence, if $R>\sigma>0$,

$$\frac{m^{2i}(z', g^{2i})}{z} = \int_{-\infty}^{+\infty} \frac{d\psi^{2i}(u, g^{2i})}{1+zu} = \int_{-R-\sigma}^{R+\sigma} \frac{-\,ud\psi^{2i}(1/u, g^{2i})}{z+u}$$

$$= \int_{-R-\sigma}^{R+\sigma} -\,u\left\{\frac{1}{z} - \frac{u}{z^2} + \cdots\right\} d\psi^{2i}(1/u, g^{2i}).$$

Since the series within the braces converges uniformly over the interval $-R-\sigma \leqq u \leqq R+\sigma$, if $|z| > R+\sigma$, it may be integrated term by term and hence (81) converges for $|z| > R$ and is equal to $m^{2i}(z', g^{2i})/z$ there. We state the following theorem.

THEOREM 11. *Let $\mathfrak{P}(z)$ be indeterminate, and suppose S_{2i-1} contains an infinite subsequence of approximants*:

$$[n_p, n_p + 2i - 1] \qquad (p = 1, 2, \cdots),$$

with limit $f(z)$. Then if $f(z) \not\equiv S_{2i}$ we have the following expansion:

$$f(z) = \mathfrak{P}_{2i} + z^{2i}\left\{\frac{c_1^{(i)}}{z} - \frac{c_2^{(i)}}{z^2} - \frac{c_3^{(i)}}{z^3} - \cdots\right\},$$

which is convergent for $|z| > R$, where R is the distance from the origin to the most remote pole of $f(z)$, and is always finite. The numbers $c_k^{(i)}$ are given by (80), *and thus, for every k,*

(82) $$c_{2k}^{(i)} - c_{2k-1}^{(i)}z + \cdots - c_1^{(i)}z^{2k-1} + c_{2i}z^{2k} - \cdots$$

is an extension of $\mathfrak{P}^{2i}(z)$ of order k.

9. **An existence theorem for Case II.** We shall apply the work of the preceding paragraph to show that, in Case II, *for some i the files S_{-1}, S_1, $S_2, \cdots, S_{2i}$ may have a common limit while the succeeding odd files, S_{2i+1}, S_{2i+3}, $S_{2i+5}, \cdots$, may either converge or diverge.* To construct such examples, suppose $\mathfrak{P}(z)$ to be indeterminate, and let all the odd files of the associated Padé table be divergent. Let us now form the series (82) taking $k=i$. The series so formed will be determinate by Theorem 10, and in the corresponding Padé table, $S_{-1}, S_0, \cdots, S_{2i}$ will be convergent, while the subsequent odd files must clearly diverge. If, on the other hand, the odd files in the table for $\mathfrak{P}(z)$ had been convergent, the same would have been true in the table for the constructed series.

If, in Case II, S_{2i+2} is the first file different from $S_{2i}(i \geqq 0)$, then S_{2i+2} is a meromorphic function of $1/z$. The like is true of S_{2i+1} when the latter is convergent. It is readily shown that *the common limit*, S_0, *of the preceding files is a meromorphic function of* $1/z$.

In fact, the series $\mathfrak{P}^{2i+2}(z)$, being indeterminate, may be extended indefinitely. One such set of extensions is made up of the determinate series $\mathfrak{P}^{2i-2k}(z)$, $k=0, 1, 2, \cdots, i$.

Now the function $m^{2i+2}(z', t)$ connected with $\mathfrak{P}^{2i+2}(z)$ may be written in the form

$$\sum_{\nu=-\infty}^{+\infty} \frac{N_\nu(t)}{z' + \lambda_\nu(t)}.$$

Let us take $t=c_{2i+1}$ (cf.(79)), and consider the function

$$\psi_1(u) = \int_{-\infty}^{u} \frac{d\psi^{2i+2}(u, c_{2i+1})}{u^{2i+2}}.$$

Clearly the integrals

$$\int_{-\infty}^{+\infty} u^i d\psi_1(u) = c_i' \qquad (i = 0, 1, 2, \cdots)$$

all exist and

$$c_i' = c_i,\ i = 1, 2, \cdots,\ c_0' = c_0 + \omega,\ \omega \geqq 0.$$

It follows that

$$S_0 = \omega + \frac{1}{z}\int_{-\infty}^{+\infty} \frac{d\psi_1(u)}{z' + u} = \omega + \frac{1}{z}\sum_{\nu=-\infty}^{+\infty} \frac{N_\nu}{(\lambda_\nu)^{2i+2}(z' + \lambda_\nu)},$$

where we have written N_ν, λ_ν for $N_\nu(c_{2i+1})$, $\lambda_\nu(c_{2i+1})$, respectively. Hence S_0 is a meromorphic function of $z'=1/z$.

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A PROOF OF THE GENERALIZED SECOND-LIMIT THEOREM IN THE THEORY OF PROBABILITY*

BY

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Introduction. A function $F(x)$, defined for all real x, will be called a "law of probability," if the following conditions are satisfied:

(i) $F(x)$ is monotone non-decreasing in $(-\infty, \infty)$ and continuous to the left,

(ii) $$F(-\infty) = 0, \ F(\infty) = 1.\dagger$$

A particular case is represented by $dF(x) = f(x)dx$, where $f(x)$, summable and $\geqq 0$, is the "probability density" or "law of distribution" for x.

The expression $\int_{-\infty}^{\infty} x^s dF(x)$ is called the "sth moment" of the distribution, s taking values $0, 1, 2, \cdots$.

The Second Limit-Theorem, which was the starting point of this paper, can be stated, with A. Markoff,‡ as follows:

If a sequence of laws of probability $F_k(x)$ $(k=1, 2, \cdots)$ is such that they admit moments of all orders, and if

$$\lim_{k\to\infty} \int_{-\infty}^{\infty} x^s dF_k(x) = \pi^{-1/2} \int_{-\infty}^{\infty} x^s e^{-x^2} dx \qquad (s = 0, 1, \cdots),$$

then, for all x,

$$\lim_{k\to\infty} \int_{-\infty}^{x} dF_k(x) = \pi^{-1/2} \int_{-\infty}^{x} e^{-x^2} dx.$$

Markoff's proof is rather complicated, being based on the distribution of roots and other properties of Hermite polynomials, also on the so-called Tchebycheff inequalities in the theory of algebraic continued fractions. He points out that the theorem still holds if we replace the law of probability $\pi^{-1/2}\int_{-\infty}^{x} e^{-x^2}dx$ by a more general one: $\int_{-\infty}^{x} f(x)dx$ (in which case, however, his considerations need many supplements).§

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* Presented to the Society, April 18, 1930; received by the editors August 22, 1930.

† In fact, if X is a fortuitous variable (finite, not necessarily bounded), and if $F(x)$ is the probability that $X<x$, then $F(x)$ will satisfy these conditions, provided we assume that the principle of total probabilities still holds for a *countable infinity* of inconsistent events.

‡ A. Markoff, *Theory of Probability*, 4th edition (1924, in Russian), p. 522.

§ Cf. J. Chokhate, *Sur la convergence des quadratures mécaniques dans un intervalle infini* . . . , Comptes Rendus, vol. 186 (1928), pp. 344–346.

The same theorem has recently attracted the attention of many investigators: R. von Mises,* G. Pólya,† Paul Lévy,‡ Cantelli,§ Jacob|| and others.

The object of this paper is (a) to establish a general limit-theorem, removing many restrictions imposed otherwise on the functions involved and their moments, so that the above statement dealing with the law of probability $\pi^{-1/2}\int_{-\infty}^{x} e^{-x^2}dx$ (we shall call it hereafter the "classical case") is therein included as a very special case; (b) to give an *elementary* proof, which does not use either characteristic functions or algebraic continued fractions, being based on a well known Montel-Helly theorem concerning sequences of monotonic functions.

A brief account will first be given of the "moments-problem" to which the theorem in question is closely related.

1. **The moments-problem.** *Given a certain interval* (a, b), *finite or infinite, and an infinite sequence of real constants* $c_0, c_1, \cdots$, *find a function* $\psi(x)$, *non-decreasing in* (a, b),¶ *such that*

$$\int_a^b x^s d\psi(x) = c_s \qquad (s = 0, 1, \cdots).$$

We call this the moments-problem corresponding to the data $\{c_s\}$. We may assume, without loss of generality, $\psi(a)=0$.

It is known that *if* (a, b) *be finite, then the moments-problem cannot have more than one solution* (if it has any),** if we generally agree to *consider as*

* R. von Mises, *Fundamentalsätze der Wahrscheinlichkeitsrechnung*, Mathematische Zeitschrift, vol. 4 (1919), pp. 1–97.

† G. Pólya, *Über den zentralen Grenzwertsatz der Wahrscheinlichkeitsrechnung* . . . , Mathematische Zeitschrift, vol. 8 (1920), pp. 171–181.

‡ Paul Lévy, *Calcul des Probabilités*, Paris, 1925, Chapter IV.

§ F. P. Cantelli, *Una nuova dimostrazione del secondo teorema-limite* . . . , Rendiconti di Palermo, vol. 52 (1928), pp. 151–174.

|| Jacob, *De l'application des intégrales généralisées de Fourier au calcul des probabilités*, Comptes Rendus, vol. 188 (1929), pp. 541–43, 754–56.

¶ The case of $\psi(x)$ having but a finite number of points of increase in (a, b) is trivial.

** A simple proof is the following. The existence of two solutions $\psi_1(x), \psi_2(x)$ implies

$$\int_a^b x^s dF(x) = 0 \qquad (s = 0, 1, \cdots; F(x) = \psi_1 - \psi_2; F(a) = 0),$$

$$F(b) = F(a) \quad (\text{for } s = 0), \quad \int_a^b x^l F(x)dx = 0 \quad (l = 0, 1, \cdots; \text{integration by parts}).$$

The latter relations lead to the required conclusion: $F(x)=0$ at all points of continuity in (a, b), by the following reasoning due to Stieltjes (*Correspondance d'Hermite et Stieltjes*, Paris, 1905, pp. 337–338). If such a point z exists $(a<z<b)$, where $F(z)\neq 0$, then $1-(x-z)^2/M>0$ $(a\leqq x\leqq b)$, for a sufficiently large M; hence, it is easily seen that

$$I = \int_a^b F(x)[1 - (x - z)^2/M]^n dx,$$

identical two solutions $\psi_1(x)$, $\psi_2(x)$ *which coincide at all points of continuity.** We express this property by saying that *the moments-problem for a finite interval is "determined."*

On the other hand, *the moments-problem for an infinite interval may be "indeterminate,"* i.e., *it may admit infinitely many solutions.* In fact, in the formula

$$\int_0^\infty y^{a-1}e^{-by}dy = \frac{\Gamma(a)}{b^a}$$

take

$$b = k + di \quad (k > 0),\ a = (n+1)/\lambda,\ (2n+1)/\lambda \quad (n = 0, 1, \cdots),$$

$$\frac{d}{k} = \tan \lambda\pi, \quad \tan\frac{\mu\pi}{2} \quad (\lambda, \mu \text{ defined below}), \quad y = x^\lambda,$$

and we get functions *having all moments* $=0$:†

$$\int_0^\infty x^n e^{-\kappa x^\lambda} \sin(\kappa x^\lambda \tan \lambda\pi)dx = 0 \qquad (\kappa > 0,\ 0 < \lambda < \tfrac{1}{2}),$$

$$\int_{-\infty}^\infty x^n e^{-\kappa x^\mu} \cos\left(\kappa x^\mu \tan\frac{\mu\pi}{2}\right)dx = 0$$

$$\left(\kappa > 0,\ 0 < \mu = \frac{2s}{2s+1} < 1,\ s \text{ a positive integer}\right).$$

Hence we get infinitely many non-decreasing functions

$$(1)\qquad \begin{aligned} &\int_0^x e^{-\kappa x^\lambda}[1 + h\sin(\kappa x^\lambda \tan\lambda\pi)]dx, \\ &\int_{-\infty}^x e^{-\kappa x^\mu}\left[1 + h\cos\left(\kappa x^\mu \tan\frac{\mu\pi}{2}\right)\right]dx \qquad (-1 \leqq h \leqq 1), \end{aligned}$$

---

for n very large, is different from zero, *which is impossible,* I being a linear combination of the moments of $F(x)$, all of which vanish. (We notice that such M does not exist for (a, b) infinite.) Moreover, if $\psi_i(x)$ is continuous to the left, then $\psi_1(x) = \psi_2(x)$ *everywhere in* (a, b), since $\psi_i(x-0) = \lim \psi_i(X)$, where $X(<x)$ converges to x, being always a point of continuity of $\psi_i(x)$, $i = 1, 2$.

* Also at the points a, b, if (a, b) be finite; this, however, necessarily follows from the relations

$$\psi_{1,2}(a) = 0, \quad \int_a^b d\psi_1(x) = \int_a^b d\psi_2(x) = c_0.$$

† These have been given by Adamoff (*Proof of a theorem of Stieltjes*, Proceedings of the Kazan Mathematical Society (1911, in Russian)) and by Stekloff (*Application de la théorie de fermeture* . . . , Mémoires de l'Académie des Sciences, Petrograd, vol. 33 (1915)), but the original statement is due to Stieltjes (loc. cit., p. 230).

solutions of the same moments-problem for $(0, \infty)$ and $(-\infty, \infty)$ respectively. Either of the following conditions ensures the determined character of the moments-problem for an infinite interval:

$$(2) \qquad \sum_{n=1}^{\infty} c_{2n}^{-1/(2n)} \text{ diverges*} \qquad \left(c_n = \int_{-\infty}^{\infty} x^n dF(x)\right);$$

$$(3) \qquad dF(x) = p(x)dx \quad (p(x) \geqq 0 \text{ on } (a, b))$$

with $p(x) < M|x|^{\alpha-1}e^{-\kappa|x|^\lambda}$ for $|x| \geqq x_0$, sufficiently large (M, α, κ are positive constants); $\lambda \geqq \frac{1}{2}$ for $(a, b) = (0, \infty)$,† $\lambda \geqq 1$ for $(a, b) = (-\infty, \infty)$.‡

On the other hand, the moments-problem is indeterminate if $d\psi(x) = p(x)dx$, and for $|x|$ sufficiently large (see (1))

$$p(x) > e^{-\kappa|x|^\lambda} \quad (\kappa > 0) \text{ with } \lambda < \tfrac{1}{2} \text{ for } (0, \infty),\ \lambda < 1 \text{ for } (-\infty, \infty).$$

2. **A generalized statement of the second limit-theorem.** *Given a sequence of laws of probability* $F_n(x)$ $(n = 1, 2, \cdots)$, *with the following properties*: (i) *the moments* $m_n^{(r)} = \int_{-\infty}^{\infty} x^r dF_n(x)$ *of all orders* $r = 0, 1, \cdots$ *exist for* $n = 1, 2, \cdots$, *or at least from a certain rank n on* (*possibly depending on r*); (ii) *the quantities* $m_1^{(r)}, m_2^{(r)}, \cdots$, *for any* $r = 0, 1, \cdots$, *lie, when they exist, between two fixed limits independent of n* (*but possibly dependent on r*). *Then a subsequence* $\{C_i(x) \equiv F_{n_i}(x)\}$ $(i = 1, 2, \cdots; n_1 < n_2 < \cdots; n_i \to \infty)$ *can be extracted such that* (α) $\lim_{i\to\infty} \int_{-\infty}^{\infty} x^r dC_i(x)$ *exists* $(= m_r)$,§ *for* $r = 0, 1, \cdots$; (β) *the subsequence* $\{C_i(x)\}$ *converges for any x to one fixed law of probability* $\psi(x)$, *save, perhaps, at its points of discontinuity*; (γ) $\int_{-\infty}^{\infty} x^r d\psi(x)$ *exists and* $= m_r$ $(r = 0, 1, \cdots)$.‖

The proof will be arranged in several steps.

3. **Existence of** m_r $(r = 1, 2, \cdots)$. We apply here the classical "diagonal method." The hypothesis of the uniform boundedness of $\{m_n^{(r)}\}$ for all $r = 1, 2, \cdots$, enables us to extract from the sequence $\{m_n^{(1)}\}$ a subsequence $\{m_{p_i}^{(1)}\}$ converging to a finite limit m_1. The sequence $\{m_{p_i}^{(2)}\}$ gives rise to a subsequence $\{m_{q_j}^{(2)}\}$ converging to a finite limit m_2, and so on. We thus get a sequence $\{m_{p_1}^{(r)}, m_{q_2}^{(r)}, \cdots\}$ converging to a finite limit m_r, for any

* T. Carleman, *Sur les équations intégrales singulières* . . . , Uppsala, 1923, p. 219.

† Stieltjes, *Recherches sur les fractions continues*, Oeuvres, vol. II, pp. 402–559, where (3) is given for $(0, \infty)$ only. The corresponding condition for $(-\infty, \infty)$ follows directly.

‡ It can be easily shown that, if $x_0 > 1$ and if, in $(1, x_0)$, $p(x)$ is bounded in the sense of Lebesgue, i.e., disregarding a set of zero measure, then (3) is included in (2). Furthermore, if $\psi_1(x), \psi_2(x)$ are solutions of a determined moments-problem for the interval (a, ∞) $(a = 0, -\infty)$, we can arrange so as to have $\psi_1(x) = \psi_2(x)$ *everywhere in* (a, ∞), since $\psi_1(x) = \psi_2(x)$ at all points of continuity, it being permissible, as above, to take $\psi_i(x)$ continuous to the left $(i = 1, 2)$.

§ m_0 exists and $= 1$, by definition of law of probability.

‖ It follows that there is necessarily at least one solution of the moments-problem corresponding to the data $\{m_r\}$.

$r=1,\ 2, \cdots$. The corresponding laws of probability $\gamma_1(x) \equiv F_{p_1}(x)$, $\gamma_2(x) \equiv F_{q_2}(x), \cdots$ clearly satisfy (α). The reasoning still holds if none of the $F_n(x)$ has *all* of its moments finite.

4. **Existence of a limiting law of probability $\psi(x)$.** This follows by applying to $\{F_n(x)\}$ the Montel-Helly* theorem on monotonic functions. We state it in a slightly generalized form:

If a family $\{f(x)\}$ of functions, non-decreasing on $(-\infty, \infty)$, is uniformly bounded in any finite interval (i.e. $|f(x_0)| < A(x_0)$ at any finite point x_0, $A(x_0)$ being the same for all $f(x)$), then from any infinite sequence of this family we can extract a subsequence which converges, for any x, to a non-decreasing function. Moreover, the convergence is uniform in any interval, where the limit-function is continuous.

The theorem holds, with proper modifications, for families of functions of bounded variation.

In order to prove (β), it suffices to apply this theorem to the sequence $\{\gamma_i(x)\}$, since $0 \leqq \gamma_i(x) \leqq 1$ for $-\infty \leqq x \leqq \infty$. We extract then from it a sequence $\{C_p(x)\}$ converging everywhere to a non-decreasing function $\phi(x)$, and we take $\psi(x) \equiv \phi(x-0)$.

The following remarks are important:

(i) The limit-function $\psi(x)$ of the $\{C_p(x)\}$ varies *effectively* from 0 to 1, i.e.

$$\psi(-\infty) = 0, \ \psi(\infty) = \int_{-\infty}^{\infty} d\psi(x) = 1.\dagger$$

(ii) For any $r = 0, 1, \cdots$, the convergence of

$$\lim_{a\to-\infty,\, b\to\infty} \int_a^b x^r dF_n(x) \ \text{ to } \int_{-\infty}^{\infty} x^r dF_n(x)$$

is uniform with respect to n, at least from a certain rank n on.

(iii) $\lim_{x\to\infty} x^s(1 - F_n(x)) = 0, \ \lim_{x\to-\infty} |x|^s F_n(x) = 0 \ (n = 1, 2, \cdots; s > 0 \text{ arbitrary})$.

(i) follows from the inequalities

$$(4) \quad \int_b^{\infty} x^r dF_n(x) \leqq \frac{\overset{(2r+2)}{m_n}}{b^{r+2}} \quad (b > 1); \quad \int_{-\infty}^{a} x^r dF_n(x) \leqq \frac{\overset{(2r+2)}{m_n}}{|a|^{r+2}} \quad (a < -1),$$

* (a) P. Montel, *Sur les suites infinies des fonctions*, Annales de l'École Normale Supérieure, 1907; (b) E. Helly, *Über lineare Funktionaloperationen*, Sitzungsberichte der Akademie der Wissenschaften, Wien, vol. 121 (1912), pp. 265–297.

† This is by no means obvious. Take, for example, $F_n(x) = 0 \ (x < -n)$, $= \frac{1}{2}(-n \leqq x \leqq n)$, $= 1(x > n)$. Here $\lim_{n\to\infty} F_n(x) = \psi(x) \equiv \frac{1}{2}$ for $-\infty \leqq x \leqq \infty$.

which, applied to $\{C_p(x)\}$, yield, for $r=0$ and $p\to\infty$,

$$(5)\qquad (0 \leqq) 1 - \psi(b) \leqq \frac{1+m_2}{b^2},\ 0 \leqq \psi(a) \leqq \frac{1+m_2}{a^2} \qquad (b, -a > 1),$$

and this proves (i) by letting $b\to\infty$, $a\to-\infty$.

(ii) follows directly from (4), taking into account the uniform boundedness of $\{m_n^{(2r+2)}\}$ $(n=1, 2, \cdots)$.

In order to establish (iii), we write

$$1-F_n(b) = \int_b^\infty dF_n(x) \leqq \int_b^\infty \left(\frac{x}{b}\right)^{2r} dF_n(x);\quad b^s[1-F_n(b)] \leqq \frac{m_n^{(2r)}}{b^{2r-s}}$$
$$(b > 0,\ 0 < s < 2r),$$

and a similar expression for $|a|^s F_n(a)\,(a<-1)$.

5. $\int_{-\infty}^{\infty} x^r d\psi(x)$ exists and $=m_r (r=0, 1, \cdots)$. This statement being the fundamental part of the theorem, we give for it two proofs.

Proof I. We apply the two following theorems due respectively to Hamburger* and to Helly,† the proofs of both of which are very elementary.

HAMBURGER'S THEOREM. *Suppose* (i) $\int_{-\infty}^{\infty} [1/(z+x)]d\psi(x)$ *converges for* $z=iy$ *with* $y>0$, $\psi(x)$ *denoting a function non-decreasing in* $(-\infty, \infty)$; (ii) $F(z)\equiv\int_{-\infty}^{\infty}[1/(z+x)]d\psi(x)$ *has, for* $z=iy\to\infty$, *an asymptotic expansion* (*in Poincaré's sense*) $F(z)\sim\sum_{\nu=1}^{\infty}(-1)^\nu c_\nu/z^{\nu+1}$ (c_ν *real*). *Then* $\int_{-\infty}^{\infty}x^\nu d\psi(x)$ *exists and* $=c_\nu$ $(\nu=0, 1, \cdots)$.

HELLY'S THEOREM. *Given a sequence* $\{V_n(x)\}$ *of functions of bounded variation on a finite interval* (a, b) *such that* (i) *the total variations of all* $V_n(x)$ *on* (a, b) *are uniformly bounded*; (ii) $\lim_{n\to\infty}V_n(x)=v(x)$‡ *exists for* $a\leqq x\leqq b$, *with the possible exception of a countable set of points not including* a, b. *Then* $\lim_{n\to\infty}\int_a^b f(x)dV_n(x)=\int_a^b f(x)dv(x)$, *for any continuous function* $f(x)$.

Going back to the given sequence $\{F_n(x)\}$ and to the above function $\psi(x)=\lim_{p\to\infty}\{C_p(x)\}$, we notice, first, that $\int_{-\infty}^{\infty}f(x)d\psi(x)$, $\int_{-\infty}^{\infty}f(x)dF_n(x)$ certainly exist if $f(x)$ is bounded on $(-\infty, \infty)$ and continuous on any finite interval. Furthermore, since, as we have seen, $\int_{-\infty}^{\infty}dC_p(x)$ converges *uniformly* (with respect to p), an easy application of Helly's theorem yields

* H. Hamburger, *Über eine Erweiterung des Stieltjesschen Momentenproblems*, I, II, III, Mathematische Annalen, vols. 81–82 (1920), pp. 235–319, 120–64, 168–87.

† Loc. cit.

‡ Necessarily of bounded variation, by virtue of the Montel-Helly theorem extended to this class of functions.

$$\lim_{p\to\infty}\int_{-\infty}^{\infty} f(x)dC_p(x) = \int_{-\infty}^{\infty} f(x)d\psi(x), \tag{6}$$

$$\lim_{p\to\infty}\int_{-\infty}^{\infty} \frac{dC_p(x)}{z+x} = \int_{-\infty}^{\infty}\frac{d\psi(x)}{z+x} \qquad (z = iy;\ y > 0). \tag{7}$$

Consider now the expression

$$\begin{aligned} f_p(z) &\equiv \int_{-\infty}^{\infty}\frac{dC_p(x)}{z+x} = \sum_{r=0}^{\nu-1}(-1)^r\frac{\alpha_p^{(r)}}{z^{r+1}} + \frac{(-1)^\nu}{z^{\nu+1}}I_{\nu,p} \qquad (\nu \text{ arbitrary}),\\ I_{\nu,p} &= \int_{-\infty}^{\infty} x^\nu\left(\frac{z}{z+x}\right)dC_p(x),\quad \alpha_p^{(r)} = \int_{-\infty}^{\infty} x^r dC_p(x). \end{aligned} \tag{8}$$

Letting $p\to\infty$, using (7) and the property

$$\lim_{p\to\infty}\alpha_p^{(r)} = m_r \qquad (r = 0, 1, \cdots),$$

we get

$$F(z) \equiv \int_{-\infty}^{\infty}\frac{d\psi(x)}{z+x} = \sum_{r=0}^{\nu-1}\frac{(-1)^r m_r}{z^{r+1}} + \frac{(-1)^\nu}{z^{\nu+1}}\lim_{p\to\infty} I_{\nu,p}. \tag{9}$$

Observing that $|z/(z+x)|\leqq 1$ $(-\infty\leqq x\leqq\infty,\ z=iy,\ y\geqq y_0>0)$, we get

$$\begin{aligned} |I_{2s,p}| &\leqq \alpha_p^{(2s)};\\ |I_{2s-1,p}| &\leqq \left|\int_{-1}^{1} x^{2s-1}dC_p(x)\right| + \int_{1}^{\infty} x^{2s}dC_p(x) + \int_{-\infty}^{-1} x^{2s}dC_p(x)\\ &\leqq 1 + \alpha_p^{(2s)},\\ \left|\lim_{p\to\infty} I_{2s-\sigma,p}\right| &\leqq \lim_{p\to\infty}(1+\alpha_p^{2s}) = 1 + m_{2s} \qquad (\sigma = 0, 1;\ s = 1, 2, \cdots), \end{aligned}$$

$$\lim_{z=iy\to\infty}\left|z^\nu\left[F(z) - \sum_{r=0}^{\nu-1}(-1)^r\frac{m_r}{z^{r+1}}\right]\right| = \lim_{z=iy\to\infty}\left|\frac{1}{z}\lim_{p\to\infty} I_{\nu,p}\right| = 0. \tag{10}$$

Formula (10), where ν is arbitrary, gives the asymptotic expansion of $F(z)$:

$$F(z) \sim \sum_{r=1}^{\infty}(-1)^r\frac{m_r}{z^{r+1}} \qquad (z = iy\to\infty).$$

Hence, Hamburger's theorem is applicable and proves (γ).

Proof II. We restrict ourselves to the domain of real numbers, making use of the following extension of Helly's theorem to the infinite interval $(-\infty,\ \infty)$.

Given a sequence $\{v_n(x)\}$ *defined on* $(-\infty, \infty)$ *such that* (i) $v_n(x)$ *is of bounded variation on any finite interval,* (ii) *all* $v_n(x)$ *and their total variations are uniformly bounded on any finite interval,* (iii) $\lim_{n\to\infty} v_n(x) = v(x)$ *exists for all* x, *with the possible exception of a countable set of points,* (iv) $\int_a^b f(x)dv_n(x)$ *converges uniformly (with respect to* n*) to* $\int_{-\infty}^{\infty} f(x)dv_n(x)$ $(a\to -\infty, b\to\infty)$, *if* $f(x)$ *is continuous everywhere (not necessarily uniformly). Then* $\int_{-\infty}^{\infty} f(x)dv(x)$ *exists and* $=\lim_{n\to\infty}\int_{-\infty}^{\infty} f(x)dv_n(x)$.

We notice, first, that $v(x)$ is of bounded variation (see footnote on page 538) on any finite interval, secondly (by virtue of Helly's theorem), that

$$\left|\int_b^{b'} f(x)dv(x)\right| = \left|\lim_{n\to\infty}\int_b^{b'} f(x)dv_n(x)\right| < \epsilon$$

($bb'>0$, n, $|b|$, $|b'|$ sufficiently large; $\epsilon>0$ arbitrarily small); hence, $\int_{-\infty}^{\infty} f(x)dv(x)$ *exists*. Furthermore,

$$\Delta_n = \left|\int_{-\infty}^{\infty} fdv - \int_{-\infty}^{\infty} fdv_n\right| \leqq \left|\int_{-\infty}^{a} fdv\right| + \left|\int_{-\infty}^{a} fdv_n\right| + \left|\int_{b}^{\infty} fdv\right|$$
$$+ \left|\int_{b}^{\infty} fdv_n\right| + \left|\int_a^b fdv - \int_a^b fdv_n\right| \qquad (a<0,\ b>0)$$

can be made as small as we please by taking $-a$, b, and then n sufficiently large. Hence $\lim_{n\to\infty}\Delta_n = 0$.

Remark. *The new condition* (iv) *is essential.* The following example shows that *if* (iv) *is not satisfied, the theorem may not hold*: $f(x)=x$, $v_n(x)=0$ $(x\leqq 0)$, $1-1/2^n (0<x\leqq 4^n)$, $1(x>4^n)$. Here

$$\lim_{n\to\infty} v_n(x) = v(x) = 0\ (x\leqq 0),\ 1(x>0);$$

$$\lim_{n\to\infty}\int_{-\infty}^{\infty} xdv_n(x) = \lim_{n\to\infty} 4^n\cdot\frac{1}{2^n} = \infty \neq \int_{-\infty}^{\infty} xdv(x) = 0.*$$

It suffices to apply the above theorem, with $f(x)=x^r (r=0, 1, \cdots)$, to the above sequence $\{C_p(x)\}$ which satisfies *all four conditions* stated, and (γ) is established.

6. Special case. A direct corollary is the following

THEOREM. *If* $\lim_{n\to\infty} m_n{}^r$ *exists* $(=m_r)$ *for* $r=0, 1, \cdots$, *then at least one fixed law of probability, say* $F(x)$, *exists such that* m_r *is its rth moment* $(r=0, 1, \cdots)$, *and a subsequence* $\{C_i(x)\equiv F_{n_i}(x)\}$ *can be extracted from the given*

* $v_n(x)$, $v(x)$ have each a single saltus $=1/2^n$, 1 at $x=4^n$, 0 respectively.

sequence $\{F_n(x)\}$ *of laws of probability such that* $\lim_{i\to\infty} C_i(x) = F(x)$ *for any* x. *If, in addition, the* $\{m_r\}$ *are such that the corresponding moments-problem is determined, then the sequence* $\{F_n(x)\}$ *itself converges, for* $n\to\infty$, *to* $F(x)$ *at any point of continuity of* $F(x)$.

We need a proof for the last part only. Assume that a point x_0 of continuity of $F(x)$ exists such that $\{F_n(x_0)\}$ does not converge to $F(x_0)$. Hence, a subsequence $\{C_k(x_0) \equiv F_{n_k}(x_0)\}$ can be extracted such that $C_k(x_0)$ converges, for $k\to\infty$, to a certain number $A \neq F(x_0)$. On the other hand, we have seen that the sequence $\{C_k(x)\}$ gives rise to a subsequence $\{d_i(x)\}$ which, *for any* x, converges, as $i\to\infty$, to a function $d(x)$, *having the same moments* $m_r(r=0, 1, \cdots)$ *as* $F(x)$, and therefore, since, by hypothesis, the moments-problem corresponding to $\{m_r\}$ is *determined,*

$$\lim_{i\to\infty} d_i(x_0) = d(x_0) = F(x_0) \tag{11}$$

($F(x)$ being continuous at $x=x_0$), which is impossible, $\{d_i(x_0)\}$ being a sub sequence of $\{C_k(x_0)\}$ which *converges, but not to* $d(x_0)$. We have seen also ((i), p. 537) that $F(-\infty)=0$, $F(\infty)=1$.

The condition that the moments-problem corresponding to $\{m_r\}$ be determined is not only sufficient for the validity of the limiting relation

$$\lim_{n\to\infty} F_n(x) = F(x)$$

(at any point of continuity of $F(x)$),* but *it is also necessary*. For if $F(x)$ and $\phi(x)$ be *two distinct* solutions of the moments-problem in question, then $F_n(x)$ should converge simultaneously to $F(x)$ and $\phi(x)$ at all points of continuity, while at least one such point x_0 exists where $F(x_0)\neq\phi(x_0)$.†

7. **The classical case:** $m_r = \pi^{-1/2}\int_{-\infty}^{\infty} x^r e^{-x^2}dx (r=0, 1, \cdots)$. Here

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} x^r dF_n(x) = \pi^{-1/2}\int_{-\infty}^{\infty} x^r e^{-x^2}dx \quad (r = 0, 1, 2, \cdots)$$

implies

$$\lim_{n\to\infty} \int_{-\infty}^{x} dF_n(x) = \pi^{-1/2}\int_{-\infty}^{x} e^{-x^2}dx \ (x \text{ arbitrary}).$$

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* Even everywhere in $(-\infty, \infty)$, $F(x)$ being a law of probability, hence continuous to the left (see Introduction).

† The conditions imposed by different writers on the quantities $\{m_r\}$ are such as *to ensure the determined character of the corresponding moments-problem.* In fact, one sees readily that the conditions of R. von Mises, Pólya, and Cantelli (loc. cit.)

$$m_{2n} \leqq C\left(\frac{n}{c^2e}\right)^n (C, c = \text{const.}), \lim_{n\to\infty} m_{2n}^{1/(2n)}/n \text{ is finite}, m_{2n}\frac{[\psi(2n)]^{2n}}{2n!} < 1 \ (n > n_1;\ \psi(n)_{n\to\infty}\to\infty)$$

are but special cases of Carleman's condition (2).

In fact, the moments-problem corresponding to $\{m_r\}$ is determined (by virtue of (2) or (3)), and

$$F(x) = \pi^{-1/2}\int_{-\infty}^{x} e^{-x^2}dx$$

is continuous for all x.

We see that *the classical case is but a very special case of the general second limit-theorem.*

8. **The determined character of the moments-problem in the classical case.** The conditions (2, 3) ensuring the determined character of the moments-problem have been established by means of very profound, but also complicated, considerations (continued fractions, singular integral equations). It seems of interest to give an *elementary* proof involving a simple theorem of Pólya.*

We wish to prove the following

THEOREM. *Given a law of probability $\psi_1(x)$ such that*

$$(12)\quad \begin{aligned} m_{2r} &= \int_{-\infty}^{\infty} x^{2r}d\psi_1(x) = \Gamma\left(\frac{2r+1}{\lambda}\right)\Big/\Gamma\left(\frac{1}{\lambda}\right) = \frac{\lambda}{2\Gamma(1/\lambda)}\int_{-\infty}^{\infty} x^{2r}e^{-|x|^{\lambda}}dx, \\ m_{2r+1} &= \int_{-\infty}^{\infty} x^{2r+1}d\psi_1(x) = 0 = \frac{\lambda}{2\,\Gamma(1/\lambda)}\int_{-\infty}^{\infty} x^{2r+1}e^{-|x|^{\lambda}}dx \end{aligned}$$

$$(\lambda \geqq 1;\ r = 0, 1, \cdots).$$

Then necessarily

$$\psi_1(x) = \frac{\lambda}{2\Gamma(1/\lambda)}\int_{-\infty}^{x} e^{-|x|^{\lambda}}dx$$

for any x. In other words, *$\psi_1(x)$ is uniquely determined by* (12).

Assume the existence of two such functions $\psi_1(x)$ and $\psi_2(x)$. Employing the reasoning of §1 (footnote on page 534) and using property (iii), page 537.

$$\lim_{x\to\infty} x^s(1-\psi_i(x)) = \lim_{x\to-\infty} |x|^s\psi_i(x) = 0 \quad (i = 1, 2;\ s>0 \text{ arbitrary}),$$

we conclude that

$$(13)\quad \int_{-\infty}^{\infty} x^l F(x)dx = 0 \quad (l = 0, 1, \cdots;\ F = \psi_1 - \psi_2 = (1-\psi_2) - (1-\psi_1)).$$

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* G. Pólya, *Über den Gaussschen Fehlergesetz*, Astronomische Nachrichten, vol. 208 (1919), No. 4981, pp. 185–192.

On the other hand, a reasoning similar to that of §4 leads, making use of (12) and the asymptotic expression for the Γ function, to

$$(14)\qquad \begin{aligned} &|F(x_0)| < \frac{2m_{2n}}{x_0^{2n}} && (n \text{ very large, } x_0 \text{ arbitrary}),\\ &|F(x_0)| < Ce^{(-1/2)|x_0|^\lambda} && \left(\left(\frac{2n}{\lambda}\right)^{2/\lambda+1/(2n)} < x_0^2 < \left(\frac{4n}{\lambda}\right)^{2/\lambda}\right),\\ & && C = \text{const. independent of } n \text{ and } x_0. \end{aligned}$$

Therefore,

$$\int_{-\infty}^{\infty} |F(x)|\, e^{c|x|} dx \qquad (0 < c < \tfrac{1}{2})$$

exists. But this is precisely the condition imposed by Pólya,* which leads to the required conclusion: $F(x) \equiv 0$ *at all points of continuity*.†

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* Loc. cit. p. 187. For the classical case ($\lambda = 2$) cf. M. H. Stone, *Developments on Hermite polynomials*, Annals of Mathematics, vol. 29 (1927), pp. 1–13.

† After the present paper had been prepared for publication, we came across an interesting article by A. Wintner: *Über den Konvergenzsatz der mathematischen Statistik*, Mathematische Zeitschrift, vol. 28 (1928), pp, 470–480, some of the results of which are similar to those obtained above.

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A CERTAIN TYPE OF CONTINUOUS CURVE AND RELATED POINT SETS*

BY

P. M. SWINGLE

Definition. A continuous curve every subcontinuum of which is a continuous curve will be called a *perfect continuous curve*.† In this paper, among other things, a study is made of this type of continuous curve.

An important problem in analysis situs is to determine whether a given point set is arc-wise connected. It is known that a connected subset of a perfect continuous curve is not necessarily arc-wise connected.‡ Here, however, it is shown in Theorem 5 that, if a and b are two points of a connected subset N of a perfect continuous curve, then the set common to N and the arcs ab of N' is connected. And in Theorem 7 it is shown that if N' contains but a countable number of arcs ab then N contains an arc ab.§

A generalization of the problem as to whether there exists an arc joining two points in a given point set is the following: when do there exist in a given point set where q is a given positive integer, q arcs, distinct except for end points, joining two given points; or still a further generalization would be to ask when do there exist q such arcs joining two distinct closed point sets? This problem is considered in §II of this paper. It might also be asked when do there exist q distinct arcs joining two distinct closed point sets? This problem is considered in §I.

I wish to express my thanks and to acknowledge my great indebtedness to Professor R. L. Wilder for his suggestions, criticisms, and constant encouragement.*

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* Presented to the Society, April 6, 1928; received by the editors in June, 1929, and June, 1930. Because of the delay in publishing this paper the privilege has been taken of using some recent results of other authors to abbreviate proofs and to obtain more general results than were originally obtained with respect to space and boundedness. These changes have been made in footnotes and in Section V with the exception that recent results have been used in proving Theorem 7. In general proofs which are well known in recent papers have been omitted here.

† In this paper a connected im kleinen continuum will be called a *continuous curve*. For definitions see R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, Bulletin of the American Mathematical Society, vol. 29 (1923), pp. 289–302.

‡ B. Knaster and C. Kuratowski, *A connected and connected im kleinen point set which contains no perfect subset*, Bulletin of the American Mathematical Society, vol. 33 (1927), pp. 106–109.

§ For theorems on arc-wise connected subsets of a perfect continuous curve see R. L. Wilder: *Characterizations of continuous curves that are perfectly continuous*, Proceedings of the National Academy of Sciences, vol. 15 (1929), pp. 614–621, Theorem 3; *Concerning perfect continuous curves*, Proceedings of the National Academy of Sciences, vol. 16 (1930), pp. 233–240, Theorems 1 and 2.

I. Distinct arcs in a point set

In this section the following problem will be considered: when can it be said that in a set M, for any given positive integer q, either there exist q distinct arcs† of M joining two distinct closed point sets‡ A and B or there exist $q-1$ points such that every arc of M joining A and B contains at least one of these points? If M contains no arc joining A and B it will be understood that there exist $1-1=0$ points contained in every arc of M joining A and B. With this understanding it will be shown that one of the two above possibilities holds for every set M.

Lemma 1. *If (t) is a set of arcs contained in an arc ab such that every point of ab except a and b is an interior point of an arc of (t) and both a and b are end points of at least one arc of (t), then there exists a simple chain of arcs§ of (t) joining a and b.*

This follows in a manner very similar to that used by R. L. Moore in proving Theorem 10 of his paper *On the foundations of plane analysis situs.*‖

Lemma 2. *In E_n¶ let C and K be distinct, closed, and bounded point sets, where K is connected and does not separate E_n, and let a_iw_i $(i=1, 2, \cdots, m)$ be m distinct arcs such that $a_iw_i \times K = w_i$. Then there exists a positive number d such that for any positive number r less than d there exist m distinct arcs a_iu_i of a_iw_i and m arcs u_ih distinct except for h such that (1) every point of u_ih is at a distance less than r from K, (2) the arc w_iu_i of a_iw_i has nothing in common with C, and (3) for any j, $u_jh \times ((a_iu_i)^{**}+C) = u_jh \times u_ja_j = u_j$.*

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* Since the theorems of this paper were proved a paper has appeared giving an interesting result in this connection: see N. E. Rutt, *Concerning the cut points of a continuous curve where the arc curve, AB, contains exactly N independant arcs*, American Journal of Mathematics, vol. 51 (1929), p. 218. For another interesting result see W. L. Ayres, *Concerning continuous curves in metric space*, ibid., pp. 577–594, Theorem 6. See also K. Menger, *Zur allgemeinen Kurventheorie*, Fundamenta Mathematicae, vol. 10, p. 100, Theorem β. In each of these papers the q arcs considered lie in a continuous curve in contrast to the results of this paper where the containing set may not even be connected in Theorem 1 and in Theorem 3 is connected but not necessarily closed.

† Two point sets are *distinct* when they have no common point.

‡ An arc ab will be said to join the point sets A and B if $A\times ab=a$ and $B\times ab=b$. Under these conditions the arc ba will be said to join B and A.

§ If a and b are distinct points, then a *simple chain* from a to b is a finite sequence of arcs $t_i(i=1, 2, \cdots, m)$ such that (1) t_i contains a if and only if $i=1$, (2) t_i contains b if and only if $i=m$, and (3) if $1\leqq i\leqq m$ and $1\leqq j\leqq m$, $i<j$, then t_i has a point in common with t_j if and only if $j=i+1$.

‖ These Transactions, vol. 17 (1916), pp. 131–164.

¶ The notation E_n will be used to denote a euclidean space of n dimensions.

** Throughout this paper the set $a_1+a_2+\cdots+a_m$ will be designated by (a_i) $(i=1, 2, \cdots, m)$. Thus $(a_i)+(b_i)=a_1+a_2+\cdots+a_m+b_1+b_2+\cdots+b_m$. The range of values for i, if it is not given, will refer to the last mentioned range of values for i.

The proof will first be given for E_2 as this is the most difficult case. Let z_i be the first point of $C+a_i$ on w_ia_i and let $a_iz_i=a_i$ if $z_i=a_i$. Let d be a positive number such that every point of $C+(a_iz_i)$ is at a distance greater than d from K. Let $M=K+C+(a_iz_i)$. Then for any positive number r less than d there exists* a simple closed curve J bounding a region R such that $J\times M=0$, $R\times M=K$, and every point of R is at a distance less than r from K. Let the first point of J on a_iw_i be u_i, and let a_iu_i be an arc of a_iw_i. Let h be any point of R. Then there exist in R the m arcs hu_i, distinct except for h, which together with the arcs a_iu_i have the properties described in this theorem.

For E_n, $n>2$, a connected domain R can be obtained by means of a finite set of spheres of radius r and covering K. The proof is then similar to the above.

It will simplify the proof of the main theorem of this section to prove first the following lemma.

Lemma 3. *Suppose for the integer k it is true that for every point set G, either there exist $k-1$ points such that any arc of G, joining any two distinct closed point sets, contains at least one of these points or there exist k distinct arcs of G joining these two distinct closed point sets. Then in E_n let a_ib_i $(i=1, 2, \cdots, k)$ be k distinct arcs joining two distinct closed point sets A and B and A_iB_i be k distinct arcs such that* (1) *A contains $(A_i)\times(A+B)$ and B contains $(B_i)\times(A+B)$,* (2) *$A_iB_i\times a_1b_1=A_i+B_i-(A_i+B_i)\times(A+B)$,* (3) *$A_iB_i\times(A+B)$ contains at most A_i+B_i, and* (4) *a_1A_i of a_1b_1 does not contain a point of (B_i). Then there exist k distinct arcs w_i of $M=(a_ib_i)+(A_iB_i)$ joining A and B and there exists an arc $w_{k+1}=W_1W_2$ of a_1b_1, where W_1 is contained in $(A_i)\times a_1b_1+a_1$ and W_2 in $(B_i)\times a_1b_1+b_1$, such that $(w_i)\times w_{k+1}$ contains at most W_1+W_2, a_1W_1 of a_1b_1 does not contain W_2, and there exists an arc of $(w_i)+w_{k+1}$ which contains b_1W_2 of a_1b_1.*

Consider for example the case where A_3 is the first point of $(A_i)+a_1$ on b_1a_1† and B_4 is the first point of $(B_i)+b_1$ on a_1b_1. Let p_iq_i be an arc of A_iB_i where p_i precedes q_i on A_iB_i, $p_i\neq A_i$, and $q_i\neq B_i$. Let p_iB_i and q_iA_i be arcs of A_iB_i and let a_1A_3 and b_1B_4 be arcs of a_1b_1. Let T_a be composed of $(p_iB_i)+(a_jb_j)+b_1B_4$ $(j=2, 3, \cdots, k)$ and also of q_iA_i if and only if A contains A_i; let T_b be composed of $(q_iA_i)+(a_jb_j)+a_1A_3$ and also of p_iB_i if B contains B_i. Let d be a positive number such that no point of T_a is at a distance less than d from a_1A_3 and no point of T_b is at a distance less than d from b_1B_4.

* R. L. Moore, *Concerning the separation of point sets by curves*, Proceedings of the National Academy of Sciences, vol. 11, No. 8, pp. 469–476, Theorem 1.

† The first point of a set C on an arc ab is the first point going from a to b; the first point of C on ba is the first point going from b to a.

The sets T_a and a_1A_3 are distinct, closed, and bounded point sets where a_1A_3 is connected and does not separate E_n. Take r less than $d/3$. The arcs q_iA_i, which have a point common with a_1b_1, have the same properties with reference to a_1A_3 as the arcs a_iw_i in Lemma 2 have to K. Say that there are m of these arcs q_iA_i which have a point common with a_1b_1. Then by Lemma 2 there exist m arcs and also there exist $k-m$ vacuous point sets, giving a total of k sets h_1u_{1i}, where h_1u_{1i} is vacuous if A contains A_i and consists of an arc if A does not contain A_i, and similarly there exist k arcs or vacuous sets B_iu_{1i}, where B_iu_{1i} is contained in A_iB_i. It is seen that the sets $B_iu_{1i}+u_{1i}h_1$ are distinct except for h_1, the set $(A_iu_{1i}+u_{1i}h_1)\times T_a=0$, where $A_iu_{1i}=0$ or is an arc according as A does or does not contain A_i, and every point of h_1u_{1i} is at a distance less than r from a_1A_3. From the last mentioned property it follows that the new sets h_1u_{1i} are at a distance greater than $d/3$ from b_1B_4. Then Lemma 2 can be again applied where $T_b+(h_1u_{1i})=C$ and $b_1B_4=K$ are the distinct, closed, and bounded point sets and the arcs B_ip_i, which have a point common with a_1b_1, are the m distinct arcs of that hypothesis. Thus a new set of k arcs or vacuous point sets h_2u_{2i}, distinct except for h_2, and a set of k distinct arcs or vacuous point sets p_iu_{2i}, where p_iu_{2i} is contained in p_iB_i and so in $u_{1i}B_i$, are obtained. There exists then a set of k distinct arcs $u_{1i}u_{2i}$ in (A_iB_i) where $u_{1i}=A_i$ if A contains A_i and $u_{2i}=B_i$ if B contains B_i. Thus a set of k arcs $h_1u_{1i}+u_{1i}u_{2i}+u_{2i}h_2=t_i$, distinct except for h_1 and h_2, are obtained where $(h_1u_{1i}+u_{2i}h_2)$ is at a distance less than r from $a_1A_3+B_4b_1$ and so has no points in common with (a_jb_j). Let $G=(a_jb_j)+t_i$.

Suppose that there exists a set H consisting of $k-1$ points such that every arc of G joining $A+h_1$ and $B+h_2$ contains at least one point of H. Therefore there exists one of these points on each of the $k-1$ arcs a_jb_j. Since then the points of H are on a_jb_j and since $(h_1u_{1i}+u_{2i}h_2)\times(a_jb_j)=0$, the set $H\times(h_1u_{1i}+u_{2i}h_2)=0$. Thus if there exists a point of H on each of the arcs t_i joining $A+h_1$ and $B+h_2$ there must exist a point of H on each of the k distinct arcs $u_{1i}u_{2i}$ of t_i. But as H contains only $k-1$ points this is impossible. Therefore it is necessary under our hypothesis that G contain k distinct arcs, v_i say, joining $A+h_1$ and $B+h_2$.

There exist three cases to consider having obtained the k arcs v_i: (I) the set (v_i) contains h_1+h_2; (II) it contains one and only one of these points; and (III) it contains neither of them.

(I) Under this case there exist two possibilities: one arc of the set (v_i) may contain both h_1 and h_2 or one arc may contain one of these points and another arc contain the other.

(a) Consider for example the case where v_1 contains h_1+h_2. Then v_1 con-

tains a point of (u_{1i}), u_{12} say, and a point of (u_{2i}), u_{23} say. It will now be shown that $w_1=a_1A_2+A_2u_{12}+u_{12}u_{23}+u_{23}B_3+B_3b_1$ is such that $w_1\times(v_j)=0$, where $a_1A_2+B_3b_1$ is contained in a_1b_1, A_2u_{12} in A_2B_2, $u_{23}B_3$ in A_3B_3, and $u_{12}u_{23}$ in v_1. Since $v_1\times(v_j)=0$, $u_{12}u_{23}\times(v_j)=0$. Since v_1 contains h_1+h_2, it is the only one of the arcs of (v_i) which contains a point of $h_1u_{1i}+h_2u_{2i}$. Hence the set $(a_jb_j)+(u_{2i}u_{1i})-(u_{1i})-(u_{2i})$ contains (v_j). But the set $a_1b_1\times((a_jb_j)+(u_{1i}u_{2i}))=0$ for $a_1b_1\times(a_jb_j)=0$ and as a_1b_1 contains at most A_i+B_i of A_iB_i, $a_1b_1\times(u_{1i}u_{2i})=0$. Therefore $(a_1A_2+B_3b_1)\times(v_j)=0$. It remains to prove that $(A_2u_{12}+B_3u_{23})\times(v_j)=0$. This follows from the fact that $(a_jb_j)+(u_{1i}u_{2i})-(u_{1i})-(u_{2i})$ contains (v_j). Therefore $w_1\times(v_j)=0$. Let now $w_j=v_j$ and let $w_{k+1}=A_2B_3$ of a_1b_1. As shown above, $a_1b_1\times(w_j)=0$ and so $w_{k+1}\times(w_j)=0$. And $w_{k+1}\times w_1=A_2+B_3$ while w_1 contains b_1B_3.

(b) Consider now for example the case where $v_1=h_1b$ and $v_2=h_2a$, where A contains a and B contains b. Then v_1 contains a point, u_{11} say, of (u_{1i}), and v_2 one, u_{21} say, of (u_{2i}). Here $(a_jb_j)+(u_{1i}u_{2i})-(u_{1i})-(u_{2i})$ contains $(v_f)+bu_{11}+au_{21}$ $(f=3, 4, \cdots, k)$, where v_1 contains bu_{11} and v_2 contains au_{21}. Thus $u_{11}A_1+A_1a_1+u_{21}B_1+B_1b_1$ does not contain a point of $(v_f)+bu_{11}+au_{21}$ except u_{11} and u_{21}, where A_1B_1 contains $u_{11}A_1+u_{21}B_1$, and a_1b_1 contains $a_1A_1+b_1B_1$. Let $w_f=v_f$, let $w_1=a_1A_1+A_1u_{11}+u_{11}b$, and let $w_2=b_1B_1+B_1u_{21}+u_{21}a$. Let $w_{k+1}=A_1B_1$ of a_1b_1. Now $w_1\times(w_f)=w_2\times(w_f)=0$ and $w_1\times w_2=0$. Also $w_1\times w_{k+1}=A_1$ and $w_2\times w_{k+1}=B_1$, $w_{k+1}\times(w_f)=0$ and w_2 contains b_1B_1 of a_1b_1.

(II) In considering the case where $(v_i)\times(h_1+h_2)$ contains one and only one point, there exist two cases according as h_1 or h_2 is the point contained.

(a) Consider for example the case where $v_1=h_1b$, where B contains b, and say v_1 contains u_{11}. Let $w_j=v_j$ and let $w_1=bu_{11}+u_{11}A_1+A_1a_1$, where bu_{11} is contained in v_1, $u_{11}A_1$ in A_1B_1, and A_1a_1 in a_1b_1. Let $w_{k+1}=A_1b_1$. Then the k arcs w_i are distinct, $w_{k+1}\times(w_j)=0$, and $w_{k+1}\times w_1=A_1$.

(b) Consider nc v for example the case where $v_1=ah_2$, which contains u_{23}, say, where A contains a. Let $w_j=v_j$ and let $w_1=au_{23}+u_{23}B_3+B_3b_1$, where au_{23} is contained in v_1, $u_{23}B_3$ in A_3B_3, and B_3b_1 in a_1b_1. Let $w_{k+1}=a_1B_3$ of a_1b_1. Here $(w_1+w_{k+1})\times(w_j)=0$ and $w_1\times w_{k+1}=B_3$ while the arc w_1 contains b_1B_3.

(III) For the case where $(v_i)\times(h_1+h_2)=0$ let $w_i=v_i$ and $w_{k+1}=a_1b_1$.

In this manner the lemma is proved for every possible case.

LEMMA 4. *Suppose for the integer k it is true that for every point set G, either there exist $k-1$ points such that any arc of G, joining any two distinct closed point sets, contains at least one of these points, or there exist k distinct arcs of G joining these two distinct closed point sets. Then in E_n let a_ib_i $(i=1, 2, \cdots, k)$ be k distinct arcs joining two distinct closed point sets A and B, and ay an arc such that $A\times ay=a$ and $a_1b_1\times ay=y$ and $(a_jb_j)\times ay=0$ $(j=2, 3, \cdots, k)$; also*

let A_iB_i be k distinct arcs such that (1) *A contains $(A_i)\times(A+B)$ and B contains $(B_i)\times(A+B)$,* (2) *$A_iB_i\times(a_1b_1+ay)=A_i+B_i-(A_i+B_i)\times(A+B)$,* (3) *$A_iB_i\times(A+B)$ contains at most A_i+B_i,* (4) *a_1y of a_1b_1 and ay do not contain a point of (B_i), and* (5) *b_1y of a_1b_1 does not contain a point of (A_i). Then there exist k distinct arcs w_i of $M=(a_ib_i)+(A_iB_i)+ay$ joining A and B and an arc $w_{k+1}=a_0x$, where a_0 is contained in A and x in $(B_i)\times a_1b_1+b_1$, and such that if there does not exist an arc w_i containing b_1 then $w_{k+1}=a_1b_1$ or $w_{k+1}=ay+yb_1$ and $w_{k+1}\times(w_j)=0$, but if w_1, say, contains b_1 then $w_1\times w_{k+1}=x$ and $w_{k+1}\times(w_j)=0$, and in every case $(w_i)+w_{k+1}$ contains b_1y and there exists a g such that w_g contains b_1x of a_1b_1.*

The proof is similar to that of Lemma 3.

THEOREM 1. *If M is any point set in E_n, and A and B are any two distinct closed point sets, then, for any positive integer q, either there exists a point set N containing $q-1$ points such that every arc of M joining A and B contains at least one point of N or there exist at least q distinct arcs of M joining A and B.*

If $q=1$, it is evident that the theorem is true. Assume that it is true for $q=k$. It will now be proved to be true for $q=k+1$.

There are two cases to consider according as either (I) there are $k-1$ points such that every arc of M joining A and B contains at least one of these points, or (II) there are at least k distinct arcs of M joining A and B.

(I) Consider the first case where the $k-1$ points exist. Then if any point is added to these, any arc of M joining A and B contains at least one of these k points. Hence in this case the theorem must be true for $q=k+1$ if it is for $q=k$.

(II) Consider now the case where there are at least k distinct arcs of M joining A and B. Let $a_ib_i(i=1,2,\cdots,k)$ be k such arcs. Consider any point p of a_1b_1. Either (1) there is a set N of $k-1$ points such that every arc of $M-p$ joining A and B contains a point of N or (2) there are k arcs of $M-p$ which are distinct and join A and B.

(1) For the case where the set N exists, $N+p$ is a set of k points such that every arc of M joining A and B contains at least one of these points. Thus in this case also if the theorem is true for $q=k$ it is for $q=k+1$.

(2) Consider now the remaining case where, for every point p of a_1b_1, $M-p$ contains at least k distinct arcs joining A and B. For a certain point p let $e_i(i=1,2,\cdots,k)$ be k such distinct arcs. Take for example the case where $a_1\neq p\neq b_1$. For any i there exists a region R containing p and so containing an arc t of a_1b_1 having p as an interior point, such that $R'\times e_i=0$. Let*

* If M is a point set, M' will denote M together with the limit points of M.

$(a_1b_1-t)'=t_1+t_2$. Then, if t_1 contains a_1, there exists an arc of e_i, s_i say, joining t_1+A and t_2+B. If a_1b_1 contains both end points of s_i, let r_i be the arc of a_1b_1 joining these two points; if one of these end points is in A and one in a_1b_1, let r_i be the arc of a_1b_1 joining a_1 and the end point of s_i in a_1b_1; if B contains one of these end points and a_1b_1 contains the other, let r_i be the arc of a_1b_1 joining b_1 and this end point in a_1b_1; and if A contains one end point and B contains the other, let $r_i=a_1b_1$. Thus to each arc e_i there corresponds an arc r_i of a_1b_1 having p as an interior point. Let h be the arc $r_1\times r_2\times\cdots\times r_k$ of a_1b_1. Thus for each point p of a_1b_1 there exists an arc h, having p as an interior point unless $p=a_1$ or $p=b_1$ in which latter case p is an end point, and a corresponding set, f say, consisting of k distinct arcs e_i joining A and B and having only end points of h common with h. Let (h) be the set of arcs such as h and (f) the set of sets such as f. Then by Lemma 1 there exists a simple chain, $h_1, h_2, \cdots, h_m$ of arcs of (h) joining a_1 and b_1, where h_1 contains a_1 and h_m contains b_1. Also there exists a set f_j of (f) corresponding to each h_j.

The set f_1 contains k distinct arcs $A_{1i}B_{1i}$ joining A and a closed subset of $a_1b_1-a_1+B$ where each a_1B_{1i} of a_1b_1 contains h_1, if a_1b_1 contains B_{1i}. We have here, just as we had in the hypothesis of Lemma 3, k distinct arcs a_ib_i and a set of k distinct arcs $A_{1i}B_{1i}$. Applying this lemma we obtain a set of k distinct arcs $a_{1i}b_{1i}$ joining A and B, which are the arcs w_i of that lemma, and an arc a_1y_1 of a_1b_1 joining a_1 and $(a_{1i}b_{1i})\times(B_{1i})+B$, which is the arc w_{k+1}. The new set $(a_{1i}b_{1i})+a_1y_1$ is contained in $(a_ib_i)+(A_{1i}B_{1i})$. If B contains y_1 then there exist $k+1$ distinct arcs of M joining A and B provided $a_1y_1\times(a_{1i}b_{1i})\neq b_1$; and if B does not contain y_1, as was shown in Lemma 3, there exists an arc, $a_{11}b_{11}$ say, where $b_{11}=b_1$, containing y_1b_1 of a_1b_1. Thus either there exist $k+1$ distinct arcs of M joining A and B or there exist $k+1$ arcs of M joining A and either $B+b_1$ or $B+b_1y_1$, which are distinct, with the exception that two of them contain a common end point in b_1y_1, and so in some h_j, $j>1$. If this end point is not in h_m, say for example that it is an interior point in h_2. We now have k distinct arcs $a_{1i}b_{1i}$, similar to the arcs a_ib_i of Lemma 4, and an arc a_1y_1 similar to ay. And there exist in f_2 k distinct arcs $A_{2i}B_{2i}$, similar to A_iB_i of this lemma, joining $A+a_1y_1+x_a$ and $B+x_b$, where $(a_{11}b_{11}-h_2)'=x_a+x_b$, and x_b contains y_1b_1 of a_1b_1. Thus by this lemma it follows that there exist k distinct arcs $a_{2i}b_{2i}$ joining A and B and an arc $a_{02}y_2$ joining A and $(B_{2i})\times(a_{2i}b_{2i})+B$. These $k+1$ arcs are distinct or else a new set is obtained, proceeding in the same manner as above, by the use of Lemma 4.

It is necessary then that either $k+1$ distinct arcs joining A and B be obtained or there exist an arc $a_{0g}y_g$ such that h_m of a_1b_1 contains y_g. Hence, by one further application of Lemma 4, $k+1$ distinct arcs joining A and B

must be obtained. Thus if the theorem is true for $q=k$ it is true for $q=k+1$ in this case.

In every case the theorem is true for $q=k+1$ if it is true for $q=k$. The theorem is true for $q=1$. Hence it is true for any value of q.

Corollary 1. *If M is a continuous curve in E_n and A and B are any two distinct closed point sets, then either there exist, for any q, at least q distinct arcs of M joining A and B, or there exists a point set N of $q-1$ points such that $M-N$ does not contain a connected subset which contains points of both A and B.*

II. Arcs, distinct except for end points

Here the following problem will be considered: when can it be said that in a set M either there exists a set of at least q arcs, where q is any positive integer, distinct except for possibly their end points, joining two distinct closed point sets A and B, or there exists a set N of $q-1$ points, contained in $M-A-B$, such that every arc of M joining A and B contains at least one point of N? Here also it will be understood that if M contains no arcs joining A and B, then the set N is vacuous. A complete solution of the above problem is not obtained in this paper.

In proving the next theorem the following lemma is useful and is stated here without proof.*

Lemma 5. *If, in E_2, M is a continuous curve which contains a subcontinuum which is not a continuous curve and if k is any positive integer, then M contains k distinct arcs a_ib_i $(i=1, 2, \cdots, k)$, and a sequence of distinct arcs x_jy_j $(j=1, 2, \cdots)$ having a sequential limiting set Z, such that $x_jy_j\times a_ib_i\neq 0$, $x_jy_j\times Z=0$, $x_jy_j\times a_1b_1=x_j$, and $x_jy_j\times a_kb_k=y_j$. Furthermore x_j precedes x_{j+1} on a_1b_1, y_j precedes y_{j+1} on a_kb_k, every point of a_ib_i precedes every point of $a_{i+1}b_{i+1}$ on x_jy_j, $a_ib_i\times Z=b_i$, and $a_ib_i\times x_1y_1=a_i$.*

Theorem 2. *Let q be a given positive integer greater than one. Then in order that a continuous curve M, in E_2, be perfect, it is sufficient, if L is the point set consisting of the points of any set of arcs of M joining any two distinct closed point sets A and B, that either there exist a set N of $q-1$ points, of $L-A-B$, such that every arc of L joining A and B contains at least one point of N, or there exist at least q arcs of L, distinct except for possibly their end points, joining A and B.*

Assume that M contains a subcontinuum which is not a continuous curve. Then by applying Lemma 5 the arcs of the conclusion there are obtained,

* The proof follows from the work of R. L. Wilder, Fundamenta Mathematicae, vol. 7, pp. 362–363, and from Theorem XXI by H. Hahn, Wiener Sitzungsberichte, vol. 123 (Part IIa), p. 2475.

taking $k=q+1$. On the arc x_jy_j there obtained, taking $j=2t-1$ $(t=1, 2, \cdots)$, let u_t be the subarc joining a_1b_1 and a_3b_3; and taking $j=2t$ let v_t be the subarc of x_jy_j joining a_1b_1 and a_2b_2. Let w_t be the subarc of a_2b_2 joining v_t and u_{t+1}; and let z_t be the subarc of a_1b_1 joining u_t and v_t. Let $A=(a_i)$ and $B=Z$. Then every point of $L=(u_t+v_t+w_t+z_t)+a_3b_3+a_4b_4+\cdots+a_{q+1}b_{q+1}$ is contained in an arc of L joining A and B $(t=1, 2, \cdots)$. But as $(u_t+v_t+w_t+z_t)$ does not contain an arc of L joining A and B, L contains at most $q-1$ arcs, distinct except for possibly their end points, joining A and B. However it is evident that $L-A-B$ does not contain $q-1$ points such that every arc of L joining A and B contains at least one of these points. As this is a contradiction with our hypothesis, M must be perfect.

Whether this condition is also necessary is not determined in this paper, except for $q=2$.

The proof of the following lemma can be obtained by means of a theorem by H. M. Gehman.*

LEMMA 6. *If M is a bounded continuous curve in E_2,† then a necessary (and sufficient) condition that M be perfect is the following: let Z be any closed subset of M, W any subset of M such that $Z\times W=0$ and every point of W can be joined to Z by an arc contained in W except for an end point in Z, and x a limit point of W such that $(W+Z)\times x=0$. Then there exists an arc joining x and Z which is contained in W except for its end points.*

THEOREM 3. *If $q=2$ then the condition in Theorem 2 is also necessary, if M is bounded.*‡

Suppose that there exist sets L, A, and B such that there does not exist a point which is contained in $L-A-B$ and in every arc of L joining A and B. Then there exists an arc ab joining A and B. It then follows from the above theorem by H. M. Gehman and from Lemma 6 that for any point p of ab, except possibly a and b, there exists an arc uv of L having the following properties: (1) the set $A+ap-p$ contains u and $B+bp-p$ contains v of uv, where $ap+pb=ab$, and $A+B+ab$ contains only these points of uv, and (2), unless uv joins A and B, in which case the theorem is proved, there does not exist another arc in L, having only its end points in $A+B+ab$, which joins $A+(au-u)$ and $B+(bv-v)$, where au is either an arc of ab or a, and bv is either an arc of ab or b; furthermore if $au\neq a$ there does not exist an arc of L,

* *Concerning the subsets of a plane continuous curve*, Annals of Mathematics, vol. 27 (1925), pp. 29–46, Theorem V.

† This lemma is true if "bounded" is omitted. The sufficiency is true in E_n but the necessity is not. See §V.

‡ As shown in §V, "bounded" may be omitted.

having at most one end point in ab, joining $uv-u$ and $A+au-u$ nor if $bv\neq b$ does there exist such an arc joining $uv-v$ and $B+bv-v$. Let w be the arc of ab joining au and bv. Let (uv) be the set of arcs of L such as uv and (w) the corresponding arcs of ab. Every point of $ab-a-b$ is an interior point of one and at most of two of the arcs of (w). And $a+b$ contains the limit points of the end point of the arcs of (w). Since there exist but a countable number of arcs in (w), it is readily shown that $(uv)+(w)$ contains two arcs joining A and B, which are distinct except for possibly their end points. Hence the truth of the theorem is seen.

III. Connected subsets of perfect continuous curves

It is known that a connected subset N of a perfect continuous curve M is not necessarily arc-wise connected. But for any two points a and b of N the set N' contains an arc ab. Here the nature of the point set composed of the points of N, which are also on arcs ab of N', is considered. It is shown that this set must be connected. Some preliminary lemmas will be proved.

Theorem 4. *In order that a bounded continuous curve M in E_2* be perfect it is necessary and sufficient, if p, a, and b be any three points of any subcontinuum N of M such that there does not exist a point distinct from p which is contained in every arc of N joining p and $a+b$, that there exist an arc apb of N.*

By means of Lemma 5, using a device similar to that used in the proof of Theorem 2, it is seen that the condition is sufficient. It is also necessary. For by Theorem 3 there exist two arcs of N joining p and $a+b$ which are either distinct, except for p, or form a simple closed curve, J say, containing p and a point of $a+b$. If they are distinct, they form an arc apb of N; and if they are not distinct, since N contains an arc t joining p and $a+b-J\times(a+b)$ such that $t\times(a+b)\neq a+b$, it is seen that $J+t$ contains an arc apb of N.

Lemma 7. *If p, a, and b are any three distinct points of a bounded,† perfect, continuous curve M in E_2, and if (p) is the set of points, each of which is on every arc of M joining p and $a+b$, then there exists an arc of M which joins a and b and contains a certain point p_1 of (p) and every arc of M joining a and b contains at most p_1 of (p).*

As (p) is closed it is seen that there exists a first point of (p) on an arc of M, and so on every arc of M, joining p and $a+b$. And this point, p_1 say, has the property that there exists no other point common to every arc of M

* In §V it will be shown that "bounded" may be omitted here and that the necessity is true for E_n.

† As the proof depends entirely upon Theorem 4, "bounded" can be omitted and the theorem stated for E_n.

joining it and $a+b$. Hence by Theorem 4 there exists in M an arc ap_1b. And it is seen that every arc of M joining a and b contains at most this point of (p).

LEMMA 8. *In order that a bounded* continuous curve M in E_2 be perfect it is necessary and sufficient that, if N is any connected subset of M, a and b are any two points of N, L is a subset of N' consisting of all points contained in arcs ab of N', $Q=L\times N$, p is a point of $N-Q$, and (p) is the set of all points common to every arc of N' joining p and $a+b$, then there exists a point p_1 of $(p)\times Q$ and a subset W of $N'-p_1$, which contains p and is such that $W\times(a+b)=0$, $N'-p_1=W+(N'-p_1-W)$ separate, and every point of W can be joined to p_1 by an arc of $W+p_1$.*

In showing that the condition is necessary it is seen at once, by means of Lemma 7, that the point p_1 and an arc ap_1b exist. Also there exists a set W composed of the points of N' which can be joined to p_1 by an arc but cannot be joined to $a+b$ by an arc of N' which does not contain p_1. As $N'-p_1$ is an open subset of N', since $N'-p_1$ does not contain an arc joining p and $a+b$, it does not contain a connected subset which joins p and $a+b$.† Hence p and $a+b$ are separated in the weak sense and so in the strong sense‡ in N'. Thus the condition is necessary. That it is sufficient follows from Lemma 5.

THEOREM 5. *In order that a bounded continuous curve M in E_2§ be perfect it is necessary and sufficient that, if N is any connected subset of M of which a and b are any two points, L is the point set consisting of the arcs ab of N', and $Q=L\times N$, then Q is a non-vacuous connected point set.*

To show that the condition is necessary it is seen that, since N' is a continuous curve, L and so Q is a non-vacuous set. Assume that $Q=Y+Z$ separate. By Lemma 8, for each point p of $N-Q$, there exists a point p_1 and a subset W of $N'-p_1$ such that $N'-p_1=W+(N'-p_1-W)$ separate, where $W+p_1$ contains an arc pp_1. Thus sets (W) and (p_1) are obtained. And $(W)=(W)_1+(W)_2$, where a set W of $(W)_1$ is such that the corresponding p_1 of (p_1) is in Y and otherwise W is in $(W)_2$. Let $E=Y+(W)_1$ and $F=Z+(W)_2$. Then $N=Q+(W)\times N=F\times N+F\times N$. It is possible to show then that $N=E\times N+F\times N$ separate, which is a contradiction. And that the condition is sufficient is seen by assuming that it is not, and so obtaining by Lemma 5

* "Bounded" can be omitted and the necessity proved for E_n.

† R. L. Wilder, *Characterizations of continuous curves that are perfectly continuous*, Theorem 3, loc. cit.

‡ R. L. Wilder, *A characterization of continuous curves by a property of their open subsets*, Fundamenta Mathematicae, vol. 11, pp. 127–131, Lemma 2.

§ "Bounded" may be omitted, as seen in §V, the necessity holds in E_n, as Lemma 8 does, and the sufficiency holds in E_n, as may be seen from the Moore-Wilder lemma.

$N = a_1b_1 + x_1y_1 + x_2y_2 + \cdots + b_2$, letting $a = a_1$, and $b = b_2$, and thus obtaining a contradiction

IV. Additional theorems

The main problem considered here is the following: if N is a connected subset of a perfect continuous curve M, when does there exist an arc of N joining two points of N? Also a property is obtained of a set of arcs of M which join two distinct closed point sets.

Theorem 6. *In order that a bounded* continuous curve M in E_2 be perfect it is necessary and sufficient, if L is the point set consisting of the points of any set† of arcs of M joining any two distinct closed point sets A and B, that every point p of L' be contained in an arc of $L+p$ joining A and B.*

That the condition is necessary follows from a theorem by H. M. Gehman‡ and that it is sufficient is seen by means of the Moore-Wilder lemma.§

Theorem 7. *In order that a bounded|| continuous curve M in E_2 be perfect it is necessary and sufficient, if a and b are any two points of any connected subset N of M such that N' contains but a countable number of possible arcs ab, that N contain an arc ab.*

The condition is necessary. For assume that N does not contain an arc ab. Then every arc ab of N' contains a point of $N'-N$. Since these arcs are countable in number, let (z) be a set of points obtained by taking a point of $N'-N$ from each arc ab of N'. For each point z of (z), $N'-z$ is an open subset of N', and there exist a countable infinity of such sets. As N is common to all these open sets, it is contained in the quasi-open subset, Q, of N' determined by this countable infinity of open subsets of N'. But as Q is arc-wise connected,¶ it contains an arc ab. But as this arc does not contain a point of (z) a contradiction has been obtained.

The sufficiency follows from Lemma 5.

V. Note

Many of the theorems of this paper have been stated as holding for a bounded continuous curve. This is due to the fact that a theorem** by H. M.

* As shown in §V "bounded" may be omitted here; the sufficiency holds in E_n but the necessity does not.

† It is to be noted that this set is not necessarily the set of all arcs of M joining A and B.

‡ Loc. cit., Theorem 5.

§ R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1927), p. 371, Lemma 1.

|| Bounded may be omitted here. The necessity holds for any locally compact metric space.

¶ R. L. Wilder, *Characterizations of continuous curves that are perfectly continuous*, loc. cit., Theorem 3.

** Loc. cit., Theorem 5.

Gehman is used in the proofs, which is known to be untrue for unbounded continuous curves. However the following lemma could have been used everywhere instead, and so the theorems of this paper hold whether the continuous curve is bounded or not.

LEMMA. *In order that a continuous curve M in E_2 be perfect it is necessary and sufficient that M does not contain an infinite sequence of distinct subcontinua with two sequential limit points.**

The sufficiency is evident from the Moore-Wilder lemma which is known to hold for the unbounded case in E_n. And the necessity is proved as follows: let a and b be the sequential limit points referred to in the theorem. By a well known theorem† there exists in M a bounded continuous curve K containing a and all points of M in a certain neighborhood of a, but not containing b. It is clear that there will exist then points of infinitely many of the given subcontinua that are not in K. Consequently, since K is perfect, the theorem of Gehman is violated by portions of the given continua that lie in K.

It is known from an example by G. T. Whyburn‡ that neither the theorem by H. M. Gehman nor the above lemma is true for E_3. This example further shows that neither the necessity in Theorem 6 nor Lemma 6 is true for E_3.

The necessity of the following theorem is seen by means of a theorem by W. L. Ayres§ and the sufficiency is true because of the proof of Theorem 2. The necessity holds for E_n and the sufficiency for $q>2$.

THEOREM 3′. *Let $q=2$. Then in order that a continuous curve M in E_2 be perfect it is necessary and sufficient, if L is the point set consisting of the points of any set of arcs of M joining any two distinct closed point sets A and B, that either there exist a set N of $q-1$ points of $L'-A-B$ such that every arc of L' joining A and B contains at least one point of N or there exist at least q arcs of L', distinct except for possibly their end points, joining A and B.*

It is seen that the proof of Theorem 4 will go through for E_n, if the necessity of the above theorem, which is true for E_n, is used in place of Theorem 3 there.

* A point q is said to be a *sequential limit point* of a set of continua $C_i (i=1, 2, \cdots)$ if every region containing q contains points of all except a finite number of the sets C_i.

† H. Hahn, *Mengentheoretische Characterisierung der stetigen Kurve*, Wiener Sitzungsberichte, vol. 123 (Part IIa), pp. 2433–2489; see Theorem XXI, p. 2475.

‡ Bulletin of the American Mathematical Society, vol. 34 (1928), p. 551. This paper is to appear in Mathematische Annalen. The example has been given in a paper by R. L. Wilder, Proceedings of the National Academy of Sciences, vol. 16 (1930), p. 234. Using the notation of Professor Wilder to show that Theorem 6 does not hold in E_3, let $L=N$, $a=A$, $b=B$, and $p=(0, 0, 2^{1/2}/2)$. Then there does not exist an arc apb of $L+p$. Similarly Lemma 6 is shown to be untrue in E_3.

§ Loc. cit., Theorem 6.

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THE GEOMETRIC CONFIGURATION DEFINED BY A SPECIAL ALGEBRAIC RELATION OF GENUS FOUR*

BY

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Introduction

The double binary form, $(\alpha t)^3(a\tau)^3$, of third order in digredient variables t and τ, when equated to zero, defines a space curve of the sixth order on a quadric surface with generators t and τ. This space curve is the complete intersection of the quadric surface and a cubic surface and hence is of genus four. The geometric theory connected with a general (3, 3) form has been discussed by Coble.† He introduced the form, with the interpretation also used by H. S. White,‡ that is, as the incidence condition of a point of one space cubic curve and a plane of a second space cubic curve.

It is the purpose of this paper to discuss the special form

$$F = t_1^3\tau_1^2\tau_2 + t_1^2t_2\tau_2^3 + t_1t_2^2\tau_1^3 - t_2^3\tau_1\tau_2^2 . \tag{1}$$

This form was first investigated by Gordan,§ who followed Klein in considering its relation to the solution of the equation of the fifth degree and to the icosahedral equation.

When the equation of the fifth degree has the form

$$x^5 + 5\alpha x^2 + 5\beta x + \gamma = 0,$$

then $\sum_{i=0}^{i=4}x_i = 0$ and $\sum_{i=0}^{i=4}x_i^2 = 0$, and the x_i may be taken as the pentahedral coördinates of a point in space and $\sum_{i=0}^{i=4}x_i^2 = 0$ is a quadric surface unaltered by the group of the 120 permutations of the x_i's. Klein‖ sets

$$p_i = x_0 + \epsilon^i x_1 + \epsilon^{2i}x_2 + \epsilon^{3i}x_3 + \epsilon^{4i}x_4,\quad \epsilon = e^{2\pi i/5}\quad (i = 1, 2, 3, 4),$$

or

$$5x_\nu = \epsilon^{4\nu}p_1 + \epsilon^{3\nu}p_2 + \epsilon^{2\nu}p_3 + \epsilon^{\nu}p_4 \qquad (\nu = 0, 1, 2, 3, 4),$$

* Presented to the Society, February 28, 1931; received by the editors February 4, 1931.

† Coble, *Geometric aspects of the abelian modular functions of genus four*, American Journal of Mathematics, vol. 46 (1924), pp. 143–192; *Geometric aspects of the abelian modular functions of genus four*, Part II, American Journal of Mathematics, vol. 51 (1929), pp. 495–514.

‡ White, *A variable system of sevens on two twisted cubic curves*, Proceedings of the National Academy of Sciences, vol. 2 (1916), p. 337.

§ Gordan, *Ueber die Auflösung der Gleichungen fünften Grades*, Erlangen Physikalisch-Medicinische Societät, Sitzungsberichte, vol. 9 (1877), pp. 183–186; *Ueber die Auflösung der Gleichungen vom fünften Grade*, Mathematische Annalen, vol. 13 (1878), pp. 375–404.

‖ Klein, *Vorlesungen über das Ikosaeder*, Leipzig, 1884, p. 187.

and $t_1/t_2=-p_1/p_2=p_3/p_4$, $\tau_1/\tau_2=p_1/p_3=-p_2/p_4$. Then $25\sum_{\nu=0}^{\nu=4}x_\nu^2=10(p_1p_4+p_2p_3)$. That is, the quadric $\sum_{\nu=0}^{\nu=4}x_\nu^2=0$ becomes the quadric $p_1p_4+p_2p_3=0$ with generators $t=t_1:t_2$ and $\tau=\tau_1:\tau_2$. If then

$$p_1 = -5t_1\tau_1,\quad p_2 = 5t_2\tau_1,\quad p_3 = -5t_1\tau_2,\quad p_4 = -5t_2\tau_2,$$

$\alpha=-\sum_{i=0}^{i=4}x_i^3/15$ is in terms of t and τ equal to

$$F = t_1^3\tau_1^2\tau_2 + t_1^2t_2\tau_2^3 + t_1t_2^2\tau_1^3 - t_2^3\tau_1\tau_2^2 .$$

Gordan* computed the complete system of thirty-six covariants for the form F as a form in digredient variables.

In an article referred to above,† Coble derives the complete system of covariants of degrees one, two, and three for the general (3, 3) form. He then proceeds to the figure of two cubic curves in space. The net of quadrics and the net of quadric envelopes on one space cubic curve cut out involutions I_2^6 on the other. The four I_2^6's thus determined can be interpreted as four rational planar sextics. Any one of the four determines the figure of the two cubic curves in space. Conner‡ and Coble§ also discuss the quartic Jacobian surface and the quartic surface (the symmetroid of Cayley) which are birationally related to each other and to the figure of two cubic space curves.

It is the purpose of this paper to examine this geometric situation for the special form F which admits a G_{120}. The geometric figures determined by F are therefore also invariant under a group. In particular the rational sextics admit an icosahedral G_{60} and from this point of view have been discussed by Klein,‖ and Winger.¶ The types of quartic surfaces, of rational space sextic curves, and of pairs of space cubic curves considered here, have not been examined heretofore. They define interesting configurations with remarkably simple geometric behavior and a correspondingly simple algebra and thus are useful as a check on the general case.

Part of the planar configuration can be obtained by the projection of a certain family of elliptic norm curves. This procedure is developed in I. It af-

* Mathematische Annalen, loc. cit.

† American Journal of Mathematics, vol. 46, loc. cit.

‡ J. R. Conner, *The rational sextic curve, and the Cayley symmetroid*, American Journal of Mathematics, vol. 37 (1915), pp. 29–42, §4.

§ American Journal of Mathematics, vol. 46, loc. cit., §10.

‖ Klein, *Weitere Untersuchungen über das Ikosaeder*, Mathematische Annalen, vol. 12 (1877), pp. 503–560; also *Vorlesungen über das Ikosaeder*.

¶ R. M. Winger, *Self-projective rational sextics*, American Journal of Mathematics, vol. 38 (1916), pp. 45–56; *On the invariants of the ternary icosahedral group*, Mathematische Annalen, vol. 93 (1925), pp. 210–216.

fords a simple introduction of the modular functions into the theory of the quintic equation as given by Klein.*

I. The (3, 3) form and the elliptic quintic in S_4

The (3, 3) form (1) may be introduced by means of the elliptic quintic E^5 in S_4. Klein and Fricke† have discussed the general case E^n in S_{n-1} and Bianchi‡ and Miss B. I. Miller§ have considered the special case $n=5$ in some detail.

The parametric equations of E^5 in S_4 as given by Klein and Fricke‖ become

$$X_\alpha(u) = a_\alpha \cdot \exp\left[-\left((\bar\eta_1 - 5\eta_1)/(2\omega_1)\right)u^2 + (\alpha/5)\bar\eta_1(u - \alpha\omega_1/10)\right]$$
$$\cdot\sigma\left(u - \frac{\alpha}{5}\omega_1 \,\Big|\, \omega_1, \frac{\omega_2}{5}\right) \quad (\alpha = 0, 1, 2, 3, 4).$$

The five linearly independent quadrics which contain and completely define E^5 are¶

$$(2) \qquad Q_i = \alpha_0 X_i^2 + \alpha_1 X_{i+1}X_{i-1} + \alpha_2 X_{i+2}X_{i+3} = 0 \quad (i = 0, 1, 2, 3, 4),$$

where the ratios of the α's are modular functions. Then the linear system of quadrics on E^5 is

$$(LQ) = \sum_{i=0}^{i=4} L_i Q_i = 0.$$

Make the following transformation of variables instead of that used by Miss Miller:**

$$\begin{array}{llll} Y_0 = 2X_0, & & \zeta_0 = L_0, & \\ Y_1 = X_1 + X_4, & Z_1 = X_1 - X_4, & \zeta_1 = L_1 + L_4, & \theta_1 = L_1 - L_4, \\ Y_2 = X_2 + X_3, & Z_2 = X_2 - X_3, & \zeta_2 = L_2 + L_3, & \theta_2 = L_2 - L_3. \end{array}$$

In terms of the variables Y and Z, the linear system of quadrics becomes

$$4(LQ) = F_1 + F_2 + F_3 = 0,$$

* *Vorlesungen über das Ikosaeder.*

† Klein-Fricke, *Theorie der elliptischen Modulfunktionen*, Leipzig, 1890–92.

‡ Bianchi, *Ueber die Normalformen dritter und fünfter Stufe des elliptischen Integrals erster Gattung*, Mathematische Annalen, vol. 17 (1880), pp. 234–262.

§ B. I. Miller, *A new canonical form of the elliptic integral*, these Transactions, vol. 17 (1916), pp. 259–283.

‖ Loc. cit., p. 263.

¶ Bianchi, loc. cit., p. 251.

** Loc. cit., p. 278.

where

$$F_1 = \{ \textstyle\sum \alpha_i \alpha_j Y_{j+i} Y_{j-i} \}_\zeta \qquad (i, j = 0, 1, 2);$$
$$F_2 = \textstyle\sum (\alpha\zeta)_{ij} Z_{j+i} Z_{j-i} \qquad (i, j = 0, 1, 2, \ j > i);$$
$$F_3 = \textstyle\sum \theta_i \alpha_i (Y_{i-j} Z_{i+j} + Y_{i+j} Z_{i-j}) \qquad (i = 1, 2, \ j = 0, 1, 2).$$

In F_1, $\{ \ \}_\zeta$ indicates polarization once with respect to ζ and $(\alpha\zeta)_{ij}$ in F_2 is written for $\alpha_i\zeta_j - \alpha_j\zeta_i$. The quadrics on E^5 are the coefficients of ζ and θ. Explicitly, they may be written

(3)

	α_0	α_1	α_2	α_0	α_1	α_2
ζ_0	Y_0^2	Y_1^2	Y_2^2		$-Z_1^2$	$-Z_2^2$
ζ_1	Y_1^2	Y_0Y_2	Y_1Y_2	Z_1^2		Z_1Z_2
ζ_2	Y_2^2	Y_1Y_2	Y_0Y_1	Z_2^2	$-Z_1Z_2$	
θ_1	$2Y_1Z_1$	Y_0Z_2	$-Y_1Z_2 - Y_2Z_1$			
θ_2	$2Y_2Z_2$	$-Y_1Z_2 + Y_2Z_1$	$-Y_0Z_1$			

The E^5 is invariant under a G_{50} generated by

$$u' = -u, \quad u' = u + \frac{\omega_1}{5}, \quad u' = u + \frac{\omega_2}{5}, \tag{4}$$

or in terms of the coördinates X,*

$$\pi_1 X'_\alpha = X_{n-\alpha}, \quad \pi_2 X'_\alpha = X_{\alpha-1}, \quad \pi_3 X'_\alpha = e^{2\pi i\alpha/5} X_\alpha. \tag{5}$$

The family of E^5's is invariant under a group $G_{50\cdot 60}$ generated by (5) and

$$S: \begin{cases} \omega_1' = \omega_1 + \omega_2, \\ \omega_2' = \omega_2, \end{cases} \quad \text{and} \quad T: \begin{cases} \omega_1' = -\omega_2, \\ \omega_2' = \omega_1, \end{cases}$$

or, in terms of the coördinates X,†

$$S: \ \pi X'_\alpha = \epsilon^{\alpha(\alpha-5)/2} X_\alpha, \quad \text{and} \quad T: \ \pi X'_\alpha = \textstyle\sum_\beta \epsilon^{-\alpha\beta} X_\beta.$$

The G_{50} is an invariant subgroup of the $G_{50\cdot 60}$.

It is obvious that there are two skew spaces which are pointwise invariant under the involution $u' = -u$, that is, the line $S_1(Z)$ for which the Y's are zero and the plane $S_2(Y)$ for which the Z's are zero. The factor group G_{60} of of G_{50} in $G_{50\cdot 60}$ is then represented as a collineation group in the spaces $S_1(Z)$ and $S_2(Y)$ and hence must be isomorphic with the icosahedral group. For the

* Klein-Fricke, loc. cit., vol. 2, p. 264.

† Klein-Fricke, loc. cit., vol. 2, pp. 292–93.

elliptic parameter $u=0$, $Y_i=0$. Therefore, the line $S_1(Z)$ meets each E^5 in the point $u=0$. For the parameters $u=\omega_1/2$, $\omega_2/2$, $(\omega_1+\omega_2)/2$, $Z_i=0$. That is, $S_2(Y)$ cuts each E^5 in the three half-period points. Then by planes on $S_1(Z)$ the E^5's are projected into doubly covered conics $R^2(Y)$ in $S_2(Y)$.*

Let $t_1=Z_1(0)$ and $t_2=Z_2(0)$ for a particular E^5 of the family. Then for $u=0$, $Z_i=t_i$ and $Y_i=0$. The coefficients of ζ and θ in F_1 and F_3 vanish identically. Solving F_2 for α we obtain

$$\alpha_0:\alpha_1:\alpha_2 = t_1t_2:t_2^2: -t_1^2. \tag{6}$$

Hence for given t, that is, for a given point of $S_1(Z)$, the five quadrics (2) are known and E^5 is completely determined. Since the three quadrics in F_1 and F_2 vanish for all values of ζ they necessarily vanish for $\zeta=\alpha$. But $F_2\equiv 0$ for $\zeta=\alpha$. Therefore, F_1 gives a quadric which does not contain the Z's and hence must be the conic $R^2(Y)$ in $S_2(Y)$. For $\zeta=\alpha$ and (6), F_1 becomes the conic*

$$t_1^2t_2^2Y_0^2 + t_2^4Y_0Y_2 + t_1^4Y_0Y_1 + 2t_1t_2^3Y_1^2 - 2t_1^3t_2Y_2^2 - 2t_1^2t_2^2Y_1Y_2 = 0. \tag{7}$$

The parametric representation of $S_2(Y)$ in Z and t is obtained from F_3 in (3) which contains Y linearly:

$$\rho Y_0 = 2Z_1^2t_1t_2^3 + 6Z_1Z_2t_1^2t_2^2 + 2Z_2^2t_1^3t_2,$$
$$\rho Y_1 = Z_1^2t_1^4 - Z_1Z_2t_2^4 - 2Z_2^2t_1t_2^3,$$
$$\rho Y_2 = -2Z_1^2t_1^3t_2 - Z_1Z_2t_1^4 - Z_2^2t_2^4.$$

For the point $u=0$, or $t_1:t_2=Z_1:Z_2$, we obtain in $S_2(Y)$

$$\rho Y_0 = 10t_1^3t_2^3,\quad \rho Y_1 = t_1^6 - 3t_1t_2^5,\quad \rho Y_2 = -3t_1^5t_2 - t_2^6. \tag{8}$$

This is a rational sextic which is the projection in $S_2(Y)$ of the point $u=0$ on the family of E^5's in S_4. The conics (7) are the osculant conics of the sextic (8).* If $u=\omega_1/2$, $\omega_2/2$, $(\omega_1+\omega_2)/2$, the Z's are zero and hence the locus in $S_2(Y)$ of the half period points of E^5 is obtained by eliminating α from F_1 in (3). This gives the sextic†

$$Y_0^4Y_1Y_2 - Y_0^2Y_1^2Y_2^2 - Y_0(Y_1^5 + Y_2^5) + 2Y_1^3Y_2^3 = 0. \tag{9}$$

As a quartic in t, (7) has the invariants

$$g_2 = (1/12)(Y_0^2 + 4Y_1Y_2)^2,$$
$$g_3 = (1/216)[-Y_0^6 + 42Y_0^4Y_1Y_2 - 102Y_0^2Y_1^2Y_2^2 - 54Y_0(Y_1^5+Y_2^5) + 44Y_1^3Y_2^3].$$

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* Miller, loc. cit., p. 279.

† Miller, loc. cit., p. 283.

The conic

$$Y_0^2 + 4Y_1Y_2 = 0 \tag{10}$$

is then an invariant of G_{60} in $S_2(Y)$. Moreover

$$12^3(g_2^3 - 27g_3^2) = 54[Y_0^4Y_1Y_2 - Y_0^2Y_1^2Y_2^2 + 2Y_1^3Y_2^3 - Y_0(Y_1^5 + Y_2^5)]$$
$$\cdot\{-2[-Y_0^6 + 15Y_0^4Y_1Y_2 - 75Y_0^2Y_1^2Y_2^2 - 10Y_1^3Y_2^3 - 27Y_0(Y_1^5 + Y_2^5)]\}.$$

This discriminant gives the envelope of the conics (7). Since the first factor is obviously the locus of the half-period points (9), and since the osculant conics touch the sextic (8), the second factor must be the point equation of the rational sextic (8). These sextics must be invariants of G_{60} in $S_2(Y)$.

The quadrics on E^5 in S_4 can be expressed in terms of identical covariants and the (3, 3) form (1), where $t_1:t_2$ of (1) is replaced by $t_1: -t_2$ and $\tau_1:\tau_2$ of (1) is replaced by $\theta_2:\theta_1$. With this change of variables (1) assumes the form

$$G = \theta_1\theta_2^2t_1^3 - \theta_1^3t_1^2t_2 + \theta_2^3t_1t_2^2 + \theta_1^2\theta_2t_2^3. \tag{11}$$

Put the conic (10) in parametric form

$$Y_0:Y_1:Y_2 = 2\theta_1\theta_2: -\theta_1^2:\theta_2^2. \tag{12}$$

Then

$$F_3 = 3\{\theta_1\theta_2^2t_1^3 - \theta_1^3t_1^2t_2 + \theta_2^3t_1t_2^2 + \theta_1^2\theta_2t_2^3\}_{\theta',Z'}$$

where $\{\ \}_{\theta',Z'}$ indicates polarization once with respect to θ and once with respect to Z. After the polarization, the θ's and t's must be replaced by Y's and α's from (12) and (6) respectively.

$$F_2 = (Zt)(Zt')(tt')$$

if α and ζ are expressed in terms of t and t' respectively from (6). If G is written as $(a\theta)^3(\alpha t)^3$, it has the covariant*

$$H = (aa')(a\theta)^2(a'\theta)^2(\alpha\alpha')(\alpha t)^2(\alpha' t)^2$$
$$= t_1^4\theta_1^3\theta_2 + t_1^3t_2\theta_2^4 - 3t_1^2t_2^2\theta_1^2\theta_2^2 - t_1t_2^3\theta_1^4 - t_2^4\theta_1\theta_2^3.$$

Then

$$F_1(Y, Y') = -2H(\theta^2\theta'^2, t^2t'^2) + \tfrac{1}{3}(tt')^2(\theta\theta')^2,$$

where $F_1(Y, Y')$ indicates polarization once with respect to Y and $H(\theta^2\theta'^2, t^2t'^2)$ indicates polarization twice with respect to θ and twice with respect to t. Hence the family of E^5's with a common G_{50} can be expressed algebraically in terms of the (3, 3) form F and its covariants, and identical covariants.

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* Klein, *Vorlesungen über das Ikosaeder*, p. 196.

II. The (3, 3) form and two cubic curves in space

We now interpret the (3, 3) form $F=0$ as the incidence condition of point τ of the space cubic curve $c_1(\tau)$ and plane t of the space cubic curve $c_2(t)$. Let the coördinate system be chosen so that $c_1(\tau)$ has the equations

$$x_0 = \tau^3, \quad x_1 = 3\tau^2, \quad x_2 = 3\tau, \quad x_3 = 1, \tag{1}$$

and

$$\xi_0 = 1, \quad \xi_1 = -\tau, \quad \xi_2 = \tau^2, \quad \xi_3 = -\tau^3,$$

in points and planes respectively. If $F=0$ is the incidence condition of point τ of $c_1(\tau)$ and plane t of $c_2(t)$, then

$$\xi_0 = t, \quad \xi_1 = \tfrac{1}{3}t^3, \quad \xi_2 = -\tfrac{1}{3}, \quad \xi_3 = t^2, \tag{2}$$

and

$$x_0 = t^2, \quad x_1 = 1, \quad x_2 = t^3, \quad x_3 = -t,$$

are the equations of $c_2(t)$ in planes and points respectively. However, the incidence condition of point t of $c_2(t)$ and plane τ of $c_1(\tau)$ becomes $\overline{F}=t^2-\tau+t^3\tau^2+t\tau^3=0$. But $\overline{F}\equiv F$. Hence the point τ of $c_1(\tau)$ lies on plane t of $c_2(t)$ and point t of $c_2(t)$ lies on plane τ of $c_1(\tau)$ if $F=0$.

The two cubic curves have this further property in common. They both lie in the same null system whose equations are

$$\rho u_0 = 3x_3, \quad \rho u_1 = -x_2, \quad \rho u_2 = x_1, \quad \rho u_3 = -3x_0.$$

For convenience we here identify part of the covariants given by Coble* for the general case with those given by Gordan†‡:

$$\begin{aligned}
\overset{(2)}{c}_{4,4} &= (4/9)\left[t_1^4\tau_1\tau_2^3 - t_1^3t_2\tau_1^4 - 3t_1^2t_2^2\tau_1^2\tau_2^2 + t_1t_2^3\tau_2^4 - t_2^4\tau_1\tau_2\right],\\
\overset{(2)}{c}_{2,6} &= -(2/9)\left[t_1^6\tau_1^2 - 3t_1^5t_2\tau_2^2 - 10t_1^3t_2^3\tau_1\tau_2 + 3t_1t_2^5\tau_1^2 + t_2^6\tau_2^2\right],\\
\overset{(2)}{c}_{6,2} &= -(2/9)\left[t_1^2(\tau_2^6 - 3\tau_1^5\tau_2) + 10t_1t_2\tau_1^3\tau_2^3 + t_2^2(\tau_1^6 + 3\tau_1\tau_2^5)\right],\\
\overset{(2)}{c}_{0,4} &\equiv 0, \quad \overset{(2)}{c}_{4,0} \equiv 0, \quad \overset{(2)}{c}_{0,0} = 4/3\cdot
\end{aligned} \tag{3}$$

Since there is a group of order 120 for which F is invariant, the space con-

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* American Journal of Mathematics, vol. 46, pp. 147–48.

† Mathematische Annalen, vol. 13, pp. 386–87.

‡ By using these values to check the syzygies connecting the comitants of the second degree given by Coble (American Journal of Mathematics, vol. 46, p. 155, (18)) it is found that the last one is wrong. It should read

$$24(cc')^4 \equiv 24(\gamma\gamma')^4 \equiv 180\delta^2 - (bb')^4(\beta\beta')^4.$$

The theorem should read "The forty coefficients of these syzygies furnish the forty linearly independent relations of the second degree among the two-row minors of R, etc."

figuration must be invariant under a group G_{120}. In the group on F the largest subgroup which transforms t into t' and τ into τ' is the icosahedral G_{60}. The further operations interchange t and τ. Hence in the space configuration each space cubic curve is unaltered by an icosahedral G_{60}. The operations which interchange t and τ will then interchange the two cubic curves.

The G_{60} is generated by S and T^*:

$$S:\quad \begin{aligned} t_1' &= \epsilon t_1, & \tau_1' &= \epsilon^2\tau_1,\\ t_2' &= \epsilon^4 t_2, & \tau_2' &= \epsilon^3\tau_2; \end{aligned} \tag{4}$$

$$T:\quad \begin{aligned} 5^{1/2}t_1' &= -(\epsilon^2-\epsilon^3)t_1+(\epsilon^4-\epsilon)t_2, & 5^{1/2}\tau_1' &= (\epsilon^4-\epsilon)\tau_1+(\epsilon^2-\epsilon^3)\tau_2,\\ 5^{1/2}t_2' &= (\epsilon^4-\epsilon)t_1+(\epsilon^2-\epsilon^3)t_2, & 5^{1/2}\tau_2' &= (\epsilon^2-\epsilon^3)\tau_1-(\epsilon^4-\epsilon)\tau_2; \end{aligned}$$

and, for convenience,

$$U:\quad \begin{aligned} t_1' &= -t_2, & \tau_1' &= -\tau_2,\\ t_2' &= t_1, & \tau_2' &= \tau_1. \end{aligned}$$

Then as a collineation group on the x's the operations of G_{60} are

$$S^\mu:\quad \begin{aligned} x_0' &= \epsilon^{\mu}x_0,\\ x_1' &= \epsilon^{2\mu}x_1,\\ x_2' &= \epsilon^{3\mu}x_2,\\ x_3' &= \epsilon^{4\mu}x_3; \end{aligned} \qquad S^\mu U:\quad \begin{aligned} x_0' &= -\epsilon^{4\mu}x_3,\\ x_1' &= \epsilon^{3\mu}x_2,\\ x_2' &= -\epsilon^{2\mu}x_1,\\ x_3' &= \epsilon^{\mu}x_0; \end{aligned}$$

$$\begin{aligned} S^\mu TS^\nu:\quad 5\cdot 5^{1/2}x_0' &= \epsilon^{\nu}[-(\epsilon-\epsilon^4)^3\epsilon^{\mu}x_0-5^{1/2}(\epsilon-\epsilon^4)\epsilon^{2\mu}x_1+5^{1/2}(\epsilon^2-\epsilon^3)\epsilon^{3\mu}x_2\\ &\quad +(\epsilon^2-\epsilon^3)^3\epsilon^{4\mu}x_3],\\ 5\cdot 5^{1/2}x_1' &= \epsilon^{2\nu}[-3\cdot 5^{1/2}(\epsilon-\epsilon^4)\epsilon^{\mu}x_0+(\epsilon^2-\epsilon^3)^3\epsilon^{2\mu}x_1-(\epsilon-\epsilon^4)^3\epsilon^{3\mu}x_2\\ &\quad -3\cdot 5^{1/2}(\epsilon^2-\epsilon^3)\epsilon^{4\mu}x_3],\\ 5\cdot 5^{1/2}x_2' &= \epsilon^{3\nu}[3\cdot 5^{1/2}(\epsilon^2-\epsilon^3)\epsilon^{\mu}x_0-(\epsilon-\epsilon^4)^3\epsilon^{2\mu}x_1-(\epsilon^2-\epsilon^3)^3\epsilon^{3\mu}x_2\\ &\quad -3\cdot 5^{1/2}(\epsilon-\epsilon^4)\epsilon^{4\mu}x_3],\\ 5\cdot 5^{1/2}x_3' &= \epsilon^{4\nu}[(\epsilon^2-\epsilon^3)^3\epsilon^{\mu}x_0-5^{1/2}(\epsilon^2-\epsilon^3)\epsilon^{2\mu}x_1-5^{1/2}(\epsilon-\epsilon^4)\epsilon^{3\mu}x_2\\ &\quad +(\epsilon-\epsilon^4)^3\epsilon^{4\mu}x_3]; \end{aligned}$$

$$\begin{aligned} S^\mu TS^\nu U:\quad 5\cdot 5^{1/2}x_0' &= -\epsilon^{4\nu}[(\epsilon^2-\epsilon^3)^3\epsilon^{\mu}x_0-5^{1/2}(\epsilon^2-\epsilon^3)\epsilon^{2\mu}x_1-5^{1/2}(\epsilon-\epsilon^4)\epsilon^{3\mu}x_2\\ &\quad +(\epsilon-\epsilon^4)^3\epsilon^{4\mu}x_3],\\ 5\cdot 5^{1/2}x_1' &= \epsilon^{3\nu}[3\cdot 5^{1/2}(\epsilon^2-\epsilon^3)\epsilon^{\mu}x_0-(\epsilon-\epsilon^4)^3\epsilon^{2\mu}x_1-(\epsilon^2-\epsilon^3)^3\epsilon^{3\mu}x_2\\ &\quad -3\cdot 5^{1/2}(\epsilon-\epsilon^4)\epsilon^{4\mu}x_3], \end{aligned}$$

* Gordan, Mathematische Annalen, loc. cit., p. 379.

$$5\cdot 5^{1/2}x_2' = -\epsilon^{2\nu}[-3\cdot 5^{1/2}(\epsilon-\epsilon^4)\epsilon^{\mu}x_0 + (\epsilon^2-\epsilon^3)^3\epsilon^{2\mu}x_1 - (\epsilon-\epsilon^4)^3\epsilon^{3\mu}x_2 - 3\cdot 5^{1/2}(\epsilon^2-\epsilon^3)\epsilon^{4\mu}x_3],$$

$$5\cdot 5^{1/2}x_3' = \epsilon^{\nu}[-(\epsilon-\epsilon^4)^3\epsilon^{\mu}x_0 - 5^{1/2}(\epsilon-\epsilon^4)\epsilon^{2\mu}x_1 + 5^{1/2}(\epsilon^2-\epsilon^3)\epsilon^{3\mu}x_2 + (\epsilon^2-\epsilon^3)^3\epsilon^{4\mu}x_3] \qquad (\mu, \nu = 0, 1, 2, 3, 4).$$

The operation which interchanges t and τ and leaves F invariant is

$$R:\ t_1' = \tau_1, \quad t_2' = \tau_2, \quad \tau_1' = -t_2, \quad \tau_2' = t_1,$$

where $R^2 = U$. As a collineation in space for which a point of one cubic curve goes into a point of the other, R becomes

$$3^{1/2}x_0' = -x_1, \quad 3^{1/2}x_1' = -3x_3, \quad 3^{1/2}x_2' = -3x_0, \quad 3^{1/2}x_3' = x_2.$$

The quaternary icosahedral G_{60}'s and the symmetric G_{120}'s are listed in the literature but it is not shown that an icosahedral G_{60} leaves each of two space cubic curves unaltered and a symmetric G_{120} interchanges these two curves.

This collineation group may be enlarged to a correlation group by using R to send point and plane of $c_1(\tau)$ into plane and point of $c_2(t)$ respectively and vice versa. The correlation is, then,

$$C:\ \begin{array}{llll} \xi_0' = x_2, & \xi_1' = x_0, & \xi_2' = -x_3, & \xi_3' = x_1, \\ x_0' = \xi_2, & x_1' = \xi_0, & x_2' = -\xi_3, & x_3' = \xi_1. \end{array}$$

III. The rational icosahedral sextic

The nets of point quadrics on $c_1(\tau)$ and $c_2(t)$ are, respectively,

$$Q_1:\ u_0(x_1^2 - 3x_0x_2) + u_1(x_1x_2 - 9x_0x_3) + u_2(x_2^2 - 3x_1x_3) = 0,$$

and

$$Q_2:\ u_0(x_0^2 + x_2x_3) + u_1(-x_1x_2 - x_0x_3) + u_2(x_3^2 - x_0x_1) = 0,$$

and the nets of quadric envelopes on the planes of $c_1(\tau)$ and $c_2(t)$ are, respectively,

$$\bar{Q}_1:\ u_0(\xi_2^2 - \xi_1\xi_3) + u_1(\xi_0\xi_3 - \xi_1\xi_2) + u_2(\xi_1^2 - \xi_0\xi_2) = 0,$$

and

$$\bar{Q}_2:\ u_0(\xi_3^2 - 3\xi_0\xi_1) + u_1(\xi_0\xi_3 + 9\xi_1\xi_2) + u_2(\xi_0^2 + 3\xi_2\xi_3) = 0.$$

The net Q_1 will cut $c_2(t)$ in a linear series I_2^6 determined by substituting the coördinates of $c_2(t)$ in Q_1.* This gives

$$(1) \qquad u_0(1 - 3t^5) + u_1(10t^3) + u_2(t^6 + 3t) = 0.$$

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* Coble, American Journal of Mathematics, vol. 46, p. 156.

Then (1) may be interpreted as the line sections of a rational plane sextic $S_2(t)$ whose point equations are

$$(2) \qquad x_0 = 1 - 3t^5, \quad x_1 = 10t^3, \quad x_2 = t^6 + 3t.$$

$\bar{Q}_1$ on $c_2(t)$ is found to give the same sextic, that is, $\bar{S}_2(t) \equiv S_2(t)$.* The linear series cut out on $c_1(\tau)$ by Q_2 and $\bar{Q}_2$ give the sextics $\bar{S}_1(\tau) \equiv S_1(\tau)$:

$$x_0 = \tau^6 + 3\tau, \quad x_1 = -10\tau^3, \quad x_2 = 1 - 3\tau^5.$$

That is, instead of the four sextics of the general case* there are only two for the special case. In the same plane these two sextics are also identical since both point equations are†

$$27x_1(x_0^5 - x_2^5) - 10x_0^3x_2^3 + 75x_0^2x_1^2x_2^2 + 15x_0x_1^4x_2 + x_1^6 = 0.$$

Corresponding to the ten nodes of the sextics are the ten common bisecants and the ten common axes of the curves $c_1(\tau)$ and $c_2(t)$ since the two points which come together at a node of the sextic do not determine a line section of the sextic but a pencil of line sections.

The cones of the net Q_1 are given when the constants satisfy the condition

$$(3) \qquad u_0u_2 - u_1^2 = 0.$$

That is, to the cones in Q_1 correspond the lines of a conic in the plane of the sextic. The equations of the cones become

$$u_1^2(x_1^2 - 3x_0x_2) + u_1u_2(x_1x_2 - 9x_0x_3) + u_2^2(x_2^2 - 3x_1x_3) = 0.$$

The corresponding lines in the plane satisfy (3) and

$$u_1^2(1 - 3t^5) + u_1u_2(10t^3) + u_2^2(t^6 + 3t) = 0.$$

In the plane of $S_2(t)$ there is a conic such that for every point of the sextic there are two lines of the conic and for every line of the conic there are six points of the sextic. If the equations of the conic (3) are

$$K(\tau): \begin{array}{lll} x_0 = \tau^2, & x_1 = 2\tau, & x_2 = 1, \\ u_0 = 1, & u_1 = -\tau, & u_2 = \tau^2, \end{array}$$

in points and lines respectively, this (2, 6) relation is obtained from (1) as

$$(4) \qquad (1 - 3t^5) - \tau(10t^3) + \tau^2(t^6 + 3t) = 0,$$

which is evidently $c_{2,6}^{(2)} = 0$, of (3) §II. (4) is then the equation of the sextic $S_2(t) \equiv \bar{S}_2(t)$ in Darboux coördinates; that is,

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* Coble, American Journal of Mathematics, vol. 46, p. 157.

† Winger, Mathematische Annalen, loc. cit., p. 211.

$$x_0 = \tau_1\tau_2, \quad x_1 = \tau_1 + \tau_2, \quad x_2 = 1.$$

To a pencil of lines in the plane on a point of $S_2(t)$ corresponds a pencil of quadrics in Q_1 on a bisecant of $c_1(\tau)$ on a point of $c_2(t)$. To the tangents of the conic in the pencil of lines correspond the cones in the pencil of quadrics. If a point is a node of $S_2(t)$ the pencil of quadrics is the pencil on a common bisecant of $c_1(\tau)$ and $c_2(t)$ and the cones have vertices at the points of intersection of this common bisecant with $c_1(\tau)$, say at τ_1 and τ_2. Then τ_1 and τ_2 give two tangents on $K(t)$ which intersect in a node of the sextic $S_1(\tau)$.*

The sextic $S_2(t)$ is evidently the sextic (8) §I, except for a change in notation, which is invariant under the ternary icosahedral group. The ten double points are the ten points isomorphic with the ten diagonals of the dodecahedron. The coördinates of the points in the notation used here are

$$(-\epsilon^{4i}, \delta_j^2, \epsilon^i) \qquad (i = 0, 1, 2, 3, 4; j = 1, 2),$$

where

$$\delta_1 = \epsilon + \epsilon^4, \quad \delta_2 = \epsilon^2 + \epsilon^3.$$

Winger notes the fact that the double points lie by sixes on a set of ten conics associated with the dihedral G_6's in the G_{60}.† However, they lie by sixes on another set of fifteen conics associated with the fifteen reflexions in the G_{60}.

Name the diagonals of the icosahedron 1, 2, 3, 4, 5, 6. Then the double points of the sextic which correspond to the diagonals of the dodecahedron may be named by the three adjacent icosahedral diagonals, that is, (ijk). Let the diagonals of the icosahedron be named in such a manner that the double points will be

1,	(123),	$(-\epsilon^3, \delta_2^2, \epsilon^2)$,	6,	(245),	$(-1, \delta_1^2, 1)$,
2,	(134),	$(-\epsilon^4, \delta_2^2, \epsilon)$,	7,	(356),	$(-\epsilon, \delta_1^2, \epsilon^4)$,
3,	(145),	$(-1, \delta_2^2, 1)$,	8,	(246),	$(-\epsilon^2, \delta_1^2, \epsilon^3)$,
4,	(156),	$(-\epsilon, \delta_2^2, \epsilon^4)$,	9,	(235),	$(-\epsilon^3, \delta_1^2, \epsilon^2)$,
5,	(162),	$(-\epsilon^2, \delta_2^2, \epsilon^3)$,	0,	(346),	$(-\epsilon^4, \delta_1^2, \epsilon)$.

The second column which gives the double points with reference to the icosahedral diagonals also gives an interesting number system. It is a triad system with double couples unaltered by a G_{60} given by Emch.‡

The icosahedral group may be written as a substitution group on the six diagonals generated by $S=(23456)$ and $T=(12)(36)$. Under the operations

* Coble, American Journal of Mathematics, vol. 46, p. 157.

† American Journal of Mathematics, loc. cit., p. 56.

‡ A. Emch, *Triple and multiple systems, their geometric configurations and groups*, these Transactions, vol. 31 (1929), pp. 25–42.

of period two, two of the double points are fixed and the remaining eight are separated into four pairs. If a conic passes through one of each of three pairs and the mates of two of the three, it must pass through the mate of the third from the symmetry of the configuration. In this way we find two sets of conjugate conics, one containing ten and the other fifteen, which pass through six double points.

The conics in the set of ten are the following:

(123568),	(126890),
(134569),	(456789),
(124570),	(236790),
(234580),	(157890),
(123479),	(346780).

These conics may be associated with the ten diagonals of the dodecahedron. Three edges of the dodecahedron meet at the vertex (ijk). The three adjacent vertices are named (ijl), (ikm), (jkn). These four vertices represent a set of four nodes. The remaining six lie on a conic.

The conics forming the set of fifteen are the following:

(124589),	(234567),	(256780),
(234689),	(123460),	(356890),
(123780),	(145680),	(136789),
(134578),	(123590),	(146790),
(125679),	(345790),	(247890).

These conics may be associated with the fifteen cross lines of the dodecahedron or with the fifteen reflexions in the G_{60}. The four vertices of the dodecahedron, consisting of the two on any edge and the vertices opposite this edge in each of the adjoining faces, represent a set of four nodes. From the manner in which the figure is numbered, the opposite edge gives the same set of four nodes. The remaining six nodes lie on a conic.

The numbers which give these conics also form number systems. The set of ten conics gives the arrangement of ten numbers six at a time such that each number occurs in six. The set of fifteen conics gives a number system with double couples. It gives the arrangement of ten numbers six at a time such that each number occurs nine times and each couple occurs five times. These number systems are unaltered by a group of order 120.

IV. The cubic envelopes perspective to the rational sextic

The planes of $c_1(\tau)$ cut any plane, τ_0, of $c_1(\tau)$ in a line conic which may be taken to be the conic

(1) $$K(\tau): \quad u_0u_2 - u_1^2 = 0 \quad \text{or} \quad 4x_0x_2 - x_1^2 = 0.$$

This conic may be used as a coördinate system in the plane τ_0. Any two planes, τ_1 and τ_2, of $c_1(\tau)$ intersect in an axis of $c_1(\tau)$. These two planes cut τ_0 in two lines of the conic (1). These two lines intersect in the point x where the axis cuts the plane τ_0. The coördinates of this point x will be

$$x_0 = \tau_1\tau_2, \quad x_1 = \tau_1 + \tau_2, \quad x_2 = 1. \tag{2}$$

The planes t of $c_2(t)$ cut the planes of $c_1(\tau)$ in cubic envelopes which may be projected by the axes of $c_1(\tau)$ onto a plane τ_0 of $c_1(\tau)$ to give a pencil of cubic envelopes in τ_0.* If $c_1(\tau)$ be chosen as a reference system the plane t of $c_2(t)$ is the plane

$$F = (\alpha t)^3(a\tau)^3 = t_1^3\tau_1^2\tau_2 + t_1^2t_2\tau_2^3 + t_1t_2^2\tau_1^3 - t_2^3\tau_1\tau_2^2 = 0. \tag{3}$$

The point τ_1, τ_2, τ is on this plane if

$$(a\tau_1)(a\tau_2)(a\tau)(\alpha t)^3 = 0.$$

Polarizing F in (3) with respect to τ_1 and τ_2, we obtain

$$\tau_1\tau_2(t^3 + 3t\tau) + (\tau_1 + \tau_2)(t^3\tau - 1) + (3t^2 - \tau) = 0.$$

Substituting (2), we obtain

$$x_0(t^3 + 3t\tau) + x_1(t^3\tau - 1) + x_2(3t^2 - \tau) = 0. \tag{4}$$

That is, the pencil of cubic envelopes cut out by the planes of $c_2(t)$ on a plane of $c_1(\tau)$ are

$$E_3(t): \quad \xi_0 = t^3 + 3t\tau, \quad \xi_1 = t^3\tau - 1, \quad \xi_2 = 3t^2 - \tau. \tag{5}$$

For $\tau = \tau_1$ and τ_2 in (4), we have the lines

$$x_0(t^3 + 3t\tau_1) + x_1(t^3\tau_1 - 1) + x_2(3t^2 - \tau_1) = 0,$$
$$x_0(t^3 + 3t\tau_2) + x_1(t^3\tau_2 - 1) + x_2(3t^2 - \tau_2) = 0,$$

respectively, of the cubic envelopes (5). These lines intersect in the point

$$x_0 : x_1 : x_2 = 1 - 3t^5 : 10t^3 : 3t + t^6. \tag{6}$$

Since the coördinates of this point are independent of τ, for given t all the lines of the ∞^1 cubic envelopes pass through the same point (6). For variable t this point describes the rational sextic (2) §III.

In every plane t of $c_2(t)$ there is an axis of $c_1(\tau)$. Since $F=0$ gives the points where plane t cuts $c_1(\tau)$ the Hessian of F as a cubic in τ gives the parameters of the planes which intersect in an axis in plane t. The Hessian of F is from (3) §II

* Coble, American Journal of Mathematics, vol. 46, p. 160.

$$\tau_1^2(t_1^6 + 3t_1t_2^5) - 10t_1^3t_2^3\tau_1\tau_2 + \tau_2^2(t_2^6 - 3t_1^5t_2) = 0.$$

But this is also the rational sextic (2) §III in Darboux coördinates. That is, the rational sextic (2) §III is the locus of points where axes of $c_1(\tau)$ in planes of $c_2(t)$ cut the plane τ_0 and is perspective to the cubic envelopes in τ_0 cut out by the planes of $c_2(t)$.*

The incidence condition of point t of $S_2(t)$ and line t' of $E_3(t)$ becomes (in homogeneous parameters)

$$(7)\qquad \begin{aligned}(t_1t_2' - t_2t_1')[\tau_1(- t_1^5t_2'^2 - 10t_1^4t_2t_1't_2' - 10t_1^3t_2^2t_1'^2 - 3t_2^5t_2'^2)\\ + \tau_2(3t_1^5t_1'^2 - 10t_1^2t_2^3t_2'^2 - 10t_1t_2^4t_1't_2' - t_2^5t_1'^2)] = 0.\end{aligned}$$

For a node of $S_2(t)$ the incidence condition must contain a quadratic in t. For $(-\epsilon^{4i}, \delta_j^2, \epsilon^i)$, the condition is

$$(8)\quad (-\epsilon^{3i}t_1^2 + \delta_k^2t_1t_2 + \epsilon^{2i}t_2^2)[\delta_k^2(\epsilon^{4i}t_2\tau_1 - \epsilon^it_1\tau_2) + (\epsilon^{2i}t_1\tau_1 + \epsilon^{3i}t_2\tau_2)] = 0, j \neq k.$$

The quadratic factors in t in (8) equated to zero give the nodal parameters of $S_2(t)$. If the two roots of one of these quadratic equations are located on the conic

$$K(t):\quad x_0 = t^2,\ x_1 = 2t,\ x_2 = 1,$$

the tangents at these points intersect in the point $x_0:x_1:x_2 = t_1t_2:t_1+t_2:1$. The ten quadratics in (8) give the points $(-\epsilon^{2i}, \delta_k^2, \epsilon^{3i})$, which are the nodes of $S_1(\tau)$.

Lines on two nodes cut the sextic in two more points whose parameters are given by quadratics in t obtained by eliminating τ from the bilinear forms in (8). Locating these with respect to $K(t)$ as above we have the points

$$\begin{array}{ll} (\epsilon^i, 0, 1), & [-\epsilon^i(\delta_2 + \epsilon^2), \delta_1, \epsilon^{4i}(\delta_2 + \epsilon^3)], \\ (-\epsilon^i, \delta_j - 3, \epsilon^{4i}), & [-\epsilon^i(\delta_2 + \epsilon^3), \delta_1, \epsilon^{4i}(\delta_2 + \epsilon^2)], \\ (-\epsilon^i, 2\delta_j, \epsilon^{4i}), & [-\epsilon^i(\delta_1 + \epsilon), \delta_2, \epsilon^{4i}(\delta_1 + \epsilon^4)], \\ & [-\epsilon^i(\delta_1 + \epsilon^4), \delta_2, \epsilon^{4i}(\delta_1 + \epsilon)] \end{array}$$

$$(i = 0, 1, 2, 3, 4;\ j = 1, 2).$$

Fifteen of these points, that is,

$$(\epsilon^i, 0, 1),\quad (-\epsilon^i, 2\delta_1, \epsilon^{4i}),\quad (-\epsilon^i, 2\delta_2, \epsilon^{4i})$$

in the t-plane are the fifteen points which form a conjugate set isomorphic with the fifteen cross lines of the icosahedron.

The conics on five double points cut the sextic in two other points whose parameters will be given by a quadratic in t. Out of a possible 252 conics, 150 are accounted for by the conics on six points. That is, 150 do not give new

* Coble, American Journal of Mathematics, vol. 46, p. 160.

points in the t-plane. The equation of the conic on (12345) is $(\delta_1-\delta_2)x_0x_2+(\delta_1^2-\delta_1^6)x_1^2=0$. In this equation substitute the coördinates of the sextic $S_2(t)$, divide by the ten-ics in t which give the parameters of the points of intersection of the conic and the sextic at the five double points, and obtain as quotient a quadratic in t which gives the parameters of the two remaining points of intersection. In this case the quadratic is $t_1t_2=0$ and the corresponding point in the t-plane is (0, 1, 0). Under G_{60} the conic (12345) goes into the five other conics (23680), (34679), (12790), (15689), and (45780). The corresponding points of these six conics in the t-plane are the six points which Klein calls the fundamental points. Moreover the conics on the complementary five points:

(67890), (14579), (12580), (34568), (23470), (12369),

give the same six points in the t-plane. That is, the six quadratics which give the six fundamental points with respect to $K(t)$ give two points each on $S_2(t)$ through which pass two conics which cut the sextic again in two different sets of five nodes. The conic on (12348) gives the point $(\epsilon, 3\delta_2, -\epsilon^4)$ with respect to $K(t)$ and is transformed into thirty conics by G_{60}. The conic on (56790) gives the point $(\epsilon, 3\delta_1, -\epsilon^4)$ and is transformed into thirty other conics. The conic (13567) gives the point

$$[(3\epsilon+7+5\epsilon^4),\ -(4\epsilon+4\epsilon^2+2),\ (5\epsilon+3\epsilon^4+7)]$$

and is transformed into thirty conics. This accounts for the 252 conics. Moreover, these points in the t-plane obtained from the lines on two nodes and the conics on five nodes lie on the fifteen axes conjugate under G_{60}.

V. The sextic of genus four

Consider again the form

$$(1)\qquad (\pi x)(a\tau)(\alpha t)^3 = x_0(t^3+3t\tau)+x_1(t^3\tau-1)+x_2(3t^2-\tau),$$

which, when equated to zero, gives the cubic envelopes perspective to the rational sextic (2) §III. Some of the comitants given by Coble* to within a numerical factor are as follows:

$$(2)\quad (\pi\pi'\xi)(aa')(\alpha t)(\alpha' t)^3 = (t_2^6-3t_1^5t_2)\xi_0+10t_1^3t_2^3\xi_1+(t_1^6+3t_1t_2^5)\xi_2,$$

$$(3)\quad (\pi x)(\pi' x)(a\tau)(a'\tau)(\alpha\alpha')^2(\alpha t)(\alpha' t) = (x_0x_1t_1^2-x_1x_2t_1t_2-x_0^2t_2^2)\tau_1^2$$
$$+[x_0^2t_1^2-(x_1^2+2x_0x_2)t_1t_2-x_2^2t_2^2]\tau_1\tau_2-(x_2^2t_1^2+x_0x_1t_1t_2+x_1x_2t_2^2)\tau_2^2,$$

$$(4)\quad (\pi\pi'\pi'')(a\tau)(a'\tau)(a''\tau)(\alpha\alpha')(\alpha\alpha'')(\alpha'\alpha'')(\alpha t)(\alpha' t)(\alpha'' t) = t_1^3\tau_1^2\tau_2+t_1^2t_2\tau_2^3$$
$$+t_1t_2^2\tau_1^3-t_2^3\tau_1\tau_2^2.$$

* American Journal of Mathematics, vol. 46, p. 162.

The cubic (4) is apolar to (1) and hence it gives the three cusps of the cubic envelope for any given τ. But the cubic in (4) is the original (3, 3) form and hence the cusp locus will be a curve of genus four. For a cusp (1) must be a perfect cube and hence its Hessian must vanish identically. The cusp locus is then obtained by eliminating τ from the coefficients of t_1^2, t_1, t_2, t_2^2, in (3) which is the Hessian of (1). This gives

$$x_1(x_0^5 - x_2^5) + 2x_0^3x_2^3 + x_0^2x_1^2x_2^2 + x_0x_1^4x_2 = 0, \tag{5}$$

the sextic of genus four invariant under the icosahedral group. Klein* states, without proof, that an infinite number of triangles whose vertices lie on (5) can be circumscribed about the conic $4x_0x_2 - x_1^2 = 0$ and that every point of (5) is in one such triangle. One proof is that given by Coble.† The theorem given there also means for this case that the lines joining the vertices to the points of contact of opposite sides intersect in a point of the rational icosahedral sextic.

Consider now the equation $G=0$ of §I. By means of the above configuration the solutions of the cubic equation $G=0$ are obtained as modular functions. The sextic (9) §I is the sextic (5) where $x_0:x_1:x_2$ is replaced by $Y_2:Y_0:-Y_1$. The sextic (8) §I is also the sextic (2) §III. The sextic (5) in coördinates Y is the locus in $S_2(Y)$ of the half-period points of the family of E^5's in S_4 and is of genus four. For every t there are three values of θ, say θ, θ', θ'', in $G=0$ which give tangents to the conic forming a triangle whose vertices lie on (5) and are the half-period points $u=\omega_1/2$, $\omega_2/2$, $(\omega_1+\omega_2)/2$, of an E^5 in S_4. The lines joining the vertices to the points of contact of opposite sides meet in the point $u=0$ of the E^5. For a given ω, a value of $t=t_1:t_2$ is obtained as $Z_1(0):Z_2(0)$ on $S_1(Z)$ and t is determined as a function of ω. The three corresponding values, θ, θ', and θ'', which satisfy $G=0$ are obtained by means of the half-period points of the E^5. The coördinates $Y_0:Y_1:Y_2$ of the vertices of the triangle are obtained from

$$Y_0: Y_1: Y_2 = 2\theta_1\theta_2: -\theta_1^2:\theta_2^2$$

as

$$Y_0\left(\frac{\omega_1}{2}\right) : Y_1\left(\frac{\omega_1}{2}\right) : Y_2\left(\frac{\omega_1}{2}\right) = \theta + \theta' : -\theta\theta' : 1,$$

$$Y_0\left(\frac{\omega_2}{2}\right) : Y_1\left(\frac{\omega_2}{2}\right) : Y_2\left(\frac{\omega_2}{2}\right) = \theta + \theta'' : -\theta\theta' : 1;$$

$$Y_0\left(\frac{\omega_1+\omega_2}{2}\right) : Y_1\left(\frac{\omega_1+\omega_2}{2}\right) : Y_2\left(\frac{\omega_1+\omega_2}{2}\right) = \theta' + \theta'' : -\theta'\theta' : 1.$$

* Mathematische Annalen, vol. 12, p. 542.

† American Journal of Mathematics, vol. 46, p. 179.

Then

$$\theta = \frac{1}{2}\left\{ \frac{Y_0\left(\frac{\omega_1}{2}\right)}{Y_2\left(\frac{\omega_1}{2}\right)} + \frac{Y_0\left(\frac{\omega_2}{2}\right)}{Y_2\left(\frac{\omega_2}{2}\right)} - \frac{Y_0\left(\frac{\omega_1+\omega_2}{2}\right)}{Y_2\left(\frac{\omega_1+\omega_2}{2}\right)} \right\},$$

$$\theta' = \frac{1}{2}\left\{ \frac{Y_0\left(\frac{\omega_1}{2}\right)}{Y_2\left(\frac{\omega_1}{2}\right)} - \frac{Y_0\left(\frac{\omega_2}{2}\right)}{Y_2\left(\frac{\omega_2}{2}\right)} + \frac{Y_0\left(\frac{\omega_1+\omega_2}{2}\right)}{Y_2\left(\frac{\omega_1+\omega_2}{2}\right)} \right\},$$

$$\theta'' = \frac{1}{2}\left\{ -\frac{Y_0\left(\frac{\omega_1}{2}\right)}{Y_2\left(\frac{\omega_1}{2}\right)} + \frac{Y_0\left(\frac{\omega_2}{2}\right)}{Y_2\left(\frac{\omega_2}{2}\right)} + \frac{Y_0\left(\frac{\omega_1+\omega_2}{2}\right)}{Y_2\left(\frac{\omega_1+\omega_2}{2}\right)} \right\}.$$

These expressions give the values of t and θ satisfying $G=0$ explicitly as modular functions. As t takes all values of the line $S_1(Z)$, all solutions of $G=0$ are obtained as modular functions.

VI. The quartic surfaces, the Jacobian and the symmetroid of Cayley and the Stahl quadric

The web of point quadrics apolar to $\overline{Q}_1+\overline{Q}_2$ is

(1) $z_0(2y_2y_3 - 3y_0^2) + z_1(2y_0y_1 + 3y_3^2) + z_2(2y_0y_2 + y_1^2) + z_3(2y_1y_3 + y_2^2) = 0.$

Interpret (y_0, y_1, y_2, y_3) as points of the space of $c_1(\tau)$ and $c_2(t)$ and (z_0, z_1, z_2, z_3) as points of a second space. Then the locus of points z for which (1) has a double point is

$$\Sigma_1 = -11z_0z_1z_2z_3 + 3(z_0^3z_2 - z_1^3z_3 - z_1z_2^3 + z_0z_3^3) + z_0^2z_1^2 + z_2^2z_3^2 = 0,$$

a quartic surface given by a symmetric four-row determinant whose elements are linear functions of z and hence a symmetroid.

The locus of points y where (1) has a double point is the Jacobian of the web:

$$J_1 = -9y_0y_1y_2y_3 + (-3y_0^3y_1 + y_1^3y_3 + y_0y_2^3 + 3y_2y_3^3) + 9y_0^2y_3^2 - y_1^2y_2^2 = 0,$$

a quartic surface in the space of $c_1(\tau)$ and $c_2(t)$.

The ten double points of Σ_1 are

(2) $$(-\delta_j^2\epsilon^i, \delta_j^2\epsilon^{2i}, \epsilon^{3i}, 1) \qquad (i = 0, 1, 2, 3, 4; j = 1, 2).$$

The ten lines of the Jacobian are obtained by substituting (2) in (1).

The web of quadric envelopes apolar to Q_1+Q_2 is

$$\zeta_0(2\xi_2\xi_3 - \xi_0^2) + \zeta_1(2\xi_0\xi_1 + \xi_3^2) + \zeta_2(3\xi_1^2 + 2\xi_0\xi_2) + \zeta_3(2\xi_1\xi_3 + 3\xi_2^2) = 0. \tag{3}$$

For the web (3) the Cayley symmetroid becomes

$$\Sigma_2 = -11\zeta_0\zeta_1\zeta_2\zeta_3 + 3(\zeta_0^3\zeta_2 - \zeta_1^3\zeta_3 - \zeta_1\zeta_2^3 + \zeta_0\zeta_3^3) + \zeta_0^2\zeta_1^2 + \zeta_2^2\zeta_3^2 = 0,$$

and the Jacobian is

$$J_2 = -9\xi_0\xi_1\xi_2\xi_3 + (-\xi_0^3\xi_1 + 3\xi_1^3\xi_3 + 3\xi_0\xi_2^3 + \xi_2\xi_3^3) + \xi_0^2\xi_3^2 - 9\xi_1^2\xi_2^2 = 0.$$

By (1) the points of the cubic curves $c_1(\tau)$ and $c_2(t)$ map into the planes of two rational space sextic curves:

$$\overline{R}_1(\tau):\quad \zeta_0 = 2\tau - \tau^6,\quad \zeta_1 = 2\tau^5 + 1,\quad \zeta_2 = 5\tau^4,\qquad \zeta_3 = 5\tau^2,$$

and

$$\overline{R}_2(t):\quad \zeta_0 = -5t^4,\qquad \zeta_1 = 5t^2,\qquad \zeta_2 = 2t^5 + 1,\quad \zeta_3 = -2t + t^6.$$

The planes ζ of these curves on a point z are given by the parameters of the points in which the quadrics (1) cut $c_1(\tau)$ and $c_2(t)$. The line sections of $\overline{S}_2(t)$ and $\overline{S}_1(\tau)$ give the points of intersection of $\overline{Q}_1$ and $\overline{Q}_2$ and $c_2(t)$ and $c_1(\tau)$ respectively. Since (1) is apolar to $\overline{Q}_1$ and $\overline{Q}_2$, the sextics $\overline{R}_1(\tau)$ and $\overline{R}_2(t)$ are conjugate to $\overline{S}_1(\tau)$ and $\overline{S}_2(t)$ respectively, that is, the point sections of $\overline{R}_1(\tau)$ are apolar to the line sections of $\overline{S}_1(\tau)$, and similarly for $\overline{R}_2(t)$ and $\overline{S}_2(t)$.

The point sections of $\overline{R}_2(t)$ may be written

$$z_3t^6 + 2z_2t^5 - 5z_0t^4 + 5z_1t^2 - 2z_3t + z_2 = 0. \tag{4}$$

A line t of the perspective cubic $E_3(t)$ cuts $\overline{S}_2(t)$ in a point t and five other points given by (7) §IV. Set

$$\begin{aligned}(tt')(qt')^5 = (t_1't_2 - t_2't_1)[&\tau_1(-t_1'^5t_2^2 - 10t_1'^4t_2't_1t_2 - 10t_1'^3t_2'^2t_1^2 - 3t_2'^5t_2^2) \\ &+ \tau_2(3t_1'^5t_1^2 - 10t_1'^3t_2'^2t_2^2 - 10t_1't_2'^4t_1t_2 - t_2'^5t_1^2)].\end{aligned} \tag{5}$$

If (4) is written as $(\beta z)(bt)^6=0$ the covariant $(\beta z)(bq)^5(bt)$ becomes

$$(tt')(z\zeta) = (t_1t_2' - t_2t_1')(-z_0t_2\tau_2 + z_1t_1\tau_1 - z_2t_1\tau_2 + z_3t_2\tau_1),$$

which evidently vanishes for $t=t'$. The points z on planes ζ define on $\overline{R}_2(t)$ a linear system of binary sextics which have a common apolar quintic. The planes ζ are then planes of the Stahl quadric.* The parametric equations of the Stahl quadric are

$$\zeta_0 = -t_2\tau_2,\qquad \zeta_1 = t_1\tau_1,\qquad \zeta_2 = -t_1\tau_2,\qquad \zeta_3 = t_2\tau_1,$$

* Conner, loc. cit., p. 33.

and

$$z_0 = t_1\tau_1, \quad z_1 = -t_2\tau_2, \quad z_2 = -t_2\tau_1, \quad z_3 = t_1\tau_2,$$

in planes and points respectively, or, in quaternary coördinates,

$$\zeta_0\zeta_1 - \zeta_2\zeta_3 = 0 \text{ and } z_0z_1 - z_2z_3 = 0.$$

Introduce the substitution

$$5y_i = \epsilon^{4i}z_0 - \epsilon^i z_1 + \epsilon^{3i}z_2 + \epsilon^{2i}z_3. \tag{6}$$

Then

$$5\sum y_i^2 = 2(z_2z_3 - z_0z_1).$$

The Stahl quadric must be invariant under a group of order 120 isomorphic with the symmetric group on five letters. An invariant subgroup of order 60 generated by S and T of (4) §II leaves the two systems of generators invariant. An operation of period two interchanges t and τ and hence the two systems of generators. The collineations of the subgroup of order sixty are

$$S^\mu: \begin{aligned} z_0' &= \epsilon^{3\mu}z_0, \\ z_1' &= \epsilon^{2\mu}z_1, \\ z_2' &= \epsilon^{\mu}z_2, \\ z_3' &= \epsilon^{4\mu}z_3; \end{aligned} \qquad S^\mu U: \begin{aligned} z_0' &= -\epsilon^{2\mu}z_1, \\ z_1' &= -\epsilon^{3\mu}z_0, \\ z_2' &= \epsilon^{4\mu}z_3, \\ z_3' &= \epsilon^{\mu}z_2; \end{aligned}$$

$$S^\mu TS^\nu: \begin{aligned} 5^{1/2}z_0' &= \epsilon^{3\nu}[-\epsilon^{3\mu}z_0 - \epsilon^{2\mu}z_1 - \epsilon^{\mu}\delta_2z_2 + \epsilon^{4\mu}\delta_1z_3], \\ 5^{1/2}z_1' &= \epsilon^{2\nu}[-\epsilon^{3\mu}z_0 - \epsilon^{2\mu}z_1 - \epsilon^{\mu}\delta_1z_2 + \epsilon^{4\mu}\delta_2z_3], \\ 5^{1/2}z_2' &= \epsilon^{\nu}[-\epsilon^{3\mu}\delta_2z_0 - \epsilon^{2\mu}\delta_1z_1 + \epsilon^{\mu}z_2 - \epsilon^{4\mu}z_3], \\ 5^{1/2}z_3' &= \epsilon^{4\nu}[\epsilon^{3\mu}\delta_1z_0 + \epsilon^{2\mu}\delta_2z_1 - \epsilon^{\mu}z_2 + \epsilon^{4\mu}z_3]; \end{aligned}$$

$$S^\mu TS^\nu U: \begin{aligned} 5^{1/2}z_0' &= -\epsilon^{2\nu}[-\epsilon^{3\mu}z_0 - \epsilon^{2\mu}z_1 - \epsilon^{\mu}\delta_1z_2 + \epsilon^{4\mu}\delta_2z_3], \\ 5^{1/2}z_1' &= -\epsilon^{3\nu}[-\epsilon^{3\mu}z_0 - \epsilon^{2\mu}z_1 - \epsilon^{\mu}\delta_2z_2 + \epsilon^{4\mu}\delta_1z_3], \\ 5^{1/2}z_2' &= \epsilon^{4\nu}[\epsilon^{3\mu}\delta_1z_0 + \epsilon^{2\mu}\delta_2z_1 - \epsilon^{\mu}z_2 + \epsilon^{4\mu}z_3], \\ 5^{1/2}z_3' &= \epsilon^{\nu}[-\epsilon^{3\mu}\delta_2z_0 - \epsilon^{2\mu}\delta_1z_1 + \epsilon^{\mu}z_2 - \epsilon^{4\mu}z_3] \end{aligned}$$

$$(\mu, \nu = 0, 1, 2, 3, 4).$$

It is more convenient to write an operation of period four which interchanges the two systems of generators. Such an operation is

$$R: \quad z_0' = z_2, \; z_1' = -z_3, \; z_2' = -z_1, \; z_3' = z_0,$$

where $R^2 = U$.

The Stahl quadric cuts the symmetroid Σ_1 in the curve

$$t_1^4\tau_1\tau_2^3 - t_1^3t_2\tau_1^4 - 3t_1^2t_2^2\tau_1^2\tau_2^2 + t_1t_2^3\tau_2^4 - t_2^4\tau_1^3\tau_2 = 0.$$

The bilinear factors of (8) §IV give the double points of the symmetroid.* By means of these factors the double points of the symmetroid may be numbered to correspond with the nodes of the rational sextic:

$$
\begin{array}{ll}
1 & (-\epsilon^3\delta_2^2, \epsilon\delta_2^2, \epsilon^4, 1), \\
2 & (-\epsilon^4\delta_2^2, \epsilon^3\delta_2^2, \epsilon^2, 1), \\
3 & (-\delta_2^2, \delta_2^2, 1, 1), \\
4 & (-\epsilon\delta_2^2, \epsilon^2\delta_2^2, \epsilon^3, 1), \\
5 & (-\epsilon^2\delta_2^2, \epsilon^4\delta_2^2, \epsilon, 1), \\
6 & (-\delta_1^2, \delta_1^2, 1, 1), \\
7 & (-\epsilon\delta_1^2, \epsilon^2\delta_1^2, \epsilon^3, 1), \\
8 & (-\epsilon^2\delta_1^2, \epsilon^4\delta_1^2, \epsilon, 1), \\
9 & (-\epsilon^3\delta_1^2, \epsilon\delta_1^2, \epsilon^4, 1), \\
0 & (-\epsilon^4\delta_1^2, \epsilon^3\delta_1^2, \epsilon^2, 1).
\end{array}
$$

If six nodes of the rational sextic lie on a conic the complementary four nodes of the symmetroid lie on a plane.† Hence the nodes of the symmetroid lie by fours on two sets of planes, one containing ten and the other fifteen planes. The following are the sets of planes:

(4790),	(1259),
(2780),	(3457),
(3689),	(1230),
(1679),	(1458),
(5680),	(2346),

and

(3670),	(1890),	(1349),
(1570),	(5789),	(1247),
(4569),	(2379),	(2450),
(2690),	(4678),	(2358),
(3480),	(1268),	(1356).

These sets of numbers also give interesting number systems. The set of ten is the arrangement of ten numbers four at a time such that each occurs four times. The set of fifteen is the arrangement of ten numbers four at a time such that each appears six times and every couple twice. These number systems as well as those which give the configuration of the nodes of the rational sextic are unaltered by the G_{120} which leaves the Stahl quadric invariant.

* Coble, American Journal of Mathematics, vol. 51, p. 501.

† Coble, *Algebraic Geometry and Theta Functions*, 1929, p. 256.

Corresponding to a plane section of Σ_1 is a curve of order six and genus three on J_1.* If the plane passes through four double points of Σ_1 the curve of order six on J_1 becomes the four lines corresponding to the four double points and the two lines which intersect these four lines. Hence corresponding to the twenty-five planes which contain four double points of the symmetroid, there are fifty additional lines on the Jacobian.

VII. The two space sextics and the cubic surface

By (3) §VI the planes of the two space cubic curves $c_1(\tau)$ and $c_2(t)$ map into the points of two rational space sextic curves:

$$R_1(\tau):\ z_0 = -2\tau^5 - 1,\ z_1 = -2\tau + \tau^6,\ z_2 = 5\tau^2, \qquad z_3 = 5\tau^4,$$

and

$$R_2(t):\ z_0 = -5t^2, \qquad z_1 = 5t^4, \qquad z_2 = t^6 - 2t, \quad z_3 = 2t^5 + 1.$$

These two sextics lie on the same cubic surface:

$$z_0^2 z_3 + z_1^2 z_2 + z_2^2 z_0 - z_3^2 z_1 = 0. \tag{1}$$

The six four-fold secants of $R_1(\tau)$ are the lines determined by the pencils of planes

$$z_2 + \lambda z_3 = 0, \tag{2}$$

$$[-\epsilon^{3\mu}\delta_2 z_0 - \epsilon^{2\mu}\delta_1 z_1 + \epsilon^{\mu} z_2 - \epsilon^{4\mu} z_3] + \lambda[\epsilon^{3\mu}\delta_1 z_0 + \epsilon^{2\mu}\delta_2 z_1 - \epsilon^{\mu} z_2 + \epsilon^{4\mu} z_3] = 0. \tag{3}$$

Moreover, $z_2+\lambda z_3=0$ cuts the sextic $R_1(\tau)$ in the points whose parameters are given by

$$\tau_1^2\tau_2^2(\tau_2^2 + \lambda\tau_1^2) = 0.$$

That is, the four-fold secants are double tangents of $R_1(\tau)$. The six four-fold secants of $R_2(t)$ are also double tangents and are the lines determined by the pencils of planes

$$z_0 + \lambda z_1 = 0, \tag{4}$$

$$[-\epsilon^{3\nu} z_0 - \epsilon^{2\nu} z_1 - \epsilon^{\nu}\delta_2 z_2 + \epsilon^{4\nu}\delta_1 z_3] + \lambda[-\epsilon^{3\nu} z_0 - \epsilon^{2\nu} z_1 - \epsilon^{\nu}\delta_1 z_2 + \epsilon^{4\nu}\delta_2 z_3] = 0. \tag{5}$$

Conner† stated an opinion that the two sets of four-fold secants for the most general case would form a double six on the cubic surface. In this special case it is evidently true. $z_2+\lambda z_3=0$ does not intersect $z_0+\lambda z_1=0$ but does intersect the other five lines in (5). One of the lines (3) intersects $z_0+\lambda z_1=0$,

* Coble, American Journal of Mathematics, vol. 46, p. 191.

† Loc. cit., p. 42.

intersects (5) if $\mu \neq \nu$, but does not intersect (5) if $\mu = \nu$. Similarly for the lines (4) and (5).

The Stahl quadric intersects the cubic surface (1) in the curve $F = 0$.

Introduce the substitution (6) §VI:

$$3(z_1^2 z_2 + z_0^2 z_3 + z_0 z_2^2 - z_1 z_3^2) = 25\sum y_i^3 .$$

The cubic surface (1) is the diagonal surface of Clebsch. The symmetroid satisfies the following relation:

$$(8/25)\Sigma_1 = 30\sum y_i^4 - 7(\sum y_i^2)^2 .$$

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THE THEORY OF MULTIPLICATIVE ARITHMETIC FUNCTIONS*

BY

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Introduction

1. The work the results of which are embodied in this paper was carried out during the period from August, 1927, to October, 1928, when the paper was sent for publication in these Transactions. In October, 1927, I contributed a note *On the inversion of multiplicative arithmetic functions* to the Journal of the Indian Mathematical Society, pointing out the fact (which I then believed to be new) that every multiplicative function of a single argument possesses an inverse, which is also a multiplicative function. In this note, certain of the ideas of the earlier part of this paper are presented in an undeveloped form; in particular, there occur the term "linear function" and the notion of "rational integral function." This note called forth, a year later, a paper entitled *An outline of a theory of arithmetic functions*, by E. T. Bell (Journal of the Indian Mathematical Society, October, 1928), wherein he pointed out that he had established the existence of the inverse function, for a wider class of functions than the multiplicative, and gave a general survey, with full references, of his own work on numerical functions. Such of the literature indicated by this extremely useful paper of Dr. Bell as was then available to me, showed that the particular types of problems, for which I was interested in finding a solution, had not been considered previously. When, two or three months after the paper had left my hands, his memoir *An arithmetical theory of certain numerical functions* (University of Washington Publications in Mathematical and Physical Sciences, vol. 1, No. 1, 1915) became available for reference, it became apparent that some of his ideas were in close relation with the part of this paper (namely, the first three sections) which deals specifically with functions of a single argument. (A precise account of the relation is given in §4 of this Introduction.)

Some of my main results were communicated early in 1928 to the International Congress of Mathematics, Bologna, but the abstract published in the Acts of the Congress is imperfect, and does not cover the whole ground of this paper.

* Presented to the Society, April 3, 1931; received by the editors in November, 1928, and revised, with new introduction, in December, 1930.

In revising the paper, some explanatory passages and further references have been inserted, but little new matter has been added to the text.

2. The arithmetical functions $f(N)$ which have the property that $f(MN) = f(M)f(N)$ when M, N are mutually prime, are well known, and are of paramount importance in arithmetic; the functions of r arguments which possess the corresponding property

$$f(M_1N_1, M_2N_2, \cdots, M_rN_r) = f(M_1, M_2, \cdots, M_r)f(N_1, N_2, \cdots, N_r),$$

when the two products $M_1M_2 \cdots M_r$, $N_1N_2 \cdots N_r$ are relatively prime, are less widely known.* The functions with this property have received several names, and are here called "multiplicative."† Though some of the processes and results of this paper could be stated for a wider class of arithmetic functions, it has been thought desirable to confine it strictly to the functions with this property, so that "function" used here without any qualification, means always "multiplicative arithmetic function."

Though multiplicative functions of a single argument are widely known and used, they have not been studied, as such, by any writer before Bell; this indeed may be inferred from the fact that there is no recognized name for the fundamental process relating to them, here called "composition." Bell has termed this process "ideal multiplication," referring, for distinction, to the ordinary product of two functions, as their "algebraic" or "absolute" product.‡ The process of convolution of arguments, which is the logical basis of composition, does not appear to be known at all, though I have seen it used in a solitary instance, for a function of two arguments by Ramanujan.§

In Section I fundamental concepts are defined, and certain elementary functions are introduced; the method of generating series is explained, and the independence of the elements of a multiplicative function is affirmed.

Section II studies the five fundamental processes of the calculus, multiplication, convolution, composition, inversion and compounding, the last of these being new.|| Composition is really a particular case, though a most

* I am not aware of any writer other than Bell who has treated these functions: see E. T. Bell, *A ray of numerical functions of r arguments*, Bulletin of the American Mathematical Society, vol. 32 (1926), p. 341.

† Bell uses "factorable," and refers (Journal of the Indian Mathematical Society, October, 1928) to the German and French equivalents "zerlegbar" and "régulière," which I have not seen. I have adopted "multiplicative" from Pólya and Szegö, *Aufgaben und Lehrsätze aus der Analysis*, vol. II, p. 126, where it is used in this sense.

‡ *An arithmetical theory of numerical functions* (loc. cit.) and *An outline* etc. (loc. cit.).

§ Collected Papers of Srinivasa Ramanujan, Cambridge, 1927, p. 180.

|| Bell was led, in 1915, in a purely symbolic manner, to the compounding operation, by working with "generators," without being aware of its arithmetical significance. He called it "ideal addition." The names "ideal multiplication" and "ideal addition" appear to have been chosen under the mistaken impression that the latter distributes the former; see Introduction (4).

important one, of convolution, which is logically prior to it; this is not however usually recognized, owing to the fact that in order to explain composition in terms of convolution (as is done here), one must work from the start with functions of several arguments. The theory of inversion in its widest form is due to Bell.*

Section III studies rational functions of a single argument, proves the result that multiplication and compounding are rational processes, and gives various applications.

Sections IV and V, though both of independent interest as treating of important special types of multiplicative functions of several arguments, are intended to be preparatory to Section VI, which investigates a general form of identical relation, which is satisfied by every multiplicative function. The special functions are also determined, for which this relation reduces to a "Busche-Ramanujan identity," namely, an identity which, for functions f of a single argument, is of the form

$$f(MN) = \sum f\left(\frac{M}{\delta}\right) f\left(\frac{N}{\delta}\right) F(\delta),$$

summed for all common divisors δ of M, N. The function $\sigma_a(N)$, representing the sum of the ath powers of the divisors of N, is the known instance of a function admitting an identity of this type. On the other hand, the ϕ-function of Euler also satisfies an identity of the same form, namely

$$\phi(MN) = \sum \delta\phi\left(\frac{M}{\delta}\right)\phi\left(\frac{N}{\delta}\right)$$

summed for common divisors δ of M, N, *provided* the values of M, N, are so restricted as not to contain any common prime factor to the same power.† All functions of a single argument which admit a "restricted Busche-Ramanujan identity" of this kind are also determined.

The seventh and last section finds the general form of a multiplicative function, which can constitute the general element of a determinant capable of evaluation by the same method as Smith's determinant. Two new forms of such determinants are added here to those already known, namely, the determinants whose general element a_{mn} is equal to (1) von Sterneck's func-

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* *On a certain inversion in the theory of numbers*, Tôhoku Mathematical Journal, vol. 17 (1920), p. 221; *Extension of Dirichlet multiplication and Dedekind inversion*, Bulletin of the American Mathematical Society, vol. 28 (1922), p. 111.

† This property of the ϕ-function was discovered by Mr. S. Sivasankaranarayana Pillai, while a research scholar of the Madras University; it suggested to the author the concept of "restricted identity."

tion $f(m, n)$, (2) any "integral quadratic function" of the product mn, e.g., the sum of the divisors of mn or the number of representations of mn as a sum of two squares.

3. It may perhaps help the reader to follow the paper with greater understanding, if I explain the exact manner in which it arose. *The paper is mainly an attempt to understand the well known identity*

$$\text{(A)} \qquad \sigma_a(mn) = \sum \sigma_a\left(\frac{m}{\delta}\right)\sigma_a\left(\frac{n}{\delta}\right)\delta^a\mu(\delta)$$

summed for common divisors δ *of* m, n, *and to answer the converse problem suggested by it, of finding the most general function admitting an identity of this form.*

It is also, to a lesser extent, a study of two other particular results:

$$\text{(B)} \qquad \sum_{n=1}^{\infty} \frac{\sigma_a(n)\sigma_b(n)}{n^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)} \qquad \text{(Ramanujan)}.$$

(C) The determinant $|a_{rs}|$ $(r, s=1, 2, \cdots, N)$, where a_{rs} is the least common multiple of r and s, has the value $\prod_{j=1}^{N}\phi(j)\prod(j)$, where ϕ is Euler's function and $\prod(j)$ is the product of the negatives of the prime factors of j (Cesàro).

At a fairly early stage in the work, I succeeded in proving that the most general multiplicative function $f(N)$ admitting an identity of the form (A) must be an "integral quadratic function." Thus, *it is not possible to answer the converse question in* (A), *without the concepts of "linear" and "rational integral" functions.* At the same time I had grasped the fact that the identity (A) could not be properly understood, unless $\sigma_a(MN)$ was treated explicitly as a multiplicative function of the two arguments M, N, and its right side, as the composite of two functions, each of two arguments; from this point of view $F(\delta)=\delta^a\mu(\delta)$ can only be described as "the function of one argument equivalent to a 'principal' function of two arguments." The more difficult question of the form of the corresponding identity for the general function $f(N)$ was solved much later, giving the concepts "cardinal function," and "conjugate function," so that it became possible to describe the specialization which occurred when f became integral-quadratic, as the specialization of a cardinal function into a principal function. Also, I had previously been led to the compounding process, in working at problems involving the l.c.m. and g.c.d. of divisors of a number, but had felt doubtful whether it should be taken seriously; *the occurrence of the "conjugate function" in this context convinced me that the compounding process should be given as fundamental a place as composition.* In the last stage the whole theory was generalized and stated

for functions of r arguments, thereby necessitating the concepts "cardinal and principal functions of a matrix-set of arguments." Thus in all (A) has been responsible for (1) acceptance and systematic treatment of multiplicative functions of more than one argument; (2) the concept of rational function; (3) the theory of cardinal and principal functions; (4) the systematic study of the compounding operation. The secondary line of thought indicated by the restricted Busche-Ramanujan identity joins on with (A); it had its genesis in the attempt to understand the property of the ϕ-function (already mentioned), which Mr. S. S. Pillai communicated to me in answer to the problem which I proposed to him for solution, namely, either to explain in a satisfactory manner why Euler's function does *not* satisfy an identity of the type (A), or to find some altered form of the identity which it *does* satisfy.

After the concept "rational function" had been fully formed, the result (B) was recognized as giving "the expression in rational form, of the product of the two integral quadratic functions σ_a and σ_b." This suggested the result that the product of two rational functions is rational and raised the question of expressing it in rational form. In regard to this question, it was easily proved that the "integral component" of the product-function was capable of immediate derivation from the integral components of the factors; but no result, more general than the theorem of III §4, was obtained in spite of prolonged effort, for the specification of the inverse component. The immediate extension of (B) to the product of two general integral quadratic functions was proved both from the general theory, and by arithmetical methods utilizing the compounding operation (Example 9 and III §6 (b)). The theory of simplex functions (II §5(d)) has also been suggested by (B).

The result (C) has necessitated the close study of functions of the g.c.d. and l.c.m., the former joining on with the theory of principal functions, while the latter gives rise to the new concept of "semiprincipal function." In Smith's original statement of his determinant-theorem, *any* multiplicative function of the g.c.d. of m, n can serve as the general element a_{mn} of his determinant,* whereas in Cesàro's extension,† it is only the *linear* function of the l.c.m. of m, n which can so serve. This difference in character between the l.c.m. and the g.c.d. is explained by the fact that the "semiprincipal" function assumes a special form suitable for the production of a "Smith function," only when its equivalent function is an "enumerative totient" (cf. Example 2, VII).

4. **The relation between the first three sections and Bell's memoir** ***An arithmetical theory of certain numerical functions.*** This memoir studies com-

* Dickson, *History of the Theory of Numbers*, vol. 1, pp. 122, 123.

† Dickson, ibid., p. 128.

position under the name "ideal multiplication." Its main object appears to be the selection of a suitable subclass of multiplicative functions $f(N)$, admitting a theorem of unique (compositional) factorization into primes, and the associated arithmetical theory, and the illustration therefrom of general principles relating to arithmetical structure. The "generator" of a multiplicative function $f(N)$ is a function $f(x, z)$ of two arguments, such that, for every prime p,

$$f(p, z) = f(p)z + f(p^2)z^2 + \cdots .$$

The "generating function" $F(x, z)$ of $f(N)$ is defined by

$$F(x, z) = 1 + f(x, z).$$

The functions $f(N)$, whose generating functions $F(x, z)$ are finite polynomials in *both* x and z, are called "primitive" functions, and also "positive" functions; if in addition $F(x, z)$ is an irreducible polynomial in x, z, $f(N)$ is called a "prime primitive." The inverse of a positive function is called a "negative" function, and the composite of a positive and a negative function is a "mixed function." Thus the mixed functions are those whose generating functions $F(x, z)$ are rational functions of (x, z), and it is shown that they admit of a unique factorization theorem, corresponding to the factorization of the numerator and denominator of $F(x, z)$ into their irreducible polynomial factors. The mixed functions are special types of the rational functions of this paper.* It is clear that, if there had been no insistence on the variable x in the generating function, the theory reached would have been identical with that of this paper; but in that case, the purpose of the memoir would not have been fulfilled, since rational functions admit of (compositional) factorization, in an unenumerably infinite number of ways (cf. I §3 and remarks on Theorem II). The whole difference of outlook turns upon the difference in procedure between defining $f(N)$ by means of a single function $F(x, z)$, "the generating function," and defining it by means of an infinity of generating series $F(p, z)$, where p stands for each prime in turn. In actual application there may not appear to be much difference between these, but theoretically there is this profound distinction, that *the latter definition affirms the independence of the elements of the general function f, while the former denies it.*

* The inversion in the nomenclature, indicative of the difference in view point, may be noted; namely, the "negative" and the "positive" functions of Bell are, respectively, rational integral functions and their inverses. From the view point of arithmetical structure, the functions $f(N)$ whose $F(x, z)$ is a finite polynomial are fundamental, and are therefore called positive functions by Bell. On the other hand, from the view point of the multiplicative property, the functions $f(N)$ which possess it unconditionally are the fundamental ones, and should be termed "linear integral," even though their generating series to any base is the expansion of a fraction of the form $1/(1-ar)$.

Thus, the concept of rational function is present in the memoir, but does not reach full clarity, as it is not freed from the admixture of elements extraneous to the nature of the multiplicative function.

The memoir also defines the compounding operation, under the name "ideal addition," in a purely symbolical manner, from the addition of generators, the author indeed appearing to believe that it does not possess any simple arithmetical significance.* The ideal difference is also defined in the same manner, without the concept of the conjugate function. It is stated without proof on page 32, 5.34, and repeated on page 35, 6.26, that ideal addition distributes ideal multiplication; that this is erroneous is shown by Theorem XIV of this paper, which proves that compounding is not distributive but only quasidistributive, in respect to composition.†

Section I. Preliminary

1. **Definition.** An arithmetic function $f(M_1, M_2, \cdots, M_r)$ is one which is defined for all (non-zero) positive values of its arguments. *The arithmetic function* $f(M_1, M_2, \cdots, M_r)$ *is multiplicative, if*

$$f(M_1N_1, M_2N_2, \cdots, M_rN_r) = f(M_1, M_2, \cdots, M_r)f(N_1, N_2, \cdots, N_r),$$

whenever the products $M_1M_2 \cdots M_r$, $N_1N_2 \cdots N_r$ *are relatively prime.* With the convention that unity is both prime to and a factor of every number, we see that $f(M_1, 1, \cdots, 1) = 1$; or

Theorem I. *Every multiplicative function takes the value unity, for simultaneous unit values of the arguments.*

2. **The elements of a multiplicative function.** Let the arguments $M_1, M_2, \cdots, M_r$ of $f(M_1, M_2, \cdots, M_r)$ be resolved into their prime factors, so that

$$M_i = p_1^{a_{1i}} p_2^{a_{2i}} p_3^{a_{3i}} \quad (i = 1, 2, \cdots, r;\ p_1 < p_2 < p_3 < \cdots),$$

the a_{ik} being zero, or positive integers. Then it follows from the multiplicative property, that

$$f(M_1, M_2, \cdots, M_r) = \prod_\lambda f(p_\lambda^{a_{\lambda 1}}, p_\lambda^{a_{\lambda 2}}, \cdots, p_\lambda^{a_{\lambda r}}) \qquad (\lambda = 1, 2, \cdots).$$

We shall mean by the element of the multiplicative function f to the base p_λ the aggregate of values $f(p_\lambda^{t_1}, p_\lambda^{t_2}, \cdots, p_\lambda^{t_r})$, for all zero and positive integral

* "It is clear that in no quantitative sense is an ideal sum or difference a sum or difference; the ideal sum expresses a relation between functions, which is only remotely connected with their arguments"; p. 32, 5.33.

† Introduction dated November 2, 1930.

values of $t_1, t_2, \cdots, t_r$. Thus the values of f for arbitrary arguments can be found by multiplication, if the elements of f to every prime base are known. In other words, *the elements of a multiplicative function completely determine the function.*

3. **The independence of the elements.** The definition of the multiplicative function f implies no necessary relation between its elements. In fact, not only can the system of values which constitute an element be chosen in an arbitrary manner, but also the ∞^1 elements can be chosen in entire independence of one another. It therefore appears that the general multiplicative function falls apart into a series of elements, which while unrelated to one another, generate the function by multiplication.

This independence of the elements is the most characteristic property of the multiplicative function, and our method of generating series is based directly upon it. It introduces, however, an element of indefiniteness into the multiplicative function, allowing it, as it were, an infinity of degrees of freedom. To illustrate, let f_1, f_2 be two function-types, both possessing a multiplicative property P (i.e., a property which is implied by a property of the elements of the function only). Then each of the unenumerably infinite number of functions f, whose elements with respect to certain prime bases are the corresponding elements of f_1, and with respect to the remaining bases, the elements of f_2, also possess the property P. Hence, when f_1, f_2 are solutions of the problem of determining the functions with the property P, each of the unenumerably infinite number of *crosses* f between f_1, f_2 is also a solution. Several examples of this will occur in this paper.

4. **Generating series.** With each element to a base p_λ of the multiplicative function $f(M_1, M_2, \cdots, M_r)$ we associate the power series

$$f_{(p_\lambda)}(x_1, x_2, \cdots, x_r) = \sum f(p_\lambda^{m_1}, p_\lambda^{m_2}, \cdots, p_\lambda^{m_r}) x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r}.$$

We call this power series the generating series of f to the base p_λ. From Theorem I, it follows that *the constant term in every generating series is unity.*

If the generating series is the formal expansion of a function $f_{(p_\lambda)}(x_1, x_2, \cdots, x_r)$ we call this latter the generating function of f to the base p_λ.

We shall generally use the generating series as the representative of the corresponding element of the function. The variables $x_1, x_2, \cdots, x_r$ are not to be regarded as quantities, but purely as algebraic symbols, which are in formal correspondence with the arguments, and exhibit the element in an ordered shape; thus no questions of convergence of the generating series can arise.

5. **Linear functions.** The multiplicative function $f(M_1, \cdots, M_r)$ will be called a *linear* function, if the equation

$$(1) \quad f(M_1N_1, M_2N_2, \cdots, M_rN_r) = f(M_1, M_2, \cdots, M_r)f(N_1, N_2, \cdots, N_r)$$

holds not merely when $M_1M_2 \cdots M_r$ is prime to $N_1N_2 \cdots N_r$, but for all values of M_i, $N_i (i=1, 2, \cdots, r)$.

It is clear that a linear function of r arguments can be expressed as the product of r linear functions, each of one of the arguments. Thus, from (1), we have

$$f(M_1, M_2, \cdots, M_r) = f(M_1, 1, \cdots, 1)f(1, M_2, 1, \cdots, 1) \cdots f(1, 1, \cdots, 1, M_r);$$

and $f(M_1, 1, \cdots, 1)$ is a linear function of the argument M_1.

The generating function to the base p of a linear function $f(M)$ is

$$\begin{aligned} f_{(p)}(x) &= 1 + f(p)x + f(p^2)x^2 + \cdots \\ &= 1 + f(p)x + \{f(p)\}^2x^2 + \cdots \\ &= \frac{1}{1 - ax}, \quad \text{if } f(p) = a. \end{aligned}$$

Similarly the generating function to the base p of a linear function of r arguments is of the form $(1-a_1x_1)^{-1}(1-a_2x_2)^{-1} \cdots (1-a_rx_r)^{-1}$.

6. **The elementary functions.** These are of fundamental importance, and may be divided into four groups:

(1) the E-functions,

(2) the λ-functions,

(3) the I-functions,

(4) the power units π_k, ϵ_k.

We shall first define these functions for a single argument.

$E_k(M) = k^\nu$; $\nu =$ the number of different prime factors of M. For k a positive integer, $E_k(M)$ may also be defined as the number of decompositions of M into k factors every two of which are mutually prime.

$\lambda_k(M)$ is defined to be the *linear* function, which takes the value k when M is a prime; hence $\lambda_k(M) = k^\nu$; $\nu =$ total number of prime factors of M, multiple prime factors being counted as often as their multiplicity.

The general linear function of M is obviously a cross between an infinity of functions λ_k.

$I_k(M)$ is the linear function M^k.

The power units π_k, ϵ_k are defined (for positive integral values of k) as follows:

$$\pi_k(M) = \begin{cases} 0 \text{ if } M \text{ is divisible by a } k\text{th power,} \\ 1 \text{ otherwise;} \end{cases}$$

$$\epsilon_k(M) = \begin{cases} 1 \text{ if } M \text{ is a } k\text{th power,} \\ 0, \text{ otherwise.} \end{cases}$$

The corresponding elementary functions of r arguments are defined by

$$E_k(M_1, M_2, \cdots, M_r) = E_k(M_1M_2 \cdots M_r);$$
$$= k$$

if $M_1, M_2, \cdots, M_r$ are all powers of a prime p;

$$\lambda_k(M_1, M_2, \cdots, M_r) = \lambda_k(M_1M_2 \cdots M_r) = \lambda_k(M_1)\lambda_k(M_2) \cdots \lambda_k(M_r);$$
$$I_k(M_1, M_2, \cdots, M_r) = I_k(M_1M_2 \cdots M_r) = I_k(M_1)I_k(M_2) \cdots I_k(M_r);$$
$$\pi_k(M_1, M_2, \cdots, M_r) = \pi_k(M_1) \cdots \pi_k(M_r);$$
$$\epsilon_k(M_1, \cdots, M_r) = \epsilon_k(M_1)\epsilon_k(M_2) \cdots \epsilon_k(M_r).$$

Among the E-functions, those which occur most frequently are E_0, E_1, E_{-1}, E_2. We shall write simply E for E_1. The function E_0 vanishes for all values of its arguments, excepting simultaneous unit values, for which it takes the value 1 (Theorem I). The function $E=E_1$ takes the same value 1 for all values of its arguments. Among the E-functions, E_0 and E are the only ones which are linear.

Among the λ-functions, the most important is λ_{-1}, which we shall write simply as λ. It will be noticed that

$$\pi_1 = \lambda_0 = E_0; \ \lambda_1 = \epsilon_1 = I_0 = E.$$

The function I_1 (which is equal to the product of its arguments) will be written simply I.

The generating series to the base p of these elementary functions are easily obtained; they are

$$E_{k(p)}(x) = 1 + kx + kx^2 + \cdots = \frac{1 + (k-1)x}{1-x};$$

$$E_{k(p)}(x_1, x_2, \cdots, x_r) = \frac{k}{(1-x_1)(1-x_2)\cdots(1-x_r)} - (k-1);$$

$$E_{0(p)}(x_1, x_2, \cdots, x_r) = 1;$$

$$E_{(p)}(x_1, x_2, \cdots, x_r) = \frac{1}{(1-x_1)(1-x_2)\cdots(1-x_r)};$$

$$\lambda_{k(p)}(x_1, x_2, \cdots, x_r) = \prod_i \frac{1}{1-kx_i};$$

$$\lambda_{(p)}(x_1, x_2, \cdots, x_r) = \prod_i \frac{1}{1 + x_i};$$

$$I_{k(p)}(x_1, x_2, \cdots, x_r) = \prod_i \frac{1}{1 - p^k x_i};$$

$$\epsilon_{k(p)}(x_1, x_2, \cdots, x_r) = \prod_i \frac{1}{1 - x_i^k};$$

$$\pi_{k(p)}(x_1, x_2, \cdots, x_r) = \prod_i \frac{1 - x_i^k}{1 - x_i}.$$

SECTION II. THE PROCESSES OF THE CALCULUS

The calculus of multiplicative functions consists of certain processes, which while applicable to arithmetic functions in general, have the characteristic property of yielding only multiplicative functions, when performed on multiplicative functions. These processes are the following:

(1) Multiplication (including division) of functions;

(2) Convolution of arguments;

(3) Composition of functions (including inversion);

(4) Compounding of functions.

We shall consider these in turn.

1. **Multiplication of functions.** If $f(M_1, M_2, \cdots, M_r), \phi(M_1, M_2, \cdots, M_r)$ be multiplicative functions of the same r arguments, their product

$$f(M_1, M_2, \cdots, M_r)\phi(M_1, \cdots, M_r),$$

which we shall denote by $(f\times\phi)(M_1, M_2, \cdots, M_r)$, is also a multiplicative function of $M_1, M_2, \cdots, M_r$.

If the generating series of f, ϕ to the base p are

$$f_{(p)}(x_1, x_2, \cdots, x_r) = \sum a_{m_1, m_2, \cdots, m_r} x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r},$$

$$\phi_{(p)}(x_1, x_2, \cdots, x_r) = \sum b_{m_1, m_2, \cdots, m_r} x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r},$$

then the generating series of their product $f\times\phi$ is given by

$$(f\times\phi)_{(p)}(x_1, x_2, \cdots, x_r) = \sum a_{m_1, m_2, \cdots, m_r} b_{m_1, m_2, \cdots, m_r} x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r}.$$

Also, if ϕ does not vanish for finite values of its arguments, we can define its reciprocal function $1/\phi$ by

$$\frac{1}{\phi}(M_1, M_2, \cdots, M_r) = \frac{1}{\phi(M_1, \cdots, M_r)};$$

then

$$\left(\frac{1}{\phi}\right)_{(p)}(x_1, x_2, \cdots, x_r) = \sum \frac{1}{b_{m_1, m_2, \cdots, m_r}} \cdot x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r}.$$

By taking the product of f and the reciprocal function of ϕ, we have the case of division.

We have defined the symbol $f\times\phi$ only for the case in which f and ϕ are functions of the same r arguments. We can extend the scope of the symbol by a device which will be generally useful. If f is a function of $M_1, M_2, \cdots, M_r$ while ϕ is a function of $M_1, M_2, \cdots, M_i$ only $(i<r)$, we regard ϕ as a function of $M_1, M_2, \cdots, M_r$ which vanishes unless $M_{i+1}=M_{i+2}= \cdots =M_r=1$, and is then equal to $\phi(M_1, M_2, \cdots, M_i)$; that is, we consider $\phi(M_1, M_2, \cdots, M_i)$ to be the function $\phi(M_1, M_2, \cdots, M_i)E_0(M_{i+1}, \cdots, M_r)$. The justification for considering these to be identical is that the generating series are the same for the two functions (since the generating series of E_0 to any base is simply 1). With this convention, then, $f\times\phi$ is seen to be a function of $M_1, M_2, \cdots, M_r$, which vanishes unless $M_{i+1}=M_{i+2}= \cdots =M_r=1$, that is, for all practical purposes, a function of the common arguments only, of f and ϕ. In following this convention, it should not be forgotten that we are making a distinction between the *functional* multiplication in $f\times\phi$, and the multiplication of *quantities*; thus, if $f(M)$, $\phi(N)$ are functions of different arguments, $(f\times\phi)(M, N)$ should be the same as $E_0(M, N)$ according to our interpretation, but $f(M)\times\phi(N)$ is not $E_0(M, N)$, the multiplication in the former case being functional, and in the latter, quantitative. The distinction between the two senses of multiplication will be generally evident from the fact that the arguments will appear explicitly in algebraic multiplication, while they will usually be dropped in functional multiplication.

If we take for ϕ the elementary function $E_k(M_1, M_2, \cdots, M_r)$ (Section I), we have

$$(f \times E_k)_{(p)}(x_1, x_2, \cdots, x_r) = 1 + \sum k a_{m_1, m_2, \cdots, m_r} x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r},$$

where the summation on the right is for all zero and positive integral values of $m_1, m_2, \cdots, m_r$, with the exception of simultaneous zero values. In particular,

$$f \times E = f,$$

$$f \times E_0 = E_0.$$

Thus the functions E, E_0 behave like unity and zero, with respect to functional multiplication.

The linear functions have a special property with respect to multiplication, namely:

THEOREM II. *The product of linear functions is also a linear function.*

For, if

$$K_{(p)}(x_1, x_2, \cdots, x_r) = \prod_i (1 - \alpha_i x_i)^{-1},$$

$$L_{(p)}(x_1, x_2, \cdots, x_r) = \prod_i (1 - \beta_i x_i)^{-1},$$

then it is clear that

$$(K \times L)_{(p)}(x_1, x_2, \cdots, x_r) = \prod_i (1 - \alpha_i\beta_i x_i)^{-1}.$$

In particular, the multiplicative powers of L, namely $L\times L$, $L\times L\times L$, etc., are all linear functions. Conversely, we may expect the existence of *linear* root-functions $L_1, L_2, \cdots$, such that

$$L_1 \times L_1 = L;\ L_2 \times L_2 \times L_2 = L; \cdots.$$

Supposing, for simplicity, that L is a linear function of a single argument, we can take

$$L_{(p)}(x) = 1 + \alpha x + \alpha^2 x^2 + \cdots.$$

Then, since L_1 is to be linear, we must have

$$L_{1(p)}(x) = \begin{cases} \text{either } 1 + \alpha^{1/2}x + (\alpha^{1/2})^2x^2 + \cdots = (1 - \alpha^{1/2}x)^{-1}, \\ \text{or } \quad 1 - \alpha^{1/2}x + (-\alpha^{1/2})^2x^2 + \cdots = (1 + \alpha^{1/2}x)^{-1}. \end{cases}$$

Thus each element of the root-function L_1 has two determinations, and therefore by choosing one of the two admissible elements for each prime base (which we are at liberty to do, from the independence of the elements of a multiplicative function), we can construct the root-function L_1 in an unenumerably infinite number of ways. This shows that the concept "root-function" is an unprofitable one. We shall however see later on, that we can construct a unique function called the "root-composite" from the elements of the root-function.

2. **Convolution of arguments.** Let $f(M_1, M_2, \cdots, M_r)$ be a multiplicative function of $r(>1)$ arguments. The process of convolving M_1, M_2 in f consists in forming the function

$$\phi(M, M_3, \cdots, M_r) = \sum_{M_1M_2=M} f(M_1, M_2, M_3, \cdots, M_r).$$

From the multiplicative property of f it is easy to show that ϕ is a multiplicative function of its $r-1$ arguments.

If the generating series of f to the base p be

$$f_{(p)}(x_1, x_2, \cdots, x_r) = \sum a_{m_1, m_2, \cdots, m_r} x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r},$$

then

$$\phi_{(p)}(x, x_3, \cdots, x_r) = \sum \Big(\sum_{m_1+m_2=m} a_{m_1, m_2, \cdots, m_r} \Big) x^m x_3^{m_3} \cdots x_r^{m_r}.$$

Thus the generating series $\phi_{(p)}(x, x_3, \cdots, x_r)$ is obtained by putting $x_1 = x_2 = x$ in $f_{(p)}(x_1, x_2, \cdots, x_r)$. Therefore,

THEOREM III. *Convolution of arguments is equivalent to identifying the corresponding variables in the generating series.*

The process of convolution is evidently applicable to any number s of arguments, and is equivalent to the identification of the s corresponding variables in the generating series. Hence in convolving several arguments, it is immaterial whether we convolve them all together, or in stages.

The functions obtained by convolving sets of arguments in all possible ways in $f(M_1, M_2, \cdots, M_r)$ may be called *convolutes* of f. Among these convolutes, there is one which is a function of a single argument only, namely, that in which all the arguments have been convolved together. This particular convolute may be termed "*the convolute of f.*"

We shall also use the term convolute in another special sense. If $\psi(M)$ is a multiplicative function of a single argument, we can define a multiplicative function $\psi'(M_1, M_2, \cdots, M_r)$ of r arguments by

$$\psi' \qquad (M, M, \cdots, M) = \psi(M),$$

$\psi'(M_1, M_2, \cdots, M_r) = 0$, if two of the arguments are unequal.

The function ψ' is termed the principal function of r arguments equivalent to $\psi(M)$. The convolute $\psi_r(M)$ of $\psi'(M_1, M_2, \cdots, M_r)$ will be referred to as *the rth convolute of* $\psi(M)$.

THEOREM IV. *The rth convolute $\psi_r(M)$ of $\psi(M)$ is the function defined by*

$$\psi_r(M) = \begin{cases} \psi(M^{1/r}), \textit{ if } M \textit{ is an rth power,} \\ 0, \textit{ otherwise.} \end{cases}$$

For, by the definition,

$$\psi_r(M) = \sum \psi'(M_1, M_2, \cdots, M_r),$$

where ψ' is the principal function of r arguments equivalent to $\psi(M)$, and the summation on the right is for all values of $M_1, M_2, \cdots, M_r$ such that $M_1 M_2 \cdots M_r = M$. Since ψ' vanishes whenever two of its arguments are unequal, it follows that the right side vanishes when M is not an rth power, and is equal to $\psi'(N, N, \cdots, N) = \psi(N)$, if $M = N^r$. As an illustration, the

elementary function ϵ_k is the kth convolute of E. It is obvious that the generating series of the rth convolute of $\psi(M)$ is obtained by substituting x^r for x, in the corresponding generating series of ψ.

To avoid misunderstanding, we shall use the phrase rth convolute of ψ with the second meaning of convolute, *only* for functions ψ of a single argument. The great utility of this concept will be seen from the applications in II §5 (c).

3. **Composition of functions.** Let $f_1(M_1, M_2, \cdots, M_r), f_2(N_1, N_2, \cdots, N_r)$ be two multiplicative functions of r arguments. Then $f_1(M_1, M_2, \cdots, M_r) \times f_2(N_1, N_2, \cdots, N_r)$ is a multiplicative function of $2r$ arguments. The result of convolving the r pairs of corresponding arguments (M_i, N_i) in this product is therefore a multiplicative function f of r arguments. We call f *the composite of* f_1, f_2, and denote it by the functional symbol $(f_1 \cdot f_2)$.

From the arithmetical significance of convolution, it follows that the composite can be defined arithmetically by

$$f(M_1, M_2, \cdots, M_r) = \sum f_1(\delta_1, \delta_2, \cdots, \delta_r) f_2\left(\frac{M_1}{\delta_1}, \frac{M_2}{\delta_2}, \cdots, \frac{M_r}{\delta_r}\right),$$

summed for all divisors δ_i of M_i $(i = 1, 2, \cdots, r)$.

There is a simple relation between the generating series of f_1, f_2 and their composite f. If $f_{1(p)}(x_1, x_2, \cdots, x_r), f_{2(p)}(y_1, y_2, \cdots, y_r)$ are the generating series of f_1, f_2 to the base p, the generating series of $f_1(M_1, M_2, \cdots, M_r) \times f_2(N_1, N_2, \cdots, N_r)$ to the same base is evidently $f_{1(p)}(x_1, x_2, \cdots, x_r) \times f_{2(p)}(y_1, y_2, \cdots, y_r)$. Since convolution of arguments is equivalent to identification of corresponding variables in the generating series (Theorem III), it follows that the generating series of the composite is given by

$$f_{(p)}(z_1, z_2, \cdots, z_r) = f_{1(p)}(z_1, z_2, \cdots, z_r) \times f_{2(p)}(z_1, z_2, \cdots, z_r).$$

Thus,

THEOREM V. *Composition of functions of r arguments is equivalent to the multiplication of their generating series to each base, after identifying corresponding variables.*

Hence composition of several functions of r arguments is associative and commutative.

The process of composition of f_1, f_2 implies a correspondence between their arguments. A convenient way of expressing this fact would be to say that $f(M_1, M_2, \cdots, M_r)$ is the composite of the functions f_1, f_2 of the *same* r arguments $M_1, M_2, \cdots, M_r$. With this understanding we can interpret the composite of functions of different arguments in the same way as was done in the

case of multiplication. Thus the composite of $\psi(M_1)$ and $f(M_1, M_2, \cdots, M_r)$ is to be interpreted to mean the composite of $\psi(M_1)E_0(M_2, \cdots, M_r)$ and $f(M_1, M_2, \cdots, M_r)$, which are functions of the same r arguments. In the same way, the composite of functions $f_1(M_1, M_2, \cdots, M_r), f_2(N_1, N_2, \cdots, N_r)$ without common arguments is to be interpreted as the composite of

$$f_1(M_1, M_2, \cdots, M_r)E_0(N_1, N_2, \cdots, N_r)$$

and $E_0(M_1, M_2, \cdots, M_r)f_2(N_1, N_2, \cdots, N_r)$, and is easily seen to reduce to the product $f_1(M_1, M_2, \cdots, M_r)\times f_2(N_1, N_2, \cdots, N_r)$. Theorem V is easily seen to hold when composition is extended in this manner.*

The composites of $f(M_1, M_2, \cdots, M_r)$ with itself will be indicated by exponents. Thus $f\cdot f=f^2$; $f\cdot f\cdot f=f^3$; etc. To avoid confusion, products like $f\times f$ will *not* be denoted by exponents, but will be written out in full.

The composite of $f(M_1, M_2, \cdots, M_r)$ and $E(M_1, M_2, \cdots, M_r)$ is given by

$$(f\cdot E)(M_1, M_2, \cdots, M_r) = \sum f(\delta_1, \delta_2, \cdots, \delta_r)$$

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* The object in introducing the extended sense of composition as well as functional multiplication in addition to quantitative multiplication may be explained here. Composition in the unextended sense has two defects; namely

(1) The symbol $f_1\cdot f_2$ is not completely defined by f_1 and f_2, but requires in addition a knowledge of the correspondence between the arguments in f_1, f_2.

(2) Since composition has been defined only for functions of the same *number* of arguments, we cannot speak of an expression like $\sum\psi(\delta)f(M/\delta, N)$ summed for divisors δ of M, as a composite.

Both these defects are removed if we adopt the following conventions:

(1) Corresponding arguments in the composition $f_1\cdot f_2$ shall be thought of as identical, so that composition becomes a process which is defined (in the first instance) only for two functions of the *same r* arguments.

(2) If f_1, f_2 are not functions of the same arguments, they are converted into functions F_1, F_2 of the same arguments, by multiplication by E_0 (as explained in the text), and $f_1\cdot f_2$ is defined to be $F_1\cdot F_2$.

If we accept these conventions, it follows that between any two functional symbols f_1, f_2 there exists a relation, which is expressed by one of the three statements "f_1 and f_2 are functions of the same arguments, or f_1 and f_2 have some or no common arguments." Does this relation between f_1, f_2, which, by our accepting the conventions, has become implicit in the functional symbols themselves, affect the product $f_1\times f_2$? The answer is: $f_1\times f_2$ must be interpreted as the functional product, and this is the only possible answer if we wish Theorem VII (namely $\theta\times(f_1\cdot f_2)=(\theta\times f_1)\cdot(\theta\times f_2)$, when θ is linear) to be unconditionally true. For example $\theta(M)\times(f_1(M)\cdot f_2(N))$ is equal to $\theta(M)\times(f_1(M)f_2(N))$ (by the use of the extended definition of composition), which cannot be equal to $(\theta(M)\times f_1(M))\cdot(\theta(M)\times f(N))$, unless we interpret $\times$ as functional multiplication, in which case we should have

$$\theta(M)\times(f_1(M)f_2(N)) = \theta(M)f_1(M)E_0(N),$$
$$\theta(M)\times f(N) = E_0(M, N),$$
$$(\theta(M)\times f_1(M))\cdot(\theta(M)\times f(N)) = \{\theta(M)f_1(M)E_0(N)\}\cdot E_0(M, N) = \theta(M)f_1(M)E_0(N).$$

Even without this example, functional multiplication and extended composition may be justified by using generating series, as the reader may easily verify for himself.

The reader may note the elegant application of extended composition, made in the proof of Theorem X below.

summed for all divisors δ_i of M_i $(i=1, 2, \cdots, r)$. We call $f\cdot E$ the ***numerical integral*** or simply *the integral* of f. The function E^2 is the integral of E, and is given by

$$E^2(M) = \text{number of divisors of } M,$$
$$E^2(M_1, M_2, \cdots, M_r) = E^2(M_1)E^2(M_2)\cdots E^2(M_r).$$

The function $E^k(M)$, where k is a positive integer, is equal, from the definition of composition, to the number of ways of expressing M as a product of r factors, attention being paid to the order of the factors.

The composite of f_1, f_2 has been defined as the result of convolving the r pairs M_i, N_i in the product $f_1(M_1, M_2, \cdots, M_r)\times f_2(N_1, N_2, \cdots, N_r)$. Now, we have seen that the result of a series of convolutions is independent of the order in which they are performed. Hence the result of performing any convolution on the composite of f_1, f_2 is the same as that of first performing this convolution on f_1 and f_2, and then taking their composite. Hence

THEOREM VI. *Convolution is distributed by composition; or, explicitly, $\Omega(f_1\cdot f_2)=(\Omega f_1)\cdot(\Omega f_2)$, where f_1, f_2 are functions of r arguments, and Ω represents any convolution of the arguments.*

In particular the rth convolute of the composite of $f(M)$, $\phi(M)$ is the composite of their rth convolutes.

We have already observed that, given a linear function L, we can always find a linear function L', such that $L'\times L'\times\cdots$ (to r factors) $=L$, and that each element of L' has r determinations. Therefore we can find r linear functions $L_i (i=1, 2, \cdots, r)$ no two of which have any elements in common, such that

$$L_i\times L_i\times\cdots (r \text{ factors}) = L, \qquad i = 1, 2, \cdots, r.$$

It is clear that the elements of L_i to any base p are the r determinations of the elements of L' to the same base. Hence, even though the functions L_i can be chosen in an unenumerably infinite number of ways, yet their continued composite $L_1\cdot L_2\cdot L_3\cdots L_r$ is the same function f in every case; we call f *the rth root-composite* of the linear function L. An important property of the linear function is

THEOREM VI(a). *The rth root-composite of a linear function is also its rth convolute.*

For, if

$$L_{(p)}(x) = \frac{1}{1-ax},$$

then

$$f_{(p)}(x) = \frac{1}{(1 - a^{1/r}x)(1 - \omega a^{1/r}x) \cdots (1 - \omega^{r-1}a^{1/r}x)} = \frac{1}{1 - ax^r},$$

ω being a primitive rth root of unity. But $1/(1-ax^r)$ is clearly the generating series, to the base p, of the rth convolute of L; hence the theorem.

The function E_0 plays a special rôle in composition. For, since its generating series to any base is 1, we must have

$$f \cdot E_0 = f; \quad E_0{}^k = E_0.$$

It was observed that E_0 behaves like zero with respect to functional multiplication. Accepting the analogy, the fact that $f \cdot E_0 = f$ suggests that composition must be considered analogous to addition. Like addition, composition is associative and commutative, but unlike addition it does not distribute (functional) multiplication. It has however a restricted distributive property given by

THEOREM VII. *The compositional operation distributes multiplication, whenever the multiplier is a linear function.*

For, let $\theta(M_1, M_2, \cdots, M_r)$ be a linear function multiplying the composite $(f\cdot\phi)(M_1, M_2, \cdots, M_r)$. We have

$$\{(\theta \times f)\cdot(\theta \times \phi)\}(M_1, M_2, \cdots, M_r)$$
$$= \sum \theta(\delta_1,\delta_2, \cdots,\delta_r)f(\delta_1,\delta_2, \cdots,\delta_r)\theta\left(\frac{M_1}{\delta_1}, \frac{M_2}{\delta_2}, \cdots, \frac{M_r}{\delta_r}\right)\phi\left(\frac{M_1}{\delta_1}, \cdots, \frac{M_r}{\delta_r}\right)$$
$$= \sum \theta(M_1, M_2, \cdots, M_r)f(\delta_1, \cdots, \delta_r)\phi\left(\frac{M_1}{\delta_1}, \cdots, \frac{M_r}{\delta_r}\right), \text{ since } \theta \text{ is linear,}$$
$$= \{\theta \times (f\cdot\phi)\}(M_1, M_2, \cdots, M_r).$$

It is easy to see that the same result will hold even if some of the arguments in θ are different from those in the composite.

4. **Inversion.** From each generating series $f_{(p)}(x_1, x_2, \cdots, x_r)$ of a multiplicative function $f(M_1, M_2, \cdots, M_r)$, we can determine uniquely a second power series $f_{(p)}^{-1}(x_1, x_2, \cdots, x_r)$ such that, on term-by-term multiplication, we get

$$f_{(p)}(x_1, x_2, \cdots, x_r) \times f_{(p)}^{-1}(x_1, x_2, \cdots, x_r) = 1$$

(apart from any questions of convergence which are irrelevant). The series $f_{(p)}^{-1}$ have all the constant term unity, and are therefore the generating series of a determinate multiplicative function f^{-1}, which we call *the inverse of f*. The relation between the functions f, f^{-1} is clearly symmetrical, each being the inverse of the other.

Since every generating series of $E_0(M_1, M_2, \cdots, M_r)$ is 1, the composite of f and its inverse function f^{-1} is E_0. This property may be taken as the definition of the inverse function. It also follows from this that in a compositional equation we can transpose any term from one side to the other, provided we replace it by its inverse. Thus from

$$f_1 \cdot f_2 = \phi_1 \cdot \phi_2,$$

we have

$$\phi_1^{-1} \cdot f_1 \cdot f_2 = \phi_1^{-1} \cdot \phi_1 \cdot \phi_2 = E_0 \cdot \phi_2 = \phi_2.$$

The only function identical with its inverse is the function E_0. Since $E(M)$ has the generating function $1/(1-x)$ to any base, the generating function of $E^{-1}(M)$ to any base is $1-x$; hence $E^{-1}(M)$ is the same as Mertens' function $\mu(M)$ which is equal to zero if M has a squared factor, and to $(-1)^\nu$ if M is the product of ν different primes. The function $E^{-1}(M_1, M_2, \cdots, M_r)$ is equal to $E^{-1}(M_1)E^{-1}(M_2) \cdots E^{-1}(M_r)$.

THEOREM VIII. *Inversion is permutable with convolution.*

For the equation $f_{(p)}(x_1, x_2, \cdots, x_r) \times f_{(p)}^{-1}(x_1, x_2, \cdots, x_r) = 1$ continues to be true if we put $x_1 = x_2 = x$. Hence if Ω represents any convolution or series of convolutions of the arguments, $\Omega(f^{-1})$ must be identical with $(\Omega f)^{-1}$. It also follows from this theorem, that the inverse of the rth convolute of $f(M)$ is the rth convolute of its inverse.

THEOREM IX. *Inversion is distributed by composition; that is, the inverse of the composite of any number of functions is identical with the composite of their inverses.*

For, the generating series to base p of the inverse of the composite of f, f' is

$$\{f_{(p)}(x_1, x_2, \cdots, x_r) \times f'_{(p)}(x_1, x_2, \cdots, x_r)\}^{-1},$$

while the generating series to the same base of the composite of their inverses is

$$\{f_{(p)}(x_1, x_2, \cdots, x_r)^{-1} \times f'_{(p)}(x_1, x_2, \cdots, x_r)^{-1}\}.$$

Since these series are the same, the theorem follows.

COROLLARY. *It follows from this theorem that $(f^{-1})^r = (f^r)^{-1}$. We may accordingly denote each of these by f^{-r}, so that*

$$f^{-r} = (f^{-1})^r = (f^r)^{-1},$$

$$f^{r+s} = f^r \cdot f^s, \textit{ for positive or negative integers } r, s;$$

$$f^0 = E_0.$$

It was already observed that the product of two functions

$$f_1(M_1, M_2, \cdots, M_r), f_2(N_1, N_2, \cdots, N_r)$$

without common arguments is the composite of the two functions $f_1(M_1, M_2, \cdots, M_r)E_0(N_1, N_2, \cdots, N_r)$ and $E_0(M_1, M_2, \cdots, M_r)f_2(N_1, N_2, \cdots, N_r)$. Now, it is easily seen that the inverse of $f_1(M_1, M_2, \cdots, M_r)E_0(N_1, \cdots, N_r)$ is $f_1^{-1}(M_1, M_2, \cdots, M_r)E_0(N_1, \cdots, N_r)$. Hence, from Theorem IX we have

THEOREM X. *The inverse of the product of functions without common arguments is the product of their inverses.*

An important property of the linear functions with respect to inversion is given by

THEOREM XI. *If θ is a linear, and f an arbitrary, function, the inverse of $\theta \times f$ is $\theta \times f^{-1}$.*

For

$$\begin{aligned}(\theta \times f)\cdot(\theta \times f^{-1}) &= \theta \times (f \cdot f^{-1}) \quad \text{(Theorem VII)}\\ &= \theta \times E_0\\ &= E_0.\end{aligned}$$

COROLLARY. *Since $\theta = \theta \times E$, the inverse of θ is $\theta \times E^{-1}$.*

5. **Some applications to functions of a single argument.** (a) **Examples.** The following functions of the argument N are of frequent occurrence:

(1) $\tau(N)$ = number of divisors of $N = E^2(N)$; $\sigma(N)$ = sum of the divisors of $N = (I \cdot E)(N)$; $\sigma_k(N)$ = sum of the kth powers of the divisors of $N = (I_k \cdot E)(N)$; $\mu(N) = E^{-1}(N)$.

(2) $\phi(N)$ = Euler's function representing the number of numbers less than and prime to $N = (I \cdot E^{-1})(N)$; $\phi_k(N)$ = Jordan's function* representing the number of sets of k numbers not greater than N, whose greatest common divisor is prime to N, $= (I_k \cdot E^{-1})(N)$; $\tau_k(N) = \tau(N^k) = (E_k \cdot E)(N)$.

(3) $I_{r,s}(N) = N^r$, if N is an sth power, and $=0$, otherwise; in other words $I_{r,s}$ is the sth convolute of I_r.

To prove the statements in (2), we note that the number of numbers not greater than N which have the greatest common divisor δ with it, is $\phi(N/\delta)$. Hence

$$\sum_{\delta/N} \phi\left(\frac{N}{\delta}\right) = N,$$

* See Dickson's *History of the Theory of Numbers*, vol. 1, p. 147. This book will be hereafter quoted by the author's name.

or $\phi \cdot E = I$, so that $\phi = I \cdot E^{-1}$. An exactly similar type of grouping shows that $\phi_k = I_k \cdot E^{-1}$, and also proves Cesàro's general result*

$$\sum_{j=1}^{N} f(d_j) = (f\cdot\phi)(N),$$

where d_j denotes the greatest common divisor of j and N.

To prove that $\tau_k = E_k \cdot E$, we note that $E_k(N)$ is equal to the number of divisors of N^k which do not divide δ^k, where $\delta(<N)$ is any divisor of N. For, if $N = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_\nu^{\alpha_\nu}$, the divisors in question are the terms of the product

$$(p_1^{(\alpha_1-1)k+1} + p_1^{(\alpha_1-1)k+2} + \cdots + p^{\alpha_1 k})(p_2^{(\alpha_2-1)k+1} + \cdots + p_2^{\alpha_2 k}),$$

and are hence k^ν in number. Now to each divisor D of N^k we can make correspond a unique divisor δ of N, such that D divides δ^k, but does not divide the kth power of any other divisor of δ. Hence $(E_k \cdot E)(N)$ enumerates all the divisors of N^k, or $\tau_k = E_k \cdot E$.

As applications of Theorem VII, we have

Example 1. If L is a linear function, $L^k = L \times E^k$, where k is a positive or negative integer. For

$$\begin{aligned} L^k = L\cdot L\cdot L \cdots &= (L \times E)\cdot(L \times E)\cdots \\ &= L \times (E\cdot E \cdots) \text{ (Theorem VII) } = L \times E^k. \end{aligned}$$

For the case of a negative integer, $-k$,

$$\begin{aligned} (L \times E^k)\cdot(L \times E^{-k}) &= L \times (E^k \cdot E^{-k}) \text{ (Theorem VII)} \\ &= L \times E_0 = E_0. \end{aligned}$$

Hence

$$L^{-k} = (L^k)^{-1} = (L \times E^k)^{-1} = L \times E^{-k} \text{ (Theorem XI)}.$$

As an example,

$$\sigma\cdot\sigma = I\cdot E\cdot I\cdot E = (I \times E^2)\cdot E^2,$$

or, explicitly,

$$\sum_{\delta/N} \sigma(\delta)\sigma\left(\frac{N}{\delta}\right) = \sum_{\delta/N} \delta\tau(\delta)\tau\left(\frac{N}{\delta}\right) \text{ (Liouville)}\dagger.$$

Example 2.

$$\sum_{\delta/N} \delta^{\mu-\nu}\sigma_{\nu+\lambda}(\delta)\sigma_{\mu+\rho}\left(\frac{N}{\delta}\right) = \sum_{\delta/N} \delta^{\mu-\nu}\sigma_{\nu+\rho}(\delta)\sigma_{\mu+\lambda}\left(\frac{N}{\delta}\right) \text{ (Liouville)}\ddagger.$$

* Dickson, p. 127, Note 57.

† Dickson, p. 285, 25.

‡ Dickson, p. 286, 30.

For the left side

$$= \{I_{\mu-\nu} \times (I_{\nu+\lambda}\cdot E)\}\cdot I_{\mu+\rho}\cdot E$$
$$= I_{\mu+\lambda}\cdot I_{\mu-\nu}\cdot I_{\mu+\rho}\cdot E \text{ (Theorem VII)},$$

and the symmetry of this in λ, ρ proves the result.

(b) **Relations between the elementary functions.** The following relations between the elementary functions are fundamental:

(1) $$\epsilon_k\cdot\pi_k = E;$$

(2) $$E_{1-k} = E\cdot\lambda_k^{-1};$$

in particular,

(3) $$E_2\cdot\lambda = E;$$

(4) $$E\cdot\lambda = \epsilon_2;$$

(5) $$E_2\cdot\epsilon_2 = E^2;$$

(6) $$E_2\times\lambda = E_2^{-1}.$$

These relations are immediately evident from the generating series and are also capable of direct arithmetical proof. For instance the truth of (1) follows from the fact that a number can be expressed in only one way as the product of two factors, the first of which is an exact kth power, and the second is not divisible by any kth power. To prove (3), we observe that $E\cdot\lambda^{-1}(N)$ enumerates all the divisors of N which possess no squared factor, and is therefore equal to $E_2(N)$. The relation (4) expresses the fact that the second convolute ϵ_2 of E is also its second root-composite (Theorem VI (a)); for, the linear functions E, λ have no common elements, and each of them yields E on multiplication with itself, and therefore the second root-composite of E is $E\cdot\lambda$. To prove (5), we note that the number of divisor-pairs δ, N/δ of N with a given greatest common divisor δ_1 is zero if δ_1^2 does not divide N, and $E_2(N/\delta_1^2)$ otherwise. Hence $(E_2\cdot\epsilon_2)(N)$ is equal to the total number of divisors of N. The relation (6) is a consequence of (3); for, by (3),

$$E_2 = E\cdot\lambda^{-1},$$

therefore

$$E_2\times\lambda = \lambda\times(E\cdot\lambda^{-1}) = \lambda\cdot(\lambda\times\lambda^{-1}) \text{ (Theorem VII)}$$
$$= \lambda\cdot E^{-1} = E_2^{-1} \text{ (Theorem IX)}.$$

Example 3. $E_2\cdot E^2 = E^2\times E^2$. For $(E^2\times E^2)(N)$ is the number of divisor-pairs of N. Let δ_1, δ_2 be two divisors of N, l their least common multiple, and t their greatest common divisor. Group the divisor-pairs in such a way that, for each group, l/t is a fixed divisor δ of N. It is clear that, for each group, t

is an arbitrary divisor of N/δ, while δ_1, δ_2 must be of the form tl_1, tl_2, where l_1, l_2 are relatively prime, and $l_1l_2=\delta$. Thus the number of divisor-pairs in the group specified by δ is $\tau(N/\delta)E_2(\delta)$. Thus $E_2\cdot E^2=E^2\times E^2$.

As an alternative proof, the number of divisor-pairs of N, which have δ for their greatest common divisor, and N for their least common multiple, is $E_2(N/\delta)$. Hence $(E_2\cdot E)(N)$ is the number of factor-pairs which have N for their least common multiple. Hence $E_2\cdot E\cdot E=(E_2\cdot E^2)(N)$ is equal to the total number of factor-pairs of N, that is, to $(E^2\times E^2)(N)$.

The following results bear on this and previous theorems:

(1) $$(\tau\times\tau)\cdot(E_2\times\lambda) = (E^2\times E^2)\cdot E_2^{-1} = E^2, \text{ or}$$

$$\sum\{\tau(\delta)\}^2\lambda\left(\frac{N}{\delta}\right)E_2\left(\frac{N}{\delta}\right) = \tau(N) \quad \text{(Liouville)*;}$$

(2) $$(\lambda\times\tau_2)\cdot\sigma_\mu = (\lambda\times E_2)\cdot\lambda\cdot E\cdot I_\mu \quad \text{(Theorem VII)}$$
$$= E_2^{-1}\cdot\lambda\cdot E\cdot I_\mu = \lambda\cdot\lambda\cdot I_\mu = (\lambda\times E^2)\cdot I_\mu; \text{ or}$$

$$\sum\lambda(\delta)\tau(\delta^2)\sigma_\mu\left(\frac{N}{\delta}\right) = \sum\lambda(\delta)\tau(\delta)\left(\frac{N}{\delta}\right)^\mu \quad \text{(Liouville)†;}$$

(3) $$\tau_2\cdot\lambda = E_2\cdot E\cdot\lambda = E^2, \text{ or}$$
$$\sum\tau(\delta^2)\lambda(N/\delta) = \tau(N) \quad \text{(Liouville).‡}$$

Example 4. $\sum\tau(\delta_1)\tau(\delta_2)$ summed for all pairs of numbers δ_1, δ_2 with the least common multiple N is equal to $\{\tau(N)\}^3$.

To prove this, let us introduce the term "block-factor of N" to denote a factor δ in which each prime factor has the same exponent as in N, that is, a factor δ which is relatively prime to N/δ. Two factors δ_1, δ_2 which have N for their least common multiple, can be evidently put into the form

$$\delta_1 = PQr,$$
$$\delta_2 = PRq,$$

where P, Q, R are block-factors such that $PQR=N$, and q, r are respectively factors of Q, R, having no common block-factors with them. Hence

$$\tau(\delta_1)\tau(\delta_2) = \tau(P)\tau(Q)\tau(r)\tau(p)\tau(R)\tau(q)$$
$$= \tau(N)\tau(P)\tau(q)\tau(r).$$

Thus $\sum\tau(\delta_1)\tau(\delta_2)$ is of the form $\tau(N)\sum\tau(\delta)$, where each δ occurs as many times as N/δ can be expressed as the product of relatively prime factors. That is, $\sum\tau(\delta_1)\tau(\delta_2)=\tau\times(\tau\cdot E_2)=E^2\times E^2\times E^2$ (Example 3).

* Dickson, p. 285, 27.
† Dickson, p. 286, 29.
‡ Dickson, p. 285, 27.

If we make N vary over the factors of a given number M, and sum each side of this result, we obtain Liouville's theorem:*

$$\left\{ \sum_{\delta/M} \tau(\delta) \right\}^2 = \sum_{\delta/M} \{\tau(\delta)\}^3, \text{ or } E^3 \times E^3 = (E^2 \times E^2 \times E^2) \cdot E.$$

(c) **Applications of Theorem VI.** The following results due to Liouville and Gegenbauer serve as applications of the theorem that the rth convolute of a composite is the composite of the rth convolutes:

$$(1) \qquad \sum \phi(D) \tau\left(\frac{N}{D^2}\right) = \sum D E_2\left(\frac{N}{D^2}\right);\dagger$$

$$(2) \qquad \sum E_2(D) \tau\left(\frac{N}{D^2}\right) = \sum \tau(D^2) E_2\left(\frac{N}{D^2}\right);$$

$$(3) \qquad \sum \lambda(D) \tau\left(\frac{N}{D^2}\right) = \sum E_2\left(\frac{N}{e^4}\right);$$

where the summations extend over the square divisors D^2 and the biquadrate divisors e^4 of N.

Writing conv f and conv$_r f$ for the second and the rth convolutes of f, we have

$$(1) \qquad \begin{aligned} \text{conv } \phi \cdot \tau &= \text{conv } I \cdot \text{conv } E^{-1} \cdot \tau \\ &= \text{conv } I \cdot \epsilon_2^{-1} \cdot E^2 \\ &= \text{conv } I \cdot E_2, \end{aligned}$$

which proves (1);

$$(2) \qquad \begin{aligned} \text{conv } E_2 \cdot \tau &= \text{conv } E_2 \cdot \epsilon_2 \cdot E_2 \\ &= \text{conv } E_2 \cdot \text{conv } E \cdot E_2 \\ &= \text{conv } (E_2 \cdot E) \cdot E_2 \\ &= \text{conv } \tau_2 \cdot E_2, \text{ which proves (2);} \end{aligned}$$

$$(3) \qquad \begin{aligned} \text{conv } \lambda \cdot E^2 &= \text{conv } \lambda \cdot \epsilon_2 \cdot E_2 \\ &= \text{conv } (E \cdot \lambda) \cdot E_2 = \text{conv } \epsilon_2 \cdot E_2 = \epsilon_4 \cdot E_2, \text{ which proves (3).} \end{aligned}$$

In what follows, d_t denotes a divisor of N such that N/d_t is a tth power, d denotes an arbitrary divisor of N, $\rho_{k,t}(N) = \sum d_t^k$, and the summation is for integers m, n such that $N = mn^p$.

$$(4) \qquad \sum \sigma_k(m) \rho_{0,2}(n) = \sum \rho_{0,2t}(d) \rho_{k,t}\left(\frac{N}{d}\right). \ddagger$$

* Dickson, p. 286, 28.

† These results are taken from Dickson, p. 285, 27.

‡ Results 4–12 are taken from Dickson, p. 298, 72, and p. 299, 73.

For the right side is $E\cdot\epsilon_{2t}\cdot I_k\cdot\epsilon_t$, while the left side is $I_k\cdot E\cdot\mathrm{conv}_t\,(E\cdot\epsilon_2) = I_k\cdot E\cdot\epsilon_t\cdot\epsilon_{2t}$.

(5)
$$\sum_{m,n}\rho_{\nu,t}(m)\phi_{tk}(n) = N^k\rho_{\nu-k,t}(N).$$

For $I_\nu\cdot\epsilon_t\cdot\mathrm{conv}_t\,(I_{tk}\cdot E^{-1}) = I_\nu\cdot\epsilon_t\cdot\epsilon_t^{-1}\cdot\mathrm{conv}_t\,I_{tk} = I_\nu\cdot(I_k\times\epsilon_t) = I_k\times(I_{\nu-k}\cdot\epsilon_t) = I_k\times\rho_{\nu-k,t}$.

(6)
$$\sum\sigma_{\nu-k}(m)\tau(n)m^k = \sum\rho_{k,t}(d)\rho_{\nu,t}(N/d).$$

For $(I_k\times\sigma_{\nu-k})\cdot\mathrm{conv}_t\,(E^2) = I_\nu\cdot I_k\cdot\epsilon_t\cdot\epsilon_t = \rho_{\nu,t}\cdot\rho_{k,t}$.

(7)
$$\sum\rho_{k,t}(m)\lambda(n) = \rho_{k,2t}(N).$$

For
$$I_k\cdot\epsilon_t\cdot\mathrm{conv}_t\,(\lambda) = I_k\cdot\mathrm{conv}_t\,(E\cdot\lambda) = I_k\cdot\mathrm{conv}_t\,\epsilon_2 = I_k\cdot\epsilon_{2t}.$$

In the following, h is put for $(N/d_2)^{1/2}$.

(8)
$$\sum\mu^2(d_4) = \sum\lambda(h).$$

For
$$\epsilon_4 = \mathrm{conv}\,\epsilon_2 = \mathrm{conv}\,(E\cdot\lambda) = \epsilon_2\cdot\mathrm{conv}\,\lambda.$$

Hence $\epsilon_4\cdot\lambda^{-1} = \lambda^{-1}\cdot\epsilon_2\cdot\mathrm{conv}\,\lambda = E\cdot\mathrm{conv}\,\lambda$.

(9)
$$\sum E_2(h)\mu^2(d_2) = \sum\mu^2(h).$$

For
$$\lambda^{-1}\cdot\mathrm{conv}\,E_2 = \lambda^{-1}\cdot\mathrm{conv}\,(E\cdot\lambda^{-1}) = \lambda^{-1}\cdot\epsilon_2\cdot\mathrm{conv}\,\lambda^{-1} = E\cdot\mathrm{conv}\,\lambda^{-1}.$$

(10)
$$\sum\tau(h^2)\mu^2(d_2) = \sum E_2(h).$$

To prove this, note that the kth convolute of $F(N)=f(N^k)$ is $f\times\epsilon_k$. Hence
$$\mathrm{conv}\,(E_2\cdot E) = \mathrm{conv}\,\tau_2 = E^2\times\epsilon_2.$$

Therefore
$$(E^2\times\epsilon_2)\cdot\lambda^{-1} = \mathrm{conv}\,(E_2\cdot E)\cdot\lambda^{-1} = \mathrm{conv}\,E_2\cdot\epsilon_2\cdot\lambda^{-1} = E\cdot\mathrm{conv}\,E_2.$$

(11)
$$\sum\tau(d_2)\mu(h) = E_2(N).$$

For $E^2\cdot\mathrm{conv}\,E^{-1} = E^2\cdot\epsilon_2^{-1} = E_2$.

(12)
$$\sum\mu^2(d)\phi_k\left(\frac{N}{d}\right) = \sum d_2{}^k\mu(h).$$

For
$$\lambda^{-1}\cdot\phi_k = \lambda^{-1}\cdot E^{-1}\cdot I_k = \mathrm{conv}\,E^{-1}\cdot I_k.$$

(d) **Zeta-series and simplex functions.** When the necessary conditions of convergence are satisfied, we obtain by Dirichlet multiplication

$$\sum_{N=1}^{\infty} \frac{(f\cdot F)(N)}{N^s} = \left\{\sum \frac{f(N)}{N^s}\right\}\left\{\frac{F(N)}{N^s}\right\},$$

$$\sum_{N=1}^{\infty} \frac{f^{-1}(N)}{N^s} = \left\{\sum \frac{f(N)}{N^s}\right\}^{-1}.$$

Hence

(1) $$\sum E^k(N)/N^s = \{\zeta(s)\}^k,$$

(2) $$\sum \epsilon_k(N)/N^s = \zeta(ks),$$

(3) $$\sum I_k(N)/N^s = \zeta(s-k).$$

More generally, if $I_{r,t}$ is the tth convolute of I_r, we have

$$\sum I_{r,t}(N)/N^s = \zeta(ts-r); \quad \sum I_{r,t}^{-1}(N)/N^s = \frac{1}{\zeta(ts-r)}\cdot$$

We shall refer to the functions $I_{r,t}$, $I_{r,t}^{-1}$ (where r is unrestricted, and t is a positive integer) as *the elementary simplex functions*. A composite of α functions of the type $I_{r,t}$ and β functions of the type $I_{r,t}^{-1}$ will be called *a simplex function of structure* (α, β).

THEOREM. *The composite of simplex functions, and the inverse of a simplex function, are simplex. Also, every convolute of a simplex function f is a simplex function with the same structure as f.*

If f is a simplex function, $\sum f(N)/N^s$, supposed convergent, can be evaluated as a product of quantities of the form $\zeta(ts+k)^{\pm 1}$, where k is arbitrary and t is a positive integer; and conversely.

The first part follows from the definition. To prove the second part of the theorem, we observe that the kth convolute of a simplex function must be the composite of the kth convolutes of elementary simplex functions (Theorem VI). Since the pth convolute of the qth convolute of f is its pqth convolute, we have

$$\mathrm{conv}_k\, I_{r,t} = I_{r,kt}; \quad \mathrm{conv}_k\, I_{r,t}^{-1} = I_{r,kt}^{-1}.$$

Thus any convolute of a simplex function $f(N)$ is a composite of elementary simplex functions of the same type as the components of f, and so is a simplex function of the same structure as f.

The generating series of a simplex function f to any base must clearly be of the form

$$f_{(p)}(x) = \frac{(1 - p^{r_1}x^{s_1})(1 - p^{r_2}x^{s_2}) \cdots}{(1 - p^{\rho_1}x^{\sigma_1})(1 - p^{\rho_2}x^{\sigma_2}) \cdots}.$$

Since $I_0 = E$, it follows that, among the elementary functions, the I-functions, as well as the functions E and $E^{\pm k}$, are simplex. The power units ϵ_k, being convolutes of E, are simplex, while the power units π_k are simplex because $\pi_k \cdot \epsilon_k = E$. Also, since $\lambda = \epsilon_2 \cdot E^{-1}$, it follows that λ is simplex; and lastly, E_2 is simplex, since $E_2 = E \cdot \lambda^{-1}$. It is easy to verify that no others from among the elementary functions are simplex.

The values of the zeta-series determined by π_k, E_2, λ are

$$\sum \pi_k(N)/N^s = \zeta(s)/\zeta(ks), \tag{1}$$

$$\sum \lambda(N)/N^s = \zeta(2s)/\zeta(s), \tag{2}$$

$$\sum E_2(N)/N^s = (\zeta(s))^2/\zeta(2s). \tag{3}$$

The product of simplex functions is not in general simplex. Ramanujan's result that

$$\sum \frac{\sigma_a(N)\sigma_b(N)}{N^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)} \,*$$

shows that the product of the simplex functions σ_a, σ_b is also simplex. It does not appear to be easy to obtain any general theorem to cover such exceptional cases. The following results in this direction seem to be worthy of notice:

(1) The product of two elementary simplex functions is either an elementary simplex function, or a simplex function of structure (1, 1).

For

(a) $I_{r,t} \times I_{r',t'} = I_{\rho,\tau}$, where τ is the least common multiple of t, t', and $\rho = (r\tau/t + r'\tau/t')$.

(b) $I_{r,t} \times I^{-1}_{r',t'} = I^{-1}_{kr+r',t'}$ or E_0, according as $t' = kt$, or is not a multiple of t.

(c) $I^{-1}_{r,t} \times I^{-1}_{r',t'} = I_{2(r+r'),2t} \cdot I^{-1}_{r+r',t}$ or E_0 according as t is or is not equal to t'.

(2) More generally, the product of two simplex functions of structure (1, 1) is also a simplex function of the same structure.

It follows from this, that $f \times \lambda$ is a simplex function when f is simplex. For, λ being linear, multiplication by it can be distributed to the simplex components of f (Theorem VII); and since λ is a simplex function of structure (1, 1), its product with an elementary simplex function must be simplex, by the present result.

(3) If f is any simplex function of structure (2, 0) or (1,1), $f \times \epsilon_2$ is simplex, and therefore $f(N^2)$ is simplex.

* Collected Papers, p. 135, = Messenger of Mathematics, 1916, p. 83.

For example $\sigma_a(N^2)$ and $\phi_k(N^2)$ are simplex functions of N.

(e) **Crosses between elementary functions.** Some of the most important functions of arithmetic are crosses between elementary functions. We mention a few such below.

(1) The function $T(N)=(-1)^{N-1}$ is easily verified to be a multiplicative function of N. For the base 2 it has the same element as E_{-1}, and for the remaining bases it has the same elements as E. We may conveniently represent it by the notation

$$T(N) = \left\{ \begin{matrix} 2 & E_{-1} \\ * & E \end{matrix} \right\} (N).$$

Hence, if ϕ is Euler's function,

$$(\phi\cdot T)(N) = \left\{ \begin{matrix} 2 & \phi\cdot E_{-1} \\ * & \phi\cdot E \end{matrix} \right\} (N).$$

Now $\phi\cdot E=I$ and $\phi\cdot E_{-1}=I\cdot E^{-1}\cdot E_{-1}=I\cdot\lambda_2^{-1}$; but $\lambda_2^{-1}(2^m)=I^{-1}(2^m)$, and so $(\phi\cdot E_{-1})(2^m)=E_0(2^m)$. Hence, we have the result of Liouville*

$$\sum(-1)^{d-1}\phi\left(\frac{N}{d}\right) = 0 \text{ or } N, \text{ according as } N \text{ is even or odd.}$$

(2) Let $R(N)$ denote the number of representations of N as the sum of two squares, all representations being counted twice, with the exception of those of the form 0^2+M^2, M^2+M^2, which are counted only once. It is known that $R(N)$ is equal to the excess of the number of divisors of N, of the form $4k+1$, over the number of those of the form $4k-1$. Thus $R(N)=(E\cdot\mathfrak{E})(N)$, where $\mathfrak{E}(N)$ is the cross between E, λ, E_0, given by

$$\mathfrak{E} = \left\{ \begin{matrix} 2 & E_0 \\ 4k+1 & E \\ 4k-1 & \lambda \end{matrix} \right\}.$$

6. **Compounding of functions of a single argument.** We confine ourselves in this subsection to functions of a single argument; the compounding of functions of several arguments is treated in Section V.

By a block-factor δ of M we shall mean a factor δ which is relatively prime to M/δ. If δ is a block-factor, the complementary factor M/δ is also a block-factor. We have to consider 1 and M also as complementary block-factors of M. If $f(M)$, $F(M)$ are given multiplicative functions, the sum

$$\sum f(\delta)F(M/\delta)$$

* Dickson, p. 121, 29.

extended to all the block-factors δ of M, is easily seen to be a multiplicative function Φ of M. We call Φ the compound of f and F, and denote it by the symbol $f \oplus F$.

If the generating series of f, F are

$$f_{(p)}(x) = 1 + a_1x + a_2x^2 + \cdots,$$

$$F_{(p)}(x) = 1 + b_1x + b_2x^2 + \cdots,$$

it follows immediately that the generating series of their compound is given by

$$(f \oplus F)_{(p)}(x) = 1 + (a_1 + b_1)x + (a_2 + b_2)x^2 + \cdots.$$

The constant term here is not 2, but 1, as must be the case with every multiplicative function. Similarly the generating series of the compound of r functions is obtained by adding the corresponding generating series of the functions, and replacing the constant term r by 1, in the sum. If we say that two power series in the same variables are *equivalent* (symbol $\sim$), when they differ only in their constant terms, we have

THEOREM XII. *The generating series of a compound is equivalent to the sum of the generating series, to the same base, of the functions compounded.*

Hence compounding is associative and commutative.

THEOREM XIII. *The compounding operation distributes multiplication.*

For, let the functions f_1, f_2, f have the generating series

$$f_{1(p)}(x) = 1 + a_1x + a_2x^2 + \cdots,$$

$$f_{2(p)}(x) = 1 + b_1x + b_2x^2 + \cdots,$$

$$f_{(p)}(x) = 1 + c_1x + c_2x^2 + \cdots.$$

Then

$$\{f \times (f_1 \oplus f_2)\}_{(p)}(x) = 1 + c_1(a_1 + b_1)x + c_2(a_2 + b_2)x^2 + \cdots,$$

while

$$\{(f \times f_1) \oplus (f \times f_2)\}_{(p)}(x) = 1 + (c_1a_1 + c_1b_1)x + (c_2a_2 + c_2b_2)x^2 + \cdots,$$

which establishes the distributive property.

THEOREM XIV. *Compounding is quasidistributive with respect to composition.*

For, consider the composite of f with the compound of $f_1, f_2, \cdots, f_r$. We have

$$\{f\cdot(f_1 \oplus f_2 \oplus \cdots \oplus f_r)\}_{(p)}(x)$$
$$= f_{(p)}(x) \times \{f_{1(p)}(x) + f_{2(p)}(x) + \cdots + f_{r(p)}(x) + 1 - r\} \quad \{\text{Theorem XII}\}$$
$$= \sum_i f_{(p)}(x) f_{i(p)}(x) + (1 - r) f_{(p)}(x)$$
$$= \sum_i (f\cdot f_i)_{(p)}(x) + (E_{1-r} \times f)_{(p)}(x) - r$$
$$\sim \sum_i (f\cdot f_i)_{(p)}(x) + (E_{1-r} \times f)_{(p)}(x).$$

Hence

$$f\cdot (f_1 \oplus f_2 \oplus \cdots \oplus f_r) = \sum_i (f\cdot f_i) \oplus (E_{1-r} \times f).$$

This formula is fundamental. The term "quasidistributive" in the enunciation refers to the occurrence of the additional term $\oplus\ (E_{1-r}\times f)$, which marks the failure of the distributive property.

The compound of two *linear* functions is capable of simple expression as a composite.

THEOREM XV. *The compound of two linear functions L_1, L_2 is the composite of L_1, L_2 and the inverse of the second convolute (or root-composite) of $L_{12}=(L_1\times L_2)$.*

More generally, the compound of the rth convolute of L_1 and the sth convolute of L_2, is their continued composite with the $(r+s)$th convolute of $\overset{-1}{L_{12}}$.

For let

$$L_{1(p)}(x) = \frac{1}{1-\alpha x}, \quad L_{2(p)}(x) = \frac{1}{1-\beta x},$$

so that

$$(\text{conv}_r\, L_{12})_{(p)}(x) = \frac{1}{1-\alpha\beta x^r}.$$

Then

$$(L_1 \oplus L_2)_{(p)}(x) = \frac{1}{1-\alpha x} + \frac{1}{1-\beta x} - 1 = \frac{1-\alpha\beta x^2}{(1-\alpha x)(1-\beta x)}$$
$$= (L_1\cdot L_2\cdot \text{conv}\ \overset{-1}{L_{12}})_{(p)}(x);$$

$$(\text{conv}_r\, L_1 \oplus \text{conv}_s\, L_2)_{(p)}(x) = \frac{1}{1-\alpha x^r} + \frac{1}{1-\beta x^s} - 1 = \frac{1-\alpha\beta x^{r+s}}{(1-\alpha x^r)(1-\beta x^s)}$$
$$= (\text{conv}_s\, L_1\cdot \text{conv}_s\, L_2\cdot \text{conv}_{r+s}\, \overset{-1}{L_{12}})_{(p)}(x).$$

As a corollary from this theorem it follows that the compound of *simplex* linear functions is also simplex. For instance, if

$$S_{a,b} = I_a \oplus I_b = I_a \cdot I_b \cdot \overset{-1}{I}_{a+b,2},$$
$$S_a' = I_a \oplus \lambda = I_a \cdot \lambda \cdot \operatorname{conv} (I_a \times \lambda)^{-1} = I_a \cdot \lambda \cdot I_{a,2} \cdot \overset{-1}{I}_{2a,4},$$

then $S_{a,b}$, S_a' are simplex functions, and

$$\sum_N \frac{S_{a,b}(N)}{N^t} = \frac{\zeta(t-a)\zeta(t-b)}{\zeta(2t-a-b)},$$
$$\sum_N \frac{S_a'(N)}{N^t} = \frac{\zeta(2t)\zeta(t-a)\zeta(2t-a)}{\zeta(t)\zeta(4t-2a)}.$$

We note also that E_2 is a particular case of $S_{a,b}$, namely, $S_{0,0}$.

The *conjugate* of $f(M)$ is the function conj f defined by

$$f \oplus \operatorname{conj} f = E_0.$$

If the generating series of f to any base is $1+\sum_{m=1} a_m x^m$, the generating series of conj f to the same base is evidently $1-\sum_{m=1} a_m x^m$. Hence an alternative definition of conj f is

$$\operatorname{conj} f = f \times E_{-1}.$$

The conjugate function has evidently the properties

$$\operatorname{conj} \operatorname{conj} f = f; \quad \operatorname{conj} (f \times \phi) = \operatorname{conj} f \times \phi = f \times \operatorname{conj} \phi;$$

conjugate of a compound = compound of the conjugates.

The E-functions play an important part in relation to the compounding operation. In fact, their algebra under compounding and functional multiplication is isomorphic with the algebra of ordinary numbers under addition and multiplication, so that any identity between ordinary numbers can be translated into a relation between E-functions. Thus

$$E_r \oplus E_s = E_{r+s}; \quad E_r \times E_s = E_{rs}; \quad \operatorname{conj} E_r = E_{-r};$$
$$f \oplus f \oplus \cdots (\text{to } r \text{ terms}) = f \times E_r.$$

We give below several results involving the E-functions.

Example 5. $f \cdot (E_r \times \phi) = f \cdot (\phi \oplus \phi \oplus \cdots) = \{(f \cdot \phi) \times E_r\} \oplus (E_{1-r} \times f)$ (Theorem XIV). As particular cases, we have

(1) $\operatorname{conj} (f \cdot \phi) = \{f \cdot \operatorname{conj} \phi\} \oplus (E_{-2} \times f) \qquad \{r = -1\};$

(2) $f \cdot E_r = \{(f \cdot E) \times E_r\} \oplus E_{1-r} \times f \qquad \{\phi = E\};$

(3) $f \cdot (E_r \times f^{-1}) = E_{1-r} \times f \qquad \{\phi = f^{-1}\};$

(4) $f \cdot \operatorname{conj} f^{-1} = E_2 \times f \qquad \{\phi = f^{-1}; r = -1\}.$

Example 6.

$$(E_r \times f)\cdot(E_s \times f^{-1}) = \{f \oplus f \oplus \cdots\}\cdot\{f^{-1} \oplus f^{-1} \oplus \cdots\}$$
$$\doteq \{E_{r(1-s)} \times f\} \oplus \{E_{s(1-r)} \times f^{-1}\}.$$

Putting $r=s=-1$, we have $\operatorname{conj} f\cdot \operatorname{conj} f^{-1} = E_{-2}\times(f\oplus f^{-1})$.

Example 7.

(1) $$E^r\cdot E_s = (E^{r+1}\times E_s)\oplus(E^r\times E_{1-s}).$$

In particular, putting $r=1$, $E^2\times E_s = (E_s\cdot E)\oplus E_{s-1}$.

(2) $$E_r^{-1}\times E_{r-1} = \lambda_{1-r}\times E_r.$$

For, since $E_r\cdot\lambda_{1-r}=E$ (II §5 (b)),

$$\lambda_{1-r}\times E_r = (E_r^{-1}\cdot E)\times E_r$$
$$= (E_r^{-1}\cdot E_r)\oplus(E_{r-1}\times E_r^{-1}) \text{ (Example 5) } = E_r^{-1}\times E_{r-1}.$$

In particular, $E_2^{-1}=E_2\times\lambda$, as has been already proved.

Example 8.

(1) $$E_r\cdot E_s = \tau_{rs}\oplus E_{(1-r)(s-1)};$$

(2) $$\tau_{2r}\cdot E = \tau\times\tau_r.$$

For

$$E_r\cdot E_s = \{(E_r\cdot E)\times E_s\}\oplus(E_{1-s}\times E_r)$$
$$= (E^2\times E_r\times E_s)\oplus(E_{1-r}\times E_s)\oplus(E_{1-s}\times E_r)$$
$$= (E^2\times E_{rs})\oplus E_{r+s-2rs}$$
$$= (E_{rs}\cdot E)\oplus E_{rs-1}\oplus E_{r+s-2rs} \text{ (Example 5,(2))}$$
$$= \tau_{rs}\oplus E_{(1-r)(s-1)};$$

and

$$\tau\times\tau_r = E^2\times(E\cdot E_r) = E^2\times\{(E^2\times E_r)\oplus E_{1-r}\} \text{ (Example 5)}$$
$$= (E^2\times E^2\times E_r)\oplus(E^2\times E_{1-r})$$
$$= \{(E^2\cdot E_2)\times E_r\}\oplus(E^2\times E_{1-r}) \text{ (Example 3)}$$
$$= E^2\cdot(E_2\oplus E_2\oplus\cdots \text{ to } r \text{ terms}) \text{ (Theorem XIV)}$$
$$= E^2\cdot E_{2r} = \tau_{2r}\cdot E.^*$$

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* The results in Examples 5–8 are proved here only for the case in which r, s are positive integers; they may be easily proved for arbitrary values of r, s, by considering the generating series. It is also possible to deduce their truth for arbitrary values of r, s, from their truth when r, s are integers, by purely logical considerations.

Example 9.

(a) $\sum f_1(d_1)f_2(d_2)\cdots f_k(d_k)$ summed for all sets of divisors $d_1, d_2, \cdots, d_k$ of N, such that every two are relatively prime, is equal to $\{(f_1\oplus f_2\oplus\cdots\oplus f_k)\cdot E\}(N)$.

(b) $\sum f_1(d_1)f_2(d_2)\cdots f_k(d_k)$ summed for all sets of divisors $d_1,d_2,\cdots, d_k$ of N, with the least common multiple N, is equal to $[\{(f_1\cdot E)\times(f_2\cdot E)\times\cdots\times(f_k\cdot E)\}\cdot E^{-1}](N)$.

The first part is obvious. To prove (b), we observe that if the required sum be $F(N)$, then $(F\cdot E)(N)$ is equal to $\sum f_1(d_1)f_2(d_2)\cdots f_k(d_k)$ summed for *all* sets of divisors $d_1, d_2, \cdots, d_k$ of N; that is, to $\{(f_1\cdot E)\times(f_2\cdot E)\cdots\times(f_k\cdot E)\}(N)$.

Putting $k=2$, we can deduce an important result from these two theorems. Let d_1, d_2 be divisors having the least common multiple N. If their greatest common divisor is δ, we can write $d_1=\delta t_1$; $d_2=\delta t_2$, where t_1, t_2 are relatively prime, and $t_1t_2=N/\delta$. Hence, if L_1, L_2 are linear functions.

$$\begin{aligned}\sum L_1(d_1)L_2(d_2) &= \sum L_1(\delta)L_2(\delta)L_1(t_1)L_2(t_2)\\ &= \sum L_{12}(\delta)L_1(t_1)L_2(t_2), \text{ where } L_{12} = L_1\times L_2,\\ &= \{L_{12}\cdot(L_1\oplus L_2)\}(N) = \{L_{12}\cdot L_1\cdot L_2\cdot \operatorname{conv} \overset{-1}{L_{12}}\}(N)\end{aligned}$$

(Theorem XV).

But the left side is, by the present theorem, equal to $\{(L_1\cdot E)\times(L_2\cdot E)\}\cdot E^{-1}$. Hence, if L_1, L_2 be linear functions,

$$(L_1\cdot E)\times(L_2\cdot E) = L_{12}\cdot L_1\cdot L_2\cdot E\cdot\operatorname{conv}\overset{-1}{L_{12}}.$$

In particular, $(I_a\cdot E)\times(I_b\cdot E)=I_{a+b}\cdot I_a\cdot I_b\cdot E\cdot I^{-1}_{a+b,2}$, which proves Ramanujan's result (II §5(d)) that

$$\sum_N \frac{\sigma_a(N)\sigma_b(N)}{N^s} = \frac{\zeta(s)\zeta(s-a)\zeta(s-b)\zeta(s-a-b)}{\zeta(2s-a-b)}.$$

Section III. Rational functions of one argument

1. **The concept of rational function.** A function of N which is the composite of r linear functions will be called a *rational integral function* of degree r*. If $f(N)$ be such a function,

$$f_{(p)}(x) = \frac{1}{(1-a_1x)(1-a_2x)\cdots(1-a_rx)};$$

$$\overset{-1}{f_{(p)}}(x) = (1-a_1x)(1-a_2x)\cdots(1-a_rx),$$

* The notion, though not the actual expression "rational integral function," as well as Theorem XVI, occur in the last part of my note *On the inversion of multiplicative arithmetic functions*, Journal of the Indian Mathematical Society, October, 1927.

so that $f^{-1}(p^n)=0$, if $n>r$. Hence

THEOREM XVI. *The condition that $f(N)$ be a rational integral function of degree r is the vanishing of $f^{-1}(N)$ for all values of N divisible by an $(r+1)$th power.*

The composite of a rational integral function P_r of degree r, and the inverse of a rational integral Q_s of degree s, will be called a *rational function* of degree (r, s); P_r will be called *the integral component*, and Q_s^{-1}, *the inverse component*, of the rational function. The linear components of P_r and Q_s will also be called, respectively, the *linear* and the *inverse-linear* components of the rational function $P_r \cdot Q_s^{-1}$. Also, a rational function will be said to be expressed in *rational form*, when its integral and inverse components are put in explicit evidence.

It follows from the above that the generating series $f_{(p)}(x)$ of a rational function f of degree (r, s) is the expansion of a rational function of x, whose denominator and numerator are of degrees r, s respectively; $f_{(p)}(x)$ is thus a recurring series of order r according to the usual definition.

Rational functions of degree $(1, 1)$ will be called *totients*.

The fundamental theorem in the theory of rational functions may be stated as follows:

THEOREM XVII. *All the processes of our calculus, excepting division,* are rational processes, in the sense that, when performed on rational functions, they yield only rational functions.*

The truth of this theorem is obvious from the definition, so far as composition and inversion are concerned. For, the inverse of the rational function $P_r \cdot Q_s^{-1}$ of degree (r, s) is the rational function $Q_s \cdot P_r^{-1}$ of degree (s, r); and the composite of the rational functions $(P_r \cdot Q_s^{-1})$, $(P_\rho \cdot Q_\sigma^{-1})$ is the rational function $(P_r \cdot P_\rho) \cdot (Q_s \cdot Q_\sigma)^{-1}$, which is in general of degree $(r+\rho, s+\sigma)$. It remains, therefore, to prove the fundamental theorem for multiplication and compounding.

We shall prove in a general manner that the product of two rational functions is a rational function. Let f, ϕ be two rational functions, whose integral components are of degree r, ρ, respectively. Then the generating series

$$f_{(p)}(x) = 1 + a_1x + a_2x^2 + \cdots,$$

$$\phi_{(p)}(x) = 1 + \alpha_1x + \alpha_2x^2 + \cdots$$

* That division is not a rational process may be seen from an example. The series $\sum nx^n$ is a recurring series, but the series $\sum x^n/n$ is not a recurring series. Thus the quotient of two rational functions need not be rational. (Cf. the form of the generating series of a quotient obtained in II §1.)

are recurring series of orders r, ρ, respectively. To prove that $(f\times\phi)$ is a rational function, we have to show that the series

$$(f \times \phi)_{(p)}(x) = 1 + a_1\alpha_1 x + a_2\alpha_2 x^2 + \cdots$$

is also a recurring series (in fact, of order $r\rho$, as we shall see). To prove this, observe that when m is sufficiently large, the two sets of quantities

$$a_m,\ a_{m-1},\ \cdots,\ a_{m-t},$$

$$\alpha_m,\ \alpha_{m-1},\ \cdots,\ \alpha_{m-t}$$

are connected respectively by $t-r+1$ and $t-\rho+1$ linear equations. On multiplying each of the linear equations between the a's by each α, and each of the equations between the α's by each a, we obtain $(t+1)(2t-r-\rho+2)$ linear equations connecting the $(t+1)^2$ quantities $a_\mu\alpha_\nu$. Of these, $(t-r+1)\cdot(t-\rho+1)$ equations are redundant, since we obtain this number on multiplication of each of the $t-r+1$ equations between the a's with each of the $t-\rho+1$ equations between the α's. The remaining $(t+1)^2-r\rho$ equations between the products $a_\mu\alpha_\nu$ are in general linearly independent; if the $(t+1)^2-(t+1)$ products $a_\mu\alpha_\nu(\mu\neq\nu)$ are to be just capable of elimination from these, we must have

$$(t + 1)^2 - r\rho = (t + 1)^2 - (t + 1) + 1,$$

or

$$t = r\rho.$$

Hence the series $1+a_1\alpha_1x+a_2\alpha_2x^2+\cdots$ is a recurring series of order $r\rho$, as was to be proved.

This result may be looked upon as a particular case of Hadamard's theorem on the multiplication of singularities,* according to which, the singularities of the analytic function represented by the power series $\sum a_rb_rx^r$ are included among the products $\alpha_\lambda\beta_\mu(\lambda, \mu=1, 2, \cdots)$, the α's and β's being the singularities of the analytic functions represented by the respective series $\sum a_rx^r$ and $\sum b_rx^r$; for, if $\sum a_rx^r$ and $\sum b_rx^r$ are both rational functions, then the α's and β's, as well as their products, must be finite in number, so that $\sum a_rb_rx^r$ has only a finite number of isolated singularities and is therefore a rational function. This theorem shows also that the integral components of the product-function $f\times\phi$ may be specified in terms of the integral components of f and ϕ. We shall indicate below a second proof of this theorem, which has the advantage of specifying the integral component of the product-function.

* Bieberbach, Encyklopädie, Band II, Teil 3, erste Hälfte, p. 464.

The behavior of inverses of integral functions under multiplication, composition and compounding is noteworthy; we have

THEOREM XVIII. *The product, composite, or compound of inverses of integral functions is also the inverse of an integral function.*

To prove this, we have only to observe that the generating series of inverses of integral functions to the base p, are all finite polynomials $1+\sum a_m x^m$, $1+\sum b_m x^m$, etc. Hence each of the power series $1+\sum a_m b_m x^m$, $(1+\sum a_m x^m)\cdot(1+\sum b_m x^m)$ and $1+\sum(a_m+b_m)x^m$ is a finite polynomial.

We may also note that the composite of two rational integral functions is a rational integral function. Thus, composition is not only a rational, but also an integral, process. It will appear that it is the only one of our processes which is integral.

2. **Properties of totients.** Totients being rational functions of degree (1, 1), are functions of the form $L_1\cdot L_2^{-1}$, where L_1, L_2 are linear. Two special types of totients are of great importance. The totients of the form $L_1\cdot E^{-1}$ (including Euler's and Jordan's functions as special cases) will be called *enumerative totients*. It follows from this definition, that the integral of an enumerative totient is a linear function.

The inverse of an enumerative totient $T=L_1\cdot E^{-1}$ is of the form $T^{-1}=E\cdot L_1^{-1}$, and will be called a *level totient*. The property of the level totient which gives it its name is $T^{-1}(p)=T^{-1}(p^2)=\cdots=T^{-1}(p^n)$, where p is any prime. The level totient is consequently the integral (=composite with E, II §3) of the inverse of a linear function. Since $E_k(p)=E_k(p^2)=\cdots$ it follows that the elementary functions E_k are level totients, though of a special type, since the value of $E_k(p)$ is independent of the prime p. Hence, also, the general level totient may be regarded as a cross between (in general, an infinite number of) E-functions. Consequently, from the independence of the elements of a multiplicative function, we can deduce properties of level totients from known theorems on E-functions. For example

Example 10. The product and compound of level totients are level totients.

To deduce the truth of this for level totients, from its truth for E-functions, we observe that

$$E_k = \overset{-1}{\lambda}_{1-k}\cdot E;\quad E_k\times E_{k'} = E_{kk'} = \overset{-1}{\lambda}_{1-kk'}\cdot E;$$

$$\overset{-1}{\lambda}_{1-kk'} = \overset{-1}{\lambda}_{1-k}\oplus\overset{-1}{\lambda}_{1-k'}\oplus(\overset{-1}{\lambda}_{1-k}\times\overset{-1}{\lambda}_{1-k'}).$$

Hence

$$(\overset{-1}{\lambda}_{1-k}\cdot E)\times(\overset{-1}{\lambda}_{1-k'}\cdot E) = \left\{\overset{-1}{\lambda}_{1-k}\oplus\overset{-1}{\lambda}_{1-k'}\oplus(\overset{-1}{\lambda}_{1-k}\times\overset{-1}{\lambda}_{1-k'})\right\}\cdot E.$$

Now, if $T_1=L_1^{-1}\cdot E$, $T_2=L_2^{-1}\cdot E$ are two given level totients, and if T_1, T_2 have the same elements as E_k, $E_{k'}$, for the base p, then for the same base, L_1, L_2 should have the same elements as λ_{1-k}, $\lambda_{1-k'}$. Hence the two functions

$$(L_1^{-1}\cdot E)\times(L_2^{-1}\cdot E),$$
$$\{L_1^{-1}\oplus L_2^{-1}\oplus(L_1^{-1}\times L_2^{-1})\}\cdot E$$

have identical elements for every base, and are therefore identical. Since the part within braces is the inverse of a linear function (Theorem XVIII), this expresses the product of the level totients in rational form as a level totient.

In the same way, since

$$E_r\oplus E_s = E_{r+s} = \lambda_{1-r-s}^{-1}\cdot E = (\lambda_{1-r}^{-1}\oplus\lambda_{1-s}^{-1}\oplus\lambda^{-1})\cdot E,$$

it follows that the compound of the level totients $L_1^{-1}\cdot E$, $L_2^{-1}\cdot E$ is the level totient given by

$$(L_1^{-1}\oplus L_2^{-1}\oplus\lambda^{-1})\cdot E.$$

Example 11. It was proved that

$$(f\cdot\phi)\times E_r = f\cdot(\phi\times E_r)\oplus(E_{r-1}\times f)\ \text{(Example 5)}.$$

Hence, if T is a level totient,

$$(f\cdot\phi)\times T = f\cdot(\phi\times T)\oplus\{f\times(T\oplus E_{-1})\}.$$

It follows from this, that if f is a rational function, $f\times T$ is also a rational function, with the same integral component as f. For, if $f=P\cdot Q^{-1}$, where P, Q are rational integral functions, then

$$\begin{aligned} P^{-1}\cdot(f\times T) &= \{(P^{-1}\cdot f)\times T\}\oplus\{P^{-1}\times(\operatorname{conj} T\oplus E)\}\\ &= \{Q^{-1}\times T\}\oplus\{P^{-1}\times(\operatorname{conj} T\oplus E)\}.\end{aligned}$$

Since the expression on the right is the inverse of an integral function, this expresses $f\times T$ in rational form.

The enumerative totients do not form, like the level totients, a closed system under multiplication, or compounding, but they possess this property for a different process of combination. It was proved that the function $f(N)$ defined by

$$f(N) = \sum f_1(d_1)f_2(d_2)\cdots f_k(d_k),$$

where the summation is for all sets of divisors $d_1d_2\cdots d_k$ having the least common multiple N, is equal to $[\{(f_1\cdot E)\times(f_2\cdot E)\times\cdots\times(f_k\cdot E)\}\cdot E^{-1}](N)$ (Example 9). The function f may be called the combinant of $f_1, f_2, \cdots, f_k$. From the expression for the combinant, we immediately have

THEOREM XIX. *The combinant f of any number of enumerative totients* $f_1, f_2, \cdots, f_k$ *is also an enumerative totient; the linear component of f is the product of the linear components of* $f_1, f_2, \cdots, f_k$.

As an example, the Jordan function ϕ_k is the combinant of k functions each identical with the Euler totient ϕ.*

The general totient may be investigated by means of a canonical form, which expresses it as the product of a level totient and a linear function.

THEOREM XX. *The totient* $L_1 \cdot L_2^{-1} = T$ *is the product of its integral component* L_1 *by the level totient* $L \cdot E^{-1}$, *where L is the linear function defined by* $L(N) = L_2(N)/L_1(N)$.

For, let

$$L_{1(p)}(x) = \frac{1}{1 - \alpha x}, \quad L_{2(p)}(x) = \frac{1}{1 - \beta x};$$

then

$$T_{(p)}(x) = \frac{1 - \beta x}{1 - \alpha x} = 1 + \frac{\alpha - \beta}{\alpha} \cdot \alpha x + \frac{\alpha - \beta}{\alpha} \cdot \alpha^2 x^2 + \cdots = (L_1 \times K)_{(p)}(x)$$

where

$$K_{(p)}(x) = 1 + \frac{\alpha - \beta}{\alpha}(1 + x + x^2 + \cdots) = \frac{1 - \frac{\beta}{\alpha} x}{1 - x} = (L^{-1} \cdot E)_{(p)}(x),$$

where

$$L_{(p)}(x) = \frac{1}{1 - \frac{\beta}{\alpha} x}; \text{ or } L(N) = \frac{L_2(N)}{L_1(N)} \cdot$$

The theorem will however become invalid if L_1 vanishes for finite values of its argument, that is, if L_1 has common elements with E_0.

We may call the totient $K = L \cdot E^{-1}$ the *associated level totient* of T. We also observe that the general totient can be obtained by multiplication of the general level totient and the general linear function.

We shall use this expression for the totient to prove

THEOREM XXI. *The product of the totients* T_1, T_2 *is a totient* T, *whose linear component is the product of the linear components of* T_1, T_2.

For, let K_1, K_2 be the associated level totients of T_1, T_2, and let

* This property of the Jordan function is due to von Sterneck; see Dickson, p. 151, 215.

$$T_1 = L_1 \cdot \overset{-1}{P_1}; \; T_2 = L_2 \cdot \overset{-1}{P_2}.$$

Then

$$T = T_1 \times T_2 = L_1 \times L_2 \times K_1 \times K_2 = L \times K,$$

where $L = L_1 \times L_2$, and K is the level totient which is the product of the level totients K_1, K_2. Thus the product is a totient whose linear component is $L_1 \times L_2 = L$. The inverse-linear component of T may be easily shown to be

$$(L_1 \times \overset{-1}{P_2}) \oplus (L_2 \times \overset{-1}{P_1}) \oplus (\overset{-1}{P_1} \times \overset{-1}{P_2}).$$

This proof will, however, become invalid, if L_1 or L_2 has the same element as E_0 for certain prime bases p. For such prime bases, however, either T_1 or T_2 or both must reduce to the inverse of a linear function, and therefore, also, their product. Since the inverse of a linear function may be considered as a degenerate case of the totient, the theorem is always true.

As a special case of the theorem, we may note that the conjugate T' of a totient T, being equal to the product $T \times E_{-1}$, is also a totient. By Example 11, the linear components of T, T' are the same.

3. **Compound of rational functions.** Let $F = A \cdot B^{-1}$, $\Phi = C \cdot D^{-1}$; A, B, C, D being rational integral functions. Also let L be *the least common composite* of A, C, that is, the integral function of least degree, which is of each of the forms $A \cdot L_1$, $C \cdot L_2$, where L_1, L_2 are rational and integral. Then

$$\begin{aligned} F \oplus \Phi &= (A \cdot B^{-1}) \oplus (C \cdot D^{-1}) \\ &= \{(A \cdot L_1) \cdot (B \cdot L_1)^{-1}\} \oplus \{(C \cdot L_2) \cdot (D \cdot L_2)^{-1}\} \\ &= L \cdot \{D \cdot (B \cdot L_1)^{-1} \oplus (D \cdot L_2)^{-1}\} \oplus L \\ &= L \cdot \{(B \cdot L_1)^{-1} \oplus (D \cdot L_2)^{-1}\} \oplus (E_2 \times L) \oplus (E_{-1} \times L). \end{aligned}$$

Now

$$E_2 \times L = L \cdot \operatorname{conj} L^{-1} \quad \text{(Example 5(4))}.$$

Hence

$$F \oplus \Phi = L \cdot \{(B \cdot L_1)^{-1} \oplus (D \cdot L_2)^{-1} \oplus \operatorname{conj} L^{-1}\}.$$

This shows that the compound of the rational functions F, Φ is a rational function, and (since the expression within the braces is the inverse of an integral function) also expresses it in rational form.

4. **Product of compositional powers of linear functions.** The following theorem which expresses the product $L_1^r \times L_2^s$, where L_1, L_2 are linear functions, in rational form, is of fundamental importance.

THEOREM XXII. *If L_1, L_2 are linear functions, and $L_{12}=L_1\times L_2$,*

$$L_1^r \times L_2^s = L_{12}^{r+s-1}\cdot(L_1^{-(r-1)} \times L_2^{-(s-1)}),$$

r and s being positive integers.

To prove this, let

$$L_{1(p)}(x) = \frac{1}{1-\alpha x}, \quad L_{2(p)}(x) = \frac{1}{1-\beta x}\cdot$$

Also, let $\overset{H}{r_n}$ denote the coefficient of x^n in $(1-x)^{-r}$, and $\binom{r}{n}$, the usual binomial coefficient. Then

$$(L_1^r \times L_2^s)_{(p)}(x) = \sum_n \overset{H}{r_n}\cdot \overset{H}{s_n}(\alpha\beta)^n x^n,$$

and

$$\{L_{12}^{r+s-1}\cdot(L_1^{-(r-1)} \times L_2^{-(s-1)})\}_{(p)}(x)$$
$$= \frac{1}{(1-\alpha\beta x)^{r+s-1}}\cdot\left\{\sum\binom{r-1}{n}\binom{s-1}{n}(\alpha\beta)^n x^n\right\}.$$

We have therefore to show that

$$F(r, s, n) \equiv \overset{H}{r_n}\cdot\overset{H}{s_n} - \binom{r+s-1}{1}\overset{H}{r}_{n-1}\cdot\overset{H}{s}_{n-1} + \cdots = \binom{r-1}{n}\binom{s-1}{n}.$$

To prove this, we note that when $n>r+s-1$, $F(r, s, n)$ is a polynomial in n of degree $r+s-2$, and represents $\Delta_n^{r+s-1}(\overset{H}{r_n}\cdot\overset{H}{s_n})$, where the operator Δ_n is defined by

$$\Delta_n f(n) = f(n) - f(n-1).$$

Hence it follows that

$$F(r, s, n) = 0, \text{ if } n \geqq r+s-1,$$

since $\overset{H}{r_n}\cdot\overset{H}{s_n}$ is a polynomial of degree $r+s-2$ in n. Also, when $n<r+s-1$,

$$0 = \Delta_n^{r+s-1}(\overset{H}{r_n}\cdot\overset{H}{s_n})$$
$$= F(r, s, n) + (-1)^{n+1}\left\{\overset{H}{r}_{-1}\cdot\overset{H}{s}_{-1}\cdot\binom{r+s-1}{n+1} - \overset{H}{r}_{-2}\cdot\overset{H}{s}_{-2}\cdot\binom{r+s-1}{n+2} + \cdots\right\}.$$

Now $\overset{H}{r}_{-k}=0$ unless $k\geqq r$; hence the part within the braces will contain no significant terms if $r+s-1-n<r$, or if $r+s-1-n<s$. Hence $F(r, s, n)=0$ if n is greater than either $r-1$ or $s-1$. Also when $n<r+s-1$, $F(r, s, n)$ is a polynomial in r, and also in s, of degree n, and we have just proved that it vanishes

for the n values $r-1=0, 1, 2, \cdots, n-1$, and also for the n values $s-1=0, 1, \cdots, n-1$. Hence we must have, identically.

$$F(r, s, n) = \psi(n)\binom{r-1}{n}\binom{s-1}{n}.$$

By putting $r=s=0$, we immediately find that $\psi(n)=1$. The theorem is thus established.

We may note that the relation $E^2 \times E^2 = E_2 \cdot E^2$ (Example 3) is a very special case of this theorem.

5. **Regular rational functions.** By an *elementary rational function*, we shall mean a function which is either the inverse of a rational integral function, or is of the form $T\times L^r$, where T is a level totient, and L a linear function (r being a positive integer).

A rational function will be said to be *regular*, if no two of its distinct linear components have any common elements.

THEOREM XXIII. *A regular rational function can be expressed as the compound of elementary rational functions.*

For, let the integral component of the rational function f be $L_1^{r_1}\cdot L_2^{r_2}\cdots L_i^{r_i}$, $L_1, L_2, \cdots, L_i$ being the distinct linear components of f. Also let

$$f_{(p)}(x) = \frac{\phi(x)}{(1-a_1x)^{r_1}(1-a_2x)^{r_2}\cdots(1-a_ix)^{r_i}},$$

where $\phi(x)$ is a polynomial. Now, if the function f is regular, then no two of the linear functions $L_1, L_2, \cdots, L_i$ have common elements, and therefore the quantities $a_1, a_2, \cdots, a_i$ will be unequal for every prime p. Hence for every prime p, we can express $f_{(p)}(x)$ in partial fractions in the form

$$f_{(p)}(x) = \phi_1(x) + \sum_{t=1}^{r_1}\frac{b_{1t}}{(1-a_1x)^t} + \sum_{t=1}^{r_2}\frac{b_{2t}}{(1-a_2x)^t} + \cdots + \sum_{t=1}^{r_i}\frac{b_{it}}{(1-a_ix)^t},$$

$\phi_1(x)$ being a certain polynomial. That is to say, the function f can be expressed as the compound of the inverse of an integral function, and functions of the type $T\times L_j^t$, where T is a level totient, $j=1, 2, \cdots, i$, and $t=1, 2, \cdots, r_j$.

If, however, two of the functions L, say L_1, L_2, have identical elements for the base q (so that q is an *irregular base* of the function f), then $f_{(q)}(x)$ is not of the above form, the quantities $b_{1s}, L_{2t} (s=1, 2, \cdots, r_1; t=1, 2, \cdots, r_2)$ now becoming infinite with certain mutual relations. We may however deal with this as a limiting case when $a_2 \to a_1$. The limiting process will show

generally that results proved for regular functions may be interpreted as true for irregular functions as well.

THEOREM XXIV. *The product of regular rational functions is a rational function.*

For, a regular function is the compound of elementary rational functions. Hence the product of two regular rational functions can be expressed as a compound of products of pairs of elementary rational functions. Now, the product of the inverse of an integral function by any other function is also the inverse of an integral function. And the product of the elementary functions $T_1 \times L_1^r$, $T_2 \times L_2^s$ is $T_1 \times T_2 \times f'$, where f' is the rational function $L_{12}^{r+s-1} \cdot (L_1^{-(r-1)} \times L_2^{-(s-1)})$ (Theorem XXII). Also T_1 and T_2 being level totients, their product $T_1 \times T_2$ is also a level totient, and $T_1 \times T_2 \times f'$ is a rational function with the integral component L_{12}^{r+s-1} (Example 11). Thus the product-function can be expressed as a compound of rational functions with known integral components, and is therefore a rational function.

To specify the integral component of the product-function, let the integral components of the regular functions F_1, F_2 be

$$L_1^{r_1} \cdot L_2^{r_2} \cdots L_i^{r_i},$$

$$K_1^{s_1} \cdot K_2^{s_2} \cdots K_j^{s_j},$$

respectively, the L's and K's being distinct linear functions. By what has preceded, $F_1 \times F_2$ is the compound of rational functions, whose integral components are of the form

$$L_{\mu\nu}^{\rho_\mu+\sigma_\nu-1}, \ \rho_\mu \leqq r_\mu, \ \sigma_\nu \leqq s_\nu; \ L_{\mu\nu} = L_\mu \times K_\nu \quad (\mu = 1, 2, \cdots, i; \ \nu = 1, 2, \cdots, j).$$

The integral component of $F_1 \times F_2$ is therefore the least common composite of these (III §3); but this latter is evidently identical with the least common composite of the ij functions $L_{\mu\nu}^{r_\mu+s_\nu-1}$. This least common composite will coincide with the continued composite, if the ij linear functions $L_{\mu\nu}$ are all distinct.

If n_1, n_2, n are the degrees of the integral components of F_1, F_2 and $F_1 \times F_2$, so that

$$n_1 = r_1 + r_2 + \cdots + r_i,$$

$$n_2 = s_1 + s_2 + \cdots + s_j,$$

it is easy to see that

$$n = jn_1 + in_2 - ij.$$

These results will hold generally, even when the functions are irregular, provided we consider that the product function has a certain number of elements in common with E_0 for the irregular bases.

6. **Some special cases.** (a) **The rational form of $f\times L^r$, where L is linear, and f is rational.** We can immediately find $f\times L$, where L is linear; for

$$(f \times L)_{(p)}(x) = f_{(p)}(lx), \text{ where } L_{(p)} = l.$$

Since $L^r = L\times E^r$, it follows that it is sufficient to evaluate $f\times E^r$. If

$$f_{(p)}(x) = 1 + a_1x + a_2x^2 + \cdots,$$

then

$$(f \times E^r)_{(p)}(x) = 1 + a_1\cdot \overset{H}{r_1}\, x + a_2\cdot \overset{H}{r_2}\, x^2 + \cdots$$
$$= \frac{1}{(r-1)!}\,\frac{d^{r-1}}{dx^{r-1}}(x^{r-1}f_{(p)}(x)).$$

When f is rational, this can be evaluated in finite terms. For example,

$$E^2 \times E^2 = E^2\cdot E_2 \text{ (Example 3)}.$$

Hence

$$(E^2 \times E^2)_{(p)}(x) = \frac{1+x}{(1-x)^3}.$$

Therefore

$$(E^2 \times E^2 \times E^2)_{(p)}(x) = \frac{d}{dx}\left\{\frac{x(1+x)}{(1-x)^3}\right\} = \frac{1+4x+x^2}{(1-x)^4},$$

so that

$$E^2 \times E^2 \times E^2 = E^4\cdot(E^{-2} \times E^{-2}) = E^{-1}\cdot(E^3 \times E^3) \text{ (Theorem XXII)}.$$

This gives a second proof of Liouville's theorem (Example 4):

$$\sum_{d/N}\{\tau(d)\}^3 = \{\textstyle\sum\tau(d)\}^2.$$

(b) **Product of integral quadratic functions.** If the L's are linear functions, $L_{ij} = L_i\times L_j$, and $L_{1234} = L_1\times L_2\times L_3\times L_4$, then

$$(L_1\cdot L_2) \times (L_3\cdot L_4) = L_{13}\cdot L_{14}\cdot L_{23}\cdot L_{24}\cdot \text{conv } \overset{-1}{L_{1234}}.$$

This may be proved directly from the generating series, or by the method of grouping factors according to their greatest common divisor and least common multiple, employed in Example 9. Let the function $f(N)$ be defined by

$$f(N) = \sum L_1(d_1)L_2\left(\frac{N}{d_1}\right)L_3(d_2)L_4\left(\frac{N}{d_2}\right),$$

where the summation is for all pairs of numbers d_1, d_2 having the least common multiple N. We shall evaluate $f(N)$ in two different ways. First group the terms in the sum according to the greatest common divisor δ of d_1, d_2. Writing

$$d_1 = \delta t_1;\ d_2 = \delta t_2,$$

so that

$$N = \delta t_1 t_2,$$

we have

$$\begin{aligned} f(N) = \sum L_1(d_1)L_2\left(\frac{N}{d_1}\right)L_3(d_2)L_4\left(\frac{N}{d_2}\right) &= \sum L_1(\delta)L_1(t_1)L_2(t_2)L_3(\delta)L_3(t_2)L_4(t_1) \\ &= \sum L_{13}(\delta)L_{14}(t_1)L_{23}(t_2) \\ &= \left\{L_{13}\cdot(L_{14}\oplus L_{23})\right\}(N) \\ &= \left\{L_{13}\cdot L_{14}\cdot L_{23}\cdot \operatorname{conv} \overset{-1}{L_{1234}}\right\}(N) \end{aligned}$$

(Theorem XV).

Secondly, let us evaluate $f\cdot L_{24}$. We have

$$\begin{aligned} f(\delta)L_{24}\left(\frac{N}{\delta}\right) &= \sum L_1(\delta_1)L_2\left(\frac{\delta}{\delta_1}\right)L_3(\delta_2)L_4\left(\frac{\delta}{\delta_2}\right)L_{24}\left(\frac{N}{\delta}\right) \\ &= \sum L_1(\delta_1)L_2\left(\frac{N}{\delta_1}\right)L_3(\delta_2)L_4\left(\frac{N}{\delta_2}\right), \end{aligned}$$

the summation extending to all pairs of numbers δ_1, δ_2 having the least common multiple δ. Summing both sides of this over all divisors δ of N, we see that $(f\cdot L_{24})(N)$ is equal to

$$\sum L_1(\delta_1)L_2\left(\frac{N}{\delta_1}\right)L_3(\delta_2)L_4\left(\frac{N}{\delta_2}\right),$$

summed for *every* pair of divisors δ_1, δ_2 of N, that is, to $\{(L_1\cdot L_2)\times(L_3\cdot L_4)\}(N)$. Combining the two relations, we have immediately

$$(L_1\cdot L_2)\times(L_3\cdot L_4) = L_{13}\cdot L_{14}\cdot L_{23}\cdot L_{24}\cdot \operatorname{conv} \overset{-1}{L_{1234}}.$$

As a special case of this, we have the result already proved (Example 9):

$$\sigma_a \times \sigma_b = E\cdot I_a\cdot I_b\cdot I_{a+b}\cdot \overset{-1}{I_{a+b,2}}.$$

As another special case:

$$\begin{aligned} (L_1\cdot L_2)\times(L_1\cdot L_2) &= L_{11}\cdot L_{22}\cdot \overset{2}{L_{12}}\cdot \operatorname{conv}(L_{12}\times L_{12})^{-1} \\ &= L_{11}\cdot L_{22}\cdot L_{12}\cdot(L_{12}\times\lambda^{-1}). \end{aligned}$$

Section IV. The cardinal, principal, and semi-principal functions

1. Derivates of a function of r arguments. The derivate of the multiplicative function $f(M_1, M_2, \cdots, M_r)$ with respect to any subset $M_1, M_2, \cdots, M_i$ of its arguments, is defined to be the function of the remaining $r-i$ arguments, obtained on putting $M_1 = M_2 = \cdots = M_i = 1$ in $f(M_1, M_2, \cdots, M_r)$. This derivate is clearly a multiplicative function, and may be denoted by the symbol $D_{M_1 M_2 \cdots M_i}(f)$.

If $f_{(p)}(x_1, x_2, \cdots, x_r)$ is the generating series of f to the base p, the generating series of $D_{M_1 M_2 \cdots M_i}(f)$ to the same base is clearly $f_{(p)}(0, 0, \cdots, x_{i+1}, x_{i+2}, \cdots, x_r)$. From this we immediately deduce

Theorem XXV. *If f, ϕ be two multiplicative functions of $M_1, M_2, \cdots, M_r$, then*

$$\text{(a)} \qquad D_{M_1 M_2 \cdots M_i}(f) \times D_{M_1 M_2 \cdots M_i}(\phi) = D_{M_1 M_2 \cdots M_i}(f \times \phi),$$

$$\text{(b)} \qquad D_{M_1 M_2 \cdots M_i}(f) \cdot D_{M_1 M_2 \cdots M_i}(\phi) = D_{M_1 M_2 \cdots M_i}(f \cdot \phi),$$

$$\text{(c)} \qquad D_{M_1 M_2 \cdots M_i}(f^{-1}) = \left\{ D_{M_1 M_2 \cdots M_i}(f) \right\}^{-1}.$$

Or briefly, the product or compound of derivates is the derivate of the product or compound, and the derivate of the inverse is the inverse of the derivate.

It will be seen in the next section, that the derivates also possess a similar property with respect to the compounding operation.

Theorem XXVI. *The necessary and sufficient condition that a multiplicative function f of the r arguments $M_1, M_2, \cdots, M_r$, be also a multiplicative tunction of a subset $M_1, M_2, \cdots, M_i$ of its arguments, is*

$$D_{M_1 M_2 \cdots M_i}(f) \equiv E(M_{i+1}, M_{i+2}, \cdots, M_r).$$

The condition is obviously necessary, since a multiplicative function of $M_1, M_2, \cdots, M_i$ must be equal to 1, when $M_1 = M_2 = \cdots = M_i = 1$. To prove that it is sufficient, we observe that, when $D_{M_1 M_2 \cdots M_i}(f) = E(M_{i+1}, \cdots, M_r)$, the value of $f(M_1, M_2, \cdots, M_r)$ is not affected either by introducing into, or removing from, the last $r-i$ arguments, such factors as are relatively prime to $M_1 M_2 \cdots M_i$. For, if p be any prime which does not divide $M_1 M_2 \cdots M_r$,

$$\begin{aligned} &f(M_1, \cdots, M_i, M_{i+1}p^{\alpha}, M_{i+2}p^{\beta}, \cdots) \\ &= f(M_1, M_2, \cdots, M_r) \times f(1, 1, \cdots, p^{\alpha}, p^{\beta}, \cdots) \\ &= f(M_1, M_2, \cdots, M_r) = f(M_1, M_2, \cdots, M_i, M_{i+1}p^{\alpha'}, M_{i+2}p^{\beta'}, \cdots). \end{aligned}$$

Hence, writing

$$M'_{i+1} = M_{i+1}p^\alpha; \quad M'_{i+2} = M_{i+2}p^\beta; \text{ etc.},$$

$$\alpha' - \alpha = a; \qquad \beta' - \beta = b; \text{ etc.},$$

we have

$$f(M_1, M_2, \cdots, M_i, M'_{i+1}, \cdots, M'_r) = f(M_1, \cdots, M_i, M'_{i+1}p^a, M'_{i+2}p^b, \cdots),$$

where p is any prime not occurring in $M_1, M_2, \cdots, M_i$. This proves our statement.

To prove that $f(M_1, M_2, \cdots, M_r)$ is a multiplicative function of $M_1, M_2, \cdots, M_i$, that is, that, whenever $M_1M_2 \cdots M_i$ is relatively prime to $N_1N_2 \cdots N_i$,

$$f(M_1N_1, M_2N_2, \cdots, M_iN_i, M_{i+1}, \cdots, M_r)$$
$$= f(M_1, M_2, \cdots, M_r) \times f(N_1, \cdots, N_i, M_{i+1}, \cdots, M_r),$$

suppose that M'_{i+k}, M''_{i+k} are the maximum factors of M_{i+k}, which contain only prime factors occurring in $M_1M_2 \cdots M_i$, $N_1N_2 \cdots N_i$, respectively. It follows that M'_{i+k}, M''_{i+k} are relatively prime, and therefore $M_{i+k} = M'_{i+k}M''_{i+k}T_{i+k}$, where T_{i+k} is relatively prime to $M_1M_2 \cdots M_i N_1N_2 \cdots N_i$. Now, since M'_{i+k}, M''_{i+k} are respectively prime to $N_1N_2 \cdots N_i$, $M_1M_2 \cdots M_i$, it follows that

$$f(M_1, M_2, \cdots, M_r) = f(M_1, M_2, \cdots, M_i, M'_{i+1}, M'_{i+2}, \cdots, M'_r),$$

$$f(N_1, N_2, \cdots, N_i, M_{i+1}, \cdots, M_r) = f(N_1, N_2, \cdots, N_i, M''_{i+1}, M''_{i+2}, \cdots, M''_r).$$

Now, however, $M_1M_2 \cdots M_iM'_{i+1} \cdots M'_r$ is relatively prime to $N_1N_2 \cdots N_i M''_{i+1} \cdots M''_r$. Hence, since $f(M_1, M_2, \cdots, M_r)$ is a multiplicative function of its r arguments, it follows that the product of the left sides is equal to

$$f(M_1N_1, M_2N_2, \cdots, M_iN_i, M'_{i+1}M''_{i+1}, \cdots, M'_rM''_r).$$

Here, the factors T_{i+k}, which are relatively prime to the first i arguments of this function, may be introduced into the last $r-i$ arguments, without affecting the value. Thus

$$f(M_1, \cdots, M_i, M_{i+1}, \cdots, M_r)f(N_1, \cdots, N_i, M_{i+1}, \cdots, M_r)$$
$$= f(M_1N_1, \cdots, M_iN_i, T_{i+1}M'_{i+1}M''_{i+1}, \cdots)$$
$$= f(M_1N_1, \cdots, M_iN_i, M_{i+1}, \cdots, M_r),$$

establishing the multiplicative property in respect to $M_1, M_2, \cdots, M_i$.

As an illustration of the theorem, any multiplicative function $f(g)$ of the greatest common divisor g of $M_1, M_2, \cdots, M_r$ is seen to be a multiplicative function, not only of $M_1, M_2, \cdots, M_r$, but of any subset of them as well; for when any of the arguments $M_1, M_2, \cdots, M_r$ becomes equal to unity, g and therefore also $f(g)$ take the value unity.

2. **Cardinal functions of r arguments.** A cardinal function of $M_1, M_2, \cdots, M_r$ is defined to be a function f, all of whose derivates are identical with E_0; it is clearly necessary and sufficient for this that the r derivates $D_{M_i}(f)$ $(i=1, 2, \cdots, r)$ be identical with E_0. This implies that the generating series (f_p) $(x_1, x_2, \cdots, x_r)$ of a cardinal function f, is of the form $1+x_1x_2\cdots x_r$ $F(x_1, x_2, \cdots, x_r)$, where $F(x_1, x_2, \cdots, x_r)$ is some power series. The arithmetical property of the cardinal function f, implied by this, is evidently that $f(M_1, M_2, \cdots, M_r)=0$, whenever M_i admits a factor relatively prime to M_j, or in other words, that $f(M_1, M_2, \cdots, M_r)=0$, unless each of the arguments has the same distinct prime factors.

From Theorem XXIV, we now have, immediately,

THEOREM XXVII. *The product of a cardinal function and an arbitrary function of the same arguments, is also a cardinal function. The composite of a cardinal function and an arbitrary function f has all its derivates identical with the corresponding derivates of f. Lastly, the inverse of a cardinal function is also a cardinal function.*

To prove the last statement, let F be a cardinal function. Then,

$$\begin{aligned} D_{M_i}(F)\cdot D_{M_i}(F^{-1}) &= D_{M_i}(F\cdot F^{-1}) \\ &= D_{M_i}(E_0) = E_0. \end{aligned}$$

But, since F is a cardinal function, $D_{M_i}(F)=E_0$. Hence $D_{M_i}(F^{-1})=E_0$, or F^{-1} is also a cardinal function.

The converse of this theorem is of great importance; it is as follows:

THEOREM XXVIII. *If ϕ_1, ϕ_2 are functions of $M_1, M_2, \cdots, M_r$, whose corresponding derivates are identical, then each of them is the composite of the other with a cardinal function; also, each is the product of the other and the integral of a cardinal function.*

For, if

$$\phi_1 = \phi_2\cdot f = \phi_2 \times (F\cdot E),$$

then

$$D_{M_i}(\phi_1) = D_{M_i}(\phi_2)\cdot D_{M_i}(f) = D_{M_i}(\phi_2) \times (D_{M_i}(F)\cdot E) \text{ (since } D_{M_i}(E) = E).$$

Since $D_{M_i}(\phi_1) = D_{M_i}(\phi_2)$ by hypothesis, it follows that

$$D_{M_i}(f) = E_0,$$

$$D_{M_i}(F)\cdot E = E, \text{ or } D_{M_i}(F) = E_0.$$

Thus both f and F are cardinal functions.

COROLLARY 1. *If ϕ_1, ϕ_2 are functions of $M_1, M_2, \cdots, M_r$ which are known to be equal to one another, whenever two of the arguments $M_1, M_2, \cdots, M_r$ are relatively prime, it is thereby implied that the functions are also equal when any of the arguments is equal to unity. Thus ϕ_1, ϕ_2 have identical derivates, and the theorem applies to them.*

COROLLARY 2. *The most general function $f(M_1, M_2, \cdots, M_r)$ with the property of being a multiplicative function of every subset of its arguments is the integral* (II §3) *of an arbitrary cardinal function.*

3. **Cardinal functions of a matrix-set of arguments.** A function f of the matrix-set of $r\times s$ arguments $|M_{i1}, M_{i2}, \cdots, M_{ir}|$ $(i=1, 2, \cdots, s)$, is said to be a *cardinal function*, if the derivate of f with respect to the s arguments of any column (or, briefly, if each *column-derivate* of f) is the function E_0. This reduces to our earlier definition for $s=1$. It follows that a cardinal function of the matrix-set $|M_{i1}, M_{i2}, \cdots, M_{ir}|$ $(i=1, 2, \cdots, s)$ vanishes if any of the r column-products $M_{1j}M_{2j}\cdots M_{sj}$ $(j=1, 2, \cdots, r)$ contains a prime factor not dividing one of the others. We have the following obvious analogue of Theorem XXVIII:

THEOREM XXIX. *If ϕ_1, ϕ_2 are two functions of the matrix set $|M_{i1}, M_{i2}, \cdots, M_{ir}|$ $(i=1, 2, \cdots, s)$, which have identical column-derivates, then each is the composite of the other with a cardinal function, and also the product of the other with the integral of a cardinal function of the matrix-set.*

The proof is on the same lines as the proof of Theorem XXVIII. As in Corollary 1 of that theorem, this also applies to two functions of the matrix-set, which are known to be equal to each other, whenever any two column-products are relatively prime.

From the definition of the cardinal function of a matrix-set $|M_{ij}|$, it is easy to specify the peculiarity of its generating series to any base. Let x_{ij} be the variable corresponding to the argument M_{ij}. Since every column-derivate of the cardinal function is E_0, it follows that the generating series must reduce to 1, if we put the variables corresponding to any column equal to zero. Thus the peculiarity of the generating series is that every non-vanishing

term of it contains at least one representative from each column of the matrix $|x_{ij}|$, or in other words, the terms which involve none of the variables of a column vanish.

4. **Principal functions.** These constitute a very important subclass of cardinal functions. The function $f(M_1, M_2, \cdots, M_r)$ is called *a principal function*, if it vanishes whenever two of its arguments are unequal. The function

$$\psi(M) = f(M, M, \cdots, M)$$

is called the function of one argument *equivalent* to the principal function f. It is clear that ψ determines f completely; for, if

$$\psi_{(p)}(x) = 1 + a_1 x + a_2 x^2 + \cdots,$$

then

$$f_{(p)}(x_1, x_2, \cdots, x_r) = 1 + a_1(x_1 x_2 \cdots x_r) + a_2(x_1 x_2 \cdots x_r)^2 + \cdots.$$

This will be indicated by writing

$f = \text{princ}_r\, \psi$ (f is the principal function of r arguments, equivalent to ψ).

More generally, a function f of the matrix-set $|M_{i1}, M_{i2}, \cdots, M_{ir}|$ $(i=1, 2, \cdots, s)$ will be called a principal function, if it vanishes whenever two arguments in the same row are unequal. The function $\psi(M_1, M_2, \cdots, M_s)$ to which f reduces when $M_{i1} = M_{i2} = \cdots = M_{ir} = M_i$ $(i=1, 2, \cdots, s)$, is called *the function of s arguments equivalent to the principal function f*. It is clear that ψ determines f, and that the generating series of f is obtained from $\psi_{(p)}(x_1, x_2, \cdots, x_s)$, by substituting $x_{i1}x_{i2} \cdots x_{ir}$ for $x_i (i=1, 2, \cdots, s)$. We also call f the principal function of $r \times s$ arguments *equivalent* to $\psi(M_1, M_2, \cdots, M_s)$, and denote it by $\text{princ}_r\, \psi$.

THEOREM XXX. *If ψ, ψ' are functions of s arguments, and* princ ψ, princ ψ' *the equivalent principal functions of $r \times s$ arguments, then*

(a) $$\text{princ}\, \psi \cdot \text{princ}\, \psi' = \text{princ}\, (\psi \cdot \psi'),$$

(b) $$(\text{princ}\, \psi)^{-1} = \text{princ}\, (\psi^{-1}).$$

These relations follow immediately from the fact that the generating series of princ ψ is obtained from the generating series of ψ by substituting $x_{i1}x_{i2} \cdots x_{ir}$ for $x_i (i=1, 2, \cdots, s)$.

5. **Functions of the greatest common divisor (g.c.d.).** We prove the following theorem.

THEOREM XXXI. *The necessary and sufficient condition that $f(M_1, M_2, \cdots, M_r)$ be a function $\psi(g)$ of the greatest common divisor g of $M_1, M_2, \cdots, M_r$, is that f be the integral of the principal function of r arguments, equivalent to $(\psi\cdot E^{-1})(M)$.*

The necessary and sufficient condition that $f(|M_{i1}, M_{i2}, \cdots, M_{ir}|)$ $(i=1, 2, \cdots, s)$ be a function $\psi(g_1, g_2, \cdots, g_s)$ of the g.c.d.'s g_i of $M_{i1}, M_{i2}, \cdots, M_{ir}$, is that f be the integral of the principal function of $r\times s$ arguments, equivalent to $(\psi\cdot E^{-1})(M_1, M_2, \cdots, M_s)$.

This theorem illustrates Theorem XXIX; for, if two column-products are relatively prime, it is clear that $g_1, g_2, \cdots, g_s$ all become unity, and so $f=1=E$. Hence, by Theorem XXIX, f is the composite of E and a cardinal function. The present theorem identifies the cardinal function as a certain principal function.

To prove the first part, let

$$f_{(p)}(x_1, x_2, \cdots, x_r) = \sum a(m_1, m_2, \cdots, m_r)x_1^{m_1}x_2^{m_2}\cdots x_r^{m_r},$$

so that

$$(f\cdot E^{-1})_{(p)}(x_1, x_2, \cdots, x_r) = \sum b(m_1, m_2, \cdots, m_r)x_1^{m_1}x_2^{m_2}\cdots x_r^{m_r},$$

where

$$\begin{aligned} b(m_1, m_2, \cdots, m_r) = a(m_1, m_2, \cdots, m_r) &- \sum a(m_1 - 1, m_2, \cdots, m_r) \\ &+ \sum a(m_1 - 1, m_2 - 1, m_3, \cdots, m_r) - \cdots, \end{aligned}$$

terms with negative suffixes not appearing in the summations. Since f is given to be a function of the g.c.d. of its arguments, $a(m_1, m_2, \cdots, m_r)$ must depend only on the least of its suffixes. Suppose that m is the least of $m_1, m_2, \cdots, m_r$, and that

$$m_1 = m_2 = \cdots = m_i = m;\ m_{i+1},\ m_{i+2},\ \cdots, m_r > m;$$

$$a(m_1, m_2, \cdots, m_r) = A(m).$$

Now group the expression for $b(m_1, m_2, \cdots, m_r)$ in such a way that in each group the values of the first i suffixes remain the same. If $m=0$, then, since negative suffixes are not permitted, there will be only one group, and the value of $b(m_1, m_2, \cdots, m_r)$ will be 1, or 0, according as all the m's do, or do not, vanish. (This result might of course have been anticipated, since we know that $f\cdot E^{-1}$ must be a cardinal function.) If $m>0$, then there are 2^i groups of terms, and we have

$$b(m_1, m_2, \cdots, m_r) = A(m)(1-1)^{r-i} - \binom{i}{1}A(m-1)(1-1)^{r-i}$$

$$+\binom{i}{2}A(m-1)(1-1)^{r-i} - \text{etc.}$$

$$= \{A(m) - A(m-1)\}(1-1)^{r-i}.$$

This vanishes unless $r=i$, and is then equal to $A(m)-A(m-1)$. Thus $b(m_1, m_2, \cdots, m_r)$ vanishes unless all the m's are equal, and is equal to $A(m)-A(m-1)$, if $m_1=m_2=\cdots=m_r=m$. Now the series

$$1 + \sum_{m=1}\{A(m) - A(m-1)\}x^m$$

is evidently the generating series to the base p, of the function $\psi\cdot E^{-1}$. Thus $f\cdot E^{-1}$ is the principal function equivalent to the function $(\psi\cdot E^{-1})(M)$.

The same method will apply to the case of a matrix-set of arguments. For convenience of writing we may take $s=2$, so that

$$f\left(\begin{vmatrix} M_{11}, M_{12}, \cdots, M_{1r} \\ M_{21}, M_{22}, \cdots, M_{2r}\end{vmatrix}\right) = \psi(g_1, g_2).$$

Let

$$\psi_{(p)}(x, y) = \sum A(m, n)x^m y^n,$$

$$f_{(p)}\begin{pmatrix} x_1, & x_2, & \cdots, & x_r \\ y_1, & y_2, & \cdots, & y_r\end{pmatrix} = \sum a\begin{pmatrix} m_1, & m_2, & \cdots, & m_r \\ n_1, & n_2, & \cdots, & n_r\end{pmatrix} x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r} y_1^{n_1} y_2^{n_2} \cdots y_r^{n_r},$$

so that, as before,

$$a\begin{pmatrix} m_1, & m_2, & \cdots, & m_r \\ n_1, & n_2, & \cdots, & n_r\end{pmatrix} = A(m, n),$$

where m is the least of $m_1, m_2, \cdots, m_r$, and n, the least of $n_1, n_2, \cdots, n_r$. If

$$(f\cdot E^{-1})_{(p)}\begin{pmatrix} x_1, & x_2, & \cdots, & x_r \\ y_1, & y_2, & \cdots, & y_r\end{pmatrix}$$
$$= \sum b\begin{pmatrix} m_1, & m_2, & \cdots, & m_r \\ n_1, & n_2, & \cdots, & n_r\end{pmatrix} x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r} y_1^{n_1} \cdots y_r^{n_r},$$

then

$$b\begin{pmatrix} m_1, & m_2, & \cdots, & m_r \\ n_1, & n_2, & \cdots, & n_r\end{pmatrix} = a\begin{pmatrix} m_1, & m_2, & \cdots, & m_r \\ n_1, & n_2, & \cdots, & n_r\end{pmatrix} - \sum a\begin{pmatrix} m_1-1, & m_2, & \cdots, & m_r \\ n_1, & n_2, & \cdots, & n_r\end{pmatrix}$$
$$+ \sum a\begin{pmatrix} m_1-1, & m_2-1, & m_3, & \cdots, & m_r \\ n_1, & n_2, & n_3, & \cdots, & n_r\end{pmatrix} - \text{etc.},$$

where the kth summation sign indicates that every set of k suffixes from among $m_1, m_2, \cdots, m_r, n_1, \cdots, n_r$ are to be diminished by unity, terms with negative suffixes being omitted.

As before, let

$$m_1 = m_2 = \cdots = m_i = m; \; m_{i+1}, m_{i+2}, \cdots, m_r > m;$$
$$n_1 = n_2 = \cdots = n_j = n; \quad n_{j+1}, n_{j+2}, \cdots, n_r > n.$$

Group the terms in the expression for

$$b\begin{pmatrix} m_1, \cdots, m_r \\ n_1, \cdots, n_r \end{pmatrix}$$

in such a way, that, in each group, the first i suffixes in the top row and the first j suffixes in the bottom row have the same values. Then it is easy to see that, on evaluating each group separately, we obtain

$$\begin{aligned} b\begin{pmatrix} m_1, m_2, \cdots, m_r \\ n_1, n_2, \cdots, n_r \end{pmatrix} &= (1-1)^{2r-i-j}\Big\{A(m,n) + A(m-1,n)\sum_{\mu=1}^{i}(-1)^{\mu}\binom{i}{\mu} \\ &\quad + A(m, n-1)\sum_{\nu=1}^{j}(-1)^{\nu}\binom{j}{\nu} \\ &\quad + A(m-1, n-1)\sum(-1)^{\mu+\nu}\binom{i}{\mu}\binom{j}{\nu}\Big\} \\ &= (1-1)^{2r-i-j}\{A(m,n) - A(m-1,n) - A(m,n-1) \\ &\quad + A(m-1,n-1)\}. \end{aligned}$$

Thus

$$b\begin{pmatrix} m_1, m_2, \cdots, m_r \\ n_1, n_2, \cdots, n_r \end{pmatrix}$$

is equal to zero, if two of the m's or two of the n's are unequal, and to $A(m,n) - A(m-1,n) - A(m,n-1) + A(m-1,n-1)$, if $m_1 = m_2 = \cdots = m_r = m$; $n_1 = n_2 = \cdots = n_r = n$. Thus $f \cdot E^{-1}$ is the principal function equivalent to $\psi \cdot E^{-1}$.

Corollary 1. *The generating function to the base p of the multiplicative function of $M_1, M_2, \cdots, M_r$, which is equal to their g.c.d., must, by the present theorem, be*

$$\frac{1 - x_1x_2\cdots x_r}{(1-x_1)(1-x_2)\cdots(1-x_r)(1-px_1x_2\cdots x_r)}.$$

This may be easily verified by direct expansion.

COROLLARY 2. *The composite of $F(M_1, M_2, \cdots, M_r)$ and a principal function can be expressed as the composite of $F\cdot E^{-1}$ and a function $K(g)$ of the g.c.d. g of $M_1, M_2, \cdots, M_r$.*

For

$$\begin{aligned} F\cdot \operatorname{princ}\psi &= F\cdot E^{-1}\cdot E\cdot \operatorname{princ}(\psi\cdot E\cdot E^{-1}) \\ &= F\cdot E^{-1}\cdot(\psi\cdot E)(g). \end{aligned}$$

As an illustration we may show that

$$\sum \phi_k\left(\frac{M}{\delta_1}\right)\phi_k\left(\frac{N}{\delta_2}\right)\tau(\delta_{12}) = M^k N^k \frac{\sigma_{2k}(g)}{g^{2k}},$$

where ϕ_k is the Jordan function, and the summation is extended to all divisors δ_1 of M, and δ_2 of N, δ_{12} being the g.c.d. of δ_1, δ_2, and g the g.c.d. of M, N. For, if we write $\Phi_k(M,N) = I_k(M,N)\cdot E^{-1}(M,N)$, so that $\Phi_k(M,N) = \phi_k(M)\phi_k(N)$, the left side is

$$\begin{aligned} \Phi_k\cdot E^2(g) &= \Phi_k\cdot E\cdot \operatorname{princ}(E^2\cdot E^{-1}) \\ &= I_k\cdot \operatorname{princ} E \\ &= M^k N^k \textstyle\sum \delta^{-2k} \text{ summed for common divisors } \delta, \\ &= M^k N^k \sigma_{2k}(g)/g^{2k}. \end{aligned}$$

6. **Semiprincipal functions and functions of the least common multiple** (l.c.m.). The semiprincipal functions are analogous to the principal functions, in that they are determined by functions of a smaller number of arguments; but they are not cardinal functions.

The function $f(M_1, M_2, \cdots, M_r)$ is said to be *the semiprincipal function equivalent to* $\psi(M)$, if, for any prime p, $f(p^{t_1}, p^{t_2}, \cdots, p^{t_r})$ vanishes, whenever any two non-zero exponents t are unequal, and is equal to $(-1)^{r-\mu+1}\psi(p^\tau)$, if μ of the exponents t are zero, and the remaining are equal to τ. Otherwise expressed, the semiprincipal function of $M_1, M_2, \cdots, M_r$, equivalent to $\psi(M)$, vanishes unless the g.c.d. g_{ij} of M_i, M_j is relatively prime both to M_i/g_{ij}, and to M_j/g_{ij} $(i, j = 1, 2, \cdots, r)$, and is equal, when this condition is satisfied, to

$$\frac{\psi(l)E_{-1}(M_1)E_{-1}(M_2)\cdots E_{-1}(M_r)}{E_{-1}(l)},$$

where l is the least common multiple of $M_1, M_2, \cdots, M_r$.

It follows from this definition that if

$$\psi_{(p)}(x) = 1 + a_1x + a_2x^2 + \cdots,$$

then

$$\begin{aligned} f_{(p)}(x_1, x_2, \cdots, x_r) &= 1 + a_1\{ \sum x_1 - \sum x_1x_2 + \sum x_1x_2x_3 - \cdots \} \\ &\quad + a_2\{ \sum x_1^2 - \sum x_1^2 x_2^2 + \sum x_1^2 x_2^2 x_3^2 - \cdots \} + \cdots \\ &= \sum_i \psi_{(p)}(x_i) - \sum_{i,j} \psi_{(p)}(x_i x_j) + \sum_{i,j,k} \psi_{(p)}(x_i x_j x_k) - \cdots . \end{aligned}$$

We may denote f by the symbol semiprinc ψ, or if we wish to put the number of arguments in evidence, by semiprinc$_r$ ψ. It will be noticed that the derivate of semiprinc$_r$ ψ with respect to s of its arguments is simply semiprinc$_{r-s}$ ψ.

An analogous definition holds for the case of a matrix-set of arguments. $f(|M_{i1}, M_{i2}, \cdots, M_{ir}|)$ $(i=1, 2, \cdots, s)$ is called the semiprincipal function of $r\times s$ arguments, equivalent to $\psi(M_1, M_2, \cdots, M_s)$, if, for any prime p, $f(|p^{t_{i1}}, p^{t_{i2}}, \cdots, p^{t_{ir}}|)$ vanishes unless all the non-zero numbers among $t_{i1}, t_{i2}, \cdots, t_{ir}$ are equal to each other $(i=1, 2, \cdots, s)$, and is equal to $(-1)^{\sigma_1+\sigma_2+\cdots+\sigma_s+s}\psi(p^{\tau_1}, p^{\tau_2}, \cdots, p^{\tau_s})$, if all but σ_i of the exponents $t_{i1}, t_{i2}, \cdots, t_{ir}$ are zero, and these equal to $\tau_i (i=1, 2, \cdots, s)$. As an alternative definition, the semiprincipal function of $r\times s$ arguments M_{ij}, equivalent to $\psi(M_1, M_2, \cdots, M_s)$, vanishes whenever the g.c.d. g_{ijk} of M_{ij}, M_{ik} is not relatively prime to both M_{ij}/g_{ijk} and $M_{ik}/g_{ijk} (j, k=1, 2, \cdots, r; i=1, 2, \cdots, s)$, and is otherwise equal to

$$\frac{\psi(l_1, l_2, \cdots, l_s) \prod_{ij} E_{-1}(M_{ij})}{E_{-1}(l_1)E_{-1}(l_2) \cdots E_{-1}(l_s)},$$

where l_i is the least common multiple of $M_{i1}, M_{i2}, \cdots, M_{ir}$.

THEOREM XXXII. *The necessary and sufficient condition that $f(M_1, M_2, \cdots, M_r)$ be a function $\psi(l)$ of the l.c.m. l of $M_1, M_2, \cdots, M_r$, is that f be the integral of the semiprincipal function of r arguments equivalent to $(\psi\cdot E^{-1})(M)$.*

The necessary and sufficient condition that $f(|M_{i1}, M_{i2}, \cdots, M_{ir}|)(i=1, 2, \cdots, s)$ be a function $\psi(l_1, l_2, \cdots, l_s)$ of the l.c.m.'s l_i of $M_{i1}, M_{i2}, \cdots, M_{ir}$, is that f be the integral of the semiprincipal function of $r\times s$ arguments, equivalent to $(\psi\cdot E^{-1})(M_1, M_2, \cdots, M_s)$.

The sufficiency of the conditions may be easily proved for each case. To prove that the condition is necessary, let

$$\begin{aligned} \psi_{(p)}(x) &= \sum A(m)x^m; \\ f_{(p)}(x_1, x_2, \cdots, x_r) &= \sum a(m_1, m_2, \cdots, m_r)x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r}, \\ (f\cdot E^{-1})_{(p)}(x_1, x_2, \cdots, x_r) &= \sum b(m_1, m_2, \cdots, m_r)x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r}, \end{aligned}$$

where

$$b(m_1, m_2, \cdots, m_r) = a(m_1, m_2, \cdots, m_r)$$
$$- \sum a(m_1 - 1, m_2, \cdots, m_r) + \sum a(m_1 - 1, m_2 - 1, m_3, \cdots, m_r) - \text{etc.}$$

Since $f(M_1, M_2, \cdots, M_r)$ is the function ψ of the l.c.m. of $M_1, M_2, \cdots, M_r$, $a(m_1, m_2, \cdots, m_r) = A(m)$, where m is the greatest of the suffixes $m_1, m_2, \cdots, m_r$. As in the proof of Theorem XXXI, let

$$m_1 = m_2 = \cdots = m_i = m;\ m_{i+1}, m_{i+2}, \cdots, m_r < m.$$

Also, let t of the $r-i$ numbers $m_{i+1}, m_{i+2}, \cdots, m_r$ be equal to zero. Grouping the terms in the expression for $b(m_1, m_2, \cdots, m_r)$ in such a way, that in each group the values of the first i suffixes remain the same, and summing each group, we obtain

$$b(m_1, m_2, \cdots, m_r)$$
$$= (1-1)^{r-i-t}\left\{A(m) - A(m)\binom{i}{1} + A(m)\binom{i}{2} - \cdots \right.$$
$$\left. + (-1)^{i-1}\binom{i}{i-1}A_m + (-1)^i A(m-1)\right\}$$
$$= (-1)^{i+1}(1-1)^{r-i-t}\{A(m) - A(m-1)\}.$$

Thus $b(m_1, m_2, \cdots, m_r)$ vanishes unless all the non-zero m's are equal, and is equal to $(-1)^{i+1}(A(m)-A(m-1))$, if $r-i$ of the m's vanish, and the remaining i are all equal to m. This proves that $f\cdot E^{-1}$ is the semiprincipal function of r arguments, equivalent to $(\psi\cdot E^{-1})(M)$.

An exactly similar proof holds for the case of a matrix-set of arguments.

Example 12. The semiprincipal function of two arguments equivalent to an enumerative totient.

Let $\phi = L\cdot E^{-1}$ be the enumerative totient, and let T denote the totient

$$\phi^{-1} \times L = L \times (L^{-1}\cdot E) = L\cdot(L \times L)^{-1}.$$

Also, let the functional symbol $\phi\phi$ denote the function $\phi(M)\phi(N)$ of two arguments. If

$$\phi_{(p)}(x) = \frac{1-x}{1-ax},$$

$$L_{(p)}(x) = \frac{1}{1-ax},$$

then

$$T_{(p)}(x) = \{L\cdot(L \times L)^{-1}\}_{(p)}(x) = \frac{1-a^2x}{1-ax}.$$

Now

$$\{\text{semiprinc}_2\,(\phi)\}_{(p)}(x, y) = \frac{1-x}{1-ax} + \frac{1-y}{1-ay} - \frac{1-xy}{1-axy}$$
$$= \frac{(1-a^2xy)(1-x)(1-y)}{(1-ax)(1-ay)(1-axy)}.$$

Hence,

$$\text{semiprinc}\,\phi = \phi\phi\cdot\text{princ}\,T.$$

As a consequence of this result, it follows that, if l is the l.c.m. of M, N, then

$$L(l) = E\cdot\text{semiprinc}\,(L\cdot E^{-1})$$
$$= E\cdot\phi\phi\cdot\text{princ}\,(L\cdot(L\times L)^{-1})$$
$$= LL\cdot\text{princ}\,(L\cdot(L\times L)^{-1}),$$

or

$$L(l)\cdot\text{princ}\,(L\times L) = LL\cdot\text{princ}\,L.$$

In particular, it also follows that

$$\{I_a(l)\}_{(p)}(x, y) = \frac{1-p^{2a}xy}{(1-p^a x)(1-p^a y)(1-p^a xy)}.$$

We add two illustrative results. On putting $L = I_a$, it follows, from the expression for the semiprincipal function, that

(A) $$I_a(l)\cdot\text{princ}\,I_{2a} = I_a\cdot\text{princ}\,I_a.$$

Composing both sides of this with E, we obtain

$$I_a(l)\cdot E\cdot\text{princ}\,(I_{2a}\cdot E\cdot E^{-1}) = I_a\cdot E\cdot\text{princ}\,(I_a\cdot E\cdot E^{-1})$$

or

$$\sum\left\{l\left(\frac{M}{\delta_1}, \frac{N}{\delta_2}\right)\right\}^a \sigma_{2a}(g(\delta_1, \delta_2)) = \sum\left(\frac{MN}{\delta_1\delta_2}\right)^a \sigma_a(g(\delta_1, \delta_2)).$$

Here, the summation extends over all divisors δ_1 of M, and δ_2 of N, and l and g denote functions, whose values are respectively the l.c.m. and the g.c.d. of their arguments.

Composing both sides of (A) with $E\cdot\text{princ}\,E^{-1}$, we have

$$E\cdot I_a(l)\cdot \text{princ}\,\phi_{2a} = I_a\cdot E\cdot \text{princ}\,\phi_a,$$

where the ϕ's are Jordan functions; or

$$\sum\left\{l\left(\frac{M}{\delta_1},\frac{N}{\delta_2}\right)\right\}^a\{g(\delta_1,\delta_2)\}^{2a} = \sum\left(\frac{MN}{\delta_1\delta_2}\right)^a\{g(\delta_1,\delta_2)\}^a.$$

Section V. Compounding of functions of several arguments

1. The formula for the compound. We define the compound $f\oplus\phi$ of two functions $f(M_1, M_2, \cdots, M_r)$, $\phi(M_1, M_2, \cdots, M_r)$, by

$$\{f\oplus\phi\}(M_1, M_2, \cdots, M_r) = \sum f(\delta_1,\delta_2,\cdots,\delta_r)\phi\left(\frac{M_1}{\delta_1},\frac{M_2}{\delta_2},\cdots,\frac{M_r}{\delta_r}\right)$$

summed for all block-factors δ_i of $M_i (i=1, 2, \cdots, r)$.

It follows from this definition that the compounding operation performed on several functions is both associative and commutative; also, if

$$f_{(p)}(x_1, x_2, \cdots, x_r) = \sum a(m_1, m_2, \cdots, m_r)x_1^{m_1}x_2^{m_2}\cdots x_r^{m_r},$$

$$\phi_{(p)}(x_1, x_2, \cdots, x_r) = \sum b(m_1, m_2, \cdots, m_r)x_1^{m_1}x_2^{m_2}\cdots x_r^{m_r},$$

then

$$(f\oplus\phi)_{(p)}(x_1, x_2, \cdots, x_r) = \sum c(m_1, m_2, \cdots, m_r)x_1^{m_1}x_2^{m_2}\cdots x_r^{m_r},$$

where

$$c(m_1,m_2,\cdots,m_r) = a(m_1,m_2,\cdots,m_r) + \sum a(0,m_2,\cdots,m_r)b(m_1,0,0,\cdots,0)$$
$$+ \sum a(0,0,m_3,\cdots,m_r)b(m_1,m_2,0,\cdots,0) + \text{etc.}$$

Here on the right, the non-zero suffixes among $m_1, m_2, \cdots, m_r$ are partitioned in every possible way between a and b, so that the expression for $c(m_1, m_2, \cdots, m_r)$ contains 2^i terms, when $r=i$ of $m_1, m_2, \cdots, m_r$ are zero.

The expression for $c(m_1, m_2, \cdots, m_r)$ shows that

Theorem XXXIII. *The derivate of the compound of any number of functions is the same as the compound of the corresponding derivates of the functions.*

Another expression may be given to the compound of f and ϕ. Write

$$\psi_0(x_1, x_2, \cdots, x_r) = f_{(p)}(x_1, x_2, \cdots, x_r)$$
$$+ \sum f_{(p)}(0, x_2, \cdots, x_r)\phi_{(p)}(x_1, 0, \cdots, 0)$$
$$+ \sum f_{(p)}(0,0,x_3,\cdots,x_r)\phi_{(p)}(x_1,x_2,0,\cdots,0) + \text{etc.},$$

$$\psi_0(x_1, x_2, \cdots, x_i) = f_{(p)}(x_1, x_2, \cdots, x_i, 0, \cdots, 0)$$
$$+ \sum f_{(p)}(0, x_2, \cdots, x_i, 0, \cdots, 0)\phi_{(p)}(x_1, 0, \cdots, 0)$$
$$+\text{etc.},$$
$$\psi_i(x_1, x_2, \cdots, x_r) = \sum \psi_0(x_1, x_2, \cdots, x_i),$$

summed for every set of i of the variables.

Then

$$(f \oplus \phi)_{(p)}(x_1, x_2, \cdots, x_r) = \psi_0 - \psi_1 + \psi_2 - \psi_3 + \cdots .$$

For the coefficient of $x_1^{m_1}x_2^{m_2} \cdots x_k^{m_k}$ $(m_1, m_2, \cdots, m_k > 0)$ on the left is

$$C_k = a(m_1, \cdots, m_k, 0, \cdots, 0)$$
$$+ \sum a(0, m_2, \cdots, m_k, 0, \cdots, 0)b(m_1, 0, \cdots, 0) + \sum \text{etc.},$$

while its coefficient on the right is seen to be

$$c_k\left(2^{r-k} - \binom{r-k}{1}2^{r-k-1} + \cdots\right) = (2-1)^{r-k}c_k = c_k.$$

This formula for the generating series of the compound is more complicated than the corresponding formula for functions of one argument. But it is important to notice that the two formulas are of the same type, if either f or ϕ is a cardinal function. For, if f is a cardinal function, then every coefficient a with a zero suffix vanishes, and the expression for c becomes simply

$$c(m_1, m_2, \cdots, m_r) = a(m_1, m_2, \cdots, m_r) + b(m_1, m_2, \cdots, m_r).$$

The generating series of the compound of functions, all but one of which are cardinal functions, is therefore obtained by adding the generating series of all the functions, and replacing the constant term by unity.

It follows that, while the distributive and quasidistributive properties of the compounding operation (Theorems XIII and XIV) do not hold in general for functions of several arguments, the former continues to hold in the same form, when all but one of the functions compounded are cardinal functions, and the latter, when in addition, the function which enters into composition with the compound is also a cardinal function. Thus if f is a cardinal function, and all but one of $f_1, f_2, \cdots, f_k$ are cardinal functions, then

$$f\cdot(f_1 \oplus f_2 \oplus \cdots \oplus f_k) = (f\cdot f_1) \oplus (f\cdot f_2) \oplus \cdots \oplus (f\cdot f_k) \oplus (E_{1-k} \times f),$$

where E_{1-k} is the elementary function of r arguments, which takes the value $1-k$ when the arguments are powers of the same prime. We notice that $E(M_1, M_2, \cdots, M_r)$ compounded with itself $k-1$ times does not give $E_k(M_1, M_2, \cdots, M_r)$ but $E_k(M_1)E_k(M_2) \cdots E_k(M_r)$.

From the fact that the derivate of the compound is the compound of the

derivates (Theorem XXXIII), it readily follows that theorems like XXVIII and XXIX should hold with respect to compounding also; namely

THEOREM XXXIV. *If two functions of r arguments have identical derivates, each is the compound of the other with a cardinal function.*

If two functions of a matrix-set of arguments have identical column-derivates each is the compound of the other with a cardinal function of the matrix-set.

2. **The conjugate function.** We define the conjugate function by

$$\operatorname{conj} f \oplus f = E_0.$$

We shall determine the conjugate of a given function, and thereby show that it exists and is unique. Let

$$f_{(p)}(x_1, x_2, \cdots, x_r) = \sum a(m_1, m_2, \cdots, m_r) x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r};$$

$$(\operatorname{conj} f)_{(p)}(x_1, x_2, \cdots, x_r) = \sum b(m_1, m_2, \cdots, m_r) x_1^{m_1} x_2^{m_2} \cdots x_r^{m_r}.$$

Since the derivate of a compound is the compound of the corresponding derivates, it follows from the definition of the conjugate function that any derivate of $\operatorname{conj} f$ is the conjugate of the corresponding derivate of f. Hence from the formula for the conjugate of a function of one argument (II §6), it follows that

$$b(m_1, 0, 0, \cdots, 0) = -a(m_1, 0, \cdots, 0) \qquad (m_1 \neq 0).$$

Similarly, by taking the derivate of $\operatorname{conj} f \oplus f$ with respect to $r-2$ of the arguments, we have

$$b(m_1, m_2, 0, \cdots, 0) + b(0, m_2, 0, \cdots, 0)a(m_1, 0, \cdots, 0)$$
$$+ b(m_1, 0, \cdots, 0)a(0, m_2, 0, \cdots, 0) + a(m_1, m_2, 0, \cdots, 0) = 0 \quad (m_1, m_2 \neq 0).$$

Therefore, using the previous result,

$$b(m_1, m_2, 0, \cdots, 0) = -a(m_1, m_2, 0, \cdots, 0)$$
$$+ 2!a(m_1, 0, \cdots, 0)a(0, m_2, 0, \cdots, 0).$$

This process might be continued. We shall prove by induction that, generally,

$$\text{(A)}\ b(m_1, m_2, \cdots, m_r)$$
$$= \sum_i (-1)^i i! \sum a(m_1, m_2, 0, \cdots, 0)a(0, 0, m_3, m_4, m_5, 0, \cdots, 0)a(\cdots),$$

where, on the right, the suffixes of the a's constitute a partition of the non-vanishing m's into i parts, and the second summation is for all such distinct partitions.

Assume this formula to be true for conjugates of functions of $r-1$ arguments. Then by taking derivates of both sides of the relation

$$\operatorname{conj} f \oplus f = E_0,$$

it follows that the formula is also true for functions of less than $r-1$ arguments. To establish its truth for the conjugate of a function f of r arguments, we observe that it follows from the definition of the conjugate that

$$\text{(B)}\quad 0 = b(m_1, m_2, \cdots, m_r) + \sum b(0, m_2, \cdots, m_r)a(m_1, 0, \cdots, 0) \\ + \cdots + a(m_1, m_2, \cdots, m_r).$$

If any of the m's is zero, then the expression for b follows from the proved truth of formula (A) for functions of less than $r-1$ arguments. We may therefore take all the m's to be different from zero. Substituting on the right of (B) for b's with zero suffixes, from formula (A), it becomes evident that the expression for $b(m_1, m_2, \cdots, m_r)$ is a sum of products of a's, whose non-zero suffixes constitute a partition of $m_1, m_2, \cdots, m_r$, multiplied by certain numerical coefficients. To find the numerical coefficients of any a-product corresponding to an i-part partition, say $a(m_1, m_2, 0, \cdots, 0)a(0, 0, m_3, m_4, m_5, 0, \cdots, 0) \cdots$, we observe that this product could have come from exactly i terms on the right of (B), namely

$$a(m_1, m_2, 0, \cdots, 0)b(0, 0, m_3, \cdots, m_r),$$
$$a(0, 0, m_3, m_4, m_5, 0, \cdots, 0)b(m_1, m_2, 0, 0, 0, m_6, \cdots, m_r),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Substituting for the b's from (A), it follows that the product in question occurs on the right of (B) with the coefficient $i\cdot(-1)^{i-1}(i-1)!$. Hence it occurs in the expression for $b(m_1, m_2, \cdots, m_r)$ with the coefficient $(-1)^i i!$, proving the truth of (A) for functions of r arguments. Since (A) has been proved to be true for functions of 2 arguments, the induction is complete.

In particular, for the conjugate of a cardinal function f, we have the result that if

$$f_{(p)}(x_1, x_2, \cdots, x_r) = 1 + \sum a(m_1, m_2, \cdots, m_r)x_1^{m_1}x_2^{m_2}\cdots x_r^{m_r} \\ (m_1, m_2, \cdots, m_r = 1, 2, \cdots),$$

then

$$(\operatorname{conj} f)_{(p)}(x_1, x_2, \cdots, x_r) = 1 - \sum a(m_1, m_2, \cdots, m_r)x_1^{m_1}x_2^{m_2}\cdots x_r^{m_r}.$$

Thus the formula which gives the conjugate of a cardinal function is analogous to the formula for conjugates of functions of a single argument.

3. **The compounding of functions of a matrix-set.** The two theories of the compounding process, which have been stated for functions of one argument and of several arguments, respectively, have yielded formally different results, even though they have the same arithmetical basis. The reason

for the difference lies in the fact that, while in $(f\oplus\phi)(M)$ a term of the type $f(1)\phi(M)$ occurs only once, there are in $(f\oplus\phi)(M_1, M_2, \cdots, M_r)$ (2^r-1) terms of the type $f(\delta_1, \delta_2, \cdots, \delta_r)\times\phi(M_1/\delta_1, M_2/\delta_2, \cdots, M_r/\delta_r)$, in which one or more of the block-factors δ are equal to unity. It is of course open to us to develop a theory of the compound of functions of several arguments, which is formally similar to that for a single argument; we could to this by using the definition

$$(f \oplus \phi)(M_1, M_2, \cdots, M_r) = \sum f(\delta_1, \delta_2, \cdots, \delta_r)\phi\left(\frac{M_1}{\delta_1}, \frac{M_2}{\delta_2}, \cdots, \frac{M_r}{\delta_r}\right),$$

where the summation is for all sets $\delta_1, \delta_2, \cdots, \delta_r$ of block-factors, which have the property that no factor of any δ is prime to any other δ (in other words, that the distinct prime factors of every δ are the same), with a similar property for the set M_i/δ_i. It is clear that the compound defined thus is a multiplicative function, and indeed (as will be seen below) a cardinal function, of $M_1, M_2, \cdots, M_r$. *We shall describe this definition of the compound, by saying that, in it, the argument-group* $(M_1, M_2, \cdots, M_r)$ *behaves like a single argument.* The possibility of this alternative definition suggests a more general view of the compounding operation, which embraces the two theories as aspects of itself.

Let a given set of arguments be divided into mutually exclusive groups in any manner. The most general definition of the compound of two functions of these arguments is then that *the compound is formed in the usual way, with the modification that each group of arguments behaves like a single argument, in the sense defined above.*

This generalized view of the compound will always be understood, whenever there is any indication of grouping in the arguments. For example, when we talk of the compound of two functions of a matrix set of arguments, it will be understood, by convention, that the arguments in each column of the matrix have been grouped together, so as to behave like a single argument in the above sense.

It is easy to see that the generating series, to any base, of the compound of two functions f, ϕ of the matrix $|M_{i1}, M_{i2}, \cdots, M_{ir}|$ $(i=1, 2, \cdots, s)$, need not be worked out ab initio, but may be seen at a glance, by a proper interpretation of the formula (V §1) for the case of r arguments, viz.

$$\begin{aligned} c(m_1, m_2, \cdots, m_r) = {} & a(0, 0, \cdots, 0)b(m_1, \cdots, m_r) \\ & + \sum a(m_1, 0, \cdots, 0)b(0, m_2, \cdots, m_r) + \text{etc.} \end{aligned}$$

To do this, we consider that $\sum a(m_1, m_2, \cdots, m_r)x_1^{m_1}\cdots x_r^{m_r}$ and $\sum b(m_1, \cdots, m_r)x_1^{m_1}\cdots x_r^{m_r}$ still represent the generating series of the functions f, ϕ of the

matrix set, on the understanding that each $x_j^{m_j}$ stands for $x_{1j}^{m_{1j}}x_{2j}^{m_{2j}}\cdots x_{sj}^{m_{sj}}$, and each m_j in the coefficients stands for the group of indices $m_{1j}, m_{2j}, \cdots, m_{sj}$. If, further, we interpret the occurrence of a zero in place of m_j to mean $m_{1j}=m_{2j}=\cdots=m_{sj}=0$, then the above formula determines the generating series of the compound. For, the expression for the compound of two functions f, ϕ, when regarded as functions of the matrix $|M_{ij}|$, differs from the corresponding expression when they are regarded as functions of the $r\times s$ arguments M_{ij}, in the suppression of all those terms $f(\delta_{ij})\phi(M_{ij}/\delta_{ij})$, for which the elements in any column of $|\delta_{ij}|$ or $|M_{ij}/\delta_{ij}|$ do not all involve the same prime factors; hence it follows that the expression for the general coefficient $c(m_{ij})$ of the generating series of the compound in the former case, differs from its expression in the latter in the suppression of those terms $a(m_{ij}), b(m_{ij})$, in which some only of the elements of a column of $|m_{ij}|$ are zero.

In particular, it follows from the expression for $c(m_{ij})$, that $c(m_{ij})=0$, when one element from each column of $|m_{ij}|$ is zero, without all the m's vanishing simultaneously.

Definition. A function F of the matrix $|M_{i1}M_{i2}\cdots M_{ir}|$ $(i=1,2,\cdots,s)$ will be called a *transcardinal function*, if its derivate with respect to *every* set of r arguments, chosen one from each column, is the function E_0.

The following points may be noted in connection with this definition and the correlated definition of the cardinal function of a matrix:

(1) A transcardinal function of $|M_{ij}|$ is necessarily a cardinal function of the transposed matrix $|M_{ji}|$, but the converse is not true.

(2) A transcardinal function of the "long matrix" $|M_1, M_2, \cdots, M_r|$ is any arbitrary function of $M_1, M_2, \cdots, M_r$; a transcardinal function of the "deep matrix"

$$|M_i| = \begin{vmatrix} M_1 \\ M_2 \\ \cdot \\ \cdot \end{vmatrix}$$

is a cardinal function of $M_1, M_2, \cdots, M_r$.

(3) A cardinal function of the long matrix $|M_1, \cdots|$ is a cardinal function of $M_1, M_2, \cdots, M_r$, a cardinal function of the deep matrix $|M_i|$ is an arbitrary function of $M_1, M_2, \cdots, M_r$.

THEOREM XXIII(a). *The compound of any two functions of a matrix $|M_{ij}|$ is a transcardinal function of $|M_{ij}|$.*

This may be seen directly from the definition of the compound, or thus.

By definition, a transcardinal function vanishes when one argument in each of the columns of $|M_{ij}|$ is equal to 1, unless all the other arguments are also equal to 1. Hence, the coefficient $c(m_{ij})$ of the general term of its generating series vanishes when one element from each column of $|m_{ij}|$ is zero, unless all the remaining m's are also zero. But this property was noted above to hold for the compound of two functions of a matrix, hence the theorem.

From this theorem it follows that if f is a function of the matrix which is not transcardinal, $f\oplus E_0$ cannot be equal to f; as a matter of fact we see directly that $f\oplus E_0$ is that transcardinal function of the matrix, which is equal to f for all sets of values of the arguments, except those for which it vanishes in virtue of being a transcardinal function. We shall write

$$f \oplus E_0 = \operatorname{tcrd} f,$$

and call $\operatorname{tcrd} f$ *the transcardinal part of* f.

THEOREM XXIII(b). *The compound of two functions f, ϕ of a matrix depends only on the transcardinal parts of f and ϕ.*

For, if $f\oplus\phi=\psi$, then ψ is transcardinal by the previous theorem, so that $\operatorname{tcrd}\psi=\psi$. Hence

$$\begin{aligned} f \oplus \phi = \psi = \operatorname{tcrd}\psi &= \psi \oplus E_0 \\ &= f \oplus \phi \oplus E_0 = f \oplus \operatorname{tcrd}\phi. \end{aligned}$$

The conjugate of the function f of the matrix $|M_{ij}|$ is defined to be another function $\operatorname{conj} f$ of the same matrix, such that

$$f \oplus \operatorname{conj} f = E_0.$$

Since the formula for the generating series of the conjugate function is simply a consequence of that for a compound, it follows that we can deduce the generating series of $\operatorname{conj} f$ from that of f, from the result (V §2)

$$b(m_1, m_2, \cdots, m_r) = \sum(-1)^i i! \sum a(m_1, m_2, 0, \cdots)a(0, 0, m_3, \cdots) \cdots,$$

by proceeding in the same manner as was done in the case of the compound. In particular it follows that the conjugate of a cardinal function f of the matrix is another cardinal function of the same matrix, whose generating series to any base is obtained from that of f, by changing the sign of all the terms except the constant term.

4. **Rational functions of several arguments.** (Cf. III.) The function $f(M_1, M_2, \cdots, M_r)$ is said to be a *rational function*, if

$$f_{(p)}(x_1, x_2, \cdots, x_r) = \frac{F_{(p)}(x_1, x_2, \cdots, x_r)}{\phi_{(p)}(x_1, x_2, \cdots, x_r)},$$

where $F_{(p)}$ and $\phi_{(p)}$ are polynomials for every p, whose degrees ρ_i, σ_i in x_i have finite upper bounds as $p \to \infty$ $(i = 1, 2, \cdots, r)$; the functions whose generating series are $\phi_{(p)}$ and $F_{(p)}$ are, respectively, the *integral* and *inverse component* of f.

Since polynomials in more than one variable are not necessarily factorable, a rational function can not in general be expressed as a composite of functions of lower degrees.

The composite of rational functions f_1, f_2 is clearly a rational function, whose integral and inverse components are, respectively, the composites of the integral and inverse components of f_1, f_2. Thus composition is a rational and integral process. It is easy to show that the compound of f_1, f_2 is a rational function whose integral component is the composite of the integral components of f_1, f_2, and of all products of complementary derivates of f_1, f_2. The product of rational functions is also a rational function. This may be proved by a method similar to the one adopted for functions of a single argument. It could be inferred directly, if Hadamard's multiplication theorem can be extended to power series in several variables; the extension appears however to have been carried out so far only to two variables.*

We have to distinguish between a *linear function* of r arguments, and a rational integral function of degree 1; the former is a product of linear functions of single arguments, and is a particular case of the latter. We define a *totient* as the composite of a linear function and the inverse of a linear function; thus the totient is, like the linear function, a *separable function*, being the product of totients of single arguments (a function of r arguments is *separable*, if it is the product of functions of fewer arguments). Thus a totient of r arguments is not the most general rational function of degree (1, 1) in each of the arguments.

SECTION VI. THE IDENTICAL EQUATION OF THE MULTIPLICATIVE FUNCTION

1. The cardinal functions associated with a function of r arguments. The multiplicative property of $f(M_1, M_2, \cdots, M_r)$ implies that

$$f(M_1N_1, M_2N_2, \cdots, M_rN_r) = f(M_1, M_2, \cdots, M_r)f(N_1, N_2, \cdots, N_r),$$

whenever the products $M_1M_2 \cdots M_r$, $N_1N_2 \cdots N_r$ are relatively prime. Now the functions on both sides of this equation are functions of the matrix-

* Cf. U. S. Haslam-Jones, *An extension of Hadamard's multiplication theorem*, Proceedings of the London Mathematical Society, (2), vol. 27.

set of $2\times r$ arguments $|M_i, N_i|$ $(i=1, 2, \cdots, r)$, and the equation asserts the equality of the functions, whenever the column-products are relatively prime. Hence it follows from Theorems XXIX, and XXXIV, that $f(M_1N_1, M_2N_2, \cdots, M_rN_r)$ is

(1) the composite of ff and a cardinal function C_1 of the matrix $|M_i, N_i|$ (where we use ff as the functional symbol for $f(M_1, M_2, \cdots, M_r)f(N_1, N_2, \cdots, N_r)$),

(2) the product of ff and the integral of a cardinal function C_2 of the same matrix,

(3) and, lastly, the compound of ff and a cardinal function C_3, of the same matrix.

It will be convenient to refer to C_1, C_2, C_3, as the *first*, *second*, and *third cardinal functions* associated with $f(M_1, M_2, \cdots, M_r)$. Unlike the second and third cardinal functions, the first cardinal function can be expressed in a simple manner in terms of f, by means of the processes of the calculus. Namely, when F is a function of a matrix-set of arguments, let us denote by crd F (*the cardinal of* F) that cardinal function of the matrix-set, which is equal to F, except for those values of the arguments for which it vanishes in virtue of being a cardinal function. It is easy to see that the generating series of crd F is obtained from the corresponding generating series of F, by suppressing all those terms, in which the variables corresponding to a column of the matrix-set are entirely absent. With this notation, it will now be shown that the first cardinal function of $f(M_1, M_2, \cdots, M_r)$ is the conjugate* of crd $f^{-1}(M_1N_1, M_2N_2, \cdots, M_rN_r)$, where f^{-1} represents the inverse function of r arguments, of f.

2. The expression for the first cardinal function. The first cardinal function $C_1(|M_i, N_i|)$ associated with f is given by

$$f(M_1N_1, M_2N_2, \cdots, M_rN_r) = ff\cdot C_1.$$

Now, the inverse of ff is $f^{-1}f^{-1}$ (Theorem X). Hence

$$C_1 = f(M_1N_1, \cdots, M_rN_r)\cdot f^{-1}f^{-1}.$$

To show that this is equal to

$$\text{conj crd } f^{-1}(M_1N_1, M_2N_2, \cdots, M_rN_r),$$

we shall, for convenience of writing, take $r=2$, the method being the same for any value of r. Let

* Notice that this is the conjugate of a function of a *matrix*, and must therefore be understood not in the sense of V §2, but in the generalized sense of V §3.

$$f_{(p)}(x_1, x_2) = \sum a(m_1, m_2) x_1^{m_1} x_2^{m_2},$$

$$\overset{-1}{f}_{(p)}(x_1, x_2) = \sum b(m_1, m_2) x_1^{m_1} x_2^{m_2},$$

so that

$$\text{(A)} \qquad \sum b(m_1 - \mu_1, m_2 - \mu_2) a(\mu_1, \mu_2) = 1 \text{ or } 0,$$

according as m_1, m_2 are, or are not, simultaneously zero ($\mu_1 = 0, 1, \cdots, m_1$; $\mu_2 = 0, 1, \cdots, m_2$). Then,

$$\{f(M_1N_1, M_2N_2)\}_{(p)}(x_1, y_1, x_2, y_2) = \sum a(m_1 + n_1, m_2 + n_2) x_1^{m_1} y_1^{n_1} x_2^{m_2} y_2^{n_2};$$

$$\{f^{-1}(M_1, M_2) f^{-1}(N_1, N_2)\}_{(p)}(x_1, x_2, y_1, y_2) = \sum b(m_1, m_2) b(n_1, n_2) x_1^{m_1} y_1^{n_1} x_2^{m_2} y_2^{n_2}.$$

Hence, if

$$C_{1(p)}(x_1, x_2, y_1, y_2) = \sum c(m_1, m_2, n_1, n_2) x_1^{m_1} x_2^{m_2} y_1^{n_1} y_2^{n_2},$$

we must have

$$c(m_1, m_2, n_1, n_2) = \sum a(\mu_1 + \nu_1, \mu_2 + \nu_2) b(m_1 - \mu_1, m_2 - \mu_2) b(n_1 - \nu_1, n_2 - \nu_2)$$

$$(\mu_1 = 0, 1, \cdots, m_1;\ \mu_2 = 0, 1, \cdots, m_2;\ \nu_1 = 0, 1, \cdots, n_1;\ \nu_2 = 0, 1, \cdots, n_2)$$

$$= \sum_{\nu_1, \nu_2} b(n_1 - \nu_1, n_2 - \nu_2) \sum_{\mu_1, \mu_2} a(\mu_1 + \nu_1, \mu_2 + \nu_2) b(m_1 - \mu_1, m_2 - \mu_2).$$

Since C_1 is a cardinal function, $c(m_1, m_2, n_1, n_2)$ vanishes when m_1, m_2 (or n_1, n_2) are simultaneously zero. We may therefore suppose that m_1, m_2 are not both zero, so that $m_1 + \nu_1$ and $m_2 + \nu_2$ are not simultaneously zero, for any admissible values of ν_1, ν_2. Hence, by (A) above,

$$\sum_{\mu_1, \mu_2} a(\mu_1 + \nu_1, \mu_2 + \nu_2) b(m_1 - \mu_1, m_2 - \mu_2)$$

$$= - \sum_{k_1, k_2} a(\nu_1 - k_1, \nu_2 - k_2) b(m_1 + k_1, m_2 + k_2)$$

$$(k_1 = 0, 1, \cdots, \nu_1;\ k_2 = 0, 1, \cdots, \nu_2;\ k_1 \text{ and } k_2 \text{ not both zero}).$$

Therefore,

$$c(m_1, m_2, n_1, n_2) = \sum - b(n_1 - \nu_1, n_2 - \nu_2) a(\nu_1 - k_1, \nu_2 - k_2) b(m_1 + k_1, m_2 + k_2)$$

$$= \sum_{k_1, k_2} - b(m_1 + k_1, m_2 + k_2) \sum_{\nu_1, \nu_2} b(n_1 - \nu_1, n_2 - \nu_2) a(\nu_1 - k_1, \nu_2 - k_2)$$

$$(\nu_1 = k_1, k_1 + 1, \cdots, n_1;\ \nu_2 = k_2, k_2 + 1, \cdots, n_2).$$

It is clear that here the value of the inner sum is zero, unless $n_1 - k_1, n_2 - k_2$ are simultaneously zero, in which case it is 1. Now $k_1 = n_1, k_2 = n_2$ is a set of values within the range of k_1, k_2, only when n_1, n_2 are not both zero. We see, therefore, that

$$c(m_1, m_2, n_1, n_2) = 0, \text{ if either } m_1, m_2, \text{ or } n_1, n_2 \text{ are both zero,}$$
$$= -b(m_1 + n_1, m_2 + n_2), \text{ otherwise.}$$

But the cardinal function, the generating series of which has $-b(m_1+n_1, m_2+n_2)$ for the coefficient of $x_1^{m_1}x_2^{m_2}y_1^{n_1}y_2^{n_2}$, is obviously conj crd $f^{-1}(M_1N_1, M_2N_2)$.* The same method shows that, generally,

$$c_1 = \text{conj crd } f^{-1}(M_1N_1, M_2N_2, \cdots, M_rN_r).$$

The equation

$$f(M_1N_1, M_2N_2, \cdots, M_rN_r) = ff\cdot \text{conj crd } f^{-1}(M_1N_1, M_2N_2, \cdots, M_rN_r)$$

is thus identically satisfied by any multiplicative function f, and will be referred to as *the identical equation of f*.

For functions of one argument, the identical equation takes the form

$$f(uv) = \{f(u)f(v)\}\cdot\{\text{conj crd } f^{-1}(uv)\}.$$

3. **The identical equation of a totient.** Let $T = L_1\cdot L_2^{-1}$ be a totient-function of a single argument, so that its identical equation is

$$T(MN) = T(M)T(N)\cdot \text{conj crd } T^{-1}(MN).$$

Let

$$T_{(p)}(x) = \frac{1-\beta x}{1-\alpha x}; \quad \overset{-1}{T}_{(p)}(x) = \frac{1-\alpha x}{1-\beta x}.$$

Then

$$\{\text{conj crd } T^{-1}(MN)\}_{(p)}(x, y) = 1 - \beta^2\left(1 - \frac{\alpha}{\beta}\right)xy\{\textstyle\sum\beta^{m+n}x^my^n\}$$
$$= \frac{1-\beta(x+y)+\alpha\beta xy}{(1-\beta x)(1-\beta y)}.$$

Hence

$$\text{conj crd } T^{-1}(MN) = L_2(M)L_2(N)\cdot\{\overset{-1}{L_2}(\text{l.c.m of } M, N)\times \overset{-1}{L_1}(\text{g.c.d of } M, N)\}.$$

Thus the identical equation for T becomes

$$T(MN) = L_1(M)L_1(N)\cdot\overset{-1}{L_2}(M)\overset{-1}{L_2}(N)\cdot L_2(M)L_2(N)\cdot\{\overset{-1}{L_2}(l)\times\overset{-1}{L_1}(g)\}$$
$$= L_1(M)L_1(N)\cdot\{\overset{-1}{L_2}(l)\times\overset{-1}{L_1}(g)\}$$
$$= L_1(M)L_1(N)\cdot\{L_2(l)\times L_1(g)\times\mu(M)\mu(N)\}$$

where g, l are, respectively, the g.c.d. and the l.c.m. of M, N.

For example, the identical equation of the totient $\phi_{r,k} = I_r\cdot\lambda_k^{-1}$ (which

* Compare V §3.

reduces to Jordan's function for $k=1$, and for $k=$ a positive integer, is identical with Schemmel's extension of Euler's function,* when the argument has no prime factors equal to, or less than k) is

$$\phi_{r,k}(MN) = \sum\left(\frac{MN}{\delta_1\delta_2}\right)^r \lambda_k\{l(\delta_1, \delta_2)\}\{g(\delta_1, \delta_2)\}^r \mu(\delta_1)\mu(\delta_2)$$

or

$$\text{(A)} \qquad \frac{\phi_{r,k}(MN)}{(MN)^r} = \sum E_k(\delta_1\delta_2)\cdot\frac{\{g(\delta_1, \delta_2)\}^r}{(\delta_1\delta_2)^r}\mu(\delta_1)\mu(\delta_2).$$

If T is an enumerative totient, then $L_2=E$, and the identical equation takes the form

$$T(MN) = L_1(M)L_1(N)\cdot\{L_1(g)\mu(M)\mu(N)\}.$$

In particular, for the Jordan function

$$\frac{\phi_r(MN)}{M^rN^r} = \sum\frac{\{g(\delta_1, \delta_2)\}^r}{(\delta_1\delta_2)^r}\mu(\delta_1)\mu(\delta_2),$$

a result which may also be derived from (A).

4. **The Busche-Ramanujan identity.** E. Busche† stated that

$$\sigma_a(M)\sigma_a(N) = \sum d^a\sigma_a\left(\frac{MN}{d^2}\right),$$

summed over the common divisors d of M and N; Ramanujan has utilised this result (for the case $a=0$), as well as its inverse form:‡

$$\sigma_a(MN) = \sum\sigma_a\left(\frac{M}{d}\right)\sigma_a\left(\frac{N}{d}\right)d^a\mu(d).$$

We shall say generally that a function $f(M_1, M_2, \cdots, M_r)$ admits *a Busche-Ramanujan identity*, if we have identically

$$f(M_1N_1, M_2N_2, \cdots, M_rN_r)$$
$$= \sum f\left(\frac{M_1}{\delta_1}, \frac{M_2}{\delta_2}, \cdots, \frac{M_r}{\delta_r}\right) f\left(\frac{N_1}{\delta_1}, \frac{N_2}{\delta_2}, \cdots, \frac{N_r}{\delta_r}\right) F(\delta_1, \delta_2, \cdots, \delta_r),$$

* Dickson, p. 147.
† Dickson, p. 319, Note 147.
‡ Collected Papers, p. 134.

summed for common divisors δ_i of M_i, N_i $(i=1, 2, \cdots, r)$, F being some (necessarily multiplicative) function of r arguments.

A relation of this form asserts that $f(M_1N_1, M_2N_2, \cdots, M_rN_r)$ is the composite of ff and a principal function of the matrix-set $|M_i, N_i|$ $(i=1, 2, \cdots, r)$. Hence a function f admits a Busche-Ramanujan identity, if, and only if, its first cardinal function is a principal function; that is, only if conj crd $f^{-1}(M_1N_1, M_2N_2, \cdots, M_rN_r)$, and therefore also crd $f^{-1}(M_1N_1, M_2N_2, \cdots, M_rN_r)$, becomes a principal function. To find the condition that this may be the case, let

$$\overset{-1}{f}_{(p)}(x_1, x_2, \cdots, x_r) = \sum b(m_1, m_2, \cdots, m_r)x_1^{m_1}x_2^{m_2}\cdots x_r^{m_r},$$

so that

$$\begin{aligned}\{\operatorname{crd} f^{-1}(M_1N_1, M_2N_2, \cdots, M_rN_r)\}_{(p)}(x_1, \cdots, x_r, y_1, \cdots, y_r)&\\ = 1 + \sum b(m_1+n_1, \cdots, m_r+n_r)x_1^{m_1}y_1^{n_1}\cdots x_r^{m_r}y_r^{n_r}&,\end{aligned}$$

where, on the right, simultaneous zero values of $m_1, m_2, \cdots, m_r$ and $n_1, n_2, \cdots, n_r$ are excepted from the summation. The condition that this may be a principal function is the vanishing of $b(m_1+n_1, m_2+n_2, \cdots, m_r+n_r)$ when any $m_i \neq n_i$ (with the understanding that neither all the m's nor all the n's vanish together). In other words $b(\lambda_1, \lambda_2, \cdots, \lambda_r)=0$, whenever the λ's admit a partition $\lambda_i = m_i+n_i$ $(i=1, 2, \cdots, r)$, in which at least one $m_i\neq n_i$, neither the m's nor the n's being all zero. Hence

(1) $b(\lambda_1, \lambda_2, \cdots, \lambda_r)=0$, if any $\lambda>2$. For if $\lambda_i=3$ (say), we can write $\lambda_i=m_i+n_i$; $m_i=1$, $n_i=2$; $m_i\neq n_i$; $m_i\neq 0$, $n_i\neq 0$.

(2) $b(\lambda_1, \lambda_2, \cdots, \lambda_r)=0$, if at least two of the λ's do not vanish. For, if $\lambda_1\neq 0$, $\lambda_2\neq 0$, we can write $\lambda_1=m_1+n_1$; $\lambda_2=m_2+n_2$; $m_1=\lambda_1$, $n_1=0$; $m_2=0$, $n_2=\lambda_2$.

It results that $b(\lambda_1, \lambda_2, \cdots, \lambda_r)$ can have only $2r$ non-vanishing values b_i, c_i, corresponding to $\lambda_i=1$ or 2; $\lambda_j=0$ for $j\neq i$. Thus it follows that

$$\begin{aligned}\overset{-1}{f}_{(p)}(x_1, x_2, \cdots, x_r) = 1 + b_1x_1 + b_2x_2 + \cdots&\\ + b_rx_r + c_1x_1^2 + c_2x_2^2 + \cdots + c_rx_r^2&.\end{aligned}$$

Therefore

$$f_{(p)}(x_1, x_2, \cdots, x_r) = \frac{1}{1+\sum b_ix_i + \sum c_ix_i^2}.$$

Hence $f(M_1, M_2, \cdots, M_r)$ is an integral function of a special type, quadratic in each of its arguments, its special property being that $f^{-1}(M_1, M_2, \cdots, M_r)$

vanishes unless *every* two of its arguments are relatively prime.* Also, when f is a function of this form,

$$\{\text{conj crd} f^{-1}(M_1N_1, M_2N_2, \cdots, M_rN_r)\}_{(p)}(x_1, \cdots, x_r, y_1, \cdots, y_r)$$
$$= 1 - \sum c_i x_i y_i.$$

Now the derivate of $f(M_1, M_2, \cdots, M_r)$ with respect to the $r-1$ arguments other than M_i, is evidently an integral quadratic function of M_i, which we may denote by $L_i \cdot L_i'$; we have

$$(L_i \cdot L_i')_{(p)}(x_i) = \frac{1}{1 + b_i x_i + c_i x_i^2} \quad (i = 1, 2, \cdots, r);$$

$$(L_i \times L_i')^{-1}_{(p)}(x_i) = 1 - c_i x_i.$$

It therefore follows that conj crd $f^{-1}(M_1N_1, \cdots, M_rN_r)$ is a principal function, princ F (say), where

$$F(M_1, M_2, \cdots, M_r) = 0, \text{ if two of the arguments have a common factor,}$$
$$= \overset{-1}{K_1}(M_1)\overset{-1}{K_2}(M_2) \cdots \overset{-1}{K_r}(M_r), \text{ otherwise,}$$

where

$$K_i = L_i \times L_i' \qquad (i = 1, 2, \cdots, r).$$

We have thus reached the following theorem:

THEOREM XXXV. *The only functions of r arguments which admit a Busche-Ramanujan identity are the integral quadratic functions, whose inverses vanish unless every two of the arguments are relatively prime. The identity for a function f of this type has the form*

$$f(M_1N_1, \cdots, M_rN_r) = ff \text{ princ } F,$$

where $F(M_1, M_2, \cdots, M_r) = 0$, *if any two of its arguments have a common factor, and is otherwise equal to* $K_1^{-1}(M_1)K_2^{-1}(M_2) \cdots K_r^{-1}(M_r)$; $K_i = L_i \times L_i'$, *and* $L_i \cdot L_i'(M_i)$ *is the derivate of f with respect to the* $r-1$ *arguments other than* M_i.

In particular, the functions f of one argument, which admit a Busche-Ramanujan identity, are the integral quadratic functions and these only. When $f = L_1 \cdot L_2$ *the identity has the form*

$$f(MN) = f(M)f(N) \cdot \text{princ } (L_1 \times L_2)^{-1}.$$

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* Functions with this property are, in a sense, the exact opposites of cardinal functions; they may be called "anti-cardinal functions."

As illustrations of the identity for functions of one argument, we have

(1) since $\sigma_a = I_a \cdot E$,

$$\sigma_a(MN) = \sum \sigma_a\left(\frac{M}{\delta}\right)\sigma_a\left(\frac{N}{\delta}\right)\delta^a \mu(\delta),$$

summed for common divisors δ, of M and N;

(2) in particular,

$$\sigma_a(N^2) = \sum \left\{\sigma_a\left(\frac{N}{\delta}\right)\right\}^2 \delta^a \mu(\delta), \text{ summed for divisors } \delta, \text{ of } N.$$

Generally, if $f = L_1 \cdot L_2$, and $L_{12} = L_1 \times L_2$, then when M is a multiple of N,

$$f(MN) = \sum f\left(\frac{M}{\delta}\right) f\left(\frac{N}{\delta}\right) \overset{-1}{L}_{12}(\delta),$$

$$f(M)f(N) = \sum f\left(\frac{MN}{\delta^2}\right) L_{12}(\delta),$$

the summation extending to divisors δ, of N, in each case.

In particular, if $M = N^k$,

$$f(N^{k+1}) = \sum f\left(\frac{N^k}{\delta}\right) f\left(\frac{N}{\delta}\right) \overset{-1}{L}_{12}(\delta);$$

$$f(N^k)f(N) = \sum f\left(\frac{N^{k+1}}{\delta^2}\right) L_{12}(\delta).$$

A different form may also be given to the Busche-Ramanujan identity for $f(M) = (L_1 \cdot L_2)(M)$. Namely, we have

$$\begin{aligned} f(MN) &= f(M)f(N) \cdot \text{conj crd } f^{-1}(MN) \\ &= ff \cdot \overset{-1}{E} \cdot\ E \cdot \text{princ } (\overset{-1}{L}_{12} \cdot E \cdot \overset{-1}{E}) \\ &= ff \cdot \overset{-1}{E} \cdot \left\{ (\overset{-1}{L}_{12} \cdot E)(g) \right\}, \end{aligned}$$

where g is the g.c.d. of M, N.

For example,

$$\begin{aligned} \sigma_a(MN) &= I_a \cdot \left\{ (\overset{-1}{I_a} \cdot E)(g) \right\} \\ &= \sum \left(\frac{MN}{\delta_1 \delta_2}\right)^a F\left\{ g(\delta_1\, \delta_2) \right\}\ (\delta_1/M,\ \delta_2/N), \end{aligned}$$

where $g(\delta_1, \delta_2)$ denotes the g.c.d. of δ_1, δ_2 and $F(N) =$ the product $\Pi(1 - p^a)$ extended over all the prime factors of N.

An alternative form of the same result is

$$f(M)f(N) = f(MN)\cdot E^{-1}\cdot E\cdot \text{princ } (L_{12}\cdot E\cdot E^{-1})$$
$$= f(MN)\cdot E^{-1}\cdot \{(L_{12}\cdot E)(g)\}$$

or

$$(f\cdot E)(M)(f\cdot E)(N) = \sum f\left(\frac{M\,N}{\delta_1\,\delta_2}\right)(L_{12}\cdot E)(g(\delta_1, \delta_2)),$$

which is valid, when f is any integral quadratic function.

5. **The second and third cardinal functions.** We shall next investigate the form of the functions, whose second, or third, cardinal functions become principal functions.

The second cardinal function $C_2(|M_i, N_i|)$ of $f(M_1, M_2, \cdots, M_r)$ is given by

$$f(M_1N_1, M_2N_2, \cdots, M_rN_r) = ff \times (C_2\cdot E).$$

Writing

$$f_{(p)}(x_1, x_2, \cdots, x_r) = \sum a(m_1, m_2, \cdots, m_r)x_1^{m_1}x_2^{m_2}\cdots x_r^{m_r},$$

it follows that

$$(C_2\cdot E)_{(p)}(x_1, \cdots, x_r, y_1, \cdots, y_r)$$
$$= \sum \frac{a(m_1 + n_1, m_2 + n_2, \cdots, m_r + n_r)}{a(m_1, m_2, \cdots, m_r)a(n_1, n_2, \cdots, n_r)} x_1^{m_1}x_2^{m_2}\cdots x_r^{m_r}y_1^{n_1}\cdots y_r^{n_r}.$$

This determines C_2. If C_2 is to be a principal function, $C_2\cdot E$ must be a function of the g.c.d.'s g_i of M_i, N_i (Theorem XXXI). Hence

$$\frac{a(m_1 + n_1, m_2 + n_2, \cdots, m_r + n_r)}{a(m_1, m_2, \cdots, m_r)a(n_1, n_2, \cdots, n_r)}$$

must depend only on the smaller element of each of the pairs (m_1, n_1), (m_2, n_2), $\cdots$. Hence, if $m_1 \geqq n_1$,

$$\frac{a(m_1 + k + n_1, m_2 + n_2, \cdots, m_r + n_r)}{a(m_1 + k, m_2, \cdots, m_r)a(n_1, n_2, \cdots, n_r)}$$

is independent of k, and is equal to

$$\frac{a(m_1 + n_1, \cdots, m_r + n_r)}{a(m_1, \cdots, m_r)a(n_1, n_2, \cdots, n_r)}.$$

It follows from this that if $m_i > 0$,

$$a(m_1, m_2, \cdots, m_r) = \lambda_i^{m_i-1}a(m_1, m_2, \cdots, m_{i-1}, 1, m_{i+1}, \cdots, m_r)(i = 1, 2, \cdots, r).$$

Also

$$\frac{a(1,1,0,\cdots,0)}{a(0,1,0,\cdots,0)a(1,0,\cdots,0)}=\frac{a(0,1,0,\cdots,0)}{a(0,1,0,\cdots,0)a(0,0,\cdots,0)}=1,$$

since in both cases the lesser of m_i, n_i is the same. Hence

$$a(1,1,0,\cdots,0)=a(0,1,0,\cdots,0)a(1,0,\cdots,0).$$

Similarly

$$a(1,1,\cdots,1)=a(1,0,0,\cdots,0)\alpha(0,1,0,\cdots,1,0)\cdots.$$

Thus

$$a(m_1,m_2,\cdots,m_i,0,\cdots,0)=\{\lambda_1^{m_1-1}a(1,0,\cdots,0)\}\{\lambda_2^{m_2-1}a(0,1,0,\cdots,0)\}\cdots,$$

where

$$m_1,m_2,\cdots,m_i>0.$$

Writing

$$a(1,0,\cdots,0)=\lambda_1-a_1;\quad a(0,1,0,\cdots,0)=\lambda_2-a_2,\text{ etc.},$$

it follows that

$$f_{(p)}(x_1,x_2,\cdots,x_r)=\frac{(1-a_1x_1)(1-a_2x_2)\cdots(1-a_rx_r)}{(1-\lambda_1x_1)(1-\lambda_2x_2)\cdots(1-\lambda_rx_r)}.$$

Thus f is a totient. Also it is easy to see that $(C_2\cdot E)(|M_i,N_i|)$ now becomes $T'(g_1,g_2,\cdots,g_r)$, where T' is the level totient such that $T'\times f=$ the linear component of f. Hence

THEOREM XXXVI. *The totients are the only functions of r arguments, whose second cardinal function is a principal function. Also, if $T(M_1,M_2,\cdots,M_r)$ is a totient, and T' is the level totient such that $T\times T'$ is a linear function,*

$$T(M_1N_1,M_2N_2,\cdots,M_rN_r)$$
$$=T(M_1,M_2,\cdots,M_r)T(N_1,N_2,\cdots,N_r)T'(g_1,g_2,\cdots,g_r).$$

For instance, if $\phi_k(M)$ is the Jordan function,

$$\frac{\phi_k(MN)}{\phi_k(M)\phi_k(N)}=\frac{g^k}{\phi_k(g)}.$$

More generally, if $\phi_{k,r}=I_k\cdot\lambda_r^{-1}$ is the Jordan-Schemmel totient (VII §3),

$$\frac{\phi_{k,r}(MN)}{\phi_{k,r}(M)\phi_{k,r}(N)}=\frac{g^k}{\phi_{k,r}(g)}.$$

The third cardinal function is less interesting than the other two, and becomes a principal function only in a trivial case.

THEOREM XXXVII. *If the third cardinal function C_3 of f is a principal function, it is the principal function E_0, and f must be a linear function.*

For, with our previous notation, it follows that when C_3 is a principal function, we must have,

$$a(m_1 + n_1, m_2 + n_2, \cdots, m_r + n_r) - a(m_1, m_2, \cdots, m_r)a(n_1, n_2, \cdots, n_r) = 0,$$

whenever any $m_i \neq n_i$ $(i=1, 2, \cdots, r)$. Hence writing

$$a_1 = a(1, 0, \cdots, 0); \; a_2 = a(0, 1, 0, \cdots, 0) \text{ etc.},$$

we have

$$a(m_1, 0, \cdots, 0) = a(2, 0, \cdots, 0)a^{m_1-2} \qquad (m_1 \geqq 2),$$

and similar equations. Also, if $m_1 \neq 2$, $n_1 \neq 2$, $m_1 \neq n_1$,

$$\begin{aligned} \{a(2, 0, \cdots, 0)\}^2 a_1^{m_1+n_1-4} &= a(m_1, 0, \cdots, 0)a(n_1, 0, \cdots, 0) \\ &= a(m_1 + n_1, 0, \cdots, 0) \\ &= a(2, 0, \cdots, 0)a_1^{m_1+n_1-2}. \end{aligned}$$

Hence $a(2, 0, \cdots, 0) = a_1^2$, so that $a(m_1, 0, \cdots, 0) = a_1^{m_1}(m_1 = 0, 1, 2, \cdots)$. Also, if $m_1, m_2, \cdots, m_i \neq 0$,

$$\begin{aligned} &a(m_1, \cdots, m_i, 0, \cdots, 0) \\ &\quad = a(m_1, 0, \cdots, 0)a(0, m_2, 0, \cdots, 0) \cdots a(0, \cdots, 0, m_i, 0, \cdots, 0) \\ &\quad = a_1^{m_1} a_2^{m_2} \cdots a_i^{m_i}. \end{aligned}$$

Thus f is the linear function whose generating series

$$f_{(p)}(x) = \frac{1}{\Pi(1 - a_i x_i)}.$$

6. **The restricted Busche-Ramanujan identity.** If $F(M, N)$, $F'(M, N)$ are two functions with identical derivates, then Theorems XXVIII and XXXIV state that each of F, F' is the composite of the other with a cardinal function, and also the compound of the other with a cardinal function. We can however state the relation between F, F' in the more general form, that each of F, F' can be obtained from the other by composing it first with a cardinal function, and then compounding the composite with a second cardinal function; that is,

$$F' = (F \cdot C_1) \oplus C_2.$$

Now, if $f(M)$ be a function of a single argument, $f(MN)$ and $f(M)f(N)$ have identical derivates. The relation between $f(MN)$ and $f(M)f(N)$ can therefore be put in the general form

$$f(MN) = (ff\cdot C_1) \oplus C_2,$$

where C_1, C_2 are cardinal functions of M, N. It is clear that, in this relation, either of the cardinal functions C_1, C_2 may be arbitrarily chosen, and that the other becomes then determinate. We enquire, in regard to this equation, when it can happen that both C_1 and C_2 are principal functions.

The arithmetical significance of the assumption that C_1 and C_2 are simultaneously principal functions is clearly that f satisfies a Busche-Ramanujan identity for certain restricted values of M, N. For it is evident that the compound of $F(M, N)$ and a principal function of M, N has the same value as $F(M, N)$, whenever M, N do not both contain any prime factor p raised to the same power. Thus, when C_1, C_2 are principal functions, f satisfies the Busche-Ramanujan identity

$$f(MN) = ff\cdot C_1,$$

for all such values of M, N as have no common block-factor. We say in this case that f admits a *restricted Busche-Ramanujan identity*.

To investigate when C_1, C_2 are both principal functions, take

$$\begin{aligned} f_{(p)}(x) &= 1 + a_1x + a_2x^2 + \cdots, \\ \{f(MN)\}_{(p)}(x, y) &= \sum a_{m+n}x^my^n, \\ C_{1(p)}(x, y) &= 1 + t_1xy + t_2x^2y^2 + \cdots, \\ C_{2(p)}(x, y) &= 1 + k_1xy + k_2x^2y^2 + \cdots. \end{aligned}$$

Then

$$\begin{aligned} \sum a_{m+n}x^my^n = (\textstyle\sum a_mx^m)(\sum a_ny^n)&(1 + t_1xy + t_2x^2y^2 + \cdots) \\ &+ k_1xy + k_2x^2y^2 + \cdots \end{aligned}$$

(V §1). Hence, if $m \neq n$,

$$\text{(A)} \qquad a_{m+n} = a_ma_n + t_1a_{m-1}a_{n-1} + t_2a_{m-2}a_{n-2} + \cdots.$$

Hence, if $m > n+2$,

$$\begin{aligned} a_ma_{n+2} &+ a_{m-1}a_{n+1}t_1 + \cdots + a_{m-n-2}t_{n+2} \\ &= a_{m+n+2} \\ &= a_1a_{m+n+1} + a_{m+n}t_1 \text{ (by (A))} \end{aligned}$$

$$= a_1(a_m a_{n+1} + a_{m-1}a_n t_1 + \cdots + a_{m-n-1}t_{n+1})$$
$$+ t_1(a_m a_n + a_{m-1}a_{n-1}t_1 + \cdots + a_{m-n}t_n)$$
$$= a_m a_{n+2} + a_{m-1}a_{n+1}t_1 + \cdots + a_{m-n}t_n(a_1^2 + t_1) + a_1 a_{m-n-1}t_{n+1},$$

so that

(1) $$a_{m-n-2}t_{n+2} + t_n a_{m-n}(a_2 - a_1^2 - t_1) = 0 \quad (m > n+2).$$

By varying m in this equation, it follows that

(2) $$\frac{a_r}{a_{r-2}} = \text{a constant } \lambda = \frac{t_2}{a_1^2 + t_1 - a_2} \quad (r > 2).$$

Similarly, by varying n, we have

(3) $$t_{n+2} = t_n t_2.$$

Now from (2),

$$a_3 = a_1 a_2 + a_1 t_1 = a_1 \lambda,$$

therefore

(4) $$\text{either } a_1 = 0, \text{ or } \lambda = a_2 + t_1.$$

Again from (2), $a_1 a_4 = a_2 a_3$. Hence

$$a_2 a_3 + t_1(a_1 a_2 + a_1 t_1) = a_1 a_4 + a_3 t_1 = a_5 = a_2 a_3 + a_1 a_2 t_1 + a_1 t_2.$$

Hence,

(5) $$\text{either } a_1 = 0, \text{ or } t_2 = t_1^2.$$

Now, the relation $a_{m+1} - a_m a_1 - a_{m-1}t_1 = 0$ holds from $m = 2$ onwards, on account of (A). Hence the series $f_{(p)}(x)$ is a recurring series of the second order, its value, in finite terms, being evidently

$$f_{(p)}(x) = \frac{1 + (a_2 - a_1^2 - t_1)x^2}{1 - a_1 x - t_1 x^2}.$$

There are now three possibilities to be considered.

Case 1. $a_2 - a_1^2 - t_1 = 0$. For this case the element of f to the base p is that of an integral quadratic function; from (2) and (3), we easily see that $t_n = 0$ for $n > 1$. We also see that C_2 reduces to E_0, so that our restricted identity becomes the unrestricted identity, which, we know, is satisfied by every integral quadratic function.

Case 2. $a_2 - a_1^2 - t_1 \neq 0$, $a_1 \neq 0$. For this case it follows from (4), (5), that

$$t_2 = t_1^2, \quad \frac{t_2}{a_1^2 + t_1 - a_2} = \lambda = a_2 + t_1.$$

These imply the vanishing of $a_2^2 - a_1^2(a_2+t_1)$, which is the resultant of the numerator and denominator of $f_{(p)}(x)$. Thus the element of f reduces to that of a totient. Conversely, when f is the totient $L_1 \cdot L_2^{-1}$,

$$f_{(p)}(x) = \frac{1-\alpha x}{1-\beta x};$$

$$\{f(MN)\}_{(p)}(x, y) = \frac{1-\alpha(x+y)+\alpha\beta xy}{(1-\beta x)(1-\beta y)}$$

$$= \frac{(1-\alpha x)(1-\alpha y)}{(1-\beta x)(1-\beta y)(1-\alpha\beta xy)} - \frac{\alpha^2 xy}{1-\alpha\beta xy}.$$

Hence $f = L_1 \cdot L_2^{-1}$ has the restricted identity

$$f(MN) = \{ff \cdot \operatorname{princ}(L_1 \times L_2)\}.$$

Case 3. $a_2 - a_1^2 - t_1 \neq 0$, $a_1 = 0$. Now

$$f_{(p)}(x) = \frac{1+(a_2-t_1)x^2}{1-t_1x^2},$$

so that f is the convolute of a totient. Conversely, when f is the second convolute of a totient,

$$f_{(p)}(x) = \frac{1+\alpha x^2}{1-\beta x^2};$$

$$\{f(MN)\}_{(p)}(x, y) = \frac{1+\alpha(x^2+y^2)-\alpha\beta x^2y^2+(\alpha+\beta)xy}{(1-\beta x^2)(1-\beta y^2)}$$

$$= \frac{(1+\alpha x^2)(1+\alpha y^2)(1+\beta xy)}{(1-\beta x^2)(1-\beta y^2)(1+\alpha\beta x^2y^2)} + \frac{\alpha xy(1-\alpha xy)}{1+\alpha\beta x^2y^2}.$$

Hence if f is the second convolute of the totient $T = L_1 \cdot L_2^{-1}$, it has the restricted identity

$$f(MN) = ff \cdot \operatorname{princ} F,$$

valid whenever M, N have no common block-factor, where

$$F(N) = \frac{L_1(N)L_2(D)}{L_1(D)},$$

D being the greatest number whose square divides N. Combining these results, we have

THEOREM XXXVIII. *The only functions of M which admit a restricted Busche-Ramanujan identity* (*namely an identity valid when M, N have no common block-factor*) *are* (1) *the integral quadratic functions*, (2) *totients*, (3) *second convolutes of totients, and* (4) *crosses between these types.*

As illustrations of the theorem, we have the following identities, valid when M, N have no common block-factor:*

(1) $$\phi(MN) = \sum\phi\left(\frac{M}{\delta}\right)\phi\left(\frac{N}{\delta}\right)\delta,$$

summed for common divisors δ, of M, N, where ϕ is Euler's function.

(2) $$\text{conv }\phi(MN) = \sum \text{conv }\phi\left(\frac{M}{\delta}\right)\text{conv }\phi\left(\frac{N}{\delta}\right)F(\delta),$$

where $F(N)$ is the least divisor of N, which is divisible by its complementary divisor.

Section VII. The theory of Smith's determinant

1. **Ordinal functions of r arguments.** A function $f(M_1, M_2, \cdots, M_r)$ of r arguments may be called generally an ordinal function, if it vanishes whenever certain prescribed inequalities of the form $M_i > M_j$ hold between the arguments. Since we restrict ourselves to multiplicative functions, this property may be shown to imply a more specialized property of f.

Theorem XXXIX. *If a multiplicative function of $M_1, M_2, \cdots, M_r$ vanishes whenever $M_i > M_j$, it necessarily vanishes whenever M_i is not a factor of M_j.*

For by hypothesis, $f(p^{m_1}, p^{m_2}, \cdots, p^{m_r})$ vanishes, for any prime p, whenever $m_i > m_j$; that is, unless p^{m_i} is a factor of p^{m_j}. Hence $f(M_1, M_2, \cdots, M_r)$ vanishes if any prime p occurs to a higher power in M_i than in M_j, that is, unless M_i is a factor of M_j.

We shall concern ourselves only with two types of ordinal functions; namely, the functions $f(M_1, M_2, \cdots, M_r)$, which vanish whenever a prescribed argument M_i is greater than any of the remaining arguments, and functions $F(M_1, M_2, \cdots, M_r)$ which vanish whenever a prescribed argument M_j is less than any of the other arguments. We call $f(M_1, M_2, \cdots, M_r)$ a *minor ordinal function* with *the minor argument* M_i, and $F(M_1, M_2, \cdots, M_r)$ *a major ordinal function* with *the major argument* M_j. We shall usually indicate the major or minor argument by writing it without the suffix.

2. **Major ordinal functions and functions with a modulus.** There exist multiplicative functions $f(M_1, M_2, \cdots, M_{r-1}, M)$, whose value is unaltered if M_i is increased by any multiple of M $(i = 1, 2, \cdots, r-1)$; such functions may be said to possess the *modulus* M. We shall now show that, with each

* These two results (the first of which is due to Mr. S. S. Pillai, as already stated) have been published as questions for solution in the Journal of the Indian Mathematical Society (December, 1928, last page, Nos. (1529) and (1530)).

function $f(M_1, M_2, \cdots, M_{r-1}, M)$ possessing the modulus M, we can associate a major ordinal function $F(M_1, M_2, \cdots, M_{r-1}, M)$ with the major argument M, in such a way that

$$f(M_1, M_2, \cdots, M_{r-1}, M) = F(g_1, g_2, \cdots, g_{r-1}, M),$$

where g_i is the g.c.d. of M_i, M $(i=1, 2, \cdots, r-1)$.

For, since $f(M_1, M_2, \cdots, M_{r-1}, M)$ has the modulus M, and is a multiplicative function of its arguments, it follows that

$$f(M_1, M_2, \cdots, M_{r-1}, 1) = f(1, 1, \cdots, 1).$$

Hence $D_M(f)=E$, so that by Theorem XXVI, f is a multiplicative function of its modulus M alone. Further, let N be any number prime to the modulus M. Then by a well known theorem, M_1, M_1+M, $M_1+2M, \cdots$, $M_1+(N-1)M$ is a complete residue system mod N. Hence there exist integers $\lambda_1, \lambda_2, \cdots, \lambda_{r-1}$, such that $M_i+\lambda_iM$ is prime to N $(i=1, 2, \cdots, r-1)$. Hence

$$\begin{aligned} f(M_1, M_2, &\cdots, M_{r-1}, M) \\ &= f(1, 1, \cdots, 1)f(M_1, M_2, \cdots, M_{r-1}, M) \\ &= f(N, 1, \cdots, 1)f(M_1 + \lambda_1M, M_2 + \lambda_2M, \cdots, M_{r-1} + \lambda_{r-1}M, M) \\ &= f(M_1N + \lambda_1MN, M_2 + \lambda_2M, \cdots, M_{r-1} + \lambda_{r-1}M, M) \end{aligned}$$

(since f is a multiplicative function of its r arguments)

$$= f(M_1N, M_2, \cdots, M_{r-1}, M)$$

(since f has the modulus M).

Hence the $r-1$ arguments $M_1, M_2, \cdots, M_{r-1}$ of a function $f(M_1, M_2, \cdots, M_{r-1}, M)$ with the modulus M can not only be increased by multiples of M, but can also be multiplied by any number prime to M, without affecting the value of the function. Hence f depends on $M_1, M_2, \cdots, M_{r-1}$ only through their g.c.d.'s with M.

Now, if $f(M_1, M_2, \cdots, M_{r-1}, M)$ be *any* multiplicative function, we can define a function $F(M_1, M_2, \cdots, M_{r-1}, M)$ by

$$F(M_1, M_2, \cdots, M_{r-1}, M) = f(M_1, M_2, \cdots, M_{r-1}, M),$$

when each M_i is a factor of M,

$$= 0, \text{ in other cases}.$$

The multiplicative character of F follows from that of f. Thus an arbitrary function $f(M_1, M_2, \cdots, M_{r-1}, M)$ defines a major ordinal function $F(M_1, \cdots,$

$M_{r-1}, M)$; but it is obvious that F does not determine f uniquely. If, however, $f(M_1, M_2, \cdots, M_{r-1}, M)$ possesses the modulus M, then from what we have proved, f both determines and is determined uniquely by F.

As an illustration, we may mention von Sterneck's function $f(N, M)$ which represents the excess of the number of partitions of N into an even number of parts mod M, over the number of those into an odd number. This function can be shown to be multiplicative, and possesses, by definition, the modulus M. Hence it must be a multiplicative function of the modulus alone, and can be expressed as a function of M and its g.c.d. with N.*

3. **Minor ordinal functions.** If $f(M, M_1, \cdots, M_{r-1})$ is a minor ordinal function with the minor argument M, we shall call $\psi(M)=f(M, M, \cdots, M)$ the *kernel* of f.

We can associate with every minor ordinal function $f(M, M_1, \cdots, M_{r-1})$ a general multiplicative function $F(M, N_1, N_2, \cdots, N_{r-1})$, of r arguments, defined by

$$F(M, N_1, N_2, \cdots, N_{r-1}) = f(M, MN_1, MN_2, \cdots, MN_{r-1}).$$

Conversely, from any multiplicative function $F(M, N_1, \cdots, N_{r-1})$, we can define a minor ordinal function $f(M, M_1, \cdots, M_{r-1})$ by

$$\begin{aligned} f(M, M_1, \cdots, M_{r-1}) &= F\left(M, \frac{M_1}{M}, \frac{M_2}{M}, \cdots, \frac{M_{r-1}}{M}\right), \text{when } M \text{ divides each } M_i, \\ &= 0, \text{ otherwise}. \end{aligned}$$

Thus the minor ordinal functions are in reversible one-to-one correspondence with all multiplicative functions. We observe, in particular, that the derivate of the function $F(M, N_1, N_2, \cdots, N_{r-1})$ with respect to $N_1, N_2, \cdots, N_{r-1}$ is the kernel of the corresponding ordinal function $f(M, M_1, \cdots, M_{r-1})$.

The case in which the associated function $F(M, N_1, \cdots, N_{r-1})$ is of the form $\psi(M)\psi(N_1, N_2, \cdots, N_{r-1})$ is of special importance. For this case we have

THEOREM XL. *When the associated function of the ordinal function $f(M, M_1, \cdots, M_{r-1})$, with the minor argument M, is of the form $\psi(M)\psi(N_1, N_2, \cdots, N_{r-1})$, then*

(1) *ψ is the kernel of f,*

(2) *the ordinal function f is the composite of* princ *ψ and a function of $M_1, M_2, \cdots, M_{r-1}$ only.*

Conversely, the composite of a principal function P of r arguments, and

* See Bachmann, *Niedere Zahlen-Theorie*, part 2, p. 230 ff.

any function of $r-1$ of the arguments, is a minor ordinal function, whose kernel is the equivalent function of P.

The first part of the theorem follows from the fact that the kernel is the derivate of the associated function. To prove the second part, let the associated function have the generating series

$$F_{(p)}(x, x_1, \cdots, x_{r-1}) = \sum a_m b_{m_1 m_2 \cdots m_{r-1}} x^m x_1^{m_1} \cdots x_{r-1}^{m_{r-1}}.$$

Then, the generating series to the same base of the minor ordinal function is evidently

$$\sum a_m b_{m_1 m_2 \cdots m_{r-1}} x^m x_1^{m+m_1} x_2^{m+m_2} \cdots x_{r-1}^{m+m_{r-1}}$$
$$= \left\{ \sum a_m (x x_1 \cdots x_{r-1})^m \right\} \left\{ \sum b_{m_1 m_2 \cdots m_{r-1}} x_1^{m_1} x_2^{m_2} \cdots x_{r-1}^{m_{r-1}} \right\},$$

which proves the theorem.

Definition. The composite of a minor ordinal function with any function of its minor argument M, will be called a Smith function in respect to M. Also, the kernel of the ordinal function will be termed the kernel of the corresponding Smith function.

It follows from this definition, that, if $S(M, M_1, \cdots, M_{r-1})$ is a Smith function in respect to M, there exists a function $\psi(M)$, such that $S\cdot\psi$ is a minor ordinal function with the minor argument M.

4. **Smith's determinant.** Let $S(M, M_1, \cdots, M_{r-1})$ be a Smith's function in respect to M. The values taken by S when its r arguments range from 1 to m may be taken as the elements of an r-dimensional matrix of the mth order, $|S(M, M_1, \cdots, M_{r-1})|$, the r arguments of S serving as the *indices* of the matrix. This matrix can be evaluated as a determinant, if we assign a *signant* or a *non-signant* character to each index.* We shall call the de-

* The modern theory of the r-dimensional determinant

$$|a(m_1, m_2, \cdots, m_r)| \qquad (m_1, m_2, \cdots, m_r = 1, 2, \cdots, n)$$

assigns either a *signant* or a *non-signant* character to each index m. The value of the determinant is

$$\frac{1}{n!}\sum \pm\, a(m_{11}, m_{21}, \cdots, m_{r1})a(m_{12}, m_{22}, \cdots, m_{r2}) \cdots a(m_{1n}, m_{2n}, \cdots, m_{rn}),$$

summed for all permutations $(m_{i1}\, m_{i2} \cdots m_{in})$ of $1, 2, \cdots, n$ $(i=1, 2, \cdots, r)$, so that there are $(n!)^r$ terms in the summation. If ϵ_i represents $+1$ or -1 according as $(m_{i1}\, m_{i2} \cdots m_{in})$ is an even or odd permutation of $1, 2, \cdots, n$, the sign of the general term is defined to be $\prod \epsilon_i$, where the product extends over those values of i for which m_i is a signant index.

The theory is due to Rice (*P-way determinants with an application to transvectants*, American Journal of Mathematics, vol. 40 (1918)) and was also discovered independently by Lecat and the present writer (*On mixed determinants*, Proceedings of the Royal Society of Edinburgh, 1925). For further information reference may be made to the works of Lecat, e.g. *Coup d'Oeil sur la Théorie des Déterminants Supérieurs*, Bruxelles, 1927, and also to a recent article of Lecat, *Le déterminant supérieur, qu'est il exactement? Les conceptions de Cayley, Gasparis, Rice et autres*, Revue Générale des Sciences, 1929.

terminant $|S(M, M_1, \cdots, M_{r-1})|$ a *Smith determinant*, if the signant character has been assigned to the index M.

THEOREM XLI. *The value of the Smith determinant* $|S(M, M_1, \cdots, M_{r-1})| = \Delta_m$, *of order* m, *is zero if the number of signant indices is odd, and is* $F(1)F(2) \cdots F(m)$ *otherwise,* F *being the kernel of* S.

The first part follows from the fact that a determinant vanishes identically, unless it has an even number of signant indices.

The elements of the Smith determinant Δ_m, in which the index M has the value k, constitute an $(r-1)$-dimensional matrix of the mth order, which we may call the kth section of Δ_m. Since M is a signant index in Δ_m, it results that we may add to the elements of a section, any the same multiples of the corresponding elements of any other section. Also, since S is a Smith function, there exists a function $\psi(M)$, such that $S \cdot \psi$ is an ordinal function with the minor argument M. Therefore if we add to the elements of the mth section $\psi(m/\delta)$ times the δth section (where δ represents successively each divisor of m other than m itself), and denote the new elements of the mth section by $S'(m, M_1, \cdots, M_{r-1})$, we have

$$\begin{aligned} S'(m, M_1, \cdots, M_{r-1}) & \\ &= (S \cdot \psi)(m, M_1, \cdots, M_{r-1}) \\ &= 0, \text{ if any } M_i \neq m, \\ &= (S \cdot \psi)(m, m, \cdots, m) \\ &= F(m), \text{ if } M_1 = M_2 = \cdots = M_{r-1} = m. \end{aligned}$$

Now, expanding the determinant in terms of the elements of the mth section, it follows immediately that

$$\begin{aligned} \Delta_m &= F(m)\Delta_{m-1} \\ &= kF(m)F(m-1) \cdots F(2)F(1), \end{aligned}$$

where k is easily seen to be zero or 1, according as the total number of signant indices is even or odd.

It will be noticed that the result does not depend on whether the signant or non-signant character is assigned to the remaining indices.

The following are applications of the theorem:

(1) Any function $F(g)$ of the g.c.d. of $M_1, M_2, \cdots, M_r$ is a Smith function in respect to any of the arguments, with the kernel $F \cdot E^{-1}$. For,

$$F(g) = E\cdot\text{princ}\,(F\cdot E^{-1}),$$

therefore

$$E^{-1}(M_i)\cdot F(g) = E(M_1, \cdots, M_{i-1}, M_{i+1}, \cdots, M_r)\cdot\text{princ}\,(F\cdot E^{-1})$$

= an ordinal function, with the minor argument M_i, and the kernel $F\cdot E^{-1}$. Hence the corresponding Smith determinant of order m and any number of dimensions has the value

$$(F\cdot E^{-1})(m) \times (F\cdot E^{-1})(m-1) \times \cdots,$$

if the number of signant indices is even.

(2) Any linear function $L(l)$ of the l.c.m. l of two arguments M, N is a Smith function with respect to M or N. For

$$\begin{aligned} L(l) &= E\cdot\text{semiprinc}\,(L\cdot E^{-1}) \\ &= E\cdot\left\{(L\cdot E^{-1})(M)(L\cdot E^{-1})N\right\}\cdot\text{princ}\left\{(E\cdot L^{-1})\times L\right\}\text{(Example 12)} \\ &= L(M)L(N)\cdot\text{princ}\left\{(E\cdot L^{-1})\times L\right\}. \end{aligned}$$

Hence

$$\begin{aligned} L^{-1}(M)\cdot L(l) &= L(N)\cdot\text{princ}\left\{(E\cdot L^{-1})\times L\right\} \\ &= \text{an ordinal function with the minor argument } M, \text{ and kernel} \\ &\quad (E\cdot L^{-1})\times L. \end{aligned}$$

Hence the Smith determinant of order m and two dimensions formed with the elements $L(g)$, has the value

$$\prod L(j)(E\cdot L^{-1})(j) \qquad (j = 1, 2, \cdots, m).$$

In particular, if $L = I$, the value is

$$\prod\phi(j)\pi(j) \qquad (j = 1, 2, \cdots, m) \quad \text{(Cesàro)*}$$

where $\Pi(j)$ is the product of the negatives of the prime factors of j.

(3) Von Sterneck's function $f(N, M)$, which is equal to the excess of the number of partitions of N into an even number of distinct parts mod M, over the number of those into an odd number (zero not being admitted as a part), is a multiplicative function with the modulus M, and may be shown to be equal to $E^{-1}(M)E(N)\cdot\text{princ}\,I$.† Hence it is a Smith function with the kernel I, the value of the corresponding Smith determinant being therefore $m!$.

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* Dickson, p. 128, 61.

† Cf. Bachmann, loc. cit.

(4) If $f(M)=(L_1 \cdot L_2)(M)$ is an integral quadratic function, $f(MN)$ is a Smith function with the kernel $(L_1 \times L_2)^{-1}$; for, from the Busche-Ramanujan identity,

$$f(MN) = f(M)f(N) \cdot \text{princ } (L_1 \times L_2)^{-1}.$$

Since the kernel is the inverse of a linear function, it vanishes for all numbers with a squared factor, and so the corresponding Smith determinant of order m vanishes unless $m<4$.

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REMARKS CONCERNING THE PAPER OF W. L. AYRES* ON THE REGULAR POINTS OF A CONTINUUM†

BY

KARL MENGER

The reading of Ayres' interesting paper suggested to me the following remarks:

1. The order of a subset of a set S in a point p‡ cannot surpass the order of S in p. Hence if S^2 denotes the set of all points of S of order 2, then S^2 has in each point of S the order 2, the order 1, or the order 0, where the terms "order 0" and "0-dimensional" are used synonymously. $S^{2(0)}$, $S^{2(1)}$, S^{II} may denote the set of all points of S in which S^2 has the order 0, 1, 2, respectively. The points of order 2 of S are also called the *ordinary points* of S, and the set S^2 of all ordinary points of S may be called the *ordinary part* of S. The set S^{II} of all ordinary points of the ordinary part of S may be designated the *ordinary kernel* of S. We have

$$S^2 = S^{2(0)} + S^{2(1)} + S^{\mathrm{II}}.$$

Ayres has proved§ that an ordinary point of a continuum C in which the set C^2 is at least one-dimensional (i.e., in our terminology a point of $C^{2(1)} + C^{\mathrm{II}}$), and which is besides a local cut point of C, is always an end point of a *free arc* of C; that is, of an arc which, after omission of its two end points, is open in C. Exactly in the same way it can be proved that a point p of C^{II}, which is a local cut point of C, lies in the interior of a free arc of C. Ayres further proves‖ that each point of $C^{2(1)}+C^{\mathrm{II}}$, and therefore especially each point of the set C^{II}, is a local cut point. It follows then *that each point of the ordinary kernel of a continuum lies in the interior of a free arc.*

2. If C is a given continuum, let us call an *ordinary element* of C a set E which is open in C, homeomorphic with either the straight line or the circle, and which is saturated with respect to these properties; that is to say, E is not a proper subset of a set E' which is open in C and homeomorphic with a straight line or a circle. It is very easy to prove that two ordinary elements of a continuum are either identical or mutually exclusive, and that the system of all ordinary elements of a continuum is at most denumerable.

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* These Transactions, vol. 33, pp. 252–262.

† Presented to the Society, June 13, 1931; received by the editors April 13, 1931.

‡ Cf. the references concerning the concept of curve and of order at the end of this paper.

§ Loc. cit., p. 257.

‖ Loc. cit., p. 259.

According to our first remark, each point of C^{II} is contained in a free arc of C and therefore in an ordinary element of C. Conversely, each point of an ordinary element of C is evidently a point of C^{II}. Hence C^{II} is the sum of all ordinary elements of C and we have proved the following

ORDINARY KERNEL THEOREM. *If C is a continuum, then the ordinary kernel C^{II} of C is either identical with C (which is the case if and only if C is homeomorphic with the circle), or C^{II} is the sum of an at most denumerable system of sets, each of which is homeomorphic with the straight line, and each two of which are mutually exclusive. Further, each of the sets is open in C and saturated with respect to the properties of being open in C and homeomorphic with the straight line.*

3. Let E be an ordinary element of the continuum C which is not identical with C, and therefore not homeomorphic with the circle, but homeomorphic with the straight line. The closed cover $\overline{E}$ of E may be homeomorphic with the circle. In this case $\overline{E}-E$ contains exactly one point which is necessarily of an order greater than 2. As an example, C may be the lemniscate with the cut point p of order 4, and E may be one of the two components of $C-p$. The set $\overline{E}$ may be an arc; for instance, C is an arc, and E, the arc diminished by its end points. It is obvious that $\overline{E}-E$ cannot contain more than two end points of $\overline{E}$, though the set $\overline{E}-E$ may contain more than two points, which are, however, necessarily irregular points of C. For an example of this consider for C the curve $y=\sin(1/x)$, $0<x\leqq 1$, and $-1\leqq y\leqq 1$, $x=0$, and for E the set $y=\sin(1/x)$, $0<x<1$.

If p is a point of $C^{2(1)}$, that is, an end point of the ordinary part of C, then p does not lie in an ordinary element of C since ordinary elements contain only ordinary points of the ordinary part of C. But p is an end point of a free arc of C and therefore a point of the closed cover of at least one ordinary element of C. As p is a point of C^2 and hence a regular point of C, it must be an end point of the closed cover of an ordinary element of C. Since the set of all ordinary elements is at most denumerable, and the closed cover of each of them contains at most two end points, we have proved the following: *The set $C^{2(1)}$ of all end points of the ordinary part of a continuum is, at most, denumerable.*

4. The set $C^{2(0)}$ being 0-dimensional, we have now a complete knowledge of the structure of the ordinary part of a continuum. We state this in the following

ORDINARY PART THEOREM. *The ordinary part of a continuum is the sum of the ordinary kernel, the, at most, denumerable set of all end points, and the, at most, 0-dimensional set of all points in which it is 0-dimensional.*

5. If S is a set and p, q, r are three distinct points of a connected subset S' of S, then it is said that q *separates* S' in S between the points p and r if $S-q$ is the sum of two separated subsets containing p and r respectively. The definition would be applicable also to the case where S' is not connected, but would not be useful in this case.

We call the point q of S a *separating point* of S if the component of S that contains q contains two points p and r such that q separates this component in S between p and r. Further, we call a point q of S a *cut point* of S if the component of S containing q is not connected when q is deleted from this set. Evidently, a separating point of S is a cut point of S, whereas a cut point of S need not be a separating point of S. If S is a continuum then the cut points and the separating points are identical.

G. T. Whyburn* calls the point q of S a *locally separating* point of S provided that there exists a neighborhood U of p such that q is a separating point of $\bar{u}\cdot S$. A point q is called a *local cut point* of S provided that there exists a neighborhood U of q such that q is a cut point of $\bar{u}\cdot S$. From the relations between separating points and cut points it is evident that each locally separating point is a local cut point, but not conversely; whereas, if S is especially a locally connected continuum, the local separating points and the local cut points are identical.†

An important result of G. T. Whyburn's mentioned paper is that for each continuum, the set of all locally separating points of an order greater than 2 is, at most, denumerable. This theorem may be expressed as follows: *All locally separating points, with the exception of an, at most, denumerable set, are ordinary points.* Ayres has proved that all points of $S^{2(1)}+S^{\text{II}}$ are local cut points. As these points lie in free arcs of the continuum, they are even locally separating. The set of all ordinary points of C which are not locally separating is therefore a subset of $C^{2(0)}$ and hence, at most, 0-dimensional. We may, therefore, say *All points of a continuum, except those of an, at most,* 0-*dimensional set, are locally separating points.* We formulate these two results (the first result somewhat weakened) in the following way:

For each continuum, the set of all ordinary points and the set of all locally separating points may be obtained from each other by omission of an, at most, 0-*dimensional set.*

6. Ayres has proved‡ the following lemma: If C is a continuum containing two end points, a and b, such that each set that separates a and b in C contains at least one ordinary point of C, then C is an arc between a and b.

* Monatshefte für Mathematik und Physik, vol. 36, pp. 305–314.

† Whyburn, loc. cit.

‡ Loc. cit., p. 257.

Evidently this condition is satisfied by each arc and therefore characteristic for the arcs among the continua. But it is of interest that it is also characteristic for the arcs among general (not closed) connected sets. *We show that in order that a connected set be an arc between the two end points a and b, it is necessary and sufficient that each set that separates a and b in C contain at least one ordinary point.*

Let C be a connected set satisfying the condition. Then each point of C that separates a and b is an ordinary point. Let $S(a, b)$ be the set of all points of C separating a and b in C. According to Whyburn the set $a+b+S(a, b)$ may be ordered in the following way: p is *before* q if there exists a decomposition of $S-q$ into two separated subsets, one of which contains a and p, while the other contains b. We prove now that the set $a+b+S(a, b)$ is closed (not merely closed in C). If p is a point of accumulation of $S(a, b)$ different from a and b, then there exists a sequence of points $\{p_n\}$ $(n=1, 2, \cdots)$ of $S(a, b)$ converging towards p and monotonic (in the sense of the order of $S(a,b)$). Let us assume the sequence to be monotonic increasing; that is, such that for each integer n the point p_n lies before p_{n+1} and before p. If $C-p_n=A_n+B_n$ is a decomposition of $C-p_n$ into two separated sets (A_n containing a, B_n containing b), then the set $A=\sum_{n=1}^{\infty}A_n$ is open in C and contains a, but not b, whereas p lies in $\bar{A}-A$. Furthermore, it is easily proved that if the set $\bar{A}-A$ should contain any point besides p, then all points of $\bar{A}-A$ would be irregular points. Hence the point b, supposed to be an end point of C, would not lie in $\bar{A}-A$. Then the set $\bar{A}-A$ would separate a and b in C, and must contain, according to the hypothesis on C, at least one ordinary point, whereas all its points (provided that it contains a point different from p) are irregular. Hence $\bar{A}-A$ contains only the point p. As C is connected, this point belongs to C. It separates a and b in C and is therefore a point of $S(a, b)$ and an ordinary point of C. Since the set $S(a, b)$ is closed, it is identical with C. The set C is therefore a continuum which, besides two end points, contains only points that separate these two end points, i.e., an arc. This completes the proof of the characterization of the arcs.

This characterization contains as a special case the characterization of the arcs among the general connected sets by the fact that besides two end points they contain only ordinary points.*

7. It seems probable to the writer that the methods of Ayres, as well as the preceding remarks, admit generalizations to higher genus. If n is an integer and α an isolated ordinal number of the first or second class, then we say that the point p of the compact space C is *of genus α and of order n* if p is

* Cf. Frankl, Fundamenta Mathematicae, vol. 11 (1928), pp. 96–104.

contained in neighborhoods, arbitrarily small, such that the $(\alpha-1)$st derivatives of their boundaries contain at most n points, whereas p is not contained in arbitrarily small neighborhoods such that the $(\alpha-1)$st derivatives of their boundaries contain less than n points.* It seems very likely that for each isolated ordinal number α the set of all points of a compact space of genus α and order n is 0-dimensional for each $n \neq 2$. It appears not impossible that even the set of all points of order n for some isolated genus is 0-dimensional if $n \neq 2$. As to the ordinary parts of higher genus, they admit decompositions analogous to the case of *genus one* treated in these remarks. As ordinary kernel of genus 2 of the continuum C we also might define the ordinary kernel of the closed set $C-C^{\mathrm{II}}$ and so on, by transfinite induction.

8. Finally, by way of more complete citation of the literature on the concepts and results of the theory of curves than is given in Ayres' paper, I quote in what follows the references concerning the concept of curve and of order of points given by the Encyklopädie der Mathematischen Wissenschaften, IIIAB 13, 1930 (Tietze-Vietoris), p. 234, footnote 276:

"K. Menger, Hinterlegung Nr. 778 (1921) bei der Wien. Akad. d. Wissensch.; Monatshefte f. Math. u. Phys. 33 (1923); Mathem. Annalen 95 (1925); Proc. Ac. Amsterdam 28 (1925); P. Urysohn, Paris C. R. 175 (1922); Fund. Math. 7 (1925)."

The characterization of the arc among the continua by the property that it contains besides two end points merely points of order 2 is mentioned without proof by Menger, Hinterlegung Nr. 778 (1921) (cf. Proceedings of the Amsterdam Academy, vol. 29, p. 1122, and Monatshefte für Mathematik und Physik, vol. 38, and by Urysohn, Paris Comptes Rendus, vol. 175 (1922). This theorem is proved by Menger, Mathematische Annalen, vol. 95 (1925), and by Urysohn, Amsterdam Verhandelingen, vol. 13 (1927).

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* Cf. Menger, Fundamenta Mathematicae, vol. 10, pp. 96–115, and Reschovsky, Fundamenta Mathematicae, vol. 14.

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THE EXISTENCE OF RATIONAL FUNCTIONS OF BEST APPROXIMATION*

BY

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1. **Introduction.** Some results have recently been proved on the approximation to arbitrary functions of a complex variable by rational functions: first, general results on the possibility of approximation with an arbitrarily small error;† and second, results on the degree of approximation, connecting the degree of approximation with the analytic or meromorphic character of the function approximated.‡ In connection with both types of results, it is of interest to know that a sequence of rational functions of *best* approximation exists, and the present paper undertakes to prove this existence in certain cases. The term *best approximation* may here be interpreted for error measured in the sense of Tchebycheff, or for error measured by an integral taken in any one of several ways.

In the study of both possibility of arbitrarily close approximation and degree of approximation, much significance is attached to the position of the poles of the approximating rational functions, and consideration of such position is of central importance in the sequel.

2. **Sequences of rational functions.** By a rational function of z of degree n we mean a function which can be written in the form

$$\frac{a_0z^n + a_1z^{n-1} + \cdots + a_n}{b_0z^n + b_1z^{n-1} + \cdots + b_n},$$

where the denominator does not vanish identically.

THEOREM I. *From any infinite family of rational functions all of the same degree n, there can be extracted a subsequence of functions converging continuously in the entire extended plane with the omission of at most $2n$ points. If the limit of the sequence is not the infinite constant on the entire extended plane (except at these $2n$ points), the limit function is a rational function of degree n, the convergence of the sequence is continuous in the entire extended plane with the omis-*

* Presented to the Society, December 30, 1930; received by the editors January 28, 1931.

† Walsh, these Transactions, vol. 31 (1929), pp. 477–502.

‡ Walsh, these Transactions, vol. 30 (1928), pp. 838–847.

Still further results, on interpolation and approximation by rational functions with assigned poles, are expected shortly to appear in these Transactions.

sion of at most n points, and these points are limit points of poles of the functions of the sequence.

A sequence is said to converge *continuously* in a region if in any closed subregion the sequence converges uniformly. In particular the limit of the sequence may be the infinite constant.

From the given family $f(z)$ (or from any infinite subsequence of it) can be extracted a new subsequence such that except in n arbitrarily small circles of the plane,* each function of the subsequence, with the possible omission of a finite number of functions, is analytic. Let a limit point of poles α_k' of the given family be the point α':

$$\lim_{k\to\infty} \alpha_k' = \alpha'.$$

Let functions $f_k'(z)$ of the given family (or subsequence) have these respective poles α_k'. We did not require that the poles α_k' should belong to distinct functions of the original set. This is a matter of indifference, for no function $f_k'(z)$ can have more than n distinct poles, and if the functions $f_k'(z)$ are not all distinct functions of the original set, we can revise the sequence $f_k'(z)$ if necessary and choose all the functions $f_k'(z)$ distinct functions of the given set. We suppose this to be done. Let a second limit point of poles α_k'' of the set $f_k'(z)$ be α'', different from α':

$$\lim_{k\to\infty} \alpha_k'' = \alpha''.$$

We assume, which is no restriction of generality, that the functions $f_k''(z)$ of the set $f_k'(z)$ which have the respective poles α_k'', are all distinct functions of that set. We consider further possible limit points of poles of the set $f_k''(z)$, and finally arrive at a subsequence $f_k^{(m)}(z)$ whose poles have only the limit points $\alpha', \alpha'', \cdots, \alpha^{(m)}$; this number m is obviously not greater than n, for if k is sufficiently large each function $f_k^{(m)}(z)$ has at least one pole near each point $\alpha^{(i)}$.

It is conceivable that an infinite number of functions of the given set should be constants. In this case the points $\alpha', \alpha'', \cdots, \alpha^{(m)}$ need not exist, but the theorem itself is obviously true. We now assume none of the functions $f_k^{(m)}(z)$ to be identically constant.

Let the circle γ_i be a neighborhood of the point α_i; we denote by D the extended plane with the omission of these circles. All but a finite number of functions of the sequence $f_k^{(m)}(z)$ are analytic in D. The family $f_k^{(m)}(z)$ is a

* If one of the points α_i used below is the point at infinity, one of these excepted regions is a neighborhood of that point, hence the *exterior* of a large circle instead of the interior of a small circle.

quasinormal family of *analytic* (i.e. holomorphic) functions in D, by virtue of the following theorem due to Montel:*

A family of functions analytic in a region D where they take on the value unity no more than p times and the value zero no more than q times is quasinormal in D, of order at most equal to the smaller of the integers p and q.

The family $f_k^{(m)}(z)$ is quasinormal in D, of order not greater than n, no matter what may be the size of the circles γ_i, for each function of the family is rational of degree n and can take on no value in the extended plane more than n times. Hence that family is quasinormal in the entire extended plane with the exception of the points $\alpha', \alpha'', \cdots, \alpha^{(m)}$; this fact is readily proved by a diagonal process, with the use of variable circles γ_i which approach the respective points $\alpha^{(i)}$.†

It will be noticed that we cannot assert that we have a *normal* family of analytic functions in D. The order n of the quasinormal family is actually attained, as is shown by the example

$$f_k^{(n)}(z) = \frac{k(z+1)(z+2)\cdots(z+n)}{(z-1-1/k)(z-2-1/k)\cdots(z-n-1/k)} \qquad (k = 1, 2, \cdots).$$

Theorem I follows directly from the fact that this family $f_k^{(m)}(z)$ is quasinormal in the extended plane with the omission of the points $\alpha^{(i)}$. The remark that the limit function of the sequence is a rational function of degree n presents no difficulty; the functions of the sequence take on no value more than n times in the extended plane and therefore the limit function if not a constant can in D take on no value more than n times, by Hurwitz's theorem on the uniform convergence of analytic functions. The limit function is analytic at every point of the extended plane other than $\alpha', \alpha'', \cdots, \alpha^{(m)}$. These points are isolated singularities and none is an essential singularity, by Picard's theorem. Hence the singularities are either poles or removable singularities. This ensures the rationality of the limit function. Since no value is taken on by this function more than n times, except perhaps in the points $\alpha', \alpha'', \cdots, \alpha^{(m)}$, the function is of degree n.

THEOREM II. *If the moduli of an infinite family of rational functions all of degree n are uniformly limited in $2n+1$ points‡ of the extended plane, then there*

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* *Familles Normales*, Paris, 1927, p. 67. The method of proof of Theorem I is analogous too to Montel's proof of his theorem. It is more convenient here for us to use the theory of quasinormal families of holomorphic functions than of quasinormal families of meromorphic functions.

† Compare Montel, loc. cit., §13.

‡ In connection with the number $2n+1$ which appears here as the number of points in which we suppose the given functions uniformly limited, it is of interest to mention the case $n=1$. There exists a function

can be extracted an infinite subsequence of these functions converging continuously in the entire extended plane with the omission of n points. The limit of this subsequence is a rational function of degree n.

We apply Theorem I directly. The limit of the subsequence cannot be the infinite constant, for if it were, the infinite constant would be the limit of the sequence at every point of the extended plane except the points $\alpha^{(i)}$ and except the possible n points corresponding to the order of the sequence as a quasinormal family. The moduli of the functions of the set are uniformly limited in $2n+1$ points, hence surely in some point of the plane different from those enumerated.*

If there is given a sequence of rational functions of degree n convergent in a given region C, or on a point set C containing at least $2n+1$ points, the hypothesis of Theorem II is naturally satisfied. Denote by $F(z)$ the limit on C of this sequence. The subsequence whose existence is ensured by Theorem II converges to some rational function $R(z)$ of degree n which naturally coincides with $F(z)$ on C. That is to say, *If a sequence of rational functions all of degree n converges to some function F(z) on a point set C consisting of at least 2n+1 points, then on C the function F(z) is a rational function of degree n.*

It is impossible to say here that the original sequence converges at every point of the extended plane, with possible exception of n points. Indeed, let us enumerate all the points on or within the unit circle with rational coördinates: $z_1, z_2, \cdots$. The sequence

$$\frac{1}{z-z_1}, \quad \frac{\frac{1}{2}}{z-z_1}, \quad \frac{\frac{1}{2}}{z-z_2}, \quad \frac{\frac{1}{3}}{z-z_1}, \quad \frac{\frac{1}{3}}{z-z_2}, \quad \frac{\frac{1}{3}}{z-z_3}, \cdots$$

converges at every point exterior to the unit circle, yet converges at none of the points $z_1, z_2, \cdots$. However, we can add the following complement to the result just proved: *If the rational function F(z) of degree n possesses effectively n poles (not neccessarily all distinct), then the given sequence converges at every point of the extended plane with the exception of these n poles. The convergence is continuous in the plane so deleted.*

In the general case treated above, from every subsequence of the given sequence can be extracted a new subsequence which converges on C to the rational function $R(z)$ of degree n and which converges continuously in the ex-

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$$w = \frac{\alpha z + \beta}{\gamma z + \delta}$$

which takes on arbitrarily prescribed distinct values, w_1, w_2, w_3, in arbitrary distinct points z_1, z_2, z_3. Limitedness or convergence of a sequence of such functions in the points z_1 and z_2 implies nothing concerning limitedness or convergence in the point z_3.

* Compare Montel, loc. cit., p. 70.

tended plane with the possible omission of n points. As we change this new subsequence, the rational function $R(z)$ cannot change, for it coincides with $F(z)$ on C,* but *the n exceptional points may vary with this subsequence*; this actually occurs in the explicit example just given. However, if $R(z)$ possesses n poles, each of them must be an exceptional point. If $R(z)$ possesses a multiple pole at a point α, the poles of the functions of the subsequence which approach α must agree in multiplicity with α as a pole of $R(z)$; we discuss this fact more in detail in §4. Hence in this case the n exceptional points cannot vary with the subsequence. From every subsequence of the original sequence can be extracted a new subsequence which converges continuously to $R(z)$ in the extended plane with the omission of the poles of $R(z)$. Hence the original sequence converges in the extended plane to the function $R(z)$ except at these poles, and the convergence is continuous in this region of convergence.

The discussion we have just given of the convergence of sequences of rational functions of degree n has been carried far beyond our needs for application to functions of best approximation. The results obtained can be much simplified if the poles of all the functions of the sequence lie in the same n points, not necessarily all distinct (this is true, for instance, if all the functions of the sequence are polynomials, and also in the situation of §10). In this case convergence in $n+1$ points different from the poles implies convergence of the given sequence on the entire extended plane, except in these poles.

3. **Application to Tchebycheff approximation.** We are now in a position to make application of Theorem II to the problem of the existence of the rational function of best approximation.

THEOREM III. *Let the function $F(z)$ be defined and continuous on a point set C which is dense in itself. Suppose there exists at least one rational function $r(z)$ of degree n such that $|F(z)-r(z)|$ is limited on C. Then there exists a rational function of degree n of best approximation to $F(z)$ on C in the sense of Tchebycheff.*

The measure of approximation of $r(z)$ to $F(z)$ on C is†

$$\overline{\text{bound}}\ [\,|F(z) - r(z)|\,,\ z \text{ on } C], \tag{1}$$

and a rational function of degree n of *best* approximation to $F(z)$ on C in the

* Two rational functions of degree n which coincide in $2n+1$ points must be identically equal, for their difference is a rational function of degree $2n$ which vanishes in $2n+1$ points.

† The symbol $\overline{\text{bound}}$ is intended to indicate the least upper bound of the quantity which follows it.

sense of Tchebycheff is that rational function of degree n or one of those rational functions for which (1) is least.

It will be noticed that the point set C need not be closed. We contemplate only finite values for $F(z)$ on C, but in effect we allow for certain infinite values, for if on a set C' the given function $F(z)$ becomes infinite at a finite number of points, we delete those points from C' and study the approximation of the function $F(z)$ on the remaining point set C. The expression (1) has the same value for any rational function $r(z)$ for the two sets C and C'.

We proceed to the proof of Theorem III. By hypothesis there exists at least one rational function $r(z)$ of degree n for which (1) is finite. The set of numbers, values of the expression (1) when all possible rational functions of degree n are substituted for $r(z)$, has a greatest lower bound b. If the numbers (1) constitute merely a finite set, the theorem is obvious. In the contrary case, we choose a sequence $\{b_k\}$ of finite numbers (1) whose limit is b:

$$\lim_{k\to\infty} b_k = b.$$

Denote by $r_k(z)$ a rational function corresponding to the value b_k:

$$\overline{\text{bound}}\,[\,|F(z) - r_k(z)|, z \text{ on } C] = b_k.$$

In each point of C the functions $r_k(z)$ are bounded, for if b_0 is the largest of the b_k we have

$$|F(z) - r_k(z)| \leqq b_0, z \text{ on } C,$$

$$|r_k(z)| \leqq b_0 + |F(z)|, z \text{ on } C.$$

It is to be remembered that we have excluded the infinite value for $F(z)$ on C. The set C, being dense in itself, contains more than $2n$ points; the functions $r_k(z)$ are naturally bounded in $2n+1$ points of C, so we can apply Theorem II. From the set $r_k(z)$ can be extracted a subsequence $r_k'(z)$ converging continuously in the extended plane with the omission of n points. Denote by $R(z)$ the rational function which is the limit of this sequence except possibly at these omitted points.

It is not yet possible to write

$$\overline{\text{bound}}\,[\,|F(z) - R(z)|, z \text{ on } C] = b, \tag{2}$$

for it is conceivable that one or more of the n exceptional points considered in Theorem II should lie on C, and hence that the equation

$$\lim_{k\to\infty} r_k'(z) = R(z) \tag{3}$$

should fail for some points z on C. Equation (3) can of course fail at no more

than n points of C. The left-hand member of (2) cannot be less than b, by the very definition of b.

Let us prove that the inequality

$$\overline{\text{bound}}\,[\,|F(z) - R(z)|\,,\, z \text{ on } C] > b \tag{4}$$

is impossible. We assume (4) to hold and reach a contradiction. Since C is dense in itself, inequality (4) holding even at a single point of C yields the inequality

$$|F(z) - R(z)| > b,\ z \text{ on } C, \tag{5}$$

on an *infinite* subset of C. Hence (5) must hold at some points of C where the sequence $r_k'(z)$ converges to the function $R(z)$, in contradiction with the inequality $|F(z)-R(z)|\leqq b$ obtained from $|F(z)-r_k'(z)|\leqq b_k$ by allowing k to become infinite. Theorem III is now completely proved.

Theorem III can be applied to the problem (1) mentioned in §1. Thus,* let M be a closed point set whose boundary consists of a finite number of Jordan arcs which do not separate the plane into more than a finite number of regions. If the function $f(z)$ is continuous on M, analytic in the interior points of M, there exists a sequence (not necessarily unique) of rational functions $r_n(z)$ of respective degrees n, which converges to $f(z)$ on M as rapidly (approximation being measured in the sense of Tchebycheff) as does any sequence of rational functions of respective degrees n.

Theorem III is not true if we omit the restriction that C shall be dense in itself. For let C be the circle $|z|=1$ together with the origin $z=0$. Let the function $F(z)$ to be approximated be zero on the unit circle and unity in the origin. We consider approximation to $F(z)$ by rational functions of the first degree. The function $r(z)=1/(mz+1)$, $m>1$, approximates to $F(z)$ on C (including the origin) with a maximum error $1/(m-1)$, which approaches zero with $1/m$. Yet there is no rational function of the first degree which approximates $F(z)$ on C with a zero error. Such a function would vanish everywhere on the unit circle and hence be identically zero, but would fail to coincide with $F(z)$ in the origin.

4. **Restriction of location of poles.** Theorems I and II admit of interesting generalizations. In each of these theorems *if the poles of the given rational functions all lie on some closed point set E, then the poles (if any) of the limit function also lie on E.* This follows from the proof as already given; the points $\alpha^{(i)}$ belong to E. If E is composed of the distinct closed point sets $E_1, E_2, \cdots, E_\nu$ containing respectively not more than $m_1, m_2, \cdots, m_\nu$ poles of the given rational functions, $m_1+m_2+\cdots+m_\nu\geqq n$, then E_i contains not more

* Compare Walsh, loc. cit., p. 479.

than m_i poles of the limit function. The proof is not difficult. Let $r_k(z)$ be the functions of the final sequence and $r(z)$ the limit function. Only points $\alpha^{(i)}$ can be poles of $r(z)$. We prove first that the multiplicity of $\alpha^{(i)}$ as a pole of $r(z)$ is not greater than the multiplicities of the poles of the functions $r_k(z)$ approaching $\alpha^{(i)}$. Let γ be a circle whose center* is $\alpha^{(i)}$ and which contains on or within it no zero of $r(z)$; if $\alpha^{(i)}$ is itself a zero of $r(z)$, nothing more needs to be proved. The circle γ is also to be chosen so small that it contains on or within it only the one point of the set $\alpha', \alpha'', \cdots, \alpha^{(m)}$. We have the relation, by the uniformity of the convergence on γ,

$$\lim_{k\to\infty}\left[\frac{-1}{2\pi i}\int_\gamma \frac{r_k'(z)}{r_k(z)}dz\right] = \frac{-1}{2\pi i}\int_\gamma \frac{r'(z)}{r(z)}\,dz.$$

The right-hand member is the multiplicity of $\alpha^{(i)}$ as a pole of $r(z)$. The left-hand member is the limit of the number of poles of $r_k(z)$ interior to γ diminished by the number of zeros of $r_k(z)$ interior to γ, which is not greater than the number of poles of $r_k(z)$ interior to γ. If the point $\alpha^{(i)}$ belongs to a point set E_j, the poles of $r_k(z)$ interior to γ must, for k sufficiently large, also belong to E_j. Hence E_j contains not more than m_j poles of $r(z)$. In particular if $m_1+m_2+\cdots+m_\nu=n$ and if the limit function has effectively n poles, the function $r_k(z)$ must have precisely m_j poles on E_j, for k sufficiently large.

The remark we have made concerning generalizations of Theorems I and II suggests a generalization of Theorem III which is now entirely obvious. *If $F(z)$ is continuous on a point set C which is dense in itself and if there exists at least one admissible rational function $r(z)$ of degree n such that $|F(z)-r(z)|$ is limited on C, then there exists an admissible rational function of degree n of best approximation in the sense of Tchebycheff to $F(z)$ on C, that is, whose poles (if any) are restricted to lie on an arbitrary closed point set E.*

The choice of E is entirely independent of the choice of C. In particular if E is the point at infinity, there are in Theorem I at most n exceptional finite points in the plane for a sequence converging to the infinite constant and no exceptional finite points in the plane for a sequence whose limit is not the infinite constant. The limit function is naturally a polynomial of degree n. In Theorem II it is sufficient to require the polynomials to be bounded in but $n+1$ finite points of the plane. The case that E is the point at infinity thus yields from Theorem III a proof of the well known existence of the polynomial of degree n of best approximation.

More generally, we remark in connection with Theorem I that from any infinite family of rational functions all of degree n and all with their poles

* The notation here assumes $\alpha^{(i)}$ to be finite. This situation can be attained by a suitable linear transformation of the complex variable if it does not present itself.

on a closed set E, there can be extracted a subsequence of functions converging continuously in that part of the extended plane exterior to E with the omission of at most n points. If the limit of this sequence is not the infinite constant, the limit function is a rational function of degree n, and the convergence of the sequence is continuous on that part of the extended plane exterior to E. If in Theorem II we admit to consideration only rational functions with their poles on some closed point set E, we need only assume that the moduli of the functions of the given family are uniformly limited in $n+1$ points exterior to E; continuous convergence of the subsequence is assured on the extended plane exterior to E.

In Theorem III *there need be no restriction whatever on the point set C if E is closed and has no point in common with C, provided that C has at least $n+1$ points. There need likewise be no restriction on the function $F(z)$ except that it be uniformly bounded on C.*

There is also an interesting case in which each of the point sets $E_1, E_2, \cdots, E_\nu$ considered above is a single point. We assume, as we may do with no loss of generality, that none of these points is the point at infinity. The given rational functions of Theorems I and II are of the form

$$\begin{aligned} r_k(z) = A_0^{(k)} &+ \frac{A_{11}^{(k)}}{z-\alpha_1} + \frac{A_{12}^{(k)}}{(z-\alpha_1)^2} + \cdots + \frac{A_{1m_1}^{(k)}}{(z-\alpha_1)^{m_1}} \\ &+ \frac{A_{21}^{(k)}}{z-\alpha_2} + \frac{A_{22}^{(k)}}{(z-\alpha_2)^2} + \cdots + \frac{A_{2m_2}^{(k)}}{(z-\alpha_2)^{m_2}} \\ &+ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &+ \frac{A_{\nu 1}^{(k)}}{z-\alpha_\nu} + \frac{A_{\nu 2}^{(k)}}{(z-\alpha_\nu)^2} + \cdots + \frac{A_{\nu m_\nu}^{(k)}}{(z-\alpha_\nu)^{m_\nu}}, \end{aligned}$$

and the limit function is of the same form without the superscripts. There are in reality two different situations according as the individual numbers m_i are prescribed, $m_1+m_2+\cdots+m_\nu=n$, or not. In either situation the generalization of Theorem III announces the existence of a minimizing rational function of the given form, provided that C contains at least $n+1$ points and contains none of the points α_i.

5. **The rational function of best approximation need not be unique.** It is interesting to notice that easily proved results, analogous to those mentioned in §1, on uniform convergence of the sequence of rational functions of best approximation, on the degree of convergence of the sequence of rational functions of best approximation, and even on overconvergence, do not depend in any way on the uniqueness of the sequence of rational functions of

best approximation, but merely on the existence of such a sequence. This remark applies not merely if best approximation is considered in the sense of Tchebycheff, but also (compare §§7 and 8) if certain integral measures of approximation are used.

We now show by examples that the sequence of rational functions of best approximation in the sense of Tchebycheff may fail to be unique. It is not our purpose here actually to determine all Tchebycheff rational functions of best approximation for the functions to be considered. We shall content ourselves with the proof that the rational function of best approximation is not unique. Moreover, we restrict ourselves in this detailed discussion to the study of best approximation in the sense of Tchebycheff. The reader can easily study the same examples and show that the rational function of best approximation fails to be unique also if an integral measure of approximation is used.

Let us approximate to the function $F(z) \equiv z$ on the point set C: $|z| = 1$. The rational functions which approximate to $F(z)$ shall be of the first degree and have their poles (if any) on the point set E: $|z| = r > 2$.

If we have best approximation to $F(z)$ on C by the function

$$\frac{\alpha z + \beta}{z - rz_0}, \quad |z_0| = 1, \tag{6}$$

then we also have best approximation to $F(z)$ on C by the function

$$\frac{\omega\alpha z + \omega^2\beta}{z - \omega r z_0}, \tag{7}$$

where ω is an arbitrary quantity whose absolute value is unity. For we have

$$\left| z - \frac{\omega\alpha z + \omega^2\beta}{z - \omega r z_0} \right| = \left| \frac{z^2 - \omega r z_0 z - \omega\alpha z - \omega^2\beta}{z - \omega r z_0} \right|.$$

Substitute now $z = \omega\zeta$, so that the above expression is

$$\left| \frac{\omega^2\zeta - \omega^2 r z_0\zeta - \omega^2\alpha\zeta - \omega^2\beta}{\omega\zeta - \omega r z_0} \right| = \left| \zeta - \frac{\alpha\zeta + \beta}{\zeta - r z_0} \right|.$$

The maximum of the first expression for $|z| = 1$ is equal to the maximum of the last expression for $|\zeta| = 1$, from which it follows that an infinity of rational functions (7) are as good an approximation to $F(z)$ on C as is (6), so the latter cannot be a unique function of best approximation.

The formulas given mask one difficulty: it is conceivable that (6) should be the constant zero, in which case (7) would also be zero, and the example

would be pointless; in every other case the two functions (6) and (7) are distinct. Rather than go through the detailed determination of a function (6) of best approximation, we merely indicate rapidly that the difficulty just raised is apparent rather than actual. The measure of approximation of the function (6) to z on C is the maximum of

$$\left| z - \frac{\alpha z + \beta}{z - rz_0} \right|, \quad |z| = 1.$$

The measure of approximation of the particular function $rz/(z+r)$, whose pole lies on E, is the maximum of

$$\left| z - \frac{rz}{z + r} \right| = \left| \frac{z^2}{z + r} \right|, \quad |z| = 1.$$

If r is greater than 2, as we suppose, this last maximum is less than unity. *Some* admissible rational function which is not identically zero is therefore a better approximation to z on C than is the rational function zero, because the measure of approximation of the latter function to $F(z)$ on C is unity. Hence the rational function whose poles lie on E of *best* approximation to $F(z)$ on C is not identically zero. The validity of the example given is now completely established.

In the example just given we might equally well have chosen for the point set C the region $|z| \leqq 1$, and/or for the point set E the region $|z| \leqq R > 2$. There is no change to be made in the reasoning given. Or we might have chosen merely two distinct points on the circle $|z| = r > 2$ as the point set E; again the rational function of best approximation is not unique. The example, moreover, considers explicitly approximation only by rational functions of the first degree. It can obviously be modified so as to apply to the approximation of the same function $F(z) \equiv z$ on the same point set $|z| = 1$ or $|z| \leqq 1$, by rational functions of an arbitrary degree n whose poles are restricted to lie on the point set E: $|z| = r > 2$ or $|z| \leqq r > 2$.

We give another example to show that the rational function of degree n of best approximation may fail to be unique, even when the point set E is the entire plane. The function $F(z) \equiv z + 1/z$ is to be approximated on the point set C: $|z| = 1$ by a rational function $r(z)$ of the first degree. The rational function $r(z)$ of best approximation is not identically zero, for we have $\max[|z+1/z|, z \text{ on } C] = 2$, but for $r(z) = 1/z$ we have $\max[|F(z) - r(z)|, z \text{ on } C] = 1$. There is no unique rational function $r(z)$ of best approximation which is identically a constant k, for we have by setting $w = -z$,

$$\begin{aligned} \max\,[\,|\,z + 1/z - k\,|, z \text{ on } C] &= \max\,[\,|\,-z - 1/z + k\,|, z \text{ on } C] \\ &= \max\,[\,|\,w + 1/w + k\,|, w \text{ on } C], \end{aligned}$$

so the rational function $r(z) = -k$ is as good an approximation to $F(z)$ on C as is $r(z) = +k$. There is no unique rational function of the first degree of best approximation, for we have, by setting $w = 1/z$,

$$\max\left[\,|\,z + 1/z - r(z)\,|,\ z \text{ on } C\right] = \max\left[\,|\,w + 1/w - r(1/w)\,|,\ w \text{ on } C\right],$$

so the rational function $r(1/z)$ is as good an approximation to $F(z)$ on C as is $r(z)$. The two functions $r(1/z)$ and $r(z)$ are distinct unless $r(z)$ is a constant; this has already been shown to be impossible.

6. **A lemma on integrals.** The following lemma is to be applied later, in the study of measures of approximation involving integrals. The integrals may be taken either in the sense of Riemann or Lebesgue, but in any case the functions $\phi_k(x)$ are naturally supposed to be integrable.

LEMMA. *Let* Γ *be an arbitrary closed point set. If we have, for the non-negative functions* $\phi_k(x)$,

$$\lim_{k\to\infty} \phi_k(x) = \phi(x) \tag{8}$$

uniformly on every closed subset of Γ *not containing* n *particular points of* Γ, *and if*

$$\lim_{k\to\infty} \int_\Gamma \phi_k(x)dx = a \tag{9}$$

exists, then

$$\int_\Gamma \phi(x)dx \tag{10}$$

exists and is equal to or less than a.

The n exceptional points of Γ are isolated, so in order to prove the existence of (10) it is sufficient to consider a neighborhood γ of one of those points, P. We assume that γ contains no other exceptional point. Let γ' be a neighborhood of P interior to γ. The inequality

$$\int_{\gamma-\gamma'} \phi_k(x)dx \leqq a + \epsilon,$$

where the positive quantity ϵ is arbitrary, obviously holds for k sufficiently large. Moreover, since equation (8) holds uniformly as stated, it holds uniformly in $\gamma - \gamma'$. Hence we have

$$\int_{\gamma-\gamma'} \phi(x)dx = \lim_{k\to\infty} \int_{\gamma-\gamma'} \phi_k(x)dx \leqq a + \epsilon,$$

from which follows immediately the existence of

$$\int_\gamma \phi(x)dx$$

and hence the existence of (10).

Let $\epsilon>0$ again be arbitrary. Choose the neighborhoods $\gamma', \gamma'', \cdots, \gamma^{(n)}$ mutually exclusive neighborhoods of the n exceptional points, in such a way that

$$\int_{\gamma^{(i)}} \phi(x)dx \leqq \frac{\epsilon}{2n} \qquad (i = 1, 2, \cdots, n).$$

There exists K such that for $k>K$ we have

$$\left| \int_{\Gamma-\gamma'-\gamma''-\cdots-\gamma^{(n)}} [\phi(x) - \phi_k(x)]dx \right| \leqq \frac{\epsilon}{2},$$

for on the point set $\Gamma-\gamma'-\gamma''- \cdots -\gamma^{(n)}$, the sequence $\{\phi_k(x)\}$ converges uniformly to the function $\phi(x)$. That is, we have

$$\int_\Gamma \phi(x)dx \leqq \int_{\Gamma-\gamma'-\gamma''-\cdots-\gamma^{(n)}} \phi_k(x)dx + \int_{\gamma'+\gamma''+\cdots+\gamma^{(n)}} \phi(x)dx + \frac{\epsilon}{2}$$
$$\leqq \int_{\Gamma-\gamma'-\gamma''-\cdots-\gamma^{(n)}} \phi_k(x)dx + \epsilon.$$

This inequality holds, once the neighborhoods $\gamma', \gamma'', \cdots, \gamma^{(n)}$ have been chosen, for all sufficiently large k. Allow k to become infinite. The last integral on the right approaches a limit, by the uniform convergence of the sequence $\phi_k(x)$. This limit is, by (9), not greater than a. Hence the integral (10) is not greater than $a+\epsilon$, hence not greater than a.

7. **Integral measures of approximation.** In the integral measures of approximation now to be considered, the weight or norm function $n(z)$ [or $n(w)$] is to be taken as integrable, non-negative, and of course not identically zero. This last restriction avoids a trivial case; if the integrals considered are taken in the sense of Lebesgue, this restriction is naturally taken to mean that the weight function is greater than zero on some point set of measure greater than zero.

THEOREM IV. *Let the function $F(z)$ be defined on the rectifiable arc, curve, or other closed point set C which is linearly measurable. Suppose there exists at least one rational function $r(z)$ of degree n such that*

$$(11) \qquad \int_C |F(z) - r(z)|^p n(z) |dz|, \ p > 0,$$

exists. Then there exists a rational function $R(z)$ of degree n of best approximation, namely such that

$$\int_C |F(z) - R(z)|^p n(z) |dz|, \quad p > 0, \tag{12}$$

is not greater than the expression (11) *formed for an arbitrary rational function $r(z)$ of degree n.*

There exists, as in the proof of Theorem III, a sequence of rational functions $r_k(z)$ all of degree n such that we have

$$\lim_{k\to\infty} \int_C |F(z) - r_k(z)|^p n(z) |dz| = b, \tag{13}$$

where b is the greatest lower bound of all expressions of type (11). From this set $r_k(z)$ there can be extracted, by Theorem I, a subsequence which converges continuously in the entire plane with the omission of at most $2n$ points. The limit of this subsequence cannot be the infinite constant, by (13); the number b is of course not the infinite constant, by the hypothesis on the $r(z)$ of the theorem.

We can apply the Lemma just proved. The subsequence of the set $r_k(z)$ converges to a function $R(z)$, which by Theorem I is a rational function of degree n. By Theorem II and the Lemma, we now have

$$\int_C |F(z) - R(z)|^p n(z) |dz| \leqq b.$$

The number b is the greatest lower bound of all expressions of type (11), so the inequality is excluded and the theorem is proved.

The argument just used is also valid if C is an arbitrary measurable set, where the integrals are taken as surface integrals.

THEOREM V. *Let the function $F(z)$ be defined on the closed measurable point set C and suppose there exists at least one rational function $r(z)$ of degree n such that*

$$\iint_C |F(z) - r(z)|^p n(z) dS, \quad p > 0, \tag{14}$$

exists. Then there exists a rational function $R(z)$ of degree n of best approximation, namely such that

$$\iint_C |F(z) - R(z)|^p n(z) dS$$

is not greater than the expression (14) *formed for an arbitrary rational function of degree n.*

In Theorems IV and V we have made no mention of the location of the poles of the functions $r(z)$ of degree n, and these poles are naturally unrestricted as to location. Here as in Theorem III it is of course possible to admit to consideration only rational functions $r(z)$ of degree n whose poles (if any) lie on an arbitrary closed point set E. It is still true that in each of these situations (Theorems IV and V) there exists an admissible rational function $R(z)$ of degree n of best approximation; that is to say, the poles of $R(z)$, if any, also lie on E. Indeed, practically the whole discussion of §4 applies here and to Theorem VI below without essential alteration.

8. **An integral measure of approximation after conformal mapping.** Two other cases are of interest where the measure of approximation is an integral, at least in the analogous situation of approximation by polynomials: (1) where C is a simply connected region which is mapped* onto the interior of the circle γ: $|w|=1$ and the integral is taken over γ, and (2) where C is an arbitrary closed point set whose complement D is simply connected, and D is mapped onto the interior or exterior of γ and the integral is taken over γ. To be sure, the conformal map is strictly defined only in the interiors of the regions mapped, but if the correspondence is extended by continuity whenever possible to the boundaries of the regions, in the w-plane say by radial approach to γ, we still have the approximated and approximating functions defined almost everywhere on γ, so the integrals on γ may be studied.† There is a possibility of complication in extending the reasoning of §7 to both (1) and (2), for to a single point of the boundary of C may correspond even an infinity of points of γ. Such a point of C is a possible point of failure of the uniform convergence of the sequence or subsequence $r_k(z)$. The Lemma permits of failure of the uniformity of convergence in only a *finite* number of points of the set over which the integrals are taken. Thus the Lemma is not applicable in the most general cases (1) and (2). A detailed study of the conformal map and its implications in these general cases would lead us too far afield from the main object of the present paper, so we content ourselves with stating the following:

THEOREM VI. *Let the function $F(z)$ be defined on the boundary B of the simply connected region C and suppose there exists at least one admissible rational function $r(z)$ of degree n such that*

* An arbitrary point interior to C is made to correspond to the center of γ and the rational function of best approximation depends on the particular point chosen. But approximation for an arbitrary choice of this point and an arbitrary weight function $n(w)$ is equivalent to a *particular* choice of this point and a suitable choice of the weight function.

A similar remark applies in case (2).

† Compare Walsh, these Transactions, vol. 32 (1930), pp. 794–816.

$$\int_\gamma |F(z) - r(z)|^p n(w) |dw|, \quad p > 0, \tag{15}$$

exists, where the region C is mapped onto the interior of the unit circle γ: $|w| = 1$. *If the region C has no boundary point which corresponds to more than a finite number of points of* γ, *or more generally if the poles of the admissible rational functions are restricted to lie on a closed point set E which contains no point of B corresponding to more than a finite number of points of* γ, *then there exists an admissible rational function* $R(z)$ *of degree n of best approximation, namely such that*

$$\int_\gamma |F(z) - R(z)|^p n(w) |dw|$$

is not greater than the integral (15) *formed for any other admissible rational function* $r(z)$ *of degree n.*

The present application of the Lemma is not essentially different from the previous ones, and will not be given in detail. One minor difference may be noted, namely that the Lemma requires uniform convergence on certain point sets, and in the present case we can assert merely uniform convergence with the possible exception of sets of measure zero. This does not of course essentially alter the validity of the application.

Theorem VI requires only that $F(z)$ should be defined on the boundary of C, and approximation is in reality measured only on this boundary. It ordinarily occurs, however, that we are interested in studying the convergence of the approximating sequence not merely on the boundary but also in a region or on a more general point set. Two distinct interpretations are possible here, as mentioned above, according to whether we consider further convergence (1) on C or (2) on the complement of C. That is to say, in effect we consider (15) as a measure of approximation (1) on C or (2) on the complement of C. These two cases are quite distinct when one studies approximation by polynomials or approximation by rational functions whose poles are restricted to lie on closed point sets E which lie (1) on the complement of C, or (2) in C, but the two cases merge into a single case when one considers approximation by rational functions with no restrictions on their poles.

So far as the present writer is aware, there has never been studied the problem of *overconvergence* of sequences of rational functions of best approximation, where the poles are restricted to lie on a closed point set E and where the measure of approximation is taken as in (11), (14), or (15). Overconvergence can be studied in all of these cases, however, by the methods already used elsewhere for polynomials and for more general rational functions.*

* See the preceding reference and also the references given in §1.

Our methods of proof of the existence of rational functions of best approximation apply, it may be added, to many measures of approximation other than those we have considered.

9. **Approximation with auxiliary conditions.** A variation of the problems of approximation already considered is to require that the rational function of best approximation (considered in any one of the several ways) shall take on prescribed values in certain points α_i. This requirement may be interpreted as restricting certain derivatives of the approximating functions in the points α_i if the latter are not taken all distinct:

$$R(\alpha_i) = m_i,\ R'(\alpha_i) = m_i',\ \cdots,\ R^{(k_i)}(\alpha_i) = m_i^{(k_i)} \qquad (i = 1, 2, \cdots, k).$$

The points α_i may or may not belong to C. The discussion we have given for the various measures of approximation is not valid in the present case, for one of the exceptional points of convergence in Theorem II may be a point α_i. The equation $r_k(\alpha_i) = m_i$ does not in this case imply $\lim_{k\to\infty} r_k(\alpha_i) = m_i$. It is true, however, that all the exceptional points in Theorem II are limit points of poles of rational functions of approximation, and if we restrict those poles so that they lie on a closed point set E which contains none of the points α_i, the auxiliary conditions satisfied by the functions $r_k(z)$ are also satisfied by the limits of those functions. This follows from the uniformity of the convergence of the sequence or subsequence $\{r_k(z)\}$ in the neighborhood of each point α_i.

Let us state the theorem resulting from this discussion if the Tchebycheff measure of approximation is used. The other theorems are so similar that they are readily supplied by the reader.

If the function $F(z)$ is defined and continuous on a point set C which is dense in itself, we consider approximation to $F(z)$ on C by rational functions $r(z)$ all of degree n which satisfy the prescribed auxiliary conditions

$$r(\alpha_i) = m_i,\ r'(\alpha_i) = m_i',\ \cdots,$$
$$r^{(k_i)}(\alpha_i) = m_i^{(k_i)} \qquad (i = 1, 2, \cdots, k),$$

and whose poles lie on some closed point set E which contains none of the points α_i. If there exists at least one admissible function $r(z)$ such that $|F(z)-r(z)|$ is limited on C, then there exists an admissible function $R(z)$ of best approximation, namely such that*

$$\overline{\text{bound}}\,|F(z) - R(z)|,\ z \text{ on } C,$$

exceeds no expression of the same form where $R(z)$ is replaced by any other admissible function.

* The existence of rational functions satisfying auxiliary conditions is considered in some detail in §10.

If we are willing to require that E shall have no point in common with C, we do not need to assume here that C is dense in itself, but merely, as in the comment on Theorem III, that C contains more than n points. Even this number may be lessened by the number of points α_i not belonging to C. The function $F(z)$ need merely be uniformly limited on C.

The theorem itself is not true if we omit the requirement that E shall contain none of the points α_i. We illustrate this by a simple example similar to the example already used in §3. Let us approximate to the function $F(z) \equiv 0$ on the point set C: $|z| = 1$ by rational functions $r(z)$ of the first degree which satisfy the single auxiliary condition $r(0) = 1$. The measure of approximation is

$$\overline{\text{bound}}\, |\, r(z)\, |, \quad z \text{ on } C.$$

The lower limit of this measure of approximation, considered for all admissible functions $r(z)$, is zero; indeed, the function $r(z) = 1/(mz+1)$, $m>1$, is admissible, yet the upper bound of its modulus on C is $1/(m-1)$, which approaches zero with $1/m$. There naturally exists no rational function $r(z)$ of the first degree which satisfies the auxiliary conditions whose least upper bound on C is zero. Such a function, being zero on C, would vanish identically and could not satisfy the auxiliary condition.

10. **Uniqueness of rational functions with preassigned poles.** There is one case, interesting for the applications, where the rational function of best approximation not only exists but is unique.

*If the rational function of degree n is required to be of the form**

$$\begin{aligned} r_n(z) = A_{00} &+ \frac{A_{11}}{z-\alpha_1} + \frac{A_{12}}{(z-\alpha_1)^2} + \cdots + \frac{A_{1m_1}}{(z-\alpha_1)^{m_1}} \\ &+ \frac{A_{21}}{z-\alpha_2} + \frac{A_{22}}{(z-\alpha_2)^2} + \cdots + \frac{A_{2m_2}}{(z-\alpha_2)^{m_2}} \\ &+ \cdots\cdots\cdots\cdots\cdots\cdots \\ &+ \frac{A_{\nu 1}}{z-\alpha_\nu} + \frac{A_{\nu 2}}{(z-\alpha_\nu)^2} + \cdots + \frac{A_{\nu m_\nu}}{(z-\alpha_\nu)^{m_\nu}}, \\ & \qquad\qquad n = m_1 + m_2 + \cdots + m_\nu, \end{aligned} \tag{16}$$

* There are two distinct points of view regarding rational functions of form (16), namely (i) to require that $r_n(z)$ should be of the form (16) where n is prescribed and the individual numbers m_i are not prescribed, although the numbers m_i may be subjected to restrictions $m_i \leqq m_i'$, $i=1, 2, \cdots, \nu$, where the m_i' are prescribed, $n < m_1'+m_2'+\cdots+m_\nu'$; (ii) to require that $r_n(z)$ should be of form (16), where n is prescribed together with each individual number m_i, $n = m_1+m_2+\cdots+m_\nu$. In §4 it is shown that in both of these cases (i) and (ii) there exists an admissible rational function $r_n(z)$ of best approximation. Only in case (ii) can it be proved that there exists a unique rational function of best approximation; compare the examples given in §5.

where the numbers $\alpha_1, \alpha_2, \cdots, \alpha_\nu, m_1, m_2, \cdots, m_\nu$ *are prescribed but the numbers* A_{ij} *are not prescribed, and if the continuous function* $F(z)$ *is approximated on a closed point set* C *containing none of the points* α_i *but containing more than* n *points, then the rational function of best approximation is unique.*

We shall proceed to prove the truth of this theorem, whether best approximation is considered in the sense of Tchebycheff, or in the sense of least weighted pth powers ($p>1$) measured by a line or surface integral, with or without conformal mapping. The theorem is also true in each of these cases, as we shall prove, for sufficiently large n even if we have prescribed auxiliary conditions relating to points distinct from the α_i, provided that if approximation is measured in the sense of Tchebycheff the auxiliary conditions are of the form $r_n(\beta_i)=(F\beta_i)$ for all points β_i belonging to C. This equation need not be taken to imply $r_n'(\beta_i)=F'(\beta_i)$ in case two or more of the β_μ fall together in β_i on C.

A case which is clearly admissible, though not covered by a literal reading of (16), is that of prescribed poles at infinity; it is no loss of generality in the proof to assume all of the points α_i finite, for that situation can be obtained by a suitable linear transformation of the complex variable.

In connection with the results to be proved, it is necessary to know that under certain conditions there exists at least one function of the form (16) satisfying given auxiliary conditions. Let us show that *there exists a unique function of type* (16) *which takes on prescribed values* B_μ *in* $\rho=n+1$ *prescribed points* β_μ *distinct from the points* α_i. We give the proof under the hypothesis that the points β_μ are all distinct, but the result holds in the more general case that two or more of them coincide. The condition is then naturally interpreted to mean the prescription of $r_n(z), r_n'(z), \cdots, r_n^{(k)}(z)$ in the points β_μ concerned, the number k depending on the number of points β_μ which are considered to coincide.

The existence and uniqueness of the function $r_n(z)$ depends on the solution of ρ obvious linear equations for the numbers A_{ij}, and this solution depends on the non-vanishing of the determinant

$$\Delta = \begin{vmatrix} 1 & \dfrac{1}{\beta_1-\alpha_1} & \dfrac{1}{(\beta_1-\alpha_1)^2} & \cdots & \dfrac{1}{(\beta_1-\alpha_1)^{m_1}} & \dfrac{1}{\beta_1-\alpha_2} & \dfrac{1}{(\beta_1-\alpha_2)^2} & \cdots & \dfrac{1}{(\beta_1-\alpha_\nu)^{m_\nu}} \\ 1 & \dfrac{1}{\beta_2-\alpha_1} & \dfrac{1}{(\beta_2-\alpha_1)^2} & \cdots & \dfrac{1}{(\beta_2-\alpha_1)^{m_1}} & \dfrac{1}{\beta_2-\alpha_2} & \dfrac{1}{(\beta_2-\alpha_2)^2} & \cdots & \dfrac{1}{(\beta_2-\alpha_\nu)^{m_\nu}} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & \dfrac{1}{\beta_\rho-\alpha_1} & \dfrac{1}{(\beta_\rho-\alpha_1)^2} & \cdots & \dfrac{1}{(\beta_\rho-\alpha_1)^{m_1}} & \dfrac{1}{\beta_\rho-\alpha_2} & \dfrac{1}{(\beta_\rho-\alpha_2)^2} & \cdots & \dfrac{1}{(\beta_\rho-\alpha_\nu)^{m_\nu}} \end{vmatrix}. \tag{17}$$

The vanishing of Δ is a necessary and sufficient condition that the homogeneous system of equations (corresponding to $B_\mu = 0$) for the determination of the numbers A_{ij} should have a non-trivial solution. The vanishing of Δ implies then the existence of a rational function of degree n which is not identically zero and which has at least $n+1$ zeros, in the points β_μ; this is impossible. Thus we have $\Delta \neq 0$, and the proof is complete.

The result just proved is of interest in the present connection as showing the existence of *at least one admissible function* $r_n(z)$, satisfying the given auxiliary conditions in ρ points β_μ different from the α_i, $\rho \leqq n+1$. Under these conditions, it follows from §§3, 4, 7, 8, 9 that there exists *at least one admissible function* $r_n(z)$ *of best approximation*. We now prove further: *The uniqueness of the function* $r_n(z)$ *of best approximation, with or without auxiliary conditions, follows directly from the inequality*

$$\left[\frac{|\alpha| + |\beta|}{2}\right]^p < \frac{1}{2}[|\alpha|^p + |\beta|^p], \quad p > 1, \quad \alpha \cdot \beta \neq 0,$$

if best approximation is measured by an integral as previously considered. For if there exist two distinct admissible rational functions $r_n(z)$ and $r_n'(z)$ of best approximation to the given function $F(z)$, then $(r_n(z)+r_n'(z))/2$ is actually an admissible function of better approximation:

$$\begin{aligned}\int n(z)\left|F(z) - \frac{r_n(z) + r_n'(z)}{2}\right|^p ds &< \frac{1}{2}\int n(z)|F(z) - r_n(z)|^p ds \\ &\quad + \frac{1}{2}\int n(z)|F(z) - r_n'(z)|^p ds \\ &= \int n(z)|F(z) - r_n(z)|^p ds,\end{aligned}$$

which is of course impossible. The proof holds whether the integral be a line integral or surface integral; the numbers α and β to which the inequality is applied must be different from zero on a point set of measure greater than zero. If $r_n(z)$ and $r_n'(z)$ both satisfy given auxiliary conditions, so also does $(r_n(z)+r_n'(z))/2$, and the proof is valid even if auxiliary conditions are prescribed.

Our proof of the uniqueness of the rational function of best approximation with or without auxiliary conditions is now complete in the case that approximation is measured by an integral. In order to complete the proof of the theorem already stated it remains for us to study further the case that approximation is measured in the sense of Tchebycheff:

Let $F(z)$ be an arbitrary function continuous on a closed point set C, which is distinct from the points $\alpha_1, \alpha_2, \cdots, \alpha_\mu$ and contains at least $n+1$ points. Then there exists a unique rational function $r_n(z)$ of form* (16) *of best approximation to $F(z)$ on C in the sense of Tchebycheff with preassigned auxiliary conditions*

$$r_n(\beta_i) = B_i \qquad (i = 1, 2, \cdots, k), \tag{18}$$

where the points β_i are distinct from the α_j, provided $n \geqq k-1$ and provided that $B_i = F(\beta_i)$ if β_i is a point of C.

The case $k=0$ is not excluded here and simply means the omission of auxiliary conditions. In case the points β_i are not all distinct, equation (18) is interpreted to define not merely $r_n(\beta_i)$ but also one or more derivatives of $r_n(z)$ at the multiple points β_i, depending on the number of points β_i which coincide; the number k is intended to denote the number of *equations* (18), not the number of distinct points β_i. The equation $B_i = F(\beta_i)$, required to hold if β_i is a point of C, is not intended necessarily to restrict the values of any derivatives of $r_n(z)$, even if β_i is a point of C.†

We have already proved the existence of at least one admissible function of best approximation. The proof of the uniqueness may be carried out by the original method due to Tonelli and used by him in the case that $r_n(z)$ is a polynomial and $k=0$, although the details for the present case follow closely a modification of Tonelli's proof recently given by the present writer.‡ The complete proof can easily be constructed by the reader. Our preliminary result on the existence and uniqueness of functions of type (16) which take on prescribed values in prescribed points is needed. Moreover, the theorem (with or without auxiliary conditions) can be extended to include even Tchebycheff approximation with a norm function, that is, the determination of a rational function $r_n(z)$ of form (16) such that the maximum value of

$$n(z) \mid F(z) - r_n(z) \mid, \; z \text{ on } C,$$

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* It is sufficient here to require that the set of points C plus the β_i contains at least $n+1$ points. Moreover, if n is sufficiently large it is allowable that some of the points α_i should coincide with points β_j and even that some points α_i should lie on C if a sufficiently large number of those points do not lie on C.

† Indeed, if we denote by ϵ_n the maximum error $|F(z)-r_n(z)|$, z on C, for the best approximation $r_n(z)$ (with the given auxiliary conditions) to $F(z)$ on C in the sense of Tchebycheff, it is sufficient to require $|B_i - F(\beta_i)| < \epsilon_n$ instead of $B_i = F(\beta_i)$.

The equation $B_i = F(\beta_i)$ or some other restriction is actually necessary for the uniqueness of the function of best approximation, as is shown by the case $n=1$, $\alpha_1 = \infty$, $C: 0 \leqq z \leqq 1$, $F(z) \equiv 1$, with the auxiliary condition $r_1(0) = 0$.

‡ These Transactions, vol. 32 (1930), pp. 335–390, §10.

is least; here $n(z)$ is supposed positive and continuous on C. This proof too follows closely the original proof due to Tonelli.* In our previous work in the present paper we did not trouble to consider Tchebycheff approximation with a norm function. The introduction of such a function throughout the paper offers not the slightest difficulty.

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* Compare Walsh, these Transactions, vol. 32 (1930), pp. 794–816; p. 797, footnote.

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ON DIRECT PRODUCTS*

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1. **Introduction.** The principal contribution of this article is a set of theorems on the structure of the direct product of a normal division algebra A over F and an algebraic field $F(\eta)$, with applications of the Galois theory of equations. The theorems are useful new tools for research on normal division algebras. In particular it is shown that it is possible to extend the reference field F of any normal division algebra A of order p^2 over F, p a prime, such that $A' = A \times F(\eta)$ is a cyclic normal divison algebra over $F(\eta)$. A new proof is given of a little known theorem of R. Brauer† which reduces the problem of determinining all normal division algebras of order n^2 over F to the case where n is a power of a prime.

2. **Known theorems.** We shall repeatedly use the theorems given in the author's paper *On direct products, cyclic division algebras, and pure Riemann matrices*, and shall refer to it by the letter T. All our algebras A will be linear associative algebras with a modulus over any non-modular field F, and all sub-algebras of A will have the same modulus, 1, and the same zero quantity, 0, as A.

The symbol $\times$ shall mean direct product, $A \times B$ the direct product of an algebra A and an algebra B over the same field F. In $A \times B$ the algebras A and B will be taken, without loss of generality,to be sub-algebras of $A \times B$.

We shall require three well known theorems.

THEOREM 1. *The direct product of two total matric algebras of orders m^2 and n^2 respectively is a total matric algebra of order $(mn)^2$ and conversely.*

THEOREM 2. *Every simple algebra‡ A over F is expressible as a direct product $B \times M$ of a division algebra B over F and a total matric algebra M over F in one and only one way in the sense of equivalence, and conversely.*

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* Presented to the Society June 13, 1931; received by the editors April 17, 1931.

† This theorem seems to be almost unknown in America. It was obtained independently by the author, and this paper was, in fact, already received by the editors of these Transactions when the author discovered Brauer's priority. However, Brauer obtained his results in three memoirs using complicated theorems on linear groups going back fundamentally to two memoirs of I. Schur, and his papers are far from self-contained. The author's independent proof is a simple application of his new theorems on direct products. For references and Brauer's theorems, see his third paper in the Mathematische Zeitschrift, vol. 30 (1929), pp. 79–107.

‡ Not a zero algebra of order one, since we assume throughout that A has the modulus 1.

THEOREM 3. *Let A contain a normal simple sub-algebra B. Then A is the direct product of B and another sub-algebra C of A.*

We shall also use the author's theorems* T 3, 4, 7, 8, 13. They are respectively

THEOREM 4. *Let A be a normal simple algebra. Then A is the direct product of a normal division algebra and a total matric algebra, and conversely.*

THEOREM 5. *Let $A = B \times C$ where B and C are normal simple algebras over F. Then A is a normal simple algebra over F.*

THEOREM 6. *Let $A = B \times C$ where A is a normal simple algebra. Then both B and C are normal simple algebras.*

THEOREM 7. *Let $A = B \times C$ where A and B are total matric algebras. Then C is a total matric algebra.*

THEOREM 8. *Let A be a normal division algebra of order m^2 over F and B be a normal division algebra of order n^2 over F such that m and n are relatively prime integers. Then $A \times B$ is a normal division algebra of order m^2n^2 over F.*

3. **Extensions of the field F.** Let A be a normal divison algebra of order n^2 over F. We shall study what happens to A when we extend F by a quantity η which is commutative with all the quantities of A and is such that $F(\eta)$ is a algebraic field of order r over F. We shall thus consider the algebra A' which has the same basis and multiplication constants as A, but which is an algebra over $F(\eta)$. It is evident that A', considered as an algebra over F, is the direct product

$$A' = A \times F(\eta).$$

Let A be as above and let x in A have† $\phi(\omega) = 0$ as its minimum equation with respect to F. We shall say in this case that $\phi(\omega) = 0$ is an equation which *belongs to A*. It is well known that when A is a normal division algebra over F every equation belonging to A is irreducible in F.

Let x in A have $\phi(\omega) = 0$ as its minimum equation with respect to F so that, if m is the degree of $\phi(\omega)$, the quantities $1, x, x^2, \cdots, x^{m-1}$ are linearly independent with respect to F. Then in a *direct product* $A \times F(\eta)$ these quantities will be linearly independent with respect to $F(\eta)$. If $\phi(\omega)$ is reducible in $F(\eta)$, then $\phi(\omega) \equiv \phi_1(\omega)\phi_2(\omega)$ and $\phi_1(x) \cdot \phi_2(x) = 0$ where the $\phi_i(x)$ are each polynomials in x of degree less than m with coefficients not all zero in $F(\eta)$. But then $\phi_1(x) \neq 0$, and $\phi_2(x) \neq 0$ while their product is zero. This is impossible in a division algebra. We have

* These Transactions, vol. 33 (1931), pp. 235–243.

† We shall use the symbol ω to represent a scalar variable and $\phi(\omega)$, $\psi(\omega)$ as polynomials in ω with coefficients in F and leading coefficient unity.

THEOREM 9. *Let A be a normal division algebra over F and $F(\eta)$ an algebraic field over F. Then*

$$A' = A \times F(\eta)$$

is not a division algebra if some $\phi(\omega)=0$ which belongs to A is reducible in $F(\eta)$.

We continue with the following well known result.*

THEOREM 10. *There exist an infinity of irreducible equations $\phi(\omega)=0$ of degree n belonging to any normal division algebra A of order n^2 over F.*

We shall henceforth in general restrict our consideration of equations $\phi(\omega)=0$ belonging to A of order n^2 over F to equations of degree n, and shall say that such a $\phi(\omega)=0$ *properly belongs to* A.

Let $\phi(\omega)=0$ properly belong to A and let ξ be a quantity such that $\phi(\xi)=0$ so that $F(\xi)$ is an algebraic field of order n over F. Consider the algebra $A'=A\times F(\xi)$ where ξ is therefore taken to be a quantity which is *scalar* with respect to all quantities of A. J. H. M. Wedderburn has proved† that A, an algebra of order n^2 over F, is equivalent to an algebra of n-rowed square matrices with elements in $F(\xi)$. The n^2 basal quantities of A are equivalent to n^2 matrices which are linearly independent in $F(\xi)$ and hence form a basis of the set of all n-rowed square matrices with elements in $F(\xi)$. Thus $A'=A\times F(\xi)=M\times F(\xi)$, where M is a total matric algebra of order n^2 over F, for this latter algebra is evidently equivalent to the set of all n-rowed square matrices with elements in $F(\xi)$.

THEOREM 11. *Let A be a normal division algebra of order n^2 over F and let $\phi(\omega)=0$ be an equation which properly belongs to A. Consider the algebra $A'=A\times F(\xi)$ where $\phi(\xi)=0$. Then*

$$A' = A \times F(\xi) = M \times F(\xi), \tag{1}$$

where M is a total matric algebra of order n^2 over F.

Let $\phi(\omega)=0$ properly belong to A of order n^2 over F. Let B be a normal simple algebra of order n^2 over F and let B contain ξ satisfying $\phi(\xi)=0$. The direct product $A\times B$ has (1) as a sub-algebra, and hence M as a sub-algebra. By Theorem 3 we have

$$A \times B = M \times C,$$

where C is a sub-algebra of $A\times B$. By Theorem 5 the algebra $A\times B$ is a normal simple algebra, so that, by Theorem 6, C is a normal simple algebra. The

* Cf. the author's *Note on an important theorem on normal division algebras*, Bulletin of the American Mathematical Society, vol. 36 (1930), pp. 649–650.

† These Transactions, vol. 22 (1921), pp. 129–135; p. 133.

order of $A\times B$ is n^4, the order of M is n^2, so that the order of C is n^2. Algebra C is obviously the set of all quantities of $A\times B$ which are commutative with all the quantities of M. Hence C contains ξ.

THEOREM 12. *Let A be a normal division algebra of order n^2 over F and let $\phi(\omega)=0$ of degree n belong to A. Suppose that B is a normal simple algebra of order n^2 over F and that ξ in B is such that $\phi(\xi)=0$. Then*

$$A \times B = M \times C, \tag{2}$$

where M is a total matric algebra of order n^2 over F and C is a normal simple algebra of order n^2 over F and contains ξ.

We shall now study the general theory of the extension of F by η. In the author's proof of his Theorem $T4$ the following result was obtained.

THEOREM 13. *Let B be an algebra over F and let there exist an algebraic field $F(\xi)$ such that $B'=B\times F(\xi)$ is a total matric algebra over $F(\xi)$. Then B is a normal simple algebra over F.*

We shall apply the above result as follows. Let $F(\eta)$ be an algebraic field over F and consider the algebra $B=A\times F(\eta)$ over $F(\eta)$. The algebra $B'=B\times K(\xi)$ is an algebra over $K(\xi)$ where $K=F(\eta)$ and, when considered as an algebra over F, contains $A\times F(\xi)=M\times F(\xi)$ as a sub-algebra. It is thus evident that $B'=M\times F(\xi,\ \eta)=M'$, a total matric algebra over $K(\xi)$. By Theorem 13, algebra B is a normal simple algebra over $K=F(\eta)$.

LEMMA 1. *Let $A'=A\times F(\eta)$ where A is a normal division algebra. Then A' is a normal simple algebra over $F(\eta)$.*

By Theorem 4 algebra $A'=A\times F(\eta)$ is expressible in the form $A'=H'\times B$ where H' is a total matric algebra of order s^2 over $F(\eta)$ and B is a normal division algebra of order t^2 over its centrum $F(\eta)$. Evidently A' is a simple algebra over F and, when considered over F, $A'=H\times B$ where H is a total matric algebra of order s^2 over F and B is a division algebra of order t^2r over F such that r is the order of $F(\eta)$. Also, since A' has order n^2 over $F(\eta)$ we have $st=n$.

LEMMA 2. *Algebra $A'=A\times F(\eta)=H\times B$, where H is a total matric algebra of order s^2 over F and B is a division algebra of order t^2r over F which is a normal division algebra of order t^2 over its centrum $F(\eta)$ of order r over F such that $s^2t^2=n^2$, the order of the normal division algebra A over F.*

Consider the equation

$$\psi(\omega) \equiv \omega^r + \alpha_1\omega^{r-1} + \cdots + \alpha_r = 0 \quad (\alpha_i \text{ in } F), \tag{3}$$

irreducible in F. It is well known that the r-rowed square matrix

$$(4) \qquad \eta = \left\| \begin{array}{ccccc} 0 & 0 \cdots 0 & - \alpha_r \\ 1 & 0 \cdots 0 & - \alpha_{r-1} \\ 0 & 1 \cdots 0 & - \alpha_{r-2} \\ \cdot & \cdot \quad \cdot \quad \cdot & \cdot \\ 0 & 0 \cdots 0 & - \alpha_2 \\ 0 & 0 \cdots 1 & - \alpha_1 \end{array} \right\|$$

has $\psi(\omega)=0$ as its characteristic equation $|\omega I-\eta|=0$, where I is the r-rowed identity matrix. Hence if $F(\eta)$ is any algebraic field of order r we may assume that η is a quantity in a total matric algebra G of order r^2 over F.

Consider the normal simple algebra $A\times G$. By Lemma 2 this algebra has $A'=H\times B$ as a sub-algebra, where H is a total matric algebra of order s^2 over F. By Theorem 3, $A\times G=H\times C$, where C is a normal simple algebra. By Theorem 4 we may write $C=D\times K$ where D is a normal division algebra and K is a total matric algebra. Hence

$$(5) \qquad A \times G = D \times (H \times K).$$

By Theorem 1 algebra $H\times K$ is a total matric algebra. By the uniqueness in Theorem 2 algebra A is equivalent to D and G to $H\times K$. If the order of K is e^2, then r^2, the order of G, is equal to s^2e^2. Hence s divides r as well as n.

THEOREM 14. *Let A be a normal division algebra of order n^2 over F, and let $F(\eta)$ be an algebraic field of order r over F. Then*

$$(6) \qquad A \times F(\eta) = H \times B,$$

where H is a total matric algebra of order s^2 over F and B is a division algebra of order t^2r over F such that

$$(7) \qquad n = st, \quad r = se,$$

and the integer s divides both r and n. Algebra B is a normal division algebra of order t^2 over $F(\eta)$.

We shall use the above theorem and its consequences repeatedly in our theory. In particular we have as immediate corollaries

THEOREM 15. *If $A\times F(\eta)=H\times F(\eta)$, then the order n^2 of A is a divisor of r^2.*

THEOREM 16. *If r and n are relatively prime, then $A\times F(\eta)$ is a division algebra.*

THEOREM 17. *If $A\times F(\eta)$ is a total matric algebra of order n^2 over $F(\eta)$ and r is prime to n, then $n=1$.*

We may now obtain two results of intrinsic interest but which will not be used in our later work. We have shown that $A\times G=H\times C$ where G is a total matric algebra of order $r^2=s^2e^2$. But then $G=H_1\times K_1$ where H_1 is equivalent to H and K_1 to K in (5), by Theorem 1. Since A is equivalent to D and K_1 to K, the algebra $A\times K_1$ is equivalent to $D\times K=C$. But C contains the sub-algebra B of Theorem 14, since C is the algebra of all quantities of $A\times G$ commutative with all the quantities of H, while $A\times F(\eta)=H\times B$. Hence we have

THEOREM 18. *Let A and $F(\eta)$ be as in Theorem 14, so that $r=es$. Then there exists a total matric algebra K of order e^2 over F such that $A\times K$ contains an algebra equivalent to B as a sub-algebra.*

In particular let $r=p$, a prime. If $A\times F(\eta)=A'$ is a division algebra, then, by Theorem 9, A contains no x whose minimum equation is that of η and hence reducible in $F(\eta)$. Conversely let A contain no x satisfying $\psi(\omega)=0$, the minimum equation of η. Since r is a prime, $s=1$ or p. If $s=p$ then $e=1$, and, by Theorem 18, A contains a sub-algebra equivalent to B and hence a quantity x satisfying $\psi(\omega)=0$, the minimum equation of η, a quantity of B. Hence $s=1$ and A' is a division algebra. As a corollary of Theorem 18 we thus have

COROLLARY. *Let A be a normal division algebra of order n^2 over F and $F(\eta)$ an algebraic field of order p, a prime. Then $A\times F(\eta)$ is a division algebra if and only if A contains no quantity satisfying the minimum equation of η with respect to F.*

4. **Applications of the Galois theory of equations.** We shall require the well known theorems*

LEMMA 1. *Every rational function η with coefficients in F of the roots of an equation having the group Γ for F belongs to a definite sub-group Δ of Γ. There exist functions belonging to any assigned sub-group of Γ.*

LEMMA 2. *If the rational function η with coefficients in F of the roots of an equation having the group Γ for F belongs to a sub-group Δ of index r under Γ, then the substitutions of Γ replace η by exactly r distinct functions called the conjugates of η under Γ. They are the roots of an equation $\psi(\omega)=0$ of degree r with coefficients in F which is irreducible in F.*

LEMMA 3. *Let Γ be the group for F of an equation $\phi(\omega)=0$ with coefficients in F. Let Δ be the sub-group to which belongs a rational function η with coefficients in F of the roots of $\phi(\omega)=0$. By the adjunction of η to F the group of $\phi(\omega)=0$ is reduced from Γ to Δ.*

* Cf. L. E. Dickson's *Modern Algebraic Theories*, Chicago, 1926, pp. 170–174, for these Lemmas.

We shall frequently use the very well known

SYLOW'S THEOREM. *If the order g of a substitution group Γ is divisible by a power p^ν of a prime p, then Γ has a sub-group Δ of order p^ν.*

Let Γ be a transitive group on p letters, where p is a prime. Then g is divisible by p and Γ has a sub-group Δ of order p. Every substitution s may be written as a product of cycles involving *different* letters and the order of s is the least common multiple of the number of letters in each cycle. If s is not the identity substitution and is in Δ of order p on p letters, the order of s is a divisor of p and must be p. Since p is the L.C.M. of the number of letters in the individual cycles of s, these cycles must all have p letters. But the letters in different cycles are different, and there are p letters altogether, so that s is a single cycle $(123 \cdots p)$. The group Δ contains the substitutions $1, s, s^2, \cdots, s^{p-1}$ and hence is composed solely of these substitutions. Thus Δ is the regular cyclic group of order p.

LEMMA 4. *Every transitive group Γ on p letters, p a prime, contains the regular cyclic group Δ of order p whose substitutions may be taken to be $1, s, s^2, \cdots, s^{p-1}$ where $s=(123 \cdots p)$.*

We may obviously apply Lemmas 1, 2, 3 to any equation belonging to A, a normal division algebra over F. In our environment they become

THEOREM 19. *Let $\phi(\omega)=0$ belong to a normal division algebra A over F and let Γ be the group for F of $\phi(\omega)=0$. Then if Δ is a sub-group of Γ of index r under Γ there exists an algebraic field $F(\eta)$ of order r over F such that in $A'=A\times F(\eta)$ over $F(\eta)$ the group of ϕ for $F(\eta)$ is Δ.*

We shall first apply this theorem to the case where A has order p^2, p a prime. Let $\phi(\omega)=0$ have degree p and belong to A, so that the group Γ of $\phi(\omega)=0$ for F is transitive and has order pr. By Lemma 4 the group Γ has a regular cyclic sub-group Δ of order p and index r. The order rp of Γ divides $p!$ so that r divides $(p-1)!$ and is prime to p. By Theorem 19 there exists an algebraic field $F(\eta)$ of order r such that the group of $\phi(\omega)=0$ for $F(\eta)$ is Δ. By Theorem 16 algebra $A'=A\times F(\eta)$ is a normal division algebra over $F(\eta)$. The quantity x in A whose minimum equation is $\phi(\omega)=0$ is in A' and satisfies an equation with cyclic regular group for $F(\eta)$. The equation $\phi(\omega)=0$ is a cyclic equation for $K=F(\eta)$, the field $K(x)$ is a cyclic normal field, and A' is a cyclic (Dickson) normal division algebra over K.

THEOREM 20. *Let A be a normal division algebra of order p^2 over F, p a prime. Then there exists an algebraic field $F(\eta)$ of order r over F such that r divides $(p-1)!$ and is prime to p and such that the algebra $A'=A\times F(\eta)$ is a cyclic (Dickson) normal division algebra of order p^2 over $F(\eta)$.*

We shall next obtain an important theorem for the case where n is not the power of a single prime. Let $n=p^eq$ where p is a prime, $e>0$, and $q>1$ is not divisible by p. Consider a normal division algebra A of order n^2 over F and let $\phi(\omega)=0$ belong properly to A, that is, have degree n and be the minimum equation for F of x in A. The group Γ of $\phi(\omega)=0$ has order g divisible by n since Γ is transitive. Hence g is divisible by p and we may write $g=p^{\nu}r_1$ where r_1 is prime to p. By Sylow's Theorem the group Γ has a sub-group Δ of order p^{ν} and, by Theorem 19, there exists a scalar η_1 of grade r_1 such that the group of $\phi(\omega)=0$ for $F(\eta)$ is Δ. By Theorem 14, $A'=A\times F(\eta)=H_1\times B_1$ where H_1 is a total matric algebra of order s^2 over F and B_1 is a normal division algebra of order t^2 over $F(\eta)$ such that $n=st$, $r=se$. Since r is prime to p so is s_1. Also A' is not a division algebra, by Theorem 9, since $\phi(\omega)=0$ belongs to A, has degree n and group Δ of order $p^e<n$ for $F(\eta_1)$, an intransitive group. Since $s_1t_1=n$ and s_1 is prime to p, we may write $t_1=p^eq_1$ where q_1 is prime to p and $q_1<q$. If $q_1>1$ we apply the same process to B_1 over $F(\eta_1)=F_1$ to obtain a scalar η_2 such that $B_1\times F_1(\eta_2)=H_2\times B_2$ where H_2 is a total matric algebra of order s_2^2 over F_1 and B_2 is a normal division algebra of order t_2^2 such that $t_2=p^eq_2$ with $1\leqq q_2<q_1$. The field $F_1(\eta_2)=F(\eta_1, \eta_2)$ has order r_1r_2 prime to p with respect to F. We define in this way a sequence of scalars $\eta_1, \eta_2, \eta_3, \cdots$, a sequence of integers $r_1, r_2, \cdots$, and a decreasing sequence of integers $q>q_1>q_2>\cdots\geqq 1$. This latter sequence must terminate at some $q_k=1$, and if $F_k=F_{k-1}(\eta_k)=F(\eta_1, \eta_2, \cdots, \eta_k)$, then $F_k=F(\eta)$, an algebraic field of order $r=r_1r_2\cdots r_k$ over F such that $A\times F(\eta)=H\times B$, where H is a total matric algebra of order $s^2=s_1^2s_2^2\cdots s_k^2=q^2$ and B is a normal division algebra of order $p^{2e}q_k^2=p^{2e}$ over $F_k=F(\eta)$.

THEOREM 21. *Let A be a normal division algebra of order n^2 over F, where*

$$n = p^eq, \quad e > 0, \quad q > 1,$$

and p is a prime not dividing q. Then there exists an algebraic field $F(\eta)$ of order r over F such that r is prime to p and

$$A \times F(\eta) = H \times B,$$

where H is a total matric algebra of order q^2 over F and B is a normal division algebra of order p^{2e} over its centrum $F(\eta)$.

We shall also require the well known theorems

LEMMA 5. *Let η and ζ be rational functions with coefficients in F of the scalar roots of an equation whose group for F is Γ, and let both η and ζ belong to the same sub-group Δ of Γ. Then each of η and ζ is expressible as a rational function with coefficients in F of the other.*

LEMMA* 6. *Let Γ be a group of order p^ν, p a prime, and let Γ_s have order p^s and be contained in Γ. Then Γ_s is an invariant sub-group of index p of a sub-group Δ of Γ.*

Let now $\phi(\omega)=0$ have degree p^e, coefficients in F and a transitive group Γ of order p^ν for F. Let η be a scalar root of $\phi(\omega)=0$ and let Γ_s of order p^s be the sub-group of Γ to which η belongs. By Lemma 6 there exists a sub-group Δ of order p^{s+1} of Γ which contains Γ_s self-conjugately. The group Δ contains a substitution g not in Γ_s and the cycle of g containing η has order q a power of p, and replaces η by a root η_2 of $\phi(\omega)=0$, a conjugate to η under Δ. The root η_2 belongs to $g^{-1}\Gamma_s g=\Gamma_s$ since Γ_s is self-conjugate under Γ_s. Hence $\eta_2=\theta(\eta)$ by Lemma 5, where $\theta(\eta)$ is a polynomial in η with coefficients in F. The substitutions $I, g, g^2, \cdots, g^{q-1}$ are all in the group Δ and replace η by $\theta^i(\eta)$ respectively. These q polynomials are all conjugates to η under Δ, while there are exactly p such conjugates to η under Δ, since the index of Γ_s is p. Since q is a power of p, q is p and

$$\theta^p(\eta) = \theta^0(\eta) = \eta, \ \theta(\eta), \ \theta^2(\eta), \cdots, \theta^{p-1}(\eta)$$

are distinct and satisfy $\phi(\omega)=0$. They are all in $F(\eta)$. Passing to any field $F(x)$ equivalent to $F(\eta)$ we have

LEMMA 7. *Let x satisfy $\phi(\omega)=0$ whose group for F is a transitive group of order p^e, p a prime. Then $F(x)$ contains a quantity $\theta(x)$ such that*

$$\theta^p(x) = \theta^0(x) = x, \ \theta(x), \cdots, \theta^{p-1}(x)$$

are p distinct roots of $\phi(\omega)=0$.

Consider now the set of all quantities of $F(x)$ symmetric in the polynomials $\theta^i(x)$. This set evidently forms a sub-field $F(y)$ of $F(x)$. The equation

$$\psi(\omega) \equiv [\omega - \theta^{p-1}(x)] \cdots [\omega - \theta(x)][\omega - x]$$

evidently has coefficients in $F(y)$ and is also evidently not reducible in $F(y)$. Hence x has grade p with respect to $F(y)$ and $F(x)$ is a cyclic field of order p over $F(y)$.

THEOREM 22. *Let $\phi(\omega)=0$ have degree p^e, coefficients in F, and a transitive group of order a power of p with respect to F where p is a prime. Then if x is a quantity satisfying $\phi(\omega)=0$ the field $F(x)$ of order p^e over F contains a sub-field $F(y)$ of order p^{e-1} over F such that $F(x)$ is a cyclic field of order p over $F(y)$.*

Let now A be a normal division algebra of order p^{2e}, p a prime, and let x in A have grade p^e and $\phi(\omega)=0$ as its minimum equation for F. Let the group of

* Cf. Burnside, *The Theory of Groups*, Cambridge, 1897, pp. 64–65.

$\phi(\omega)=0$ have order $p^\nu r_1$ where r_1 is prime to p. As in the proof of Theorem 21 there exists an algebraic field $F_1=F(\eta_1)$ of order r_1 over F such that the group of $\phi=0$ for F_1 is a group Γ of order p^ν. By Theorem 14 the algebra $A'=A\times F(\eta_1)$ is a normal division algebra over $F(\eta_1)$, so that, by Theorem 9, the equation $\phi(\omega)=0$ is irreducible in $F(\eta_1)$, and Γ is a transitive group. We may therefore apply Theorem 22 and obtain the existence of a quantity y in $F_1(x)$ such that this field is a cyclic field of order p over $F_1(y)$ of order p^{e-1}. Apply this same process to the minimum equation of y with respect to F_1. We obtain the existence of a quantity η_2 of grade r_2 prime to p with respect to $F(\eta_1)=F_1$ such that the algebra $A''=A'\times F_1(\eta_2)$ is a normal division algebra over $F_2=F(\eta_1, \eta_2)=F_1(\eta_2)$ and the field $F_2(y)$ contains a sub-field $F_2(z)$ of order p^{e-2} with respect to F_2 and is a cyclic field of order p over this field. It is evident that the field $F_2(x)$ is a cyclic field over $F_2(y)$. Continuing in this fashion we obtain

THEOREM 23. *Let A be a normal division algebra of order p^{2e} over F, p a prime, and let x in A have grade p^e with respect to F. Then there exists an algebraic field $K=F(\eta)$ of order r prime to p with respect to F such that*

$$A' = A \times F(\eta)$$

is a normal division algebra over K, the algebraic field $K(x)$ has order p^e over K, and there exist quantities

$$x = x_e, \quad x_{e-1}, \cdots, x_1$$

in $K(x)$ such that if

$$K_0 = K, \quad K_i = K(x_i) \qquad (i = 1, 2, \cdots, e),$$

then K_i has order p^i with respect to K and is a cyclic field of order p over K_{i-1}. In particular A' contains a cyclic field $K(x_1)$ of order p over K.

5. **Cyclic algebras.** Let $\phi(\omega)=0$ have degree n, coefficients in F, and have the cyclic regular group for F. Then if x is a quantity satisfying $\phi(x)=0$ the algebraic field $F(x)$ is a normal cyclic field of order n over F, and there exists a quantity $\theta(x)$ in $F(x)$ such that if we define by induction

$$\theta^0(x) = x, \quad \theta^k(x) = \theta[\theta^{k-1}(x)] \qquad (k = 1, \cdots),$$

then

$$\theta^n(x) = x,$$

and

$$\phi(\omega) \equiv [\omega - \theta^{n-1}(x)] \cdots [\omega - \theta(x)](\omega - x).$$

Consider an algebra A over F with the basis

$$x^\alpha y^\beta \qquad (\alpha, \beta = 0, 1, \cdots, n-1), \tag{8}$$

and the multiplication table

$$(9) \qquad \phi(x) = 0, \quad y^n = \gamma \text{ in } F, \quad y^\beta f(x) = f[\theta^\beta(x)]y^\beta \qquad (\beta = 0, 1, \cdots),$$

for every $f(x)$ of $F(x)$. Such an algebra is always associative and is the ***general Dickson algebra***. We shall say that a Dickson algebra is a *cyclic* algebra when it is a ***normal simple algebra***. The field F, the equation $\phi=0$, the particular polynomial $\theta(x)$ in (9), and the quantity γ in F formally define A, and we shall use the notation

$$A = F[\phi, \theta, \gamma].$$

Let $\gamma=0$ so that $y^n=0$ and $y\neq 0$ is a nilpotent quantity of A. If $a=\sum_{\alpha=0}^{n-1}a_\alpha(x)y^\alpha$ is any quantity of A then

$$(10) \qquad (ay)^n = by^n = 0.$$

For

$$y^\beta a = y^\beta \sum_{\alpha=0}^{n-1} a_\alpha(x)y^\alpha = \left(\sum_{\alpha=0}^{n-1} a_\alpha[\theta^\beta(x)]y^\alpha\right)y^\beta$$

and we may carry the n factors y in $(ay)^n$ to the right and obtain (10) with b in A. It follows that ay is nilpotent in A for every a of A. Hence y is ***properly*** nilpotent in A and A contains a ***radical***. Hence A is not even a semi-simple algebra and is not a cyclic algebra.

Let next $\gamma\neq 0$ and ξ be a scalar quantity with respect to quantities of A such that $\phi(\xi)=0$. Then Wedderburn has shown* that $A\times F(\xi)=M\times F(\xi)$ where M is a total matric algebra, so that, by Theorem 13, A is a normal simple algebra. We thus have

THEOREM 24. *A Dickson algebra is a cyclic algebra if and only if* $\gamma\neq 0$.

Professor Wedderburn in fact actually obtained formulas for the quantities of A in terms of those of M and conversely. He used a form of A reciprocal to ours and we shall use, instead, the form of L. E. Dickson.† Every quantity f of A has the form

$$f = f_0(x) + f_1(x)y + \cdots + f_{n-1}(x)y^{n-1}.$$

Let ξ be as above, and write

$$f_i \equiv f_i(\xi), \quad f_i(\theta^\alpha) \equiv f_i[\theta^\alpha(\xi)].$$

Then if we let the quantities of $M\times F(\xi)$ be represented by matrices, the quantity f of A will be represented by

* These Transactions, vol. 15 (1914), pp. 162–166.

† *Algebren und ihre Zahlentheorie*, Zurich, 1927, p. 68, equation 54.

$$
(11) \qquad \left\| \begin{matrix} f_0 & f_1 & f_2 & \cdots f_{n-1} \\ f_{n-1}(\theta)\gamma & f_0(\theta) & f_1(\theta) & \cdots f_{n-2}(\theta) \\ f_{n-2}(\theta^2)\gamma & f_{n-1}(\theta^2)\gamma & f_0(\theta^2) & \cdots f_{n-3}(\theta^2) \\ \cdot & \cdot & \cdot & \cdot \\ f_1(\theta^{n-1})\gamma & f_2(\theta^{n-1})\gamma & f_3(\theta^{n-1})\gamma & \cdots f_0(\theta^{n-1}) \end{matrix} \right\| .
$$

We require the representation of the basal quantities of M in terms of the quantities of $A\times F(\xi)$. Let then e_{ij} be the quantity of M which corresponds to the matrix with unity in the ith row and jth column and zeros elsewhere, so that the n^2 quantities e_{ij} are a basis of M with respect to F. The quantity x has the form above with $f_0(x)=1$ and all the other f's zero, while $y=0+y+0$. Hence (11) implies that

$$
(12) \qquad x = \sum_{i=1}^{n} \xi_i e_{ii}, \quad y = e_{12} + e_{23} + \cdots + e_{n-1,n} + \gamma e_{n,1},
$$

where we define

$$
(13) \qquad \xi_i = \theta^{i-1}(\xi) \qquad (i = 1, \cdots, n).
$$

By a very easy computation the quantity $(x-\xi_1)\ (x-\xi_2) \cdots (x-\xi_{i-1}) \cdot (x-\xi_{i+1}) \cdots (x-\xi_n)$ has zero for the coefficients of all the e_{kj} except e_{ii} itself, and, in fact,

$$
(14) \qquad e_{ii} = \frac{(x-\xi_1)(x-\xi_2)\cdots(x-\xi_{i-1})(x-\xi_{i+1})\cdots(x-\xi_n)}{(\xi_i-\xi_1)(\xi_i-\xi_2)\cdots(\xi_i-\xi_{i-1})(\xi_i-\xi_{i+1})\cdots(\xi_i-\xi_n)} .
$$

From (12) we obtain

$$
(15) \qquad e_{ii}y = e_{i,i+1} = ye_{i+1,i+1} \qquad (i = 1, 2, \cdots),
$$

under the assumption that subscripts are reduced modulo n when they exceed n. In particular

$$
(16) \qquad e_{11}y = ye_{22},
$$

where we shall require the form of

$$
(17) \qquad \begin{aligned} e_{11} &= \frac{(x-\xi_2)(x-\xi_3)\cdots(x-\xi_n)}{(\xi_1-\xi_2)(\xi_1-\xi_3)\cdots(\xi_1-\xi_n)}, \\ e_{22} &= \frac{(x-\xi_3)(x-\xi_4)\cdots(x-\xi_n)(x-\xi_1)}{(\xi_2-\xi_3)(\xi_2-\xi_4)\cdots(\xi_2-\xi_n)(\xi_2-\xi_1)} . \end{aligned}
$$

From y^k in (11) we obtain

$$(18)\qquad e_{11}y^k = e_{1k}, \quad \gamma e_{k1} = y^{n+1-k}e_{11},$$

where we are using always

$$(19)\qquad e_{ij}e_{jk} = e_{ik}, \quad e_{ij}e_{tk} = 0 \qquad (j \neq t;\ i, j, k, t = 1, 2, \cdots, n).$$

Since $\gamma \neq 0$ has an inverse γ^{-1} in F we have finally $e_{ij} = e_{i1}e_{1j}$ and, from (18),

$$(20)\qquad e_{ij} = \gamma^{-1}y^{n+1-i}e_{11}y^{j-1} \qquad (i, j = 1, \cdots, n).$$

We shall apply the form obtained in (20) as follows. Let $A = F[\phi, \theta, \gamma_1]$ and $B = F[\phi, \theta, \gamma_2]$ be two cyclic algebras differing only in their quantites γ. We shall use (8) as a notation for the basis of A with $y^n = \gamma$, and, correspondingly, for a basis and multiplication tables of B shall use

$$(21)\qquad B = (\xi^\alpha \eta^\beta) \qquad (\alpha, \beta = 0, 1, \cdots, n-1),$$

$$(22)\qquad \phi(\xi) = 0, \quad \eta^n = \gamma_2, \quad \eta^\beta f(\xi) = f[\theta^\beta(\xi)]\eta^\beta \qquad (\beta = 0, 1, \cdots),$$

so that if $\xi_i = \theta^{i-1}(\xi)$ then

$$(23)\qquad \eta\xi_i = \xi_{i+1}\eta \qquad (i = 1, 2, \cdots, n),$$

where $\xi_{n+1} = \xi$. We shall consider the algebra

$$(24)\qquad C = A \times B,$$

which contains $A \times F(\eta) = M \times F(\eta)$ as a sub-algebra, where M is the total matric algebra whose basis is given by (20), (17), with (16) holding. By Theorem 3 algebra C is expressible in the form

$$(25)\qquad C = M \times D,$$

where D is the algebra of all quantities of C which are commutative with all the quantities of M and obviously contains $F(\xi)$. Also D contains

$$(26)\qquad z = y\eta = \eta y.$$

For from (23) and (17) it is obvious that

$$(27)\qquad \eta e_{11} = e_{22}\eta.$$

But (16) states that $ze_{11} = y(\eta e_{11}) = ye_{22}\eta = e_{11}z$. The quantity z is commutative with e_{11} and with y, and, since γ^{-1} is in F, with all of the quantities e_{ij} of (20), a basis of M. Hence z is in D.

We have thus shown that D contains ξ and z. Since D is an algebra over F it contains all the quantities

$$(28)\qquad \xi^\alpha z^\beta \qquad (\alpha, \beta = 0, 1, \cdots, n-1),$$

where, since y is commutative with ξ and z, we have

$$z^n = y^n\eta^n = \gamma_1\gamma_2, \quad z^\beta f(\xi) = f[\theta^\beta(\xi)]z^\beta \qquad (\beta = 0, 1, \cdots), \tag{29}$$

for every $f(\xi)$ in $F(\xi)$. The quantity z is merely a scalar multiple $y\eta$ of η, with respect to B, since y is scalar with respect to B. When (21) form a basis of B, the quantities (28) are thus linearly independent in F, and in fact in $F(y)$. Algebra D has order n^2 from Theorem 3 so that (28) form a basis of D, and the multiplication table of D is (29). Since $\gamma_1 \neq 0$, $\gamma_2 \neq 0$, the quantity $\gamma_1\gamma_2 \neq 0$ and D is the cyclic algebra $F[\phi, \theta, \gamma_1\gamma_2]$. We have proved

THEOREM 25. *Let $A = F[\phi, \theta, \gamma_1]$ and $B = F[\phi, \theta, \gamma_2]$ be cyclic algebras with the same ϕ, θ, and order n^2 and thus differing possibly only in their quantities γ. Then*

$$A \times B = M \times C, \tag{30}$$

where M is a total matric algebra of order n^2 over F and C is the cyclic algebra $F[\phi, \theta, \gamma_1\gamma_2]$.

6. **Direct powers of normal simple algebras.** If A is any normal simple algebra over F and B is equivalent to A, then we can form the direct product $A \times B$ in such a fashion that A and B are sub-algebras of $A \times B$, by passing to equivalent algebras. We may evidently represent such a direct product symbolically by $A \times A$ where, in view of the direct product symbol, the two letters A mean merely *equivalent* algebras. As a generalization we may define the *direct power**

$$A^{\alpha+1} = A \times A^\alpha = A^\alpha \times A \qquad (\alpha = 1, 2, \cdots), \tag{31}$$

by an obvious induction on α, where $A = A^1$. The powers here behave like ordinary powers of integers, as in a direct product we have commutativity and associativity of the symbols representing algebras, so that

$$A^\alpha \times A^\beta = A^{\alpha+\beta}, \quad A^\alpha \times B^\alpha = (A \times B)^\alpha, \quad A \times B = B \times A, \tag{32}$$

for any normal simple algebras (in fact for any algebras) A and B. By Theorem 12 we have

$$A^2 = M \times B_1,$$

where M is as in (2) and B_1 is a normal simple algebra. By Theorem 12 algebra B_1 contains a quantity ξ whose minimum equation with respect to

* We shall use the notation of direct power so frequently in the following theory that to alter the above even slightly would make the work less clear and the printing too bulky. Hence although the symbol of (31) is customarily used in linear algebra theory to mean ordinary product, this meaning will not occur in the present paper. We shall then assume that the symbol of (31) represents throughout *direct power*.

F has degree n and belongs to A, a normal division algebra of order n^2 over F. Again by Theorem 12 we have $A^3 = M\times(A\times B_1) = M\times M\times B_2 = M^2\times B_2$, where B_2 is a normal simple algebra of order n^2 over F and also contains ξ. In general we have

$$A^\alpha = M^{\alpha-1}\times B_{\alpha-1} \qquad (\alpha = 2, 3, \cdots). \tag{33}$$

But $B_{\alpha-1}$ is a normal simple algebra and we may write

$$B_{\alpha-1} = H_\alpha \times A_\alpha, \tag{34}$$

where A_α is a normal division algebra of order t_α^2 over F and H_α is a total matric algebra of order s_α^2 over F such that $s_\alpha t_\alpha = n$.

THEOREM 26. *Let A be a normal division algebra of order n^2 over F and M be a total matric algebra of order n^2 over F. Then the direct power*

$$A^\alpha = (M^{\alpha-1}\times H_\alpha)\times A_\alpha \qquad (\alpha = 2, 3, \cdots), \tag{35}$$

where H_α is a total matric algebra of order s_α^2 over F and A_α is a normal division algebra of order t_α^2 over F such that the relation

$$s_\alpha t_\alpha = n \tag{36}$$

holds.

Suppose that A were a normal division algebra for which $t_\alpha = 1$ for some integer* α. In this case $A^\alpha = M^\alpha$, a total matric algebra. We shall say that when such an integer α exists the least integer ρ for which A^ρ is a total matric algebra is the *exponent* of A.

7. **Algebras of order p^2, p a prime.** Let first A be any cyclic algebra of order p^2, p a prime, over F. The author has shown that a necessary and sufficient condition† that A be a division algebra is that γ be not the norm, $N(a)$, of any a in $F(x)$, and also that this condition was equivalent† to the condition that $\gamma^\alpha \neq N(a)$ $(\alpha = 1, 2, \cdots, p-1)$. Hence if $A = F[\phi, \theta, \gamma]$, then the algebras $F[\phi, \theta, \gamma^\alpha]$ $(\alpha = 1, 2, \cdots, p-1)$ are also normal division algebras. But in Theorem 23 we have $A\times A = A^2 = M\times A_2$ where $A_2 = F[\phi, \theta, \gamma^2]$ is a normal division algebra. Similarly $A^3 = M^2\times A_3 = M^2\times F[\phi, \theta, \gamma^3]$, and in general

$$A^\alpha = M^{\alpha-1}\times A_\alpha,\quad A^p = M^{p-1}\times B,$$

where $B = F[\phi, \theta, \gamma^p]$ is a cyclic algebra by Theorem 22, and $A_\alpha = F[\phi, \theta, \gamma^\alpha]$ is a normal division algebra for $\alpha = 1, \cdots, p-1$. Algebra B is not a division algebra since $\gamma^p = N(\gamma)$. Since B has order p^2, p a prime, Theorem 4 implies that $B = M$. We have proved the theorem

* We shall prove the existence of such an integer for any normal division algebra A in §8.

† These results are Theorems T 18, 19.

THEOREM 27. *Every cyclic division algebra A of order p^2 over F, p a prime, has p as its exponent ρ. Thus*

$$A^\alpha = M^{\alpha-1} \times A_\alpha, \quad A^p = M^p \quad (\alpha = 2, 3, \cdots, p-1), \tag{37}$$

where A has the direct product sequence

$$A_1 = A, A_2, \cdots, A_{p-1}, A_p = F, \tag{38}$$

of normal division algebras A_α of orders p^2 over F except for A_p which has order unity.

Let next A be any normal division algebra of order p^2 over F, p a prime. By Theorem 26 we have

$$A^p = (M^{p-1} \times H_p) \times A_p, \tag{39}$$

where A_p has order t^2 and $p = st$ so that $t = 1$ or p. By Theorem 20 there exists an algebraic field $F(\eta)$ of order r prime to p such that if $A' = A \times F(\eta)$ then A' is a cyclic normal division algebra over $F(\eta)$. By Theorem 27 we have $(A')^p = (M')^p$ where $M' = M \times F(\eta)$ is a total matric algebra of order p^2 over $F(\eta)$. Evidently

$$(A')^p = (M^{p-1} \times H_p)' \times (A_p)' = (M')^p$$

in algebras over $F(\eta)$, where the meaning of the symbols is clear. But then, by Theorem 7, algebra $(A_p)'$ is a total matric algebra over $F(\eta)$, since so are $(M^{p-1} \times H_p)'$ and $(M')^p$. Since r is prime to p and the order of A_p is unity or p^2, Theorem 17 implies that $t_p = 1$, and A^p is a total matric algebra. If $A^\alpha = M^\alpha$ with $\alpha < p$, then $(A')^\alpha = (M')^\alpha$ where A' is a cyclic normal division algebra over $F(\eta)$, contrary to the proved fact that p is the exponent of A'. Hence p is the exponent of A.

THEOREM 28. *Every normal division algebra of order p^2 over F, p a prime, has p as its exponent ρ, such that*

$$A^\alpha = M^{\alpha-1} \times A_\alpha, \quad A^p = M^p \quad (\alpha = 2, 3, \cdots, p-1), \tag{40}$$

and algebra A has the direct product sequence

$$A = A_1, A_2, \cdots, A_{p-1}, A_p = F, \tag{41}$$

of normal division algebras of orders $p^2, p^2, \cdots, p^2$, 1 respectively.

8. **The general case.** Let A be a normal division algebra of order n^2 over F. We shall first treat the case $n = p^e$, p a prime. For $e = 1$ we have proved

LEMMA 1. *Let A be a normal division algebra of order p^{2e} over F, p a prime. Then if $n = p^e$, the algebra $A^n = M^n$ is a total matric algebra over F.*

Assume, then, as the basis of an induction on e that the lemma is true for $e<f$. Let then A have order p^{2f} and set $n=p^f$, $t=p^{f-1}$. We may form, from (35),

$$A^t = M^{t-1} \times H_t \times A_t,$$

where M and H_t are total matric algebras and A_t is a normal division algebra over F whose order is a power of p. By Theorem 24 there exists an algebraic field $K=F(\eta)$ of order r prime to p over F such that $A'=A\times F(\eta)$ is a normal division algebra over K and contains a quantity x of grade p with respect to K. In algebras over K we have then

$$(A')^t = (M')^{t-1} \times (H_t)' \times (A_t)',$$

and $(A_t)'$ is also a normal division algebra over K. Let ξ be a scalar root of the minimum equation of x with respect to K so that the algebra $A''=A'\times K(\xi)$ is not a division algebra over $K(\xi)$, and by Theorem 14 we have

$$A'' = G \times B,$$

where G is a total matric algebra of order p^2 and B is a normal division algebra of order p^{2f-2} over $K(\xi)$. By the assumption of our induction B^t is a total matric algebra. Hence, in algebras over $K(\xi)$,

$$(A'')^t = G^t \times B^t = (M'')^t \times (H_t)'' \times (A_t)''$$

is a total matric algebra. It follows from Theorem 7 that $(A_t)''=(A_t)'\times K(\xi)$ is a total matric algebra over $K(\xi)$. By Theorem 15 the order of $(A_t)'$ is 1 or p^2. Hence the order of A_t is 1 or p^2 and, in either case,

$$(A^t)^p = A^n = M^{pt-p} \times (H_t)^p \times (A_t)^p$$

is a total matric algebra M^n, by Theorem 28. Our induction is complete and Lemma 1 is proved.

Let next A have order n^2 where $n=p^eq$ and p is a prime, q is prime to p. If $q=1$ then $A^n=M^n$ as we have proved. Let then $q>1$ be prime to p, an arbitrary prime factor of n. Form

$$A^n = M^{n-1} \times H_n \times A_n,$$

where A_n is a normal division algebra whose order divides n^2. By Theorem 21 there exists an algebraic field $F(\eta)$ of order r prime to p such that $A'=A\times F(\eta)$ $=H\times B$ where H is a total matric algebra of order q^2 over F and B is a normal division algebra of order $t^2=p^{2e}$ over F. Then

$$(A')^n = H^n \times B^n = H^n \times (B^t)^q$$

is a total matric algebra over $F(\eta)$, by Lemma 1. Hence $(A_n)'$ is a total matric algebra. By Theorem 15 the order of A_n is a divisor of r and is prime

to p. But p was any prime factor of n and hence the order of A_n, a divisor of n^2, is prime to n and is unity. We have

LEMMA 2. *Let A be a normal division algebra of order n^2 over F. Then A^n is a total matric algebra.*

We have thus proved very simply the existence of an exponent ρ for any A where ρ is the *least* integer for which A^ρ is a total matric algebra. Let $A^\alpha = M^\alpha$ be a total matric algebra and write the positive integer α in its form

$$\alpha = \lambda\rho + \mu, \quad 0 \leqq \mu < \rho.$$

The integer $\lambda \neq 0$ since ρ is the least integer α. If $\mu \neq 0$, then $A^\alpha = M^{\lambda\rho} \times A^\mu$ is a total matric algebra and hence A^μ is a total matric algebra, a contradiction of the definition of ρ. Hence *$A^\alpha = M^\alpha$ if and only if α is divisible by ρ.*

In particular ρ divides n by Lemma 2. Let $n = tq$ where $t = p^e$, p a prime and $q > 1$ is prime to p. Then, as in the proof of Lemma 2, we have $A' = A \times F(\eta) = H \times B$ where B is a normal division algebra of order t^2 over $F(\eta)$ and H is a total matric algebra of order q^2. Then $(A')^\rho = H^\rho \times B^\rho$ is a total matric algebra since so is A^ρ. Hence B^ρ is a total matric algebra and ρ is divisible by the exponent of B, a power of p. Hence ρ is divisible by every prime factor p of n.

LEMMA 3. *Let A be a normal division algebra of order n^2 over F. Then A has an exponent ρ whose prime factors coincide with those of n and which is a divisor of n.*

Consider the sequence of algebras

$$A = A_1, A_2, \cdots, A_{\rho-1}, A_\rho = F, \tag{42}$$

a set of normal division algebras A_i of order t_i^2 over F such that t_i divides n. Let A_i and A_j be any two algebras of this sequence and let $i+j \equiv k (\text{mod } \rho)$, where $0 < k \leqq \rho$, $i+j = \lambda\rho + k$. Then since $A_i \times A_j$ is a normal simple algebra from Theorem 5,

$$A^{i+j} = M^{i-1} \times H_i \times A_i \times M^{j-1} \times H_j \times A_j = M^{i+j-2} \times H_i \times H_j \times H \times B,$$

where H is a total matric algebra and B is a normal division algebra as in Theorem 4. But

$$A^{i+j} = A^{\lambda\rho+k} = M^{\lambda\rho+k-1} \times H_k \times A_k.$$

From the uniqueness in Theorem 2 algebra B is equivalent to A_k. Hence, in the sense of equivalence,

$$A_i \times A_j = H_{i,j} \times A_k, \tag{43}$$

where $H_{i,j}$ is a total matric algebra. It is also obvious that if $\alpha=\lambda\rho+k$, where $0<k\leqq\rho$, then

$$A^\alpha = M^{\alpha-1}\times H_k \times A_k,$$

since $A^{\lambda\rho}=M^{\lambda\rho}$ and does not affect the H and A with subscripts k. We have proved

THEOREM 29. *Every normal division algebra A of order n^2 over F has an exponent ρ which divides n and which is divisible by every prime factor of n.* For every integer $\alpha\geqq 2$ we have*

$$\alpha = \lambda\rho + k,\quad 0 < k < \rho,\quad \lambda \geqq 0, \tag{44}$$

and, in the sense of equivalence,

$$A^\alpha = M^{\alpha-1}\times H_k \times A_k, \tag{45}$$

where A_k is the kth member of the direct product sequence

$$A_1 = A, A_2, \cdots, A_{\rho-1}, A_\rho = F \tag{46}$$

of A. The sequence (46) is a set of normal division algebras A_i of orders t_i^2 respectively dividing n^2. If A_i and A_j are any two members of (46) such that

$$i + j \equiv k \pmod{\rho}, \tag{47}$$

then, in the sense of equivalence,

$$A_i \times A_j = A_k \times H_{i,j}, \tag{48}$$

where $H_{i,j}$ is a total matric algebra.

Let us assume for the moment that two of the algebras A_i and A_j in (46) were equivalent. We may take $i<j$. Since A is in (46) we have

$$A_{\rho-i}\times A_i = H_i,$$

a total matric algebra in view of the fact that $A_\rho=F$. But when A_i and A_j are equivalent we have then $A_{\rho-i}\times A_j$ a total matric algebra while $\rho+j-i\not\equiv 0$ (mod ρ), a contradiction of Theorem 29. Hence the algebras of (46) are all non-equivalent.

THEOREM 30. *The direct product sequence of any normal division algebra is composed of non-equivalent algebras.*

9. **The fundamental theorem.** We shall now prove R. Brauer's principal result on normal division algebras. Let A be a normal division algebra of order n^2 and exponent ρ over F, and suppose that $n=p^eq$, where q is not divisible by the prime p and is not unity. Write $\rho=\sigma\tau$, where

* The theorem to this point was first proved by Brauer.

$$\tau = p^\beta, \quad \sigma > 1,$$

and σ is prime to p. There exists a positive integer λ such that

$$\lambda\sigma \equiv 1 \pmod{\tau},$$

since σ is prime to p. Also there exists a positive integer μ such that

$$\mu\tau \equiv 1 - \lambda\sigma \pmod{\rho}$$

since the greatest common divisor τ of ρ and τ divides $1-\lambda\sigma$. Then

$$\lambda\sigma + \mu\tau = 1 + \nu\rho$$

where evidently $\nu > 0$. We form

$$A^{1+\nu\rho} = M^{\nu\rho} \times A = A^{\lambda\sigma} \times A^{\mu\tau}.$$

But if $\delta = \lambda\sigma$, then

$$A^{\lambda\sigma} = M^{\delta-1} \times H_\delta \times A_\delta,$$

where A_δ is a normal division algebra whose order divides n^2. But $A^{\delta\tau} = A^{\lambda\rho}$ is a total matric algebra so that $(A_\delta)^\tau$ is a total matric algebra and the exponent of A_δ is a divisor of $p^\beta = \tau$ and is a power of p. By Theorem 29 the order of A_δ is also a power of p, $p^{2\alpha}$. Similarly, if $\epsilon = \mu\tau$, then

$$A^{\mu\tau} = A^\epsilon = M^{\epsilon-1} \times H_\epsilon \times A_\epsilon,$$

where A_ϵ is a normal division algebra. As before $(A_\epsilon)^\sigma$ is a total matric algebra, the exponent of A_ϵ divides σ and is prime to p, and the order of A_ϵ is s^2, a divisor of q^2. Now

$$A^{1+\nu\rho} = M^{\nu\rho} \times A = M^{\delta+\epsilon-2} \times H_\delta \times H_\epsilon \times (A_\delta \times A_\epsilon).$$

By Theorem 8 the algebra $A_\delta \times A_\epsilon$ is a normal division algebra. By Theorem 2 algebra A is equivalent to the direct product $A_\delta \times A_\epsilon$. Since n is then equal to $p^\alpha s$, we have $\alpha = e$ and $s = q$. Using Theorem 8 we obtain

Lemma 1. *Let A be a normal division algebra of order n^2 over F, $n = p^e q$, where p is a prime and $q > 1$ is prime to p. Then*

$$A = B \times C,$$

where B is a normal division algebra of order p^{2e} over F and C is a normal division algebra of order q^2 over F, and conversely.

Suppose that $A = B \times C = D \times E$ where B and D have the same order s^2 and C and E have the same order t^2 such that s and t are relatively prime integers. By Theorem T 12, there exists a normal division algebra B_0 of order s^2 over

F such that $B\times B_0$ is a total matric algebra.* Similarly there exists an algebra E_0 of order t^2 over F such that $E\times E_0$ is a total matric algebra. Then

$$A \times B_0 \times E_0 = (B \times B_0) \times (C \times E_0) = (D \times B_0) \times (E \times E_0).$$

But $C\times E_0 = H\times G$, where H is a total matric algebra and G is a normal division algebra whose order divides t^4, the order of $C\times E_0$. Similarly $D\times B_0 = Q\times S$, where Q is a total matric algebra and S is a normal division algebra whose order divides s^4. Since

$$(B \times B_0) \times H \times G = (E \times E_0) \times Q \times S,$$

it follows from Theorems 1 and 2 that G and S are equivalent and have the same order, a divisor of both s^4 and t^4. But s is prime to t so that this order is unity. Hence

$$C \times E_0 = H, \; D \times B_0 = Q, \; C \times (E_0 \times E) = H \times E,$$

where H and $E\times E_0$ are total matric algebras. By Theorem 2 algebra C is equivalent to E. Similarly algebra D is equivalent to B.

LEMMA 2. *Let A have order n^2 over F and let $n=st$ where s and t are relatively prime integers. Then A is expressible as a direct product $B\times C$ of a normal division algebra B of order s^2 and a normal division algebra C of order t^2 in only one way in the sense of equivalence.*

We now apply our two lemmas as follows. Let

$$n = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m} = p_1^{e_1} n_1, \; n_1 = p_2^{e_2} n_2, \; \cdots, \; n_{m-1} = p_{m-1}^{e_{m-1}} \cdot p_m^{e_m},$$

where the p_i are distinct primes. By Lemmas 1 and 2 we have $A = B_1\times C_1$ where B_1 has order $p_1^{2e_1}$ and C_1 has order n_1^2 in one and only one way in the sense of equivalence, and conversely. Also $C_1 = B_2\times C_2$, where B_2 has order $p_2^{2e_2}$ and C_2 has order n_2^2. Again this expression is unique, so that the expression $A = B_1\times B_2\times C_2$ is also unique. Finally we evidently obtain $C_{m-1} = B_{m-1}\times B_m$ in one and only one way in the sense of equivalence and $A = B_1\times B_2\times \cdots \times B_m$, a direct product of normal division algebras B_i of orders $p_i^{2e_i}$ respectively in one and only one way in the sense of equivalence, and conversely, by Theorem 8, every such direct product is a normal division algebra of order n^2.

FUNDAMENTAL THEOREM. *Write any positive integer n in its unique factored form*

$$n = p_1^{e_1} p_2^{e_2} \cdots p_m^{e_m}$$

* In fact if ρ is the exponent of B then B_0 is $B_{\rho-1}$.

where the p_i are distinct primes. Every normal division algebra A of order n^2 over F is expressible as a direct product

$$A = B_1 \times B_2 \times \cdots \times B_m$$

*of normal division algebras B_i of orders $p_i^{2e_i}$ respectively in one and only one way in the sense of equivalence, and conversely.**

This completely reduces the problem of the determination of all normal division algebras of order n^2 to the case where n is a power of a prime. In particular it furnishes a determination of all normal division algebras of orders 36 and 144 over F since all normal division algebras of order 16, 9, 4 are known.†

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* Brauer did not obtain the converse of the above, an immediate consequence of our Theorem 8.

† Cf. the author's *A determination of all normal division algebras of order sixteen*, these Transactions, vol. 31 (1929), pp. 253–260. The above result for the case of algebras of order 36 over F evidently replaces completely the author's partial results on *Algebras of type R_2 in thirty-six units*, American Journal of Mathematics, vol. 52 (1930), pp. 283–292, and *On normal division algebras of type R in thirty-six units*, these Transactions, vol. 33 (1931), pp. 235–243.

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CONDITIONS FOR THE SOLUBILITY OF THE DIOPHANTINE EQUATION $x^2-My^2=-1$*

BY

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1. In this paper I apply the theory of Lucas' functions† to determine conditions‡ under which the well known diophantine equation

$$x^2 - N^2Dy^2 = -1 \tag{1}$$

is soluble for given integers§ N and D.

I show first of all that it is sufficient to consider (1) in the case when N is an odd prime P and D is square-free and not divisible by P. Suppose that $N=P$. Clearly, a necessary condition that (1) be soluble is that the equation

$$x^2 - Dy^2 = -1 \tag{2}$$

be soluble. If D is not a quadratic residue of P, this condition is also sufficient for the solubility of (1). However, if D is a quadratic residue of P, the following additional restriction must hold.

Let (u,v) be the least positive integral solution of (2), and suppose that P is of the form $2^{k+1}(2M+1)+1$ with¶ $(D|P)=+1$. Then in order that (1) be soluble it is necessary and sufficient that

$$(u + vD^{1/2})^{(P-1)/2^k} \equiv -1 \pmod{P}. \tag{3}$$

In case P is of the form $8M+5$, (3) may be replaced by the following condition. Let

$$P = (a+bi)(a-bi),\ u+i = \epsilon\zeta\prod(\alpha+\beta i) \qquad (a,\ \alpha \text{ odd})$$

be the decomposition of P and $u+i$ into primary factors in the field $\mathfrak{F}(i=(-1)^{1/2})$. Here ζ is equal to 1 or $1+i$ according as the norm of $u+i$ is odd or even, and ϵ is a unit chosen so that the integers α are all odd. Then a necessary and sufficient condition that (1) be soluble is that

$$\prod(a+bi\mid\alpha+\beta i) = (\epsilon\zeta\mid a+bi). \tag{4}$$

* Presented to the Society, April 11, 1931; received by the editors March 3, 1931.

† A recent paper by D. H. Lehmer, Annals of Mathematics, (2), vol. 31 (1930), pp. 419–448, gives references to the literature on these functions.

‡ Very few general conditions are known. See Dickson's *History*, vol. 2, chapter XII.

§ On considering (1) modulo 8, it is obvious that N must be odd and D or $D/2$ odd for a solution to exist. Furthermore, every odd prime factor of N^2D must be of the form $4n+1$.

¶ $(A|B)$ denotes as usual the quadratic character of A with respect to B.

If P is of the form $8M+1$, this condition is necessary, but not sufficient, for the solubility of (1).

For a given value of D, (4) gives an easily applied criterion for the solubility of (1). Its use is illustrated in the closing sections of the paper for the case $D=5$.

2. To prove these statements, consider equations (1) and (2), where we assume that (2) is soluble and that N is odd. If (u, v) is the least positive integral solution of (2), every other solution is given by the formula

$$r_n + D^{1/2}s_n = (u + vD^{1/2})^n \qquad (n = \pm 1, \pm 3, \pm 5, \cdots).$$

Hence a necessary and sufficient condition that (1) be soluble is that there exist an odd n such that

$$s_n \equiv 0 \pmod{N}.$$

Now let $\gamma = u+vD^{1/2}$, $\delta = u-vD^{1/2}$ so that $\gamma+\delta=2u$, $\gamma-\delta=2vD^{1/2}$, $\gamma\delta=-1$. Then $r_n+s_nD^{1/2}=\gamma^n$, $r_n-s_nD^{1/2}=\delta^n$ so that $2r_n=V_n$, $s_n=vU_n$ where

$$V_n = \gamma^n + \delta^n,\ U_n = \frac{\gamma^n - \delta^n}{\gamma - \delta}$$

are the Lucas functions associated with the quadratic equation

$$x^2 - 2ux - 1 = 0.$$

Thus if $N=md$, $v=v'd$, $(m, v')=1$, $s_n\equiv 0 \pmod{N}$, when and only when $U_n\equiv 0 \pmod{m}$.

Now let $\mu(m)$ denote the least positive value of n such that $U_n\equiv 0$ (mod m). We shall refer to this number as the rank of apparition of m in $(U)_n$. Its more important properties are as follows:*

I. *U_n is divisible by m when and only when n is divisible by $\mu(m)$.*

II. *If a and b are co-prime, $\mu(ab)$ is the least common multiple of $\mu(a)$ and $\mu(b)$. Consequently*

III. *If $m=p_1^{a_1}, \cdots, p_k^{a_k}$ is the decomposition of m into its prime factors, then $\mu(m)$ is the least common multiple of $\mu(p_1^{a_1}), \cdots, \mu(p_k^{a_k})$.*

IV. *If m is a prime p, and $(D|p)$ denotes the quadratic character of D with respect to p, then $\mu(p)$ divides $p-(D|p)$ and $\mu(p^a)=p^b\mu(p)$, $b\leqq a-1$.*

V. *If m is odd, $\mu(m)$ is odd when and only when all of the V_n are prime to m.*

The first property of $\mu(m)$ gives us immediately the following theorem.

* D. H. Lehmer, paper cited.

THEOREM. *If* (u, v) *is the least positive integral solution of*

$$x^2 - Dy^2 = -1 \tag{2}$$

and $N = md$, $v = v'd$; N *odd and* $(m, v') = 1$, *then a necessary and sufficient condition that the equation*

$$x^2 - N^2Dy^2 = -1 \tag{1}$$

be soluble is that the rank of apparition of m *in the Lucas function* $(U)_n$ *associated with the quadratic equation* $x^2 - 2ux - 1 = 0$ *be odd.*

3. The question of the solubility of (1) is thus reduced to the problem of determining the parity of $\mu(m)$ for any odd m. It follows from III and IV that if

$$m = p_1^{a_1} \cdots p_k^{a_k}$$

is the decomposition of m into its prime factors, $\mu(m)$ is odd when and only when $\mu(p_1), \cdots, \mu(p_k)$ are all odd. Since if p divides D, $(D|p) = 0$ and $\mu(p)$ divides p, we can assume that m is prime to D.

Thus it is sufficient to discuss the solubility of (1) when D is square-free, and N is a prime P of the form $4n+1$ not dividing vD. We shall therefore replace (1) by

$$x^2 - P^2Dy^2 = -1, \; (P, vD) = 1, \; P \text{ a prime, } D \text{ square-free,} \tag{5}$$

where

$$P = 2^{k+1}(2M + 1) + 1, \; k \geqq 1. \tag{6}$$

From V we have immediately the following theorem.

THEOREM. *If* $x^2 - Dy^2 = -1$ *is soluble, and* P *is an odd prime, then at most one of the equations*

$$x^2 - P^2Dy^2 = -1, \; P^2x^2 - Dy^2 = -1$$

is soluble.

Suppose that $(D|P) = -1$. Since $(P+1)/2$ is odd, we see from IV that $\mu(P)$ is odd when and only when $\mu(P)$ divides $(P+1)/2$; that is, when and only when

$$U_{(P+1)/2} \equiv 0 \quad (\text{mod } P). \tag{7}$$

Referring back to the equations in §2 defining U_n, we see that (7) holds when and only when the congruence

$$\gamma^{P+1} \equiv -1 \quad (\text{mod } P)$$

holds in the Galois field of order P^2 associated with the root γ of $x^2-2ux-1=0$. But

$$\gamma^{P+1} = (u + vD^{1/2})^{P+1} \equiv u^{P+1} + v^{P+1}D^{(P+1)/2} \equiv u^2 - Dv^2 \equiv -1 \pmod{P}.$$

We thus have established the following result:

THEOREM. *If P is a prime of the form $4n+1$ such that $(D|P)=-1$, the diophantine equation $x^2-P^2Dy^2=-1$ is soluble when and only when the diophantine equation $x^2-Dy^2=-1$ is soluble.*

4. The case when $(D|P)=+1$ is considerably more difficult. Since $(P-1)/2^{k+1}$ is odd, $\mu(P)$ is odd when and only when $\mu(P)$ divides $(P-1)/2^{k+1}$. This condition is easily shown to be equivalent to the criterion stated in §1,

$$(3) \qquad (u + vD^{1/2})^{(P-1)/2^k} \equiv -1 \pmod{P}.$$

It is now necessary to discuss separately the increasingly more complicated cases* $k=1, 2, 3, 4, \cdots$, corresponding to primes of the form $8n+5$, $16n+9$, $32n+17$, $64n+33, \cdots$.

I shall confine myself here to the simplest case $k=1$ where the criterion (3) can be put into a much more manageable form.

Suppose then that

$$(8) \quad (u + vD^{1/2})^{(P-1)/2} \equiv -1 \pmod{P}, \text{ where } P = 8M + 5 \text{ and } (D|P) = +1.$$

(8) is equivalent to saying that the congruences

$$(8.1) \qquad x^2 \equiv u + vD^{1/2}, \quad \bar{x}^2 \equiv u - vD^{1/2} \pmod{P}$$

have no solutions. Now let

$$x = \kappa + \lambda D^{1/2}, \quad \bar{x} = \kappa - \lambda D^{1/2}.$$

Then the congruences (8.1) are insoluble when and only when the congruences

$$\kappa^2 + \lambda^2 D \equiv u, \quad 2\kappa\lambda \equiv v, \quad u^2 - v^2D \equiv -1 \pmod{P}$$

are insoluble. On eliminating λ and v, we obtain

$$(2\kappa^2 - u)^2 + 1 \equiv 0 \pmod{P}.$$

Hence if $w^2 \equiv -1 \pmod{P}$, $4\kappa^2 \equiv 2(u \pm w) \pmod{P}$. On recalling that P is of the form $8M+5$, we see that the congruences (8. 1) are insoluble when and only when the congruence

$$z^2 \equiv u \pm w \pmod{P}$$

* For the case $k=2$, $D=2$, $u=v=1$, see a paper by Perott, *Sur l'équation $t^2-Du^2=-1$*, Crelle's Journal, vol. 102 (1888), pp. 185–223.

is soluble. Since $(u+w)(u-w)\equiv u^2+1\equiv v^2D \pmod{P}$, $u+w$ and $u-w$ have the same quadratic character modulo P. Hence a necessary and sufficient condition that (8) should hold is that

$$\left(\frac{u+w}{P}\right) = +1 \text{ where } w^2 \equiv -1 \pmod{P}. \tag{9}$$

By passing into the field $\mathfrak{F}(i)$, $i^2=-1$, we can apply the reciprocity law* to simplify the criterion (9). Suppose that

$$P = (a+bi)(a-bi),\ a \text{ odd},$$

is the decomposition of P into primary factors in $\mathfrak{F}(i)$.

Since

$$(u+i\,|\,a-bi) = (u-i\,|\,a+bi) \quad \text{and} \quad (u+i\,|\,a+bi) = (u-i\,|\,a+bi),$$

a necessary and sufficient condition that (9) should hold is that

$$(u+i\,|\,a+bi) = +1. \tag{10}$$

Let

$$u+i = \epsilon\zeta\prod(\alpha+\beta i)$$

be the decomposition of $u+i$ into primary factors in $\mathfrak{F}(i)$ where $\zeta=1$ or $1+i$ according as the norm of $u+i=v^2D$ is even or odd and ϵ is a unit so chosen that the α are all odd. Then by the reciprocity law in $\mathfrak{F}(i)$,

$$(u+i\,|\,a+bi) = (\epsilon\zeta\,|\,a+bi)\prod(a+bi\,|\,\alpha+\beta i).$$

If P is a prime of the form $8n+1$ and $(D\,|P)=+1$, we see from (3) of the previous theorem that a necessary condition that $x^2-P^2Dy^2=-1$ be soluble is that

$$(u+vD^{1/2})^{(P-1)/2} \equiv 1 \pmod{P}.$$

On proceeding as in the previous case, we find that this condition is again equivalent to the criterion

$$(u+i\,|\,a+bi) = +1.$$

Hence we have the following theorem.

THEOREM. *Let (u, v) be the least positive integral solution of the diophantine equation*

$$x^2 - Dy^2 = -1 \tag{2}$$

and let P be a prime of the form $4n+1$ such that $(D\,|P)=+1$, $(v, P)=1$.

* Bachmann, *Kreistheilung*, Lecture 13.

Then if

$$P = (a + bi)(a - bi),\ u + i = \epsilon\zeta \prod(\alpha + \beta i)$$

(a, α odd; ϵ a unit; $\zeta = 1$ or $1+i$) are the decompositions of P and $u+i$ into primary factors in the field $\mathfrak{F}(i)$, a necessary condition that the diophantine equation

$$x^2 - P^2Dy^2 = -1 \tag{5}$$

be soluble is that

$$\prod(a + bi|\alpha + \beta i) = (\epsilon\zeta|a + bi). \tag{4}$$

If P is of the form $8M+5$, this condition is also sufficient for the solubility of (5).

5. We shall now apply our results to the case $D=5$. Here $u=2$, $v=1$ so that the norm of $u+i=5$, and $\alpha=1$, $\beta=-2$, $\zeta=1$, $\epsilon=i$. (4) becomes simply

$$(a + bi\,|\,1 - 2i) = (i\,|\,a + bi)\begin{cases} = -1,\ P \equiv 5 & (\text{mod } 8), \\ = +1,\ P \equiv 1 & (\text{mod } 8). \end{cases}$$

We easily find that the square of any integer in $\mathfrak{F}(i)$ is congruent modulo $1-2i$ to 0, ± 1 or $\pm 2i$; morever since $P=a^2+b^2$ is a quadratic residue of 5, we must have either $a\equiv 0$ or $b\equiv 0$ (mod 5). Hence

$$(a + bi\,|\,1 - 2i) = +1\begin{cases} b \equiv 0,\ a \equiv \pm 1,\ P \equiv 1 & (\text{mod } 5), \\ a \equiv 0,\ b \equiv \pm 2,\ P \equiv 4 & (\text{mod } 5), \end{cases}$$

$$(a + bi\,|\,1 - 2i) = -1\begin{cases} b \equiv 0,\ a \equiv \pm 2,\ P \equiv 4 & (\text{mod } 5), \\ a \equiv 0,\ b \equiv \pm 1,\ P \equiv 1 & (\text{mod } 5). \end{cases}$$

Now every odd prime save 5 must belong to one of the forms

$$40n + 1, 3, 7, 9, 11, 13, 17, 19, 21, 23, 27, 29, 31, 33, 37, 39.$$

Hence if P is of the form

$$40n + 3, 7, 11, 19, 23, 27, 31, 39,$$

the diophantine equation

$$x^2 - 5P^2y^2 = -1 \tag{11}$$

is insoluble, since $P\equiv 3$ (mod 4), while if P is of the form

$$40n + 13, 17, 33, 37,$$

(11) is soluble since P is then a non-residue of 5.

There are left primes of the forms

$$40n + 21,\ 40n + 29 \text{ and } 40n + 1,\ 40n + 9$$

congruent to 5 and 1 modulo 8 respectively. For such primes, we have from the results just given the following remarkable theorem:

THEOREM. *Let $P=a^2+b^2$ be the representation of any prime of the forms $40n+1$, 9, 21, 29 as the sum of two squares, where a is assumed to be odd. Then a necessary condition that the diophantine equation*

$$x^2-5P^2y^2=-1 \tag{11}$$

be soluble is given by the following table:

Residue of P (mod 40)	*Criterion for solubility*
1	$b\equiv 0 \pmod 5$
9	$a\equiv 0 \pmod 5$
21	$a\equiv 0 \pmod 5$
29	$b\equiv 0 \pmod 5$.

In the last two cases, this criterion is also sufficient for the solubility of (11).

In the concluding table, we apply this theorem to all the primes of the four forms considered less than 1000. Soluble cases are marked with S, insoluble with I and doubtful with ?. In conjunction with our previous results, we see that for the 168 primes <1000, we are left in doubt as to the solubility of (11) only in the six cases $P=89$, 401, 521, 761, 769 and 809.

Table of Primes of the Form $40n+1$, 9, 21, 29 Less than a Thousand

$P=40n+1$	a^2+b^2, a odd	$5P^2$		$P=40n+9$	a^2+b^2, a odd	$5P^2$	
41	5^2+4^2	8405	I	89	5^2+8^2	39605	?
241	15^2+4^2	290405	I	409	3^2+20^2	836405	I
281	25^2+4^2	394805	I	449	7^2+20^2	1008005	I
401	1^2+20^2	804005	?	569	13^2+20^2	1618805	I
521	11^2+20^2	1357205	?	769	25^2+12^2	2956805	?
601	5^2+24^2	1806005	I	809	5^2+28^2	3272405	?
641	25^2+4^2	2054405	I	929	23^2+20^2	4315205	I
761	19^2+20^2	2895605	?				
881	25^2+16^2	3880805	I				
$P=40n+21$				$P=40n+29$			
61	5^2+6^2	18605	S	29	5^2+2^2	4205	I
101	1^2+10^2	51005	I	109	3^2+10^2	59405	S
181	9^2+10^2	163805	I	149	7^2+10^2	111005	S
421	15^2+14^2	886205	S	229	15^2+2^2	262205	I
461	19^2+10^2	1062605	I	269	13^2+10^2	361805	S
541	21^2+10^2	1463405	I	349	5^2+18^2	609005	I
661	25^2+6^2	2184605	S	389	17^2+10^2	756605	S
701	5^2+26^2	2457005	S	509	5^2+22^2	1294055	I
821	25^2+14^2	3370205	S	709	15^2+22^2	2513405	I
941	21^2+20^2	4427405	I	829	27^2+10^2	3436205	S

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ON A SOLUTION OF LAPLACE'S EQUATION WITH AN APPLICATION TO THE TORSION PROBLEM FOR A POLYGON WITH REENTRANT ANGLES†

BY

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I. Introduction

In this paper will be found a general method of solution of the two-dimensional Laplace equation with certain boundary conditions prescribed along the sides of any rectilinear polygon. The applicability of this method to the solution of technical problems will be illustrated by the treatment of the torsion problem for an infinite T-section. The mathematical solution of this problem, and of that for a finite T, is not to be found in the existing literature.‡

It will be seen that this scheme of solving Laplace's equation can readily be applied to any region which can be mapped conformally onto the upper half-plane in such a way that the boundary of the region goes into the entire real axis while the interior of the region transforms into the upper half-plane.

Despite the considerable scientific interest in the behavior of structural members subjected to pure torsion, only a limited number of torsion problems have been brought within the range of mathematical analysis. A torsion problem is solved when one has determined a function Φ which satisfies the equation $\nabla^2\Phi=0$, and which on the boundary of the section subjected to torsion reduces to§ $\Phi^*=\frac{1}{2}(x^2+y^2)$. The determination of Φ for such simple regular sections as the circle, ellipse, equilateral triangle, and rectangle has been achieved with comparative ease.‖

In 1908, F. Kötter¶ succeeded in obtaining a solution of the torsion problem for an L-section by the use of the known solution for the rectangle and by application of the scheme of conformal transformation. Kötter's

† Presented to the Society, April 18, 1930; received by the editors February 24, 1931.

‡ For practical methods of solution, and for an extensive bibliography, see 15th Annual Report of the National Advisory Committee for Aeronautics, 1929, pp. 675–719, *The torsion of members having sections common in aircraft construction*, by W. Trayer and H. W. March. This work was also published by the Bureau of Aeronautics, Navy Department, as a separate Report No. 334, bearing the same title (U. S. Government Printing Office, 1930).

§ Love, A. E. H., *Theory of Elasticity*, pp. 318–319, 3d edition, 1920.

‖ Love, loc. cit., pp. 322–333.

¶ Sitzungsberichte der Königlich Preussischen Akademie der Wissenschaften, 1908, pp. 935–955.

method, however, does not lend itself readily to the solution of the problem involving more than one reentrant angle. Apparently the first paper proposing a general method of solution of the two-dimensional Laplace equation, with the torsion boundary conditions prescribed along the sides of a rectilinear polygon, was published in 1921 by E. Trefftz.† The method which Trefftz employed possesses a distinct disadvantage in that it makes the ultimate solution of the problem (as applied to an L-section) depend upon some graphical scheme. Moreover, the success of the Trefftz method depends upon the particular form of the boundary conditions occurring in the torsion problem. The method used in the present paper is more general, and does not restrict the choice of the boundary conditions to those of the torsion problem.

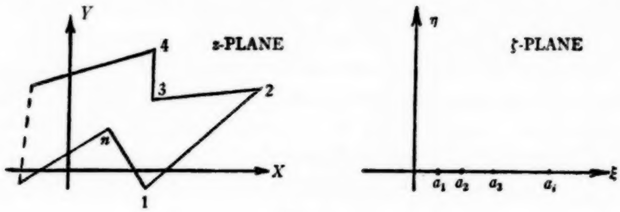


FIG. 1

II. BOUNDARY VALUE PROBLEM

It is known from a fundamental theorem of potential theory that a harmonic function is uniquely determined by the values assigned along the boundary of the region within which the harmonic function is sought, the boundary conditions and the region being subject to certain well known assumptions of continuity, connectivity, etc. In particular, when the values of the potential function are prescribed along the ξ-axis, the value of the function at any point of the upper half-plane is given by

$$\Phi(\rho, \alpha) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\Phi^*(\xi)\rho \sin \alpha d\xi}{\rho^2 - 2\xi\rho\cos\alpha + \xi^2} \tag{1}$$

where ξ is the value of ζ along the real axis of the complex ζ-plane, (ρ, α) are the polar coördinates of the point in the ζ-plane, and $\Phi^*(\xi)$ is the function prescribed on the boundary.

Consider a rectilinear polygon in the complex z-plane (Fig. 1), and denote the interior angles of the polygon at the points $1, 2, 3, \cdots, n$ by $\pi\alpha_1, \pi\alpha_2, \pi\alpha_3, \cdots, \pi\alpha_n$. It is allowable for some of the angular points of the polygon to recede to infinity. It is possible to map the z-plane conformally on

† Mathematische Annalen, vol. 82 (1921), pp. 97–112.

the ζ-plane in such a way that the boundary of the polygon transforms into the real axis of the ζ-plane, and the interior of the polygon maps into the upper half of the ζ-plane. Such a transformation was given independently by Schwartz and Christoffel† and its form is

$$z = C_1 \int \prod_{i=1}^{n} (\zeta - a_i)^{\alpha_i - 1} d\zeta + C_2, \tag{2}$$

where C_1 and C_2 are constants determinable from the orientation, scale, and position of the polygon. The points $a_1, a_2, a_3, \cdots, a_n$ are the points on the real axis of the ζ-plane corresponding to the angular points $(1, 2, 3, \cdots, n)$ of the polygon. Since it is possible to transform any three given points of the z-plane into any three desired points of the ζ-plane, we are free to prescribe the location of any three points a_i along the axis of reals, while the position of the remaining $(n-3)$ points will be determined from the dimensions of the polygon. It is necessary to remark that the order of the points $a_1, a_2, \cdots, a_n$ along the ξ-axis must be the same as that of the angular points $(1, 2, \cdots, n)$ around the polygon.

If one succeeds in integrating (2), and further if one decomposes it into its real and imaginary parts, there result two equations of transformation

$$\begin{aligned} x &= g_1(\xi, \eta), \\ y &= g_2(\xi, \eta), \end{aligned} \tag{3}$$

where $g_1(\xi, \eta)$ and $g_2(\xi, \eta)$ are real functions of the real variables ξ and η. These equations of transformation, when applied to the analytic equations representing the boundary of the polygon in the z-plane, will transform it into the real axis of the ζ-plane.

Consider now the problem of determining the harmonic function $\Phi(x, y)$ in the interior of the region bounded by the rectilinear polygon, and let the prescribed values of $\Phi(x, y)$ along the sides of this polygon be given by $\Phi^* = f(x, y)$. The application of the equations of transformation (3) gives

$$\Phi^* = f[g_1(\xi, \eta), g_2(\xi, \eta)]. \tag{4}$$

In (4), Φ^* is a function of ξ only, since, by hypothesis, the boundary of the polygon in the z-plane is transformed into the ξ-axis of the ζ-plane.

The substitution of this new boundary value function in (1) gives an expression for the determination of the value of Φ at any point (ξ, η) of the upper half of the ζ-plane. In view of the fact that the upper half of the ζ-plane corresponds to the interior of the polygon, it is clear that the integral (1),

† Schwartz, Gesammelte Werke, vol. II, pp. 65–83. Christoffel, Annali di Matematica Pura ed Applicata, (2), vol. 1 (1867), pp. 95–103. Ibid., vol. 4 (1871), pp. 1–9.

together with the equations of transformation (3), constitute the solution of the problem in parametric form.

III. Application to the torsion problem

The foregoing considerations can be applied to the solution of the torsion problem for a long prism whose cross section is in the shape of the letter T.

In this case we wish to determine a function Φ which satisfies the differential equation

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}=0 \tag{5}$$

and assumes on the boundary of the section the values given by

$$\Phi^* = \frac{x^2+y^2}{2}. \tag{6}$$

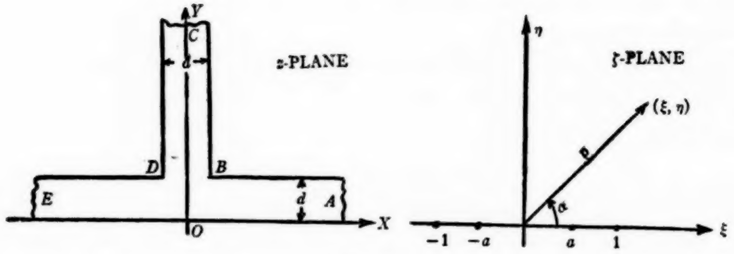


Fig. 2

Let the width of the flange and web be d. We shall consider the web and flange of the T as extending indefinitely along the X and Y axes. This assumption will lead to useful results since the behavior of Φ, at points of the flange and web sufficiently far removed from the reentrant angles, is essentially the same as that in a rectangle.†

The theory outlined in §II is applicable to the section in question. We are at liberty to transform any three desired points of the boundary of the polygon in the z-plane into any three points of the ξ-axis. Let the point B (see Fig. 2) go into the point 1 of the ξ-axis, the point $C(i\infty)$ into ∞ of the ξ-axis, and the point A into some point a of the ξ-axis. The value of a will be determined from the dimensions of the polygon. It is obvious from symmetry that the point D will go into -1, and the point E into $-a$.

† For an exposition of the torsion theory see Love, loc. cit., pp. 315–333; Trayer and March, loc. cit., pp. 10–20.

A direct substitution of the coördinates of the points a_i and of the values of the interior angles $\alpha_i\pi$ into the Schwartz integral (2) gives

$$z = C_1 \int \frac{(\zeta^2 - 1)^{1/2}}{\zeta^2 - a^2} d\zeta + C_2. \tag{7}$$

It is to be remarked that the values of the interior angles of the T-polygon are 0, $(3/2)\pi$, 0, $(3/2)\pi$, and 0, at the points A, B, C, D, and E respectively.

The integral (7) can be readily evaluated by making an elementary transformation $\zeta = \sec\theta$ and dividing the numerator of the resulting expression by the denominator. The result is

$$z = C_1 \log (\zeta + (\zeta^2 - 1)^{1/2}) + \frac{C_1}{a}(1 - a^2)^{1/2} \tan^{-1} \frac{a(\zeta^2 - 1)^{1/2}}{(1 - a^2)^{1/2}\zeta} + C_2. \tag{8}$$

The constants of integration C_1 and C_2 are complex numbers, and must be determined from the geometry of the polygon. Choosing for this purpose the points 0, -1, and 1 in the ζ-plane, and substituting the corresponding points of the z-plane in (8), one obtains with little effort the following values:

$$a = \frac{1}{5^{1/2}}, \quad C_1 = \frac{di}{\pi}, \quad C_2 = \frac{d}{2} + id,$$

where d is the width of the section. The substitution of these constants in (8) gives the explicit form of the transformation of the T-polygon into the upper half of the ζ-plane. It is

$$z = \frac{d}{\pi} i \log (\zeta + (\zeta^2 - 1)^{1/2}) - \frac{2d}{\pi} i \tan^{-1} \frac{(\zeta^2 - 1)^{1/2}}{2\zeta} + \frac{d}{2} + id. \tag{9}$$

We proceed next to decompose the equation of transformation (9) into its real and imaginary parts. By substitution of the symbols

$$A = \zeta + (\zeta^2 - 1)^{1/2} \text{ and } B = \frac{(\zeta^2 - 1)^{1/2}}{2\zeta},$$

(9) becomes

$$z = \frac{2di}{\pi}\left(\frac{1}{2} \log A + \tan^{-1} B\right) + \frac{d}{2} + id.$$

Then

$$\begin{aligned} y = \frac{1}{2i}(z - \bar{z}) &= \frac{d}{\pi}\left(\frac{1}{2} \log A + \frac{1}{2} \log \bar{A} - \tan^{-1} B - \tan^{-1} \bar{B}\right) + \\ &= \frac{d}{\pi}\left(\frac{1}{2} \log A\bar{A} - \tan^{-1} \frac{B + \bar{B}}{1 - B\bar{B}}\right) + d \\ &= \frac{d}{\pi}\left(\log |A| - \tan^{-1} \frac{\Re(B)}{1 - |B|^2}\right) + d, \end{aligned}$$

and

$$
\begin{aligned}
x = \frac{1}{2}(z+\bar{z}) &= \frac{di}{\pi}\left(\frac{1}{2}\log\frac{A}{\bar{A}} - \tan^{-1}\frac{B-\bar{B}}{1+B\bar{B}}\right) + \frac{d}{2}\\
&= \frac{di}{\pi}\left(i\arg A - \tan^{-1}\frac{2i\mathfrak{J}(B)}{1+|B|^2}\right) + \frac{d}{2}\\
&= \frac{d}{\pi}\left(-\arg A - \tanh^{-1}\frac{2\mathfrak{J}(B)}{1+|B|^2}\right) + \frac{d}{2}\cdot
\end{aligned}
$$

In order to obtain the transformed boundary value function for substitution in (1), one must compute the value of $\Phi^* = \frac{1}{2}(x^2+y^2)$ in terms of ξ. Since the boundary of the T-polygon transforms into the real axis of the ζ-plane, (6) will be a function of ξ only, and one obtains with a little effort the following equations:

$$
\begin{aligned}
x &= \pm\frac{d}{2}\quad\begin{cases} + \text{ for } \xi > 1\\ - \text{ for } \xi < -1,\end{cases}\\
y &= \frac{d}{\pi}\left(\cosh^{-1}|\xi| + 2\tan^{-1}\frac{2|\xi|}{(\xi^2-1)^{1/2}}\right), \quad \text{for } |\xi| > 1.
\end{aligned}
\tag{10}
$$

On account of the multiple-valued functions entering in (9), it is necessary to compute four sets of equations, analogous to (10), which correspond to the ranges† $(-1<\xi<-1/5^{1/2})$, $(-1/5^{1/2}<\xi<0)$, $(0<\xi<1/5^{1/2})$, and $(1/5^{1/2}<\xi<1)$. Since the equations of transformation so obtained lead to six distinct forms, it is necessary to decompose the range of the integral (1) into six parts corresponding to the different forms of the boundary function defined over these ranges. A reference to (10) partly indicates the complexity of the resulting integrals.

It will be shown next that it is possible to dispense with the task of evaluating the four integral expressions corresponding to the range $-1<\xi<1$ by means of the following expedient. Consider the function

$$
\Phi_1 = \frac{x^2-y^2}{2} + yd \tag{11}
$$

which obviously satisfies $\nabla^2\Phi = 0$. Let Φ_2 be the function which satisfies $\nabla^2\Phi = 0$, and which assumes on the boundary the value

$$
\Phi_2^* = y^2 - yd. \tag{12}
$$

† The appearance of $1/5^{1/2}$ is to be expected since, for $\zeta = 1/5^{1/2}$, (9) is not defined, inasmuch as in the period strip $\tan\zeta$ assumes every complex value except $\pm i$. The points $\zeta = \pm 1/5^{1/2}$ and $\zeta = \pm 1$ are the singular points of transformation (9).

It is clear that (12) vanishes when $y=0$, or when $y=d$, and that

$$\Phi_1 + \Phi_2^* = \frac{x^2+y^2}{2} = \Phi^*.$$

Thus, if the function $\Phi_2(x, y)$ satisfying Laplace's equation with the boundary condition (12) be found, then the function

$$\Phi = \Phi_1 + \Phi_2 \tag{13}$$

is determined. The advantage in seeking $\Phi_2(x, y)$ rather than $\Phi(x, y)$ directly from (1) lies in the fact that along the boundary of the flange (i.e. the portion corresponding to the ξ-axis between -1 and $+1$)$\Phi_2^* \equiv 0$. Consequently four of the six integral expressions vanish, since they involve the boundary value function in the numerator of the integrand.

The substitution of (10) in (12) gives for $|\xi|>1$

$$\begin{aligned}\Phi_2^* = y^2 - yd = \frac{d^2}{\pi}\left(\cosh^{-1}|\xi| + 2\tan^{-1}\frac{2|\xi|}{(\xi^2-1)^{1/2}}\right)\\ - \frac{d^2}{\pi}\left(\cosh^{-1}|\xi| + 2\tan^{-1}\frac{2\xi}{(\xi^2-1)^{1/2}}\right),\end{aligned} \tag{14}$$

and for $-1<\xi<1$,

$$\Phi_2^* = 0.$$

Using (1), and observing that (14) is an even function, we have

$$\begin{aligned}\Phi_2(\rho, \alpha) &= \int_{-\infty}^{1} \rho\frac{\sin\alpha}{\pi}[y^2(\xi) - y(\xi)d](\rho^2 - 2\xi\rho\cos\alpha + \xi^2)^{-1}\,d\xi\\ &+ \int_{1}^{\infty} \rho\frac{\sin\alpha}{\pi}[y^2(\xi) - y(\xi)d](\rho^2 - 2\xi\rho\cos\alpha + \xi^2)^{-1}d\xi\\ &= \frac{\rho}{\pi}\int_{1}^{\infty}[y^2(\xi) - y(\xi)d]\Omega(\rho, \xi)d\xi,\end{aligned} \tag{15}$$

where

$$\Omega(\rho, \xi) = \frac{\sin\alpha}{\rho^2 + 2\xi\rho\cos\alpha + \xi^2} + \frac{\sin\alpha}{\rho^2 - 2\xi\rho\cos\alpha + \xi^2}\,. \tag{16}$$

IV. Evaluation of the integrals

A brief perusal of the integrals involved in (15) is sufficient to make one abandon the hope of evaluating them in closed form. For reasons which will be made clear later, it is found advantageous to divide the range of integration from 1 to ∞ into two ranges, say from 1 to m and from m to ∞, where m is some positive number greater than 1. As will be seen, the choice of m will

depend upon the degree of accuracy desired in computing the value of $\Phi_2(\rho, \alpha)$.

Since

$$\frac{\sin \alpha}{1 - 2\dfrac{\rho}{\xi}\cos \alpha + \dfrac{\rho^2}{\xi^2}} = \sum_{n=1}^{\infty} \left(\frac{\rho}{\xi}\right)^{n-1} \sin n\alpha, \quad \text{for} \left|\frac{\rho}{\xi}\right| < 1,$$

(16) can be written as

$$(16') \qquad \Omega(\rho, \xi) = \begin{cases} \dfrac{2}{\xi^2} \displaystyle\sum_{n=0}^{\infty} \left(\frac{\rho}{\xi}\right)^{2n} \sin (2n + 1)\alpha, \text{ if } \left|\dfrac{\rho}{\xi}\right| < 1, \\ \dfrac{2}{\rho^2} \displaystyle\sum_{n=0}^{\infty} \left(\frac{\xi}{\rho}\right)^{2n} \sin (2n + 1)\alpha, \text{ if } \left|\dfrac{\xi}{\rho}\right| < 1. \end{cases}$$

Moreover, for some value of $\xi \geqq m$, (14) may be simplified, inasmuch as

$$\cosh^{-1} \xi = \log (\xi + (\xi^2 - 1)^{1/2}),$$

which is asymptotically equal to $\log 2\xi$, and

$$\tan^{-1} \frac{2\xi}{(\xi^2 - 1)^{1/2}} \sim \tan^{-1} 2.$$

With these approximations (14) reads

$$(17) \qquad \Phi_2^* = y^2 - yd = d^2(a_1 \log^2 2\xi + a_2 \log^2 \xi + a_3) + O\left(\frac{1}{\xi^2}\right),$$

where

$$a_1 = \frac{1}{\pi^2}, \quad a_2 = \frac{4 \tan^{-1} 2}{\pi^2} - \frac{1}{\pi}, \quad a_3 = \frac{4}{\pi^2}(\tan^{-1} 2)^2 - \frac{2}{\pi} \tan^{-1} 2.$$

Now define

$$\Theta(\gamma, \delta) \equiv \int_\gamma^\delta (y^2 - yd)\Omega(\rho, \xi)d\xi$$

$$(18) \qquad = \begin{cases} 2 \displaystyle\sum_{n=0}^{\infty} \rho^{2n} \sin (2n + 1)\alpha \int_\gamma^\delta (y^2 - yd)\xi^{-2n-2}d\xi, \text{ if } \rho < \gamma, \\ 2 \displaystyle\sum_{n=0}^{\infty} \frac{\sin (2n + 1)\alpha}{\rho^{2n+2}} \int_\gamma^\delta (y^2 - yd)\xi^{2n}d\xi, \text{ if } \rho > \delta. \end{cases}$$

Conditions for inversion of the order of integration and summation are clearly satisfied, if in (16) ρ is prevented from approaching ξ by an arbitrarily

small positive quantity ϵ. Moreover $\Theta(\gamma, \delta)$ is continuous in both γ and δ, when $\gamma \neq 0$, $\delta \neq 0$.

Now, if $\rho > m$,

$$
\begin{aligned}
\Phi_2(\rho, \alpha) &= \frac{\rho}{\pi}\{\Theta(1, m) + \lim_{b\to\infty} \lim_{\epsilon\to 0} [\Theta(m, \rho - \epsilon) + \Theta(\rho + \epsilon, b)]\} \\
&= \frac{\rho}{\pi}\{\Theta(1, m) + \Theta(m, \rho) + \lim_{b\to\infty} \Theta(\rho, b)\},
\end{aligned} \tag{19}
$$

since Θ is continuous.

Substituting in (19) from (18) and simplifying we have

$$
\begin{aligned}
\Phi_2(\rho, \alpha) - \frac{\rho}{\pi}\Theta(1, m) = \frac{2d^2}{\pi}\Bigg\{&\frac{\pi a_1}{2}\log^2 2\rho + \frac{\pi a_2}{2}\log 2\rho + \frac{\pi a_3}{2} + \frac{\pi^2\alpha - \pi\alpha^2}{2} a_1 \\
&- \frac{a_3 + a_2 \log 2m + a_1 \log^2 2m}{2} \tan^{-1} \frac{2m\rho \sin \alpha}{\rho^2 - m^2} \\
&+ (2a_1 \log 2m + a_2) \sum_{n=0}^{\infty} \left(\frac{m}{\rho}\right)^{2n+1} \frac{\sin (2n+1)\alpha}{(2n+1)^2} \\
&- 2a_1 \sum_{n=0}^{\infty} \left(\frac{m}{\rho}\right)^{2n+1} \frac{\sin (2n+1)\alpha}{(2n+1)^3}\Bigg\}.
\end{aligned} \tag{20}
$$

In computing (20) use was made of the equalities

$$\sum_{n=0}^{\infty} \frac{x^{2n+1}}{2n+1} \sin (2n+1)\alpha = \frac{1}{2} \tan^{-1} \frac{2x \sin \alpha}{1 - x^2}$$

and

$$\sum_{n=0}^{\infty} \frac{\sin (2n+1)\alpha}{(2n+1)^3} = \frac{\pi^2\alpha - \pi\alpha^2}{8}.$$

In like manner for $\rho < m$ we obtain

$$
\begin{aligned}
\Phi_2(\rho, \alpha) - \frac{\rho}{\pi}\Theta(1, m) = \frac{2d^2}{\pi}\Bigg\{&\frac{a_1 \log^2 2m + a_2 \log 2m + a_3}{2} \tan^{-1} \frac{2m\rho \sin \alpha}{m^2 - \rho^2} \\
&+ (2a_1 \log 2m + a_2) \sum_{n=0}^{\infty} \left(\frac{\rho}{m}\right)^{2n+1} \frac{\sin (2n+1)\alpha}{(2n+1)^2} \\
&+ 2a_1 \sum_{n=0}^{\infty} \left(\frac{\rho}{m}\right)^{2n+1} \frac{\sin (2n+1)\alpha}{(2n+1)^3}\Bigg\}.
\end{aligned} \tag{21}
$$

In order to complete the solution of the problem it remains to establish the magnitude of m and to evaluate $\Theta(1, m)$. It will be shown next that one

attains a sufficiently high degree of accuracy by choosing $m=1$, and noting that† $\Theta(1,1)=0$.

The expression for the relative error made in assuming

$$\int_1^\infty \frac{\log 2\xi + 2\tan^{-1} 2}{\xi^2 \pm 2\xi\rho\cos\alpha + \rho^2}\,d\xi \to \int_1^\infty \frac{\cosh^{-1}\xi + 2\tan^{-1}\dfrac{2\xi}{(\xi^2-1)^{1/2}}}{\xi^2 \pm 2\xi\rho\cos\alpha + \rho^2}\,d\xi$$

is

$$E = \frac{\int_1^\infty [f(\xi) - \phi(\xi)]d\xi}{\int_1^\infty f(\xi)d\xi},$$

where

$$f(\xi) = \frac{\cosh^{-1}\xi + 2\tan^{-1}\dfrac{2\xi}{(\xi^2-1)^{1/2}}}{\xi^2 \pm 2\xi\rho\cos\alpha + \rho^2}, \quad \phi(\xi) = \frac{\log 2\xi + 2\tan^{-1} 2}{\xi^2 \pm 2\xi\rho\cos\alpha + \rho^2},$$

and for m sufficiently large

$$E \to \frac{\int_1^m [f(\xi) - \phi(\xi)]d\xi}{\int_1^\infty f(\xi)d\xi}. \tag{22}$$

It can be readily established that the numerators of the integrands in the foregoing expressions for E are monotone increasing functions and that their difference is a monotone decreasing function within the limits of integration. Therefore (22) is certainly less than

$$\frac{\int_1^m [\cosh^{-1}(1+\pi) - \log 2 + 2\tan^{-1} 2](\xi^2 \pm 2\xi\rho\cos\alpha + \rho^2)^{-1}d\xi}{\int_1^\infty \left(\cosh^{-1}\xi + 2\tan^{-1}\dfrac{2\xi}{(\xi^2-1)^{1/2}}\right)(\xi^2 \pm 2\xi\rho\cos\alpha + \rho^2)^{-1}d\xi},$$

which, in turn, is less than

$$(\pi - \log 2 - 2\tan^{-1} 2)\frac{\int_1^m (\xi^2 \pm 2\xi\rho\cos\alpha + \rho^2)^{-1}d\xi}{M\int_1^\infty (\xi^2 \pm 2\xi\rho\cos\alpha + \rho^2)^{-1}d\xi}, \tag{23}$$

† A somewhat elaborate investigation of the character of the function $\Theta(1,\xi)$ for small values of ξ enabled the author to compute (20) and (21) to a higher degree of accuracy than any practical considerations would warrant. See the author's thesis, 1930, in the Library of the University of Wisconsin.

where M is the lower bound of

$$\cosh^{-1}\xi + 2\tan^{-1}\frac{2\xi}{(\xi^2-1)^{1/2}}$$

in the interval $1 \leqq \xi \leqq \infty$. Moreover, $(\xi^2 \pm 2\xi\rho\cos\alpha+\rho^2)^{-1}$ is always positive and it is clear that the quotient of the integrals in (23) is less than unity. Therefore (23) is less than

$$\frac{\pi - \log 2 - 2\tan^{-1} 2}{M} = \frac{\pi - \log 2 - 2\tan^{-1} 2}{\cosh^{-1}\xi + 2\tan^{-1}\dfrac{2\xi}{(\xi^2-1)^{1/2}}\Big|_{\xi=1}} = .0748.$$

This result is quite significant, since it states that, even allowing such crude approximations as were made above in estimating the relative error, the latter is always less than 7.48 per cent. This rough estimate of the degree of approximation gives ample justification for the development of the approximate formulas for $\Phi_2(\rho, \alpha)$ by considering† $m=1$.

Replacing m by 1 in the formulas (20) and (21) and substituting the numerical values of a_1, a_2, and a_3, leads without difficulty to the following expressions. For $\rho \leqq 1$,

$$(24)\quad \Phi_2(\rho,\alpha) = \frac{d^2}{\pi}\left\{ -.209\tan^{-1}\frac{2\rho\sin\alpha}{1-\rho^2} + 1.703\sum_{n=1}^{\infty}\frac{\sin(2n-1)\alpha}{(2n-1)^2}\rho^{2n-1} \right.$$
$$\left. + 1.272\sum_{n=1}^{\infty}\frac{\sin(2n-1)\alpha}{(2n-1)^3}\rho^{2n-1}\right\},$$

and for $\rho \geqq 1$,

$$(25)\quad \begin{aligned} \Phi_2(\rho,\alpha) = \frac{d^2}{\pi}\Bigg\{ & \log^2 2\rho + \pi\alpha - \alpha^2 + .209\tan^{-1}\frac{2\rho\sin\alpha}{\rho^2-1} \\ & + 1.703\sum_{n=1}^{\infty}\frac{\sin(2n-1)\alpha}{(2n-1)^2}\rho^{-2n+1} \\ & - 1.272\sum_{n=1}^{\infty}\frac{\sin(2n-1)\alpha}{(2n-1)^3}\rho^{-2n+1} \\ & + 1.282\log 2\rho - 2.07\Bigg\}. \end{aligned}$$

† The uniformity of the results obtained by using this approximation suggests that the actual error is in the neighborhood of one per cent.

V. Graphic expression of the results

If we use the notation of §III, and introduce a new function defined by the equation

$$\Psi = \Phi - \tfrac{1}{2}(x^2 + y^2),$$

we find that Ψ satisfies

$$\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + 2 = 0,$$

subject to the boundary condition $\Psi = 0$ on the sides of the polygon.† One can easily obtain expressions for the components of the shearing stress in terms of Ψ. They are

$$X_z = G\tau \frac{\partial\Psi}{\partial y},$$

$$Y_z = -G\tau \frac{\partial\Psi}{\partial x},$$

where τ is the angle of twist per unit length, and G is the modulus of rigidity. The tangential traction at any point of the area of the polygon is directed along the tangent to the curve of the family

$$\Psi(x, y) = \text{const.} \tag{26}$$

which passes through the point in question.‡

It will be recalled that in the case of the T-section

$$\Phi = \Phi_1 + \Phi_2 = \frac{x^2 - y^2}{2} + yd + \Phi_2,$$

and the expression for Ψ becomes

$$\Psi = \Phi_2 + yd - y^2. \tag{27}$$

Equation (27) served for computing the level lines of the shearing stress function. A sufficiently large number of these lines is shown in the accompanying drawing of the contour elevations. The width of the web and flange was taken to be unity for convenience.

It may be shown§ that the torsional rigidity of the section is equal to

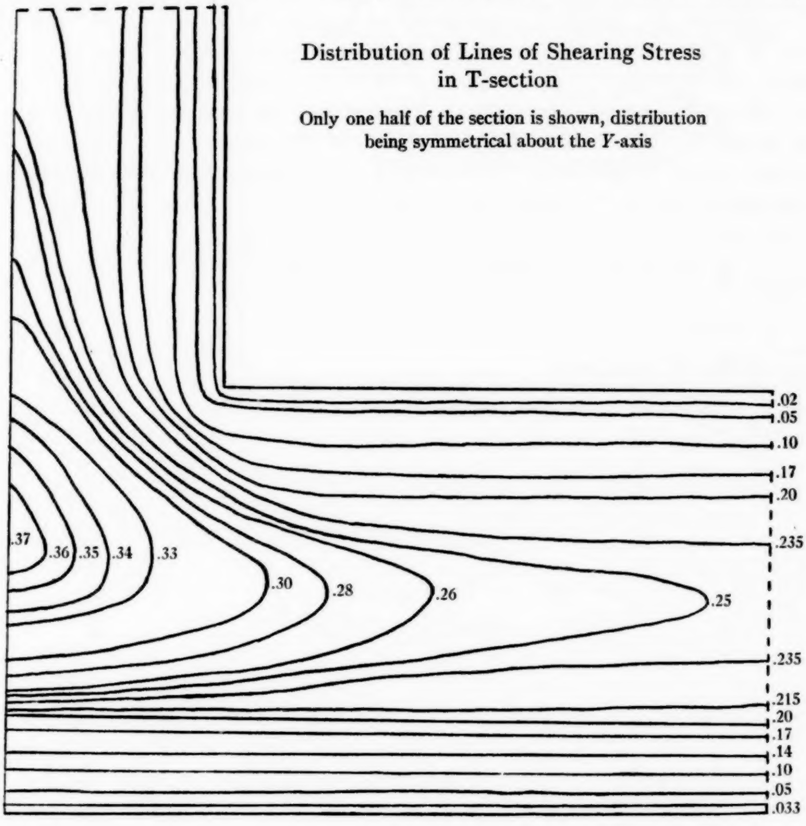
$$C = 2G \iint \Psi dxdy,$$

† Love, loc. cit., pp. 325–327. Trayer and March, loc. cit., pp. 10–13.

‡ See preceding footnote.

§ Trayer and March, loc. cit., p. 12. Love, loc. cit., pp. 325–327.

where G is the modulus of rigidity, and where the integration is carried over the area of the section. In other words, the torsional rigidity of the prism is equal to twice the product of the modulus of rigidity and the volume contained between the surface $z=\Psi(x, y)$ and the plane $z=0$. From the knowledge



of the contour elevations, the "torsion constant" $K\equiv C/G$ depending solely on the shape and the dimensions of the cross section may be easily computed.

Trayer and March† succeeded in developing a set of formulas for the torsion constants of sections whose components are rectangles, by combining results obtained from soap-film tests with known mathematical facts. The value of the torsion constant computed from their formula agrees closely with that obtained from the accompanying graph of the contour elevations.

† Loc. cit., p. 12.

VI. Concluding remarks

It is necessary to point out that the method of solution of Laplace's equation outlined and illustrated above depends neither upon the number of reentrant angles, nor the special form of the boundary conditions.

Such ingenious devices as were used by Bromwich,† Kötter,‡ and Trefftz§ do not furnish direct means for solving an important group of classical problems, the solution of which is regarded to be of considerable technical value.

It remains my pleasant duty to acknowledge that Professor H. W. March is responsible for directing my attention to this problem and for offering many helpful suggestions. To Professors R. E. Langer and Warren Weaver my thanks are due for their valuable criticisms.

† T. J. I'A. Bromwich, Proceedings of the London Mathematical Society, (2), vol. 30 (1930), pp. 165–173.

‡ Loc. cit.

§ Loc. cit.

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THE TRANSFORMATION C OF NETS IN HYPERSPACE*

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1. Introduction

It is the purpose of this paper to extend some of the ideas related to the transformation† C of nets in projective space of three dimensions to nets in hyperspace. In three dimensions two nets N_x and N_y are said to be in relation C if the developables of the congruence G of lines joining corresponding points x and y intersect the two sustaining surfaces in those nets, provided however that neither surface is a focal surface of the congruence. If N_x and N_y are in relation C, the tangents at x and y to corresponding curves of the nets intersect.

We shall say that two nets N_x and N_y in space S_n of n dimensions are *in relation* C if the two sustaining surfaces S_x and S_y are such that corresponding tangent planes intersect in a line, and if the developable surfaces of the congruence G of lines g joining corresponding points x and y intersect the two surfaces in the nets. It is to be understood that the two sustaining surfaces are not the focal surfaces of G. Not all nets N_x in S_n for $n \geqq 4$ can have a C transform N_y. A net in S_n which permits of having a C transform will be called *a C net.*

We derive necessary and sufficient conditions that a non-conjugate net be a C net. Another geometrical interpretation is given for two covariant points found by Bompiani.‡

Let S_x and S_y be two surfaces in the same space S_n of n dimensions and with their points in one-to-one point correspondence. Let the parametric equations of these surfaces be

$$x^{(i)} = x^{(i)}(u, v), \quad y^{(i)} = y^{(i)}(u, v) \qquad (i = 1, 2, 3, \cdots, n+1),$$

the parameters being so chosen that corresponding points have the same curvilinear coördinates. Suppose furthermore that the tangent planes to S_x and S_y at corresponding points intersect in a line. There exist therefore scalar functions m, s, f, A, etc., such that

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* Presented to the Society, June 13, 1931; received by the editors March 1, 1931.

† V. G. Grove, *Transformations of nets*, these Transactions, vol. 30 (1928), p. 483.

‡ E. Bompiani, *Determinazione delle superficie integrali d'un sistema di equazioni a derivate parziali lineari ed omogenee*, Rendiconti del Reale Istituto Lombardo di Scienze e Lettere, vol. 52 (1919), pp. 610–636. In particular see p. 634. Hereafter referred to as Bompiani, *Surfaces*.

$$y_u = mx_u + sx_v + fx + Ay,$$
$$(1) \qquad y_v = tx_u + nx_v + gx + By.$$

Any point z on the line g joining corresponding points x and y is defined by an expression of the form

$$z = y + \lambda x.$$

Consider the surface S_z generated by the point z, and a curve C on S_z with parametric equations

$$u = u(t), \quad v = v(t).$$

The tangent line to C at z is determined by z and $z' = dz/dt$ where

$$(2) \qquad z' = [(m + \lambda)u' + tv']x_u + [su' + (n + \lambda)v']x_v + (\ )x + (\ )y.$$

Hence the line g generates a congruence in the ordinary sense, that is, the lines of the two-parameter families of lines may be grouped into two one-parameter families of developable surfaces. The curves on S_z corresponding to these developables are defined by

$$(3) \qquad sdu^2 - (m - n)dudv - tdv^2 = 0.$$

We shall assume that these curves are not indeterminate and are distinct. By a change of parameters we may make the curves determined by (3) parametric. Let us suppose that this transformation has been made. *If two surfaces are such that their parametric nets are in relation C, the functions x and y determining the surfaces satisfy equations of the form*

$$y_u = mx_u + fx + Ay,$$
$$(4) \qquad y_v = nx_v + gx + By, \quad mn(m - n) \neq 0.$$

From (2) we find that the focal points of g are defined by

$$(5) \qquad \tau = mx - y, \quad \tau' = nx - y.$$

The nets defined by τ and τ' will be called *the focal nets* of G. The tangent to the curve $v = \text{const.}$ ($u = \text{const.}$) on the focal surface $S_{\tau'}$ (S_τ) will be called *the minus first (first) derived line of* g, and the congruences generated by them *the minus first (first) derived congruences of* G.

If we differentiate equations (4) with respect to v and u respectively, we find that x and y satisfy an equation of the form

$$(6) \qquad x_{uv} = ax_u + bx_v + cx + My$$

wherein a, b, c, M are defined by the formulas

$$(7)\qquad \begin{aligned} (m-n)a &= g + Bm - m_v, & (m-n)c &= Bf - Ag + g_u - f_v, \\ (n-m)b &= f + An - n_u, & (m-n)M &= B_u - A_v. \end{aligned}$$

Since y is not in the tangent plane to S_x at x, *the vanishing of M is a necessary and sufficient condition that the net N_x be conjugate.*

If the coefficients of the equations corresponding to (4) and (6) with the rôles of x and y interchanged are denoted by $\bar{m}$, $\bar{f}$, etc., we find that

$$\begin{aligned} &\bar{m} = 1/m,\ \bar{n} = 1/n,\ \bar{f} = -A/m,\ \bar{g} = -B/n,\ \bar{A} = -f/m,\ \bar{B} = -g/n, \\ &\bar{a} = a + m_v/m,\ \bar{b} = b + n_u/n, \\ &\bar{c} = -m[aA/m + bB/n + fB/(mn) - M - (A/m)_v], \end{aligned}$$

$$(8)\qquad \bar{M} = -m[af/m + bg/n + fg/(mn) - c - (f/m)_v].$$

2. The relation R in S_n

Denote by $R^{(v)}$ the ruled surface formed by the tangents to the curves $v=\text{const.}$ at the points where they meet a fixed curve $u=\text{const.}$ A ruled surface $R^{(u)}$ may be defined similarly.

Let l' be any line lying in the tangent plane to S_x at x, but not passing through x. Let l be a line passing through x but not lying in the tangent plane at x. The line l' intersects the tangent to the curves $v=\text{const.}$ and $u=\text{const.}$ in points r and s respectively. If the tangent planes to $R^{(v)}$ and $R^{(u)}$ at r and s respectively intersect in the line l the given lines l and l' will be said to be *in relation* R with respect to N_x.*

The points r and s are defined by expressions of the form

$$(9)\qquad r = x_u - \lambda x,\quad s = x_v - \mu x.$$

A point in the tangent plane to $R^{(v)}$ at r is

$$(10)\qquad r_v + \alpha r + \beta x = x_{uv} + \alpha x_u - \lambda x_v + (\beta - \alpha\lambda - \lambda_v)x.$$

A point in the tangent plane to $R^{(u)}$ at s is

$$(11)\qquad S_u + \alpha' s + \beta' x = x_{uv} + \alpha' x_v - \mu x_u + (\beta' - \alpha'\mu - \mu_u)x.$$

From (10) and (11) we observe that the tangent planes to $R^{(v)}$ and $R^{(u)}$ at r and s intersect in a line joining x to z defined by

$$(12)\qquad z = x_{uv} - \mu x_u - \lambda x_v.$$

* In a footnote on p. 86 of his paper *Memoir on the general theory of surfaces and rectilinear congruences*, these Transactions, vol. 20 (1919), Green defined the relation R between two lines with respect to a net in S_3. The definition we have used for the relation R between two lines with respect to a net in S_n reduces to Green's definition when $n=3$. It is to be noted however that not all lines l in S_n for $n>3$ and protruding from the surface at x have a line l' in relation R to N_x.

The line l in relation R to l' therefore joins x to the point z defined by (12).

If $M\neq 0$, we see readily from (6) that the line g' in relation R to g with respect to N_x joins the points

$$\rho = x_u - bx, \quad \sigma = x_v - ax. \tag{13}$$

The line $\bar{g}'$ in relation R to g with respect to N_y is determined by the points $\bar{\rho}$, $\bar{\sigma}$ defined by

$$\bar{\rho} = y_u - \bar{b}y, \quad \bar{\sigma} = y_v - \bar{a}y. \tag{14}$$

An examination of equations (5) readily shows that *the derived lines of g intersect the tangents to the curves of N_x and N_y in the points determining the lines g' and $\bar{g}'$ in relation R to N_x and N_y.* If N_x and N_y are in relation* F, that is, if they are in relation C and are both conjugate nets, the derived lines intersect the tangents to the curves of the nets in the focal points of these tangents.

The tangent to the curve $u=\text{const.}$ at ρ on the surface generated by that point intersects g in the point

$$(ab + c - b_v)x + My. \tag{15}$$

Similarly the tangent to $v=\text{const.}$ at σ intersects g in the point

$$(ab + c - a_u)x + My. \tag{16}$$

The points defined by (15) and (16) coincide if N_x is conjugate, or if

$$a_u - b_v = 0. \tag{17}$$

3. The third-order differential equations of the problem

Let us assume that the nets N_x and N_y are not conjugate. If we differentiate equation (6) with respect to u and v, and use (4) and (6), we find that the functions x must satisfy two differential equations of the form

$$\begin{aligned} x_{uuv} &= ax_{uu} + Ex_{uv} + Gx_u + Hx_v + Jx, \\ x_{uvv} &= E'x_{uv} + bx_{vv} + G'x_u + H'x_v + J'x, \end{aligned} \tag{18}$$

wherein

$$\begin{aligned} E &= b + A + M_u/M, & E' &= a + B + M_v/M, \\ G &= a_u + c + mM - a(E - b), & H' &= b_v + c + nM - b(E' - a), \\ H &= b_u - b(E - b), & G' &= a_v - a(E' - a), \\ J &= c_u + fM - c(E - b), & J' &= c_v + gM - c(E' - a). \end{aligned} \tag{19}$$

* L. P. Eisenhart, *Transformations of Surfaces*, Princeton University Press, 1923, p. 34.

Similar third-order differential equations are satisfied by the functions y. Hence *if two surfaces S_x and S_y are in one-to-one point correspondence with corresponding tangent planes intersecting in a line, and if the nets on S_x and S_y which are in relation C are parametric, the functions x (and y) satisfy two third-order differential equations of the type* (18). Differential equations of this type have been studied by Bompiani* and Lane.†

Conversely suppose the coördinates x defining a surface S_x satisfy a pair of differential equations of the form (18). In case the two-osculating space $S(2, 0)$ of S_x at x determined by the points $x_{uu}, x_{uv}, x_{vv}, x_u, x_v, x$ is an S_5, the integrability conditions‡ of system (18) are

$$
\begin{aligned}
aE' + G' - a^2 - a_v = 0,\ \ bE + H - b^2 - b_u &= 0,\\
bE' + E_u' + H' &= aE + E_v + G,\\
E'G + bG' + G_u' + J' &= EG' + aG + G_v,\\
EH' + aH + H_v + J &= E'H + bH' + H_u',\\
E'J + bJ' + J_u' &= EJ' + aJ + J_v.
\end{aligned}
\tag{20}
$$

We shall show that *every net N_x whose defining functions x satisfy differential equations of the form* (18) *is a C net*. Since x and y must satisfy equations of the form (4) it follows that the point y is defined by an expression of the form

$$
y = x_{uv} - \lambda x_u - \mu x_v + hx. \tag{21}
$$

Case I. Suppose that $S(2, 0)$ is an S_5. Differentiating (21) with respect to u and v we find readily that

$$
\begin{aligned}
y_u = (a - \lambda)x_{uu} &+ [G + h - \lambda_u + \lambda(E - \mu)]x_u + [H + \mu(E - \mu) - \mu_u]x_v\\
&+ [J - h(E - \mu) + h_u]x + (E - \mu)y,\\
y_v = (b - \mu)x_{vv} &+ [G' + \lambda(E' - \lambda) - \lambda_v]x_u + [H' + h - \mu_v + \mu(E' - \lambda)]x_v\\
&+ [J' - h(E' - \lambda) + h_v]x + (E' - \lambda)y.
\end{aligned}
\tag{22}
$$

Hence N_y will be in relation C to N_x if and only if $\lambda = a$, $\mu = b$, h arbitrary. The functions x and y satisfy equations of the form (4) with

$$
\begin{aligned}
m &= aA + G - a_u + h, & n &= bB + H' - b_v + h,\\
f &= J - Ah + h_u, & g &= J' - Bh + h_v,\\
A &= E - b, & B &= E' - a.
\end{aligned}
\tag{23}
$$

* Bompiani, *Surfaces*, p. 632.

† E. P. Lane, *Integral surfaces of pairs of partial differential equations of the third order*, these Transactions, vol. 32 (1930), pp. 782–793.

‡ Bompiani, *Surfaces*, p. 632.

There exists, therefore, *in this case one and only one line passing through x and not lying in the tangent plane to S_x at x, such that every point y on the line, not a focal point of the line, generates a net N_y in relation C to N_x. The space S_n is such that $n \geqq 5$.*

Case II. Suppose that $S(2, 0)$ is an S_4. It follows that the functions x satisfy an equation of the form

$$P'x_{uu} + Q'x_{uv} + R'x_{vv} + L'x_u + N'x_v + K'x = 0$$

such that not both P' and R' are zero. To fix the notation, suppose that $R' \neq 0$. The functions x therefore satisfy a system of differential equations of the form

$$\begin{aligned} x_{vv} &= Px_{uu} + Qx_{uv} + Lx_u + Nx_v + Kx, \\ x_{uuv} &= ax_{uu} + Ex_{uv} + Gx_u + Hx_v + Jx, \\ x_{uvv} &= D'x_{uu} + E'x_{uv} + G'x_u + H'x_v + J'x. \end{aligned} \tag{24}$$

Subcase (a). Suppose that x does not satisfy a differential equation of the form

$$x_{uuu} = \alpha x_{uu} + \beta x_{vv} + \delta x_u + \epsilon x_v + \phi x. \tag{25}$$

Some of the integrability conditions of system (24) are

$$P = D' = 0, \quad aE' + G' - a^2 - a_v = 0, \quad aQ + L = 0.$$

From (21) we find that

$$\begin{aligned} y_u = {} & (a - \lambda)x_{uu} + [G + h - \lambda_u + \lambda(E - \mu)]x_u + [H + \mu(E - \mu) - \mu_u]x_v \\ & + [J - h(E - \mu) + h_u]x + (E - \mu)y, \\ y_v = {} & [G' - \lambda_v - \mu L + \lambda(E' - \lambda - \mu Q)]x_u + [H' - \mu N - \mu_v + h \\ & + \mu(E' - \lambda - \mu Q)]x_v + [J' - \mu K + h_v - h(E' - \lambda - \mu Q)]x \\ & + (E' - \lambda - \mu Q)y. \end{aligned} \tag{26}$$

Hence N_y will be in relation C to N_x if and only if

$$\lambda = a, \quad \mu_u + \mu^2 - E\mu - H = 0, \quad h \text{ arbitrary}.$$

The functions x and y satisfy equations of the form (4) with

$$\begin{aligned} m &= aA + G - a_u + h, & n &= H' - \mu B - \mu N - \mu_v + h, \\ f &= J - Ah + h_u, & g &= J' - \mu K + h_v - Bh, \\ A &= E - \mu, & B &= E' - a - \mu Q. \end{aligned} \tag{27}$$

Hence, in this case, there exist lines g through x such that any point y on any one of the lines generates a net N_y in relation C to N_x. These lines g belong to a pencil

of lines with center at x and in the plane determined by the points x, x_v, $x_{uv}-ax_u$. The lines g at a point x are projectively related to the lines g through any other point of the curve $v=$const. through x. The space S_n is such that $n\geqq 5$.

Subcase (b). Suppose that x satisfies a system of differential equations of the form

$$
\begin{aligned}
x_{vv} &= Px_{uu} + Qx_{uv} + Lx_u + Nx_v + Kx,\\
x_{uuv} &= ax_{uu} + Ex_{uv} + Gx_u + Hx_v + Jx,\\
x_{uvv} &= D'x_{uu} + E'x_{uv} + G'x_u + H'x_v + J'x,\\
x_{uuu} &= \alpha x_{uu} + \beta x_{uv} + \delta x_u + \epsilon x_v + \phi x.
\end{aligned}
\tag{28}
$$

Some of the integrability conditions of system (28) are

$$
\begin{aligned}
a^2 + a_v + ED' + HP &= \alpha D' + D_u' + aE' + G,\\
D' &= L + aQ + \alpha P + P_u.
\end{aligned}
\tag{29}
$$

We may show from (21) that the net N_y is in relation C to N_x if and only if

$$
\begin{gathered}
\lambda = a,\quad D' - \mu P = 0,\quad \mu_u + \mu^2 - E\mu - H = 0,\\
G' - a_v - a^2 + aE' - \mu(L + aQ) = 0.
\end{gathered}
\tag{30}
$$

The value of μ determined by the second of (30), if substituted in the last two of (30), gives two conditions on the coefficients of system (28). These conditions are however a result of the integrability conditions (29). *Hence*, in this case, *there is a unique line joining x to*

$$y = x_{uv} - ax_u - D'x_v/P + hx$$

any point of which generates a net in relation C to N_x. The space S_n is an S_4.

The functions x and y satisfy equations of the form (4) with

$$
\begin{aligned}
m &= aA + G_u - a_u + h, & n &= H' - D'N/P - (D'/P)_v + D'B/P + h,\\
f &= J - Ah + h_u, & g &= J' - D'K/P - Bh - h_v,\\
A &= E - D'/P, & B &= E' - a - D'Q/P.
\end{aligned}
$$

We may state the results of this section as follows: *A non-conjugate net is a C net if and only if the functions x defining the net satisfy a pair of equations of the form* (18).

4. The Bompiani transforms of a C net

Suppose the functions x determining a net N_x satisfy differential equations of the form (18). Bompiani* has shown that there exist two covariant points,

* Bompiani, *Surfaces*, pp. 634–635.

one on each tangent line to the curves of the net, characterized in the following way: The point

$$\rho = x_u - bx$$

is the only point on the tangent to $v=$const. on S_x generating a surface for which the osculating plane to the curve $u=$const. at ρ lies in the S_3 determined by the points x, x_u, x_v, x_{uv} and tangent to the ruled surface $R^{(v)}$ along the generator through x. The point

$$\sigma = x_v - ax$$

has a similar characterization. We shall call the point $\rho(\sigma)$ the *minus first (first) Bompiani transform of* x, and the nets described by $\rho(\sigma)$ the *minus first (first) Bompiani transform of* N_x.

Suppose that the functions x satisfy a system of differential equations of the form (24), but no equation of the form (25). We find that the point ρ, defined by

$$\rho = x_u - \mu x,$$

generates a surface of the type described above for every value of μ. The point σ defined by

$$\sigma = x_v - ax$$

is the only point on the tangent to the curve $u=$const. at x generating a surface of the desired type. We shall say in this case that *the minus first Bompiani transform of* x (N_x) *is indeterminate*. The first Bompiani transform of x (N_x) is the point σ (net N_σ). Hence *a necessary and sufficient condition that a net permit of having Bompiani transforms is that the given net be a C net.*

We may state some of the results of this and the preceding section in the following theorem:

Let there be given a net N_x *in* S_n *whose sustaining surface* S_x *is such that its two-osculating space* $S(2, 0)$ *is an* S_4 *or an* S_5. *There exists a unique line g through x lying in the* $S(2, 0)$ *of* S_x *at x such that every point y on g generates a net* N_y *in relation C to* N_x *if and only if the given net admits of having uniquely determined Bompiani transforms. If one of the Bompiani transforms is indeterminate there exists a pencil of lines g through x with the above property. The line g is the line in relation R with respect to* N_x *to the line joining the Bompiani transforms of x.* Moreover *the minus first (first) derived line of g intersects the tangent plane of the sustaining surface in the minus first (first) Bompiani transform of x.*

The Bompiani transforms of a C net are C nets. The functions ρ and σ satisfy differential equations of the form (18). As may readily be verified the first Bompiani transform of ρ is the point $\bar{\sigma}$ defined by

$$\bar{\sigma} = \rho_v - a\rho,$$

and the minus first Bompiani transform of σ is the point $\bar{\rho}$ defined by

$$\bar{\rho} = \sigma_u - b\sigma.$$

The points $\bar{\rho}$ and $\bar{\sigma}$ lie on the line g, and coincide with the points (15) *and* (16) *respectively*. From (17) we see that these points coincide* if and only if

$$a_u - b_v = 0. \tag{17bis}$$

* Bompiani, *Surfaces*, p. 635.

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ON UNIVERSAL QUADRATIC NULL FORMS IN FIVE VARIABLES*

BY

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Introduction

1. We shall use L. E. Dickson's result:†

Theorem 1. *Every universal quadratic null form in three or more variables is equivalent to a form*

$$F = 2^e gaxy + gby^2 + cyz + gd\psi(z, w, \cdots) \qquad (e \geqq 0), \tag{1}$$

where g and a are odd, a is prime to d, c is prime to g, and the greatest common divisor of the coefficients of ψ *is* 1.

We investigate the case of five variables. In (1) let

$$\psi = \alpha(hz^2 + jzw + lw^2) + Azv + Bwv + Cv^2, \tag{2}$$

where

(3) 1 is the greatest common divisor of α, A, B, C and of h, j, l, and where, by an argument which carries over from Dickson's paper, h may be taken prime to any given odd integer. We take h prime to ga.

We shall assume that one of $N, M, P \neq 0$, where $N = j^2 - 4hl$, $M = A^2 - 4\alpha hC$, $P = 2hB - jA$. For if $N = M = P = 0$, then $4\alpha h\psi = (2\alpha hz + \alpha jw + Av)^2$, where either $\alpha jw + Av$ is identically zero or it may be taken as a product of a constant by a new variable w. Hence this case reduces to the problem for three or four variables.

2. We shall need the following lemmas:

Lemma 1. *If each of the congruences*

$$F \equiv G \quad (\mathrm{mod}\ 2^e), \qquad F \equiv G \quad (\mathrm{mod}\ gay), \tag{4}$$

with F as in (1) *and* (2), *has a solution x, y, z, w, v such that y is odd, then* $F = G$ *is solvable.*

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* Presented to the Society, June 13, 1931; received by the editors in December, 1930.

† *Universal quadratic forms*, these Transactions, vol. 31, No. 1, pp. 164–189. Subsequent references to the four-variable case refer to this paper.

LEMMA 2. *The congruence*

$$f = ax^2 + bxy + cy^2 \equiv k \pmod{p^n}, \tag{5}$$

where p is an odd prime, is solvable for every k if and only if $\Delta = b^2 - 4ac \not\equiv 0 \pmod{p}$ when $n=1$, and $(\Delta/p)=1$ when $n>1$.

Proof of Lemma 1. Since gay is odd, (4) implies $F \equiv G \pmod{Py}$, where $P = 2^e ga$. But $F = Pxy + \phi$, $\phi \equiv G \pmod{Py}$, $\phi = G + PyQ$. Hence $F = G$ when $x = -Q$.

Proof of Lemma 2. Case 1. $a \equiv 0$, $b \not\equiv 0(p)$. Let $y=1$. Then $f \equiv bx + c \equiv k(p)$, which is solvable. Assume $f = k + Hp^m$, $m \geqq 1$. Then $f \equiv k(p^{m+1})$ is solvable with x replaced by $x + Xp^m$, since $H + bX \equiv 0(p)$ has a solution X. In this case $\Delta \equiv b^2(p)$, hence $(\Delta/p) = 1$.

Case 2. $a \equiv b \equiv 0(p)$. Then $f \equiv cy^2(p)$ and $f \equiv k(p)$ is not solvable for some k. In this case $\Delta \equiv 0(p)$.

Case 3. $a \not\equiv 0(p)$. Multiplying (5) by $4a$ we get the equivalent congruence

$$Z^2 - \Delta y^2 \equiv 4ak(p^n), \tag{6}$$

where

$$Z = 2ax + by. \tag{7}$$

When $\Delta \equiv 0(p)$, (6), hence (5) also, is not solvable for some k. When $\Delta \not\equiv 0(p)$, it is well known that (6) is solvable modulo p. The solutions Z, y fix x, modulo p, by (7), so that x, y satisfy (5), modulo p.

Consider $n>1$. If $(\Delta/p) = -1$, $k = pK$ requires $y \equiv Z \equiv 0(p)$, whence (6) is not satisfied modulo p for some K. If $(\Delta/p)=1$, the pair Z, y which satisfy (6) modulo p may be chosen so that one of Z, y, hence one of $x, y \not\equiv 0(p)$. Let $f = k + Hp^m$, $m \geqq 1$, one of $x, y \not\equiv 0(p)$. Then $a(x+Xp^m)^2 + b(x+Xp^m)(y+Yp^m) + c(y+Yp^m)^2 \equiv k(p^{m+1})$ if

$$H + MX + NY \equiv 0(p), \tag{8}$$

where $M = 2ax + by$, $N = bx + 2cy$. Congruence (8) is satisfied unless $M \equiv N \equiv 0(p)$, i.e., unless $\Delta x \equiv \Delta y \equiv 0(p)$, which contradicts the hypotheses on Δ, x, and y.

THE CONGRUENCE $F \equiv G \pmod{gay}$

3. **Preliminary results. Discussion of y.** We may factor g and a as follows:

(9) $g = qr$, $a = st$, q and s have the same distinct prime factors; r and t are prime to each other and to both q and s.

Then qs, r and t are relatively prime in pairs. Thus $F \equiv G$ is solvable

modulo ga if solvable modulis qs, r and t. It is solvable modulis qs and r in the four-* and hence the five-variable case. It remains to consider

$$F = G(ty). \tag{10}$$

Let t be the product of powers p^n of distinct primes. We shall use the following theorem:

THEOREM 2. *If for each factor p^n of $t = p_1^{n_1} \cdots p_r^{n_r}$,*

(11) *$F \equiv G(p^n)$, with F as in* (1) *and* (2), *has a solution $z, w, v, y = \eta$, with $\eta = 1$ or π, where*

(12) *π is an odd prime dividing no one of g, a, d, α, h, and not dividing one of $N = j^2 - 4hl$, $M = A^2 - 4\alpha hC$, $P = 2hB - jA$, then* (10) *is solvable with y odd but not necessarily the same as in the solution of* (11).

Note that by the last paragraph of §1

(13) one of $N, M, P \neq 0$.

First we prove

LEMMA 2A. *The congruence*

(14) *$F \equiv G(\pi)$, π as in* (12), *is solvable with $y = k\pi$, where k is an arbitrary integer.*

For, dropping the terms of F containing y, and multiplying (14) by $4\alpha h$, we get the equivalent congruence

$$4\alpha hF \equiv gd(Z^2 - \alpha^2 Nw^2 - Mv^2 + 2\alpha Pwv) \equiv 4\alpha hG(\pi), \tag{15}$$

where $Z = 2\alpha hz + \alpha jw + Av$, and N, M, P are as in (12). Since $2\alpha h$ is prime to π we may take Z, w, v as new variables in place of z, w, v. If $N = M = 0$, then $P \neq 0$ by (13), and since $2gd\alpha P$ is prime to π by (12), (15) is solvable. Otherwise with $v = 0$ or $w = 0$ according as $N \neq 0$ or $N = 0$, $M \neq 0$, (15) is solvable by Lemma 2, and hence, since $4\alpha h$ is prime to π, (14) is solvable. This completes the proof of Lemma 2A.

Then by hypothesis (11) has a solution $z', w', v', y = \eta$, where $\eta = 1$ or π, and by Lemma 2A, if $\eta = \pi$ (and trivially if $\eta = 1$) $F \equiv G(\eta)$ has a solution $z'', w'', v'', y = \eta$.

By the Chinese Remainder Theorem there exist integers z, w, v such that $z \equiv z'$, $w \equiv w'$, $v \equiv v'$ (mod p^n), and $z \equiv z''$, $w \equiv w''$, $v \equiv v''$ (mod η). Then (11) and (14) have the same solution $z, w, v, y = \eta$, and hence, since η is prime to a and therefore to p,

$$F \equiv G(p^n \eta) \tag{16}$$

* With $v = w = 0$, change h to αh in line 9, p. 174, of the reference previously quoted.

has the same solution $z, w, v, y=\eta$.

For each factor p^n of $t=p_1^{n_1}\cdots p_r^{n_r}$ there are then solutions $z_i, w_i, v_i, y=\eta_i$ of $F\equiv G(p_i^{n_i}\eta_i)$, where $\eta_i=1$ or a prime π_i of type (12). Taking, by the Chinese Remainder Theorem,

$$(17)\qquad Y\equiv\eta_i,\ Z\equiv z_i,\ W\equiv w_i,\ V\equiv v_i \qquad (\mathrm{mod}\ p_i^{n_i}\eta_i)\quad (i=1,\cdots,r),$$

we have that $F\equiv G(t\zeta)$ has a solution Y, Z, W, V, where $\zeta=\eta_1\eta_2\cdots\eta_r$. Since by (17) $Y=\eta_i(1+k_ip_i^{n_i})$, where $n_i\geqq 1$, $i=1,\cdots,r$, it follows that $Y=\zeta S$, where S contains no factor p of t. We select a prime $\tau=S+X_1t$ of type (12) and not dividing $t\zeta$ from the infinite number of the form $S+Xt$, where X ranges over all integers, and take $y=\zeta\tau$. Then $F\equiv G(t\zeta)$ has a solution $Z, W, V, y=\zeta\tau=\zeta S+X_1t\zeta$, and, by Lemma 2A, $F\equiv G(\tau)$ has a solution $Z', W', V', y=\zeta\tau$. Since τ is prime to $t\zeta$ it follows from the Chinese Remainder Theorem that $F\equiv G(t\zeta\tau)$ has a solution $Z'', W'', V'', y=\zeta\tau$. This completes the proof of Theorem 2.

In applying Theorem 2 we shall proceed in practice as follows. We seek a solution of (11) $F\equiv G(p^n)$, and find that various conditions on the coefficients in F require the consideration of numerous sub-cases. As a first step we consider

$$(18)\qquad F\equiv G(p).$$

When Theorem 2 is applicable we can either set $y=1$ explicitly or show that there is a solution of (18) with $y=r\not\equiv 0(p)$. From the infinite number of primes of the form $r+Xp$, where X runs through all integral values, we select a prime π of type (12) and set $y=\pi$. Then where convenient we make the induction proving (11) solvable with y fixed. When we make the induction through y we show explicitly that y remains prime to p, so that at each stage of the induction it can be chosen a prime of type (12) selected from the infinite number of primes of the form $\eta+Xp^m$, $\eta\not\equiv 0(p)$, $m\geqq 1$.

There remain, besides the certainly non-universal cases, those for which we can prove (18) solvable but find it not convenient, and in some cases not possible, to hold $y\not\equiv 0(p)$ in the solution. In such cases, as in **12123** below, we use the following theorem:

THEOREM 2A. *Let p^n be a typical factor of $t=p_1^{n_1}\cdots p_r^{n_r}$. If, for F as in* (1) *and* (2), *$F\equiv G(p^\sigma)$ has for every $\sigma\geqq n$ a solution $z, w, v, y=p^\delta\eta$, where $\eta=1$ or a prime of type* (12) *and $\delta\geqq 0$ depends upon G, p, n, and F, but not upon σ, then* (10) *$F\equiv G(ty)$ is solvable.*

Let $s=n+\delta$. By hypothesis $F\equiv G(p^s)$ has a solution $z, w, v, y=p^\delta\eta$, where $\eta=1$ or a prime of type (12). Also, by Lemma 2A, $F\equiv G(\eta)$ has a solution

z', w', v', $y=p^\delta\eta$. Hence, applying the Chinese Remainder Theorem, $F\equiv G(p^s\eta=p^ny)$ has a solution Z, W, V, $y=p^\delta\eta$. That is, $F\equiv G(p_i^{n_i}y_i)$ has a solution z_i, w_i, v_i, $y_i=p_i^{\delta_i}\eta_i$, where $i=1, \cdots, r$ and each of the η_i is 1 or a prime of type (12).

From this point the part of the proof of Theorem 2 below (16) applies exactly if we delete the now unnecessary statement "where $\eta_i=1$ or a prime π_i of type (12)" and elsewhere replace η_i by y_i. Note that $\zeta=\eta_1\eta_2\cdots\eta_r$ becomes $\zeta=y_1y_2\cdots y_r$.

In view of Theorem 2A it will suffice in practice to prove that, excluding the (here) provably non-universal cases, congruence (11) $F\equiv G(p^n)$ has a solution z, w, v, $y=p^\delta r$, $r\not\equiv 0(p)$, where δ is unchanged in the induction from modulus p^m to modulus p^{m+1}. This will be true when y remains fixed in the induction, and will be shown explicitly when the induction is through y.

4. **The congruence** $F\equiv G$ (mod t). Since t is prime to g, d, and h, but divides a, the congruence

$$F\equiv G(p^n), \tag{19}$$

where p is a prime factor of t, is equivalent to

$$F = gby^2+cyz+gd[\alpha(hz^2+jzw+lw^2)+Azv+Bwv+Cv^2]\equiv G(p^n). \tag{20}$$

We may assume without loss of generality that

$$G\not\equiv 0(p^2). \tag{21}$$

For suppose $G=p^{2s}G_1$, where $G_1\not\equiv 0(p^2)$. If $n\leqq 2s$, (20) is solvable with $y=p^n$, $z=w=v=0$. If $n>2s$, take $y=p^sy'$, $z=p^sz'$, $w=p^sw'$, $v=p^sv'$. Division by p^{2s} yields the equivalent congruence $F(y', z', w', v')\equiv G_1(p^{n-2s})$, which justifies the assumption.

In the following scheme of subdivision, **121** and **122** are subheads of **12**, etc.

1. $C\not\equiv 0(p)$.

11. $b\equiv c\equiv D\equiv E\equiv K\equiv 0(p)$,
where $D=4\alpha hC-A^2$, $E=2\alpha jC-AB$, $K=4\alpha lC-B^2$. Then F is not universal.

Multiplying both sides of (20) by $4C$, we have

$$F_1 = 4CF = gd[V^2+Dz^2+2Ezw+Kw^2]+4C(gby^2+cyz)\equiv 4CG(p^n), \tag{22}$$

where $V=2Cv+Az+Bw$. Since our moduli are powers of p we may take V, z, w, y as new variables in place of v, z, w, y, and (22) is equivalent to (20). By the conditions of **11**, $F_1\equiv gdV^2\equiv 4CG(p)$, which is not solvable for some G.

12. One of b, c, D, E, $K\not\equiv 0(p)$.

121. $b\not\equiv 0(p)$. By (22)

(23) $$F_2 = gbF_1 = CY^2 - Lz^2 + g^2db[V^2 + 2Ezw + Kw^2] \equiv k(p^n),$$

where

$$Y = 2gby + cz,\quad L = RC + g^2dbA^2,\quad R = c^2 - 4g^2db\alpha h,\quad k = 4gbCG.$$

1211. $(-4g^2dbC/p) = (-dbC/p) = 1$. By Lemma 2, (23) is solvable with $z = w = 0$.

1212. $(-dbC/p) = -1$.

12121. One of $L, E, K \not\equiv 0(p)$. Then (23) is solvable.

Take $V = 1$. Then (23) has a solution Y, z, w, modulo p, with (1) $w = 0$, (2) $z = 0$, (3) $z = 1$ according as (1) $L \not\equiv 0$, (2) $L \equiv 0$, $K \not\equiv 0$, (3) $L \equiv K \equiv 0$, $E \not\equiv 0(p)$. Assume $F_2 = k + Hp^m$, $m \geqq 1$, $V \not\equiv 0(p)$. Then $F_2 \equiv k(p^{m+1})$ with Y, z, w (and hence y) unchanged and with V replaced by $V + p^mX \not\equiv 0(p)$, since $2g^2dbV \not\equiv 0(p)$.

12122. $L = pL_1$, $E = pE_1$, $K = pK_1$, $T = g^2dbE_1^2 + L_1K_1 \equiv 0(p)$. Then F is not universal.

Since $(-dbC/p) = -1$, the solution of (23) modulo p with $k = pk_1$ requires $Y \equiv V \equiv 0(p)$, so that, dividing out p, $F \equiv pk_1(p^2)$ reduces to $-L_1z^2 + g^2db(2E_1zw + K_1w^2) \equiv k_1(p)$, which by Lemma 2 is not solvable for some k_1.

12123. $L = pL_1$, $E = pE_1$, $K = pK_1$, $T \not\equiv 0(p)$. Then (23) is solvable.

By Lemma 2 we may fix Y and V modulo p so that

(24) $$CY^2 + g^2dbV^2 \equiv k(p),$$

hence $k - CY^2 - g^2dbV^2 = pQ$. Then (23) modulo p^2 reduces to

(25) $$-L_1z^2 + g^2db(2E_1zw + K_1w^2) \equiv Q(p),$$

which is solvable by Lemma 2. This fixes z and w, modulo p, and hence also y and v through Y and V. Assume $F_2 = k + Hp^m$, $m \geqq 2$, and assume first $V \not\equiv 0(p)$. Then $F_2 \equiv k(p^{m+1})$ with Y, z, w unchanged and with v replaced by $v + p^mv'$ (so that V is replaced by $V + 2Cp^mv'$), since $2g^2dbV \not\equiv 0(p)$.

It remains to complete the induction on (23) from $m \geqq 2$ to $m+1$ when $V \equiv 0(p)$. First assume $k \not\equiv 0(p)$, so that $Y \not\equiv 0(p)$ by (24). If $y \not\equiv 0(p)$ we complete the induction by replacing y by $y + p^my' \not\equiv 0(p)$, so that $Y = 2gby + cz$ is replaced by $Y + 2gbp^mY' \not\equiv 0(p)$. If $y \equiv 0(p)$ then $cz \not\equiv 0(p)$. We set $y = p$ and replace z by $z + p^mZ$, hence Y by $Y + cp^mY' \not\equiv 0(p)$ and $V = 2Cv + Az + Bw$ by $V + Ap^mZ \equiv 0(p)$. The induction is complete since $2(g^2dbAV + cCY) \equiv 2cCY \not\equiv 0(p)$.

Next assume $k=pk_1$, where $k_1\not\equiv 0(p)$ by (21). Then since $(-dbC/p)=-1$, $Y\equiv V\equiv 0(p)$ is required. Note that $c\equiv 0(p)$ then *requires* that $y\equiv 0(p)$. Since $k_1\not\equiv 0(p)$, $Q\not\equiv 0(p)$ by the equation just below (24), hence one of z, $w\not\equiv 0(p)$ in the solution of (25). Assume $F_2=k+Hp^m$, $m\geqq 2$, one of z, $w\not\equiv 0(p)$. Then $F_2\equiv k(p^{m+1})$ with z and w replaced respectively by $z+p^{m-1}Z$, $w+p^{m-1}W$ if $C(Y+cp^{m-1}Z)^2+g^2db(V+Ap^{m-1}Z+Bp^{m-1}W)^2\equiv CY^2+g^2dbV^2(p^{m+1})$ and $H+2\theta_1Z+2g^2db\theta_2W\equiv 0(p)$, where $\theta_1=-L_1z+g^2dbE_1w$, $\theta_2=E_1z+K_1w$. The latter congruence is solvable, since $\theta_1\equiv\theta_2\equiv 0(p)$ requires that $Tz\equiv Tw\equiv 0(p)$, contradicting the conditions on T, z, and w. The former congruence is also solvable if $Y\equiv V\equiv 0(p^2)$, $m\geqq 3$. For the case $m=2$, with z and w fixed by (25) we hold $Y\equiv V\equiv 0(p^2)$ by adjustment of y and v. The latter need not be changed thereafter, since if $Y\equiv V\equiv 0(p^2)$ then $Y+p^{m-1}cZ\equiv V+p^{m-1}(AZ+BW)\equiv 0(p^2)$ for $m\geqq 3$.

122. $b\equiv 0$, $c\not\equiv 0(p)$. By Lemma 2, (20) is solvable with $w=v=0$, since the Δ of Lemma $2=c^2-4g^2dbh\equiv c^2(p)$, hence $(\Delta/p)=1$.

123. $b=pb_1$, $c=pc_1$, $D\not\equiv 0(p)$. By (22)

$$F_3 = gdDF_1 = g^2d^2DV^2 + Z^2 + g^2d^2\theta w^2 + 4p\delta_1Cy^2 - 4p\gamma_1yw \equiv k(p^n), \tag{26}$$

where

$$Z = gdDz + gdEw + 2cCy,\quad \theta = DK - E^2,\quad \delta = g^2bdD - c^2C = p\delta_1,$$
$$\gamma = gdcCE = p\gamma_1,\quad k = 4gdCDG.$$

Applying to (26) the arguments of **121** as applied to (23), we find (26) solvable unless $\theta=p\theta_1$, $(-D/p)=-1$, and $\gamma_1^2-g^2d^2C\theta_1\delta_1\equiv 0(p)$, in which case F is not universal. To make sure that the power of p in y is not indefinitely increased in the induction we note that by the argument of **121** as obviously modified to fit the new lettering the induction is through V and Z (i.e., v and z) unless $k=pk_1$, $k_1\not\equiv 0(p)$, $V\equiv Z\equiv 0(p)$, with one of y, $w\not\equiv 0(p)$. Then if $y\equiv 0(p)$, $\theta_1w\not\equiv 0(p)$, and the induction can be made through w alone; otherwise y remains $\not\equiv 0(p)$ in the required replacement of y by $y+p^{m-1}Y$.

124. $b=pb_1$, $c=pc_1$, $D=pD_1$, $K\not\equiv 0(p)$. By (22)

$$F_4 = KF_1 = gd[W^2 + KV^2 + \theta z^2] + 4pCK(gb_1y^2 + c_1yz) \equiv 4CKG(p^n), \tag{27}$$

where

$$W = Kw + Ez,\quad \theta = DK - E^2.$$

As in **123**, (27) is solvable unless $\theta=p\theta_1$, $(-K/p)=-1$, and $c_1^2CK-g^2db_1\theta_1\equiv 0(p)$, in which case F is not universal.

125. $b=pb_1,\ c=pc_1,\ D=pD_1,\ K=pK_1,\ E\not\equiv 0(p)$.

In this case (22) is solvable by Lemma 2 with $V=0,\ y=p^n\eta$, since the Δ of Lemma $2\equiv 4E^2(p)$, hence $(\Delta/p)=1$.

2. $C\equiv 0(p)$, and one of $A, B\not\equiv 0(p)$. Then (20) is solvable.

Fix z and w so that $Az+Bw=M\not\equiv 0(p)$, and take $y=1$. Then (20) yields

$$L(v) = gd(Mv + Cv^2) \equiv k(p^n), \tag{28}$$

where k is a constant. This is solvable, modulo p. Assume $L(v)=k+Hp^m$, $m\geqq 1$, and take $v'=v+Xp^m$. Then

$$\begin{aligned} L(v') &\equiv k + p^m[H + gdMX](p^{m+1}) \\ &\equiv k(p^{m+1}) \end{aligned}$$

by choice of X, completing the induction.

3. $A=pA_1,\ B=pB_1,\ C=pC_1$. Then $\alpha\not\equiv 0(p)$ by (3).

31. $N=j^2-4hl\not\equiv 0(p)$.

Since p is odd and does not divide $gd\alpha h$, multiplication of (20) by $4gd\alpha h$ yields the equivalent congruence

$$4gd\alpha h(gby^2 + cyz) + g^2d^2\alpha^2[(2hz + jw)^2 - Nw^2] + \xi \equiv 4gd\alpha hG(p^n), \tag{29}$$

where

$$\xi = 4pg^2d^2\alpha h(A_1z + B_1w + C_1v)v. \tag{30}$$

The product of (29) by N gives

$$S(U, V, y, z, w, v) = NU^2 - V^2 + Jy^2 + 4pD(A_1zv + B_1wv + C_1v^2) \equiv k(p^n), \tag{31}$$

where

$$U = gd\alpha(2hz + jw) + cy, \tag{32}$$

$$V = Ngd\alpha w + cjy, \tag{33}$$

$$J = c^2j^2 - NR,\quad R = c^2 - 4g^2db\alpha h,\quad k = 4gd\alpha hNG,\quad D = Ng^2d^2\alpha h \not\equiv 0(p). \tag{34}$$

311. $J\not\equiv 0(p)$.

312. $(N/p)=1$.

In cases **311** and **312**, (20) is solvable with $v=0$, as shown in the four-variable case.

313. $(N/p)=-1$ and $J=pJ_1$.

The result is given in

(35) $\begin{cases} \text{when } Q \equiv 0(p), F \text{ is not universal;} \\ \text{when } Q \not\equiv 0(p), (31) \text{ is solvable, where } Q = \alpha C_1 J_1 N - c^2 h(jB_1 - 2A_1 l)^2. \end{cases}$

Since $N \not\equiv 0(p)$,

(36) one of $j, l \not\equiv 0(p)$.

Since $J \equiv 0(p)$, (31) gives

$$NU^2 - V^2 \equiv k(p). \tag{37}$$

This is solvable by Lemma 2, fixing U and V modulo p. It remains to test (31) modulo p^m, $m \geqq 2$.

3131. $C \equiv 0(p)^2$.

31311. $c \equiv 0(p)$. Then F is not universal.

For, by (37), (33), and (32), $k = pK$ requires $U \equiv V \equiv w \equiv z \equiv 0(p)$. By (31), $S \equiv Jy^2(p^2) \not\equiv pK(p^2)$ for some K.

31312. $\mu = 2A_1 l - jB_1 \equiv 0(p)$. Then F is not universal.

Eliminating y from (32) and (33) and replacing $j^2 - N$ by $4hl$, we get

$$jU - V \equiv 2gd\alpha h(jz + 2lw)(p). \tag{38}$$

Multiplying (38) by B_1 and replacing jB_1 by $2A_1 l$, we get

$$(jU - V)B_1 \equiv 4gd\alpha hlM(p), \tag{39}$$

where

$$M = A_1 z + B_1 w. \tag{40}$$

By (37) and (39) $k = pK$ requires $U \equiv V \equiv lM \equiv 0(p)$. The condition $l \equiv 0(p)$ requires $j \not\equiv 0(p)$ by (36), whence by (38) and the condition that $\mu \equiv 0(p)$, $z \equiv B_1 \equiv 0(p)$. Hence $k = pK$ requires $M \equiv 0(p)$. By (31) $S \equiv Jy^2(p^2) \not\equiv pK(p^2)$ for some K.

31313. $c\mu \not\equiv 0(p)$. Then (31) is solvable.

Since $\mu = 2A_1 l - jB_1 \not\equiv 0(p)$,

$$jB_1 = 2A_1 l + r, \text{ where } r \not\equiv 0(p). \tag{41}$$

Multiplying (38) by B_1 and replacing jB_1 by $2A_1 l + r$, we get

$$(jU - V)B_1 - 2gd\alpha hrz \equiv 4gd\alpha hlM(p^n), \text{ with } M \text{ as in (40)}. \tag{42}$$

Noting (36) and the fact that U and V are fixed, modulo p, by (37), we have three subcases:

When $l\equiv 0(p)$, $jB_1\equiv r\not\equiv 0(p)$ by (41), and z is fixed modulo p by (38). Choose w so that, by (40), $M\not\equiv 0(p)$. Then y is fixed modulo p by (32) or (33) (consistent through (38)).

When $j\equiv 0(p)$, $2A_1l\equiv -r\not\equiv 0(p)$ by (41), and w is fixed modulo p by (33). Choose z so that, by (40), $M\not\equiv 0(p)$, fixing y modulo p by (32).

When $jl\not\equiv 0(p)$, choose z so that the left side of $(42)\not\equiv 0(p)$.

In all cases we have $M=A_1z+B_1w\not\equiv 0(p)$. By (31), taking $C=p^2C_2$,

$$L(v) = 4Dp(Mv + pC_2v^2) \equiv K(p^n), \tag{43}$$

where

$$K = k - NU^2 + V^2 - Jy^2 = pK_1,$$

with K_1 independent of v. Dividing out p we find (43), hence (31), solvable by the method used for (28).

Note that (35) is satisfied in our results for **3131**.

3132. $C=pC_1$, $C_1\not\equiv 0(p)$.

31321. $c\equiv J_1\equiv 0(p)$. Then F is not universal.

By (37), (33) and (32), $k=pK$ requires $U\equiv V\equiv w\equiv z\equiv 0(p)$. By (31), noting $J=pJ_1$, $S\equiv 4pDC_1v^2(p^2)\not\equiv pK(p^2)$ for some K.

31322. $c\equiv 0$, $J_1\not\equiv 0(p)$. Then (31) is solvable.

U, V, w and z are fixed, modulo p, by (37), (33), and (32). Multiplying both sides of (31) by C_1, we get

$$pC_1J_1y^2 + pD\Delta^2 \equiv K(p^2), \tag{44}$$

where

$$\Delta = 2C_1v + A_1z + B_1w, \tag{45}$$

$$K = C_1(k - NU^2 + V^2) + pD(A_1z + B_1w)^2 = pK_1. \tag{46}$$

Dividing out p, we find (44) solvable by Lemma 2, fixing y, Δ, and v, modulo p, by (44) and (45). Thus (31) is solvable modulo p^2. The induction will be completed in **313234**.

31323. $c\not\equiv 0(p)$.

313231. $j\equiv P\equiv 0(p)$, $P=\alpha C_1J_1+c^2lA_1^2$. Then F is not universal.

By (32) we have, with $j\equiv 0(p)$,

$$c^2y^2 \equiv (U - 2gd\alpha hz)^2(p). \tag{47}$$

Multiplying (44) by c^2, replacing K and c^2y^2 by their values in (46) and (47) respectively, and then D in the coefficient of z by its value in (34), and noting that $N=j^2-4hl\equiv -4hl(p)$, we have

(48) $$pDc^2\Delta^2 + 4p\alpha h^2g^2d^2Pz^2 - pTz \equiv K_2(p^2),$$

where

(49) $$T = 4C_1J_1gd\alpha hU + 2c^2DA_1B_1w,$$

(50) $$K_2 = c^2C_1(k - NU^2 + V^2) + p(Dc^2B_1{}^2w^2 - C_1J_1U^2) = pK_3.$$

By (37), (33), and (49), $k = pK$ requires $U \equiv V \equiv w \equiv T \equiv 0(p)$, whence by (50) $K_2 \equiv pc^2C_1K(p^2)$. We then have by (48) and the fact that $P \equiv 0(p)$

$$pDc^2\Delta^2 \equiv pc^2C_1K(p^2),$$

which is not solvable for some K.

Note that $j \equiv P \equiv 0(p)$ is equivalent to $Q \equiv 0(p)$, with Q as in (35).

313232. $j \equiv 0, P \not\equiv 0(p)$. Then (31) is solvable.

U, V, and T are fixed, modulo p, by (37), (33), and (49). Completing the square in z in (48) we have

(51) $$p\delta\Delta^2 + pZ^2 \equiv K_4(p^2),$$

where

(52) $$Z = 8\alpha h^2g^2d^2Pz - T,$$

(53) $$K_4 = 16\alpha h^2g^2d^2PK_2 + pT^2 = pK_5, \quad \delta = 16\alpha h^2g^2d^2PDc^2 \not\equiv 0(p).$$

Dividing out p we find (51) solvable by Lemma 2, fixing Δ and Z, modulo p, and hence also z, v, and y by (52), (45), and (32) respectively. Thus (31) is solvable modulo p^2. The induction will be completed in **313234**.

313233. $j \not\equiv 0$, $Q \equiv 0(p)$, with Q as in (35). F is not universal.

Eliminating y and z from (44) (with K replaced as in (46)) by means of (33) and (38), we get

(54) $$p(E\Delta^2 + Lw^2 - Mw) \equiv K_6(p^2),$$

where

(55) $$E = D(2cjgd\alpha h)^2 \not\equiv 0(p),$$

(56) $$L = Dh(2gd\alpha)^2Q,$$

(57) $$M = 4gd\alpha hD[2\alpha C_1J_1V + c^2A_1(jU - V)(jB_1 - 2A_1l)],$$

(58) $$K_6 = (2gd\alpha hcj)^2C_1(k - NU^2 + V^2) + p[c^2DA_1^2(jU - V)^2 - C_1J_1(2gd\alpha hV)^2] = pK_7.$$

By (37) and (57), $k = pK$ requires $U \equiv V \equiv M \equiv 0(p)$. Since Q and hence $L \equiv 0(p)$, we get, from (54) and (58), $pE\Delta^2 \equiv p(2gd\alpha hcj)^2C_1K(p^2)$, which is not solvable for some K.

313234. $jQ\not\equiv 0(p)$, with Q as in (35). Then (31) is solvable.

U, V, and M are fixed, modulo p, by (37) and (57). Completing the square in (54), we get

$$4pLE\Delta^2 + pW^2 \equiv K_8(p^2), \tag{59}$$

where

$$W = 2Lw - M, \tag{60}$$

$$K_8 = 4LK_6 + pM^2 = pK_9. \tag{61}$$

Dividing out p we find (59) solvable by Lemma 2, fixing Δ and W, modulo p, and hence also w, y, z and v by (60), (33), (32) and (45) respectively. Thus (31) is solvable modulo p^2.

It remains to complete the induction in cases **31322, 313232,** and **313234.** This will be done in subdivisions of **313234**. For convenience we may rewrite (31) as follows:

$$S = NU^2 - V^2 + pJ_1y^2 + 4pDZv + 4pDC_1v^2 \equiv k(p^n), \tag{62}$$

where

$$Z = A_1z + B_1w,\quad Dgd\alpha hNC_1 \not\equiv 0(p),$$

and y, z, w, v are fixed, modulo p. Since (62) is solvable modulo p^2 with the exceptions noted, we may have $m\geqq 2$ in the induction.

3132341. $U\not\equiv 0(p)$.

Assume $S(z)=k+Hp^m$. Take $z'=z+Xp^m$. Then, by (32), $U'=U+2hgd\alpha Xp^m\not\equiv 0(p)$. By (62) $S(z')\equiv k+p^m[H+4gd\alpha hNX]$ (p^{m+1}), etc., completing the induction.

3132342. $U\equiv 0,\ V\not\equiv 0(p)$.

Assume $S(w)=k+Hp^m$. Take $w'=w+Xp^m$. Then by (32) and (33), $U'=U+gd\alpha jXp^m\equiv 0(p)$, $V'=V+gd\alpha NXp^m\not\equiv 0(p)$. By (62) $S(w')\equiv k+p^m[H-2gd\alpha NV'X](p^{m+1})$, etc., completing the induction.

3132343. $U\equiv V\equiv 0(p)$.

31323431. $v\not\equiv 0(p)$.

Assume $S(y, z, w, v)=k+Hp^m$. Take $z'=z+Xp^{m-1}$. Then by (32) and the definition of Z below (62), $U'=U+2hgd\alpha Xp^{m-1}\equiv 0(p)$, $Z'=Z+A_1p^{m-1}$, and $S(z')\equiv k+p^{m-1}[Hp+4L_1X](p^{m+1})$, where $L_1=NUhgd\alpha+pDA_1v$. The induction will be complete unless $L_1\equiv 0(p^2)$. Similarly, taking in succession

$y'=y+Xp^{m-1}$ (except when $y\equiv 0(p)$) and $w'=w+Xp^{m-1}$, we get in succession the two following coefficients of X:

$$L_2 = 2(NcU - cjV + pJ_1y),$$
$$L_3 = 2(gd\alpha jNU - gd\alpha NV + 2pDB_1v).$$

Hence with the proper substitution the induction is complete unless $y\equiv 0(p)$ or

$$(63) \qquad L_1 \equiv L_2 \equiv L_3 \equiv 0(p).$$

Eliminating U and V from (63) and dividing out p, we get

$$(64) \qquad 2cDv(2A_1l - jB_1) + Ngd\alpha J_1y \equiv 0(p).$$

This will be considered in conjunction with (67) below.

Again, take $z'=z(1+Xp^{m-1})$, $y'=y(1+Xp^{m-1})$, $w'=w(1+Xp^{m-1})$. Then since $U\equiv V\equiv 0(p)$, we have $U'=U(1+Xp^{m-1})\equiv 0(p)$, $V'=V(1+Xp^{m-1})\equiv 0(p)$, $Z'=Z(1+Xp^{m-1})$. Finally, take $v'=v(1+Xp^{m-1})\not\equiv 0(p)$. The induction will be valid in one of these last two cases unless

$$(65) \qquad L_4 \equiv L_5 \equiv 0(p),$$

where

$$(66) \qquad L_4 = 2(J_1y^2 + 2DZv), \quad L_5 = 4Dv(Z + 2C_1v);$$

Z is as defined below (62). Eliminating Z from (65), we get

$$(67) \qquad 4C_1Dv^2 - J_1y^2 \equiv 0(p).$$

Since $4C_1Dv\not\equiv 0(p)$, (67) is impossible if $y\equiv 0(p)$; hence in that case one of L_4, $L_5\not\equiv 0(p)$, and the induction is complete with the power of p in y unaltered.

Finally, eliminating y from (64) and (67), replacing D by $Ng^2d^2\alpha h$, and dropping common factors which we know to be prime to p (as g, d, α, h, N, C), we get $v^2Q\equiv 0(p)$, with Q as in (35), which contradicts the hypotheses on Q and v. Hence at least one of the five induction substitutions above succeeds, and the induction is complete.

31323432. $v\equiv 0(p)$.

By (21) we may assume that $G\not\equiv 0(p^2)$, hence in (62) $k=4gd\alpha hGN\not\equiv 0(p^2)$. Then by (62) $J_1y\not\equiv 0(p)$. Assume $S(y, z, w, v)=k+Hp^m$, where $U\equiv V\equiv 0$, $J_1y\not\equiv 0(p)$. Take $y'=y(1+Xp^{m-1})\not\equiv 0(p)$, $z'=z(1+Xp^{m-1})$, $w'=w(1+Xp^{m-1})$. Then $U'=U(1+Xp^{m-1})\equiv 0(p)$, $V'=V(1+Xp^{m-1})\equiv 0(p)$, $Z'=Z(1+Xp^{m-1})$.

By (62) $S(y', z', w', v') \equiv k + p^m[H + 2J_1y^2X](p^{m+1})$, etc., completing the induction. This completes **313**. Examination of the subcases reveals that (35) is a complete statement of the results.

32. $N = pN_1$. Then, since $N \equiv 0$, $h \not\equiv 0(p)$,

$$\text{either } j \equiv l \equiv 0, \text{ or } jl \not\equiv 0(p). \tag{68}$$

321. $jl \not\equiv 0(p)$. Then (31) is solvable, as shown in the four-variable case (§14, I).

322. $j \equiv l \equiv 0(p)$.

By (29),

$$\begin{aligned}\phi &= Z^2 - Ry^2 + g^2d^2\alpha[\alpha j(4hzw + jw^2) - \alpha pN_1w^2 + 4ph(A_1zv + B_1wv + C_1v^2)] \\ &\equiv k(p^n),\end{aligned} \tag{69}$$

where

$$Z = 2gd\alpha hz + cy, \tag{70}$$

$$R = c^2 - 4g^2d\alpha hb. \tag{71}$$

By (69) and **322** we have

$$Z^2 - Ry^2 \equiv k(p). \tag{72}$$

3221. $(R/p) = 1$. Then by Lemma 2, (69), hence (19), is solvable with $w = v = 0$.

3222. $R \equiv 0(p)$. Then F is not universal.

For then (72) has no solution Z for some k.

3223. $(R/p) = -1$. The result is as follows:

$$\begin{cases} \text{when } I \equiv 0(p), F \text{ is not universal;} \\ \text{when } I \not\equiv 0(p), (69) \text{ is solvable, where } I = hB_1^2 + \alpha N_1C_1. \end{cases} \tag{73}$$

Z, y and z are fixed, modulo p, by (72) and (70). It remains to determine when (69) is solvable modulis p^2 and p^n.

32231. $C_1 \equiv B_1 \equiv 0(p)$. Then F is not universal.

For, by (72) and (70), $k = pK$ requires $Z \equiv y \equiv z \equiv 0(p)$. By (69), $\phi \equiv -pg^2d^2\alpha^2N_1w^2(p^2) \not\equiv pK(p^2)$ for some K.

32232. $C_1 \equiv 0$, $B_1 \not\equiv 0(p)$. Then (69) is solvable.

Choose w so that $A_1z + B_1w = M \not\equiv 0(p)$. By (69)

$$4g^2d^2\alpha hpMv \equiv K(p^2). \tag{74}$$

where

$$K = k - Z^2 + Ry^2 - g^2d^2\alpha^2(4hjzw + j^2w^2) = pK_1 \text{ (independent of } v).$$

Dividing out p we find (74) solvable. Using (69), assume $\phi(v) = \phi(y, z, w, v) = k + p^mH$, with $m \geqq 2$. Take $v' = v + Xp^{m-1}$. Then $\phi(v') \equiv k + p^m[H + 4g^2d^2\alpha hMX](p^{m+1})$, etc., completing the induction.

32233. $C_1 \not\equiv 0$, $I \equiv 0(p)$, with I as in (73). F is not universal.

Completing the square in (69), we get

$$pEV^2 + \delta w^2 + \epsilon w \equiv K_1(p^n), \tag{75}$$

where

$$V = 2C_1v + B_1w + A_1z, \tag{76}$$

$$E = g^2d^2\alpha h \not\equiv 0(p), \tag{77}$$

$$\delta = g^2d^2\alpha(\alpha C_1j^2 - pI), \text{ with } I \text{ as in (73)}, \tag{78}$$

$$\epsilon = 2g^2d^2\alpha h(2\alpha jC_1 - pA_1B_1)z, \tag{79}$$

$$K_1 = C_1(k - Z^2 + Ry^2) + pg^2d^2\alpha hA_1^2z^2 = pK_2. \tag{80}$$

By (72) and (70), $k = pK$ requires $Z \equiv y \equiv z \equiv 0(p)$, and hence, since $j \equiv I \equiv 0(p)$, $\delta \equiv \epsilon \equiv 0(p^2)$. Then by (75) and (80),

$$pEV^2 \equiv pC_1K(p^2), \tag{81}$$

which is not solvable for some K.

32234. $C_1I \not\equiv 0(p)$, with I as in (73). Then (69) is solvable.

Z, y and z are fixed, modulo p, by (72) and (70). Writing $\delta = p\delta'$, $\epsilon = p\epsilon'$ (since $j \equiv 0(p)$), we have by (75) and (80), after dividing out p,

$$EV^2 + \delta' w^2 + \epsilon' w \equiv K_2(p^{n-1}), \tag{82}$$

where $\delta' \equiv -g^2d^2\alpha I \not\equiv 0(p)$. Since $E\delta' \not\equiv 0(p)$, we may complete the square in w and have (82) solvable modulo p, fixing V, w and v, modulo p, by (82) and (76). Hence (69) is solvable modulo p^2.

Using (69), assume $\phi = k + p^mH$, with $m \geqq 2$. By (70), if $Z \equiv 0(p)$, either $y \not\equiv 0(p)$ or $z \equiv y \equiv 0(p)$. Also, if $Z \equiv z \equiv y \equiv 0(p)$ we may assume that one of $N_1w, v \not\equiv 0(p)$, since $j \equiv 0(p)$ and by (21) k may be taken $\not\equiv 0(p^2)$. The following induction substitutions then cover all cases:

(a) $Z \not\equiv 0(p)$. Take $z' = z + Xp^m$, hence $Z' = Z + 2gd\alpha hXp^m \not\equiv 0(p)$.

(b) $Z \equiv 0$, $y \not\equiv 0(p)$. Take $z' = z(1 + Xp^m)$, $y' = y(1 + Xp^m)$, hence $Z' = Z(1 + Xp^m) \equiv 0(p)$.

(c) $Z \equiv z \equiv y \equiv B_1 w \equiv 0$, $v \not\equiv 0(p)$. Take $v' = v + Xp^{m-1}$.

(d) $Z \equiv z \equiv y \equiv N_1 \equiv 0$, $B_1 wv \not\equiv 0(p)$. Take $w' = w + Xp^{m-1}$.

(e) $Z \equiv z \equiv y \equiv B_1 v \equiv 0$, $N_1 w \not\equiv 0(p)$. Take $w' = w + Xp^{m-1}$.

(f) $Z \equiv z \equiv y \equiv 0$, $B_1 N_1 wv \not\equiv 0(p)$. Take $w' = w + Xp^{m-1}$, $v' = v + Yp^{m-1}$. Then $\phi \equiv k + p^m[H + LX](p^{m+1})$, etc., completing the induction in cases (a) to (e) inclusive, where $L \not\equiv 0(p)$ and is as follows:

(a) $4gd\alpha hZ$; (b) $-2Ry^2$; (c) $8g^2d^2\alpha hC_1 v$; (d) $4g^2d^2\alpha hB_1 v$; (e) $-2g^2d^2\alpha N_1 w$.

In case (f), $\phi \equiv k(p^{m+1})$ unless

$$S \equiv T \equiv 0(p), \tag{83}$$

where

$$S = 2hB_1 v - \alpha N_1 w, \quad T = 2C_1 v + B_1 w.$$

But on eliminating w, (83) is equivalent to $2vI \equiv 0(p)$, contradicting the hypotheses, hence (83) is impossible and the induction is complete.

The results in the subcases of **3223** satisfy (73).

We have now proved the solvability of (19), and hence, by Theorems 2 and 2A, of (10), except in cases (84) below.

Let p be any prime dividing a but not g. Let the appearance of b_1 indicate that $b = pb_1$, of D_1 that $D = pD_1$, etc. Define

$$D = 4\alpha hC - A^2, \quad E = 2\alpha jC - AB, \quad K = 4\alpha lC - B^2, \quad R = c^2 - 4g^2db\alpha h,$$
$$L = RC + g^2dbA^2, \quad \theta = DK - E^2, \quad \delta = g^2dbD - c^2C, \quad \gamma = gdcCE,$$
$$N = j^2 - 4hl, \quad J = c^2j^2 - NR, \quad I = hB_1 - \alpha N_1 C_1,$$
$$Q = \alpha C_1 J_1 N - c^2h(jB_1 - 2lA_1)^2.$$

Then F is not universal in the following cases:

(84)

1. $C \not\equiv 0(p)$, and one of **11** to **14** holds:

- **11.** $b \equiv c \equiv D \equiv E \equiv K \equiv 0(p)$;
- **12.** $(-dbC/p) = -1$, $g^2dbE_1^2 + L_1K_1 \equiv 0(p)$;
- **13.** $(-D/p) = -1$, $\gamma_1^2 - g^2d^2C\theta_1\delta_1 \equiv 0(p)$;
- **14.** $(-K/p) = -1$, $c_1^2CK - g^2d_1b\theta_1 \equiv 0(p)$;

2. $A \equiv B \equiv C \equiv 0(p)$, and one of **21**, **22** holds:

- **21.** $J \equiv Q \equiv 0(p)$, $(N/p) = -1$;
- **22.** $N \equiv j \equiv 0(p)$, and either $R \equiv 0(p)$ or $(R/p) = -1$ and $I \equiv 0(p)$.

Since y is odd we then have, by Lemma 1,

THEOREM 3. *If $e = 0$, F is universal except in cases* (84).

REMARKS ON THE CONGRUENCE $F \equiv G \pmod{2^n}$

5. **The briefer results.** The initial problem was solved completely in the writer's thesis, but the conditions upon the coefficients as found are so numerous that the bare listing of the results, except those given below for the case $e=1$, requires a prohibitive amount of space. We shall omit further proofs and close with the statements of two fundamental lemmas which give additional freedom in the choice of y, and of the theorem for the case $e=1$.

LEMMA 3. *If there exists an odd y satisfying* $(4)_2$, *it may be chosen congruent to any desired odd residue, modulo* 2^n.

LEMMA 4. *If the solution of*

$$F \equiv G(2^e) \tag{85}$$

requires the factor 2^s *in y, where* $s \geqq 0$ *depends upon G, then F is universal, subject to exceptions* (84), *if and only if*

$$F \equiv G(2^{e+s}) \tag{86}$$

is solvable with $x=0$ *for every pair G, s.*

THEOREM 4. *When* $e=1$, *F is universal subject to* (84) *unless* (a), $bcd\alpha h$ *is odd,* $l \equiv B \equiv C \equiv 0(4)$, *j and A are even, or* (b), $d=2D$, *c is even, and b or D is even. If either* (a) *or* (b) *holds, F is not universal.*

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ON THE APPROXIMATE REPRESENTATION OF A FUNCTION OF TWO VARIABLES*

BY

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The purpose of this paper is to exhibit extensions of some of the known existence theorems on the approximate representation of a function of one variable to corresponding theorems for a function of two variables and somewhat to investigate approximate representation by means of surface spherical harmonics.

For convenience the paper is divided into two sections. The material of the first section extends to a function of two variables some of the work of Professor Jackson on the approximate representation of a function of one variable given in the first chapter of his Ithaca Colloquium Lectures.† Trigonometric approximation based on an extension to two variables of Jackson's approximating integral is the foundation upon which other forms of representation are built by means of cosine transformations. These transformations are in part responsible for the essential difference between this paper and a mere rephrasing of Jackson's work. The section is concerned only with a real continuous function of two variables; its extension to a function of any finite number of variables is apparent. Moreover, it is limited to bare essentials: for simplicity, only periods of 2π and intervals of length 2 are considered; material of a superficial nature obtained by generalizing the condition of Lipschitz is omitted; no attempt is made to find small values for the absolute constants which enter—the order of approximation alone is sought; some applications of the theory paralleling those of Jackson‡ to Fourier, to Legendre, and to the corresponding mixed approximations have been omitted at this time and reserved for further extension.

The discussion in the second section is confined to the representation of a real function on the surface of a unit sphere by partial sums of Laplace's series and certain other sums of surface spherical harmonics and to the convergence of the approximating sum of surface spherical harmonics which minimizes the surface integral of a power of the absolute error. An expression for an upper bound to the absolute error in the representation by a partial sum of La-

* Presented to the Society at Chicago, March 29, 1929; received by the editors in December, 1930.

† D. Jackson, *The Theory of Approximation*, American Mathematical Society Colloquium Publications, New York, 1930; cited below as *Colloquium*.

‡ See *Colloquium*, pp. 18–32.

place's series is given in terms of the order of the sum; the expression is obtained from considerations of the mean value of the given function used in conjunction with the results of a simple lemma. The discussion of the convergence of the approximating sums in the sense of integrals parallels that of Jackson for trigonometric sums.

I. Trigonometric, polynomial, and mixed approximation

1. **The forms of the approximating functions.** The approximating functions to be used in this section are finite sums, the forms of which are herein defined and listed for reference. Let m and n be a pair of positive integers, and let (a_{ij}), (b_{ij}), (c_{ij}), and (d_{ij}), where i and j range independently over the integers from zero to m and from zero to n respectively, be sets of real constants.

By a trigonometric sum of order *at most* m in x and n in y is meant a sum of the form

$$T_{mn}(x, y) = \sum \left\{ \begin{matrix} a_{ij} \cos ix \cos jy + b_{ij} \cos ix \sin jy \\ + c_{ij} \sin ix \cos jy + d_{ij} \sin ix \sin jy \end{matrix} \right\}; \tag{1}$$

by a polynomial of degree at most m in x and n in y is meant a sum of the form

$$P_{mn}(x, y) = \sum (a_{ij} x^i y^j); \tag{2}$$

by a mixed sum of order at most m in x and degree at most n in y is meant one of the form

$$H_{mn}(x, y) = \sum [(a_{ij} \cos ix + b_{ij} \sin ix) y^j]. \tag{3}$$

2. **Trigonometric approximating functions for two variables.** Let $g(x, y)$ be a continuous periodic function of period 2π in x and in y separately. Let m and n be two positive integers; let p and q be integers such that $2p-2 \leqq m \leqq 2p$ and $2q-2 \leqq n \leqq 2q$; let $I_{pq}(x, y)$ be defined by the integral

$$I_{pq}(x, y) = h_{pq} \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} g(x + 2u, y + 2v) F_{pq}(u, v) du dv, \tag{4}$$

where

$$F_{pq}(u, v) = \left[\frac{(\sin pu)(\sin qv)}{(p \sin u)(q \sin v)} \right]^4 \tag{5}$$

and

$$1/h_{pq} = \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} F_{pq}(u, v) du dv, \text{ a positive constant.} \tag{6}$$

The integral (4) is the extension of Jackson's approximating function* to fit the case of two variables. By an argument entirely analogous to that used for one variable it follows that $I_{pq}(x, y)$ is a trigonometric sum of type (1) of order at most m in x and n in y; hence, when m and n are specified it and sums similar to it have the form of the desired approximating sum for the function $g(x, y)$.

From (6) it follows that

$$\begin{aligned}(7)\qquad & \left| g(x, y) - I_{pq}(x, y) \right| \\ & \leqq h_{pq} \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} \left| g(x + 2u, y + 2v) - g(x, y) \right| F_{pq}(u, v) du dv,\end{aligned}$$

so that the expression on the right furnishes an upper bound for the absolute value of the error in representing $g(x, y)$ by $I_{pq}(x, y)$. Evaluation of the integral on the right depends upon finding an expression for the absolute difference in the integrand through suitable restrictions on $g(x, y)$.

In anticipation of the results the following facts are noted for future reference†:

$$(8)\qquad \int_0^{\pi/2} (z \sin^4 pz)/(p^4 \sin^4 z) dz \leqq c_1'(1/p^2) \leqq c_1(1/m^2)$$

and

$$(9)\quad c_3(1/m) \leqq c_3'(1/p) \leqq \int_0^{\pi/2} (\sin^4 pz)/(p^4 \sin^4 z) dz \leqq c_2'(1/p) \leqq c_2(1/m),$$

where c_1, c_2, and c_3 are absolute positive constants.

3. **The modulus of continuity.** A suitable expression for the absolute difference in the integrand of (7) is obtained from considerations of the modulus of continuity of $g(x, y)$. It seems to be advisable at this point to lay a foundation for the remainder of this section by giving essential definitions and properties of the modulus of continuity together with demonstrations of the more involved facts.

Let $g(x, y)$ be continuous in a closed rectangular region R of the xy-plane. Define $\omega(\delta)$ to be the maximum of the absolute difference $|g(x_1, y_1) - g(x_2, y_2)|$ for all points (x_1, y_1), (x_2, y_2) in R for which $(x_1 - x_2)^2 + (y_1 - y_2)^2 \leqq \delta^2$. The

* See *Colloquium*, p. 3; also, for the case of two variables, C. E. Wilder, *On the degree of approximation to discontinuous functions by trigonometric sums*, Rendiconti del Circolo Matematico di Palermo, vol. 39 (1915), pp. 345–361; p. 358.

† See *Colloquium*, p. 5.

function $\omega(\delta)$, called the *modulus of continuity* of $g(x, y)$ in R, exists and has the following properties*:

(10) $\omega(\delta)$ is a continuous function of δ; $\omega(0)=0$;
$\omega(\delta)>0$ when $\delta>0$ unless $g(x, y)$ is constant in R;

$$\omega(\delta_1) \leqq \omega(\delta_2),\ \delta_1 \leqq \delta_2; \tag{11}$$

$$\omega(k\delta) \leqq k\omega(\delta),\ k \text{ a positive integer}; \tag{12}$$

$$\omega(k\delta) \leqq (k+1)\omega(\delta),\ k \text{ any positive number};$$

$$\omega(\delta_1+\delta_2) \leqq \omega(\delta_1)+\omega(\delta_2);$$

$$2\omega(\delta_1)/\delta_1 \geqq \omega(\delta_2)/\delta_2,\quad 0<\delta_1 \leqq \delta_2. \tag{13}$$

In case $\omega(\delta)$ does not exceed a quantitiy of the form $\lambda\delta$, λ a positive constant, $g(x, y)$ is such that $|g(x_1, y_1)-g(x_2, y_2)| \leqq \lambda[(x_1-x_2)^2+(y_1-y_2)^2]^{1/2}$ and is said to satisfy a Lipschitz condition† with parameter λ.

For present purposes the definitions are extended to include finite sets of functions. Let $g_0(x, y), g_1(x, y), g_2(x, y), \cdots, g_k(x, y)$—hereafter denoted by $\{g_k(x, y)\}$—be a finite set of functions continuous in R with moduli of continuity $\{\omega_k(\delta)\}$. Define $\Omega(\delta)$ to be the greatest of the quantities $\omega_k(\delta)$ for each value of δ; call it the *uniform modulus of continuity* for the set $\{g_k(x, y)\}$ in R. Quite obviously $\Omega(\delta)$ has the same properties (10), (11), (12), and (13) as the ordinary modulus $\omega(\delta)$. If the functions $\{g_k(x, y)\}$ satisfy Lipschitz conditions with parameters $\{\lambda_k\}$ in R they all evidently satisfy such conditions with a single parameter Λ which is the largest of the set $\{\lambda_k\}$; under such circumstances the set $\{g_k(x, y)\}$ will be said to satisfy a Lipschitz condition with parameter Λ.

The relation between the uniform modulus of continuity of the partial derivatives of specified order of a given function and the uniform modulus of continuity of the partial derivatives of the same order of its cosine transform is not only essential for the method of this paper, but it holds some interest of its own. Let $g(x, y)$ be continuous in the square region $-1 \leqq x, y \leqq 1$; let $G(\theta, \phi)=g(\cos\theta, \cos\phi)$. If the modulus of continuity of $g(x, y)$ in the region $-1 \leqq x, y \leqq 1$ is $\omega(\delta)$, $G(\theta, \phi)$ has in every finite region a modulus of continuity $w(\delta)$ such that $w(\delta) \leqq \omega(\delta)$; moreover, if $g(x, y)$ satisfies a Lipschitz condition with parameter λ in the region $-1 \leqq x, y \leqq 1$, G satisfies such a condition everywhere and with the same parameter. Furthermore, if the

* de la Vallée Poussin, *Leçons sur l'Approximation des Fonctions d'une Variable Réelle*, Paris, 1919, pp. 7–8.

† de la Vallée Poussin, op. cit., p. 9.

kth-order partial derivatives of $g(x, y)$ with respect to x and y exist and have a uniform modulus of continuity $\Omega(\delta)$ in $-1 \leqq x,\ y \leqq 1$, the kth-order partial derivatives of $G(\theta, \phi)$ with respect to θ and ϕ will have a uniform modulus of continuity $W(\delta)$ everywhere; if these derivatives of $g(x, y)$ satisfy a Lipschitz condition with parameter Λ the corresponding derivatives of $G(\theta, \phi)$ will satisfy such a condition everywhere with some parameter L. The relation of $W(\delta)$ to $\Omega(\delta)$ and of L to Λ, in general, is not so simple as in the case $k=0$. Although for particular values of k the relations are often more simple than those offered below, the latter are sufficiently acceptable generally.

Lemma I. *Let $g(x, y)$ together with all its partial derivatives of order $k \geqq 1$ and lower be continuous in the region $-1 \leqq x, y \leqq 1$; let M be the maximum of the absolute values of these derivatives in the region; let the uniform modulus of continuity of the partial derivatives of order k be $\Omega(\delta)$ for $0 \leqq \delta \leqq 2^{3/2}$, the maximum diameter of the region, and let the symbol $\Omega(\delta)$ denote the value $\Omega(2^{3/2})$ for $\delta > 2^{3/2}$. Then if $\Omega(2^{3/2}) \neq 0$, $G(\theta, \phi) = g(\cos\theta, \cos\phi)$ is a periodic function of period 2π in θ and in ϕ separately which together with its partial derivatives of order k and lower with respect to θ and ϕ is continuous everywhere, and the uniform modulus of continuity of the partial derivatives of order k does not exceed $Nk!(e^{e-1}-1)\,\Omega(\delta)$, where N is the larger of unity and $8M/\Omega(2^{3/2})$.*

Lemma II. *If the kth-order partial derivatives of $g(x, y)$, $k \geqq 1$, above satisfy a Lipschitz condition with parameter Λ, those of $G(\theta, \phi)$ satisfy the inequalities*

$$| G^{i,k-i}(\theta_1, \phi_1) - G^{i,k-i}(\theta_2, \phi_2) | \leqq L(|\theta_1 - \theta_2| + |\phi_1 - \phi_2|),$$

$i = 0, 1, 2, \cdots, k$, everywhere, where $L = N'k!\,(e^{e-1}-1)$ and N' is the larger of Λ and M, the symbol $G^{i,k-i}$ being used to denote $\partial^k G/\partial\theta^i\partial\phi^{k-i}$.

These two lemmas are of sufficient importance to warrant somewhat detailed demonstrations. Since M is the maximum of $|g^{r,s}(x, y)|$ in the region $-1 \leqq x, y \leqq 1$ for $1 \leqq r+s \leqq k$, it follows from the law of the mean that

$$(14) \qquad | g^{r,s}(x_1, y_1) - g^{r,s}(x_2, y_2) | \leqq M(|x_1 - x_2| + |y_1 - y_2|)$$

for $0 \leqq r+s \leqq k-1$ and all points of the region. Obviously, then, $g(x, y)$ and all its partial derivatives of order lower than k satisfy a Lipschitz condition. It is necessary in what follows to evaluate (so to speak) the condition (14) in terms of the uniform modulus $\Omega(\delta)$ of the partial derivatives of order exactly k. If $(x_1-x_2)^2+(y_1-y_2)^2 \leqq \delta^2$, then certainly $|x_1-x_2|+|y_1-y_2| \leqq \delta 2^{1/2}$. When $0 < \delta \leqq 2^{3/2}$, $2\,\Omega(\delta)/\delta \geqq \Omega(2^{3/2})/2^{3/2}$ by (13). Moreover, this same inequality holds without modification for $2^{3/2} \leqq \delta \leqq 2^{5/2}$ because $\Omega(\delta) = \Omega(2^{3/2})$ for these values of δ. (Obviously, the inequality can be adjusted for larger values of δ, but there is no need in this paper for values of δ greater than

$\pi 2^{1/2}$.) Consequently, when $0<\delta\leqq 2^{5/2}$, $0<\delta\leqq 2^{5/2}\Omega(\delta)/\Omega(2^{3/2})$ and $M(|x_1-x_2| + |y_1-y_2|)\leqq 8M\,\Omega(\delta)/\Omega(2^{3/2})$. Therefore, $g(x, y)$ and all its partial derivatives of order k and lower have a uniform modulus of continuity nowhere exceeding $N\,\Omega(\delta)$, where N is the larger of unity and $8M/\Omega(2^{3/2})$.

Now form the partial derivatives of $G(\theta, \phi)$:

$$G^{1,0}(\theta, \phi) = g^{1,0}(x, y)(-\sin\theta),$$
$$G^{0,1}(\theta, \phi) = g^{0,1}(x, y)(-\sin\phi),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(15)\quad G^{i,k-i}(\theta, \phi) = \begin{cases} \sum_{s=1}^{k-i}\sum_{r=1}^{i} g^{r,s}(x, y)P_{r,i}Q_{s,i}/(r!s!), & i = 1, \cdots, k-1 \geqq 1, \text{ or} \\ \sum_{s=1}^{k} g^{0,s}(x, y)Q_{s,0}/s!, & i = 0\ (k \geqq 1), \text{ or} \\ \sum_{r=1}^{i=k} g^{r,0}(x, y)P_{r,i}/r!, & i = k \geqq 1, \end{cases}$$

where $P_{r,i}$ and $Q_{s,i}$ are polynomials of degree r in $\cos\theta$ and $\sin\theta$ and of degree s in $\cos\phi$ and $\sin\phi$, and both are independent of $g(x, y)$ and of each other. It has been shown by de la Vallée Poussin* that

$$|P_{r,i}| \leqq i!(e-1)^r, \quad |Q_{s,i}| \leqq (k-i)!(e-1)^s.$$

Denote $|G^{i,k-i}(\theta_1, \phi_1)-G^{i,k-i}(\theta_2, \phi_2)|$ by D. Then if the first of the forms (15) be considered,

$$(16)\qquad D \leqq i!(k-i)! \sum_{s=1,r=1}^{k-i,i} \left\{ |g^{r,s}(x_1, y_1) - g^{r,s}(x_2, y_2)|(e-1)^{r+s}/(r!s!)\right\}.$$

In this inequality the following facts are noted: $i!(k-i)!\leqq k!$; if $(\theta_1-\theta_2)^2 + (\phi_1-\phi_2)^2\leqq\delta^2$ then $(x_1-x_2)^2+(y_1-y_2)^2\leqq\delta^2$ also, and by the conclusion reached from (14) each of the absolute differences entering does not exceed $N\,\Omega(\delta)$; finally, $\sum(e-1)^{r+s}/(r!s!) \leqq \sum_1^\infty(e-1)^k/k! \leqq e^{e-1}-1$. Therefore, $D\leqq k!N(e^{e-1}-1)\,\Omega(\delta)$. The same is true in case either of the other forms in (15) is appropriate. Thus, the first lemma is proved.

If the kth-order partial derivatives of $g(x, y)$ satisfy a Lipschitz condition with parameter Λ then conditions (14) subsist for $r+s\leqq k$ with parameter N', where N' is the larger of Λ and M. Since $|x_1-x_2|+|y_1-y_2|\leqq|\theta_1-\theta_2| + |\phi_1-\phi_2|$, it is apparent from (16) that the second lemma holds also.

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* de la Vallée Poussin, op. cit., pp. 67–68.

The preceding proofs can readily be adapted to demonstrations of the following lemmas:

LEMMA III. *Let $g(x, y)$ be a periodic function of period 2π in x alone which together with its partial derivatives of order $k \geqq 1$ and lower is continuous in the infinite strip $-\infty < x < \infty$, $-1 \leqq y \leqq 1$; let M be the maximum of the absolute values of these derivatives in the region; let the uniform modulus of continuity of the partial derivatives of order k be $\Omega(\delta)$. Then, if $\Omega(d) \neq 0$, where $d^2 = \pi^2 + 4$, $G(\theta, \phi) = g(\theta, \cos\phi)$ is a periodic function of period 2π in θ and in ϕ separately which with its partial derivatives of order k and lower is continuous everywhere, and the uniform modulus of continuity of the partial derivatives of order k nowhere exceeds $N''k!(e^{e-1}-1)\Omega(\delta)$, where N'' is the larger of unity and $2^{3/2}Md / \Omega(d)$.*

LEMMA IV. *If the kth-order partial derivatives of $g(x, y)$ in Lemma III satisfy a Lipschitz condition with parameter Λ, the kth-order partial derivatives of $G(\theta, \phi)$ satisfy relations*

$$\left| G^{i,k-i}(\theta_1, \phi_1) - G^{i,k-i}(\theta_2, \phi_2) \right| \leqq L(\left| \theta_1 - \theta_2 \right| + \left| \phi_1 - \phi_2 \right|),$$

where L is the constant of Lemma II.

The lemmas above will be used in §§5 and 6 to throw the burden of the proofs there on the theorems of §4.

4. **Degree of convergence of trigonometric approximation.** With the aid of the moduli of continuity discussed in the first part of the preceding article the function I_{pq} of §2 and functions analogous to it furnish trigonometric sums of type (1) approximating to a given periodic function. The following existence theorems exhibit the attainable degree of approximation by such sums to continuous functions and to functions having continuous partial derivatives.

THEOREM I.* *If $f(x, y)$ is a periodic function of period 2π in x and in y separately which everywhere satisfies a Lipschitz condition with parameter λ, then corresponding to every pair of positive integers m and n there exists a trigonometric sum $T_{mn}(x, y)$ of type* (1) *such that*

$$\left| f(x, y) - T_{mn}(x, y) \right| \leqq K\lambda(1/m + 1/n)$$

everywhere, where K is an absolute constant. (*The conclusion is equally valid with K replaced by a suitable constant K_1 if $f(x, y)$ is such that*

$$\left| f(x_1, y_1) - f(x_2, y_2) \right| \leqq \lambda \left| x_1 - x_2 \right| + \lambda \left| y_1 - y_2 \right|$$

everywhere.)

* The same theorem is given by Wilder, loc. cit.

The proof of this theorem is a straightforward extension of that for the corresponding theorem for one variable. In outline it is as follows: Given m and n, choose p and q as in §2 and construct the function $I_{pq}(x, y)$ for $f(x, y)$ by substituting $f(x, y)$ for $g(x, y)$ in (4), and take $T_{mn}(x, y)=I_{pq}(x, y)$; set up the difference (7); make use of the Lipschitz condition to replace the absolute difference in the integrand of (7) by $2\lambda(|u|+|v|)\geqq\lambda(4u^2+4v^2)^{1/2}$; split the resulting even integral into parts and apply (8) and (9) to each part to obtain $K\lambda(1/m+1/n)$ for an upper bound for the right-hand side of (7), where K is a combination of the c's.

THEOREM II. *If $f(x, y)$, periodic as in Theorem* I, *is merely continuous with modulus of continuity $\omega(\delta)$, then sums T_{mn} can be constructed so that*

$$|f(x, y) - T_{mn}(x, y)| \leqq K_2\omega(1/m + 1/n),$$

where K_2 is an absolute constant.

The proof of this theorem, also, will be sketched. Form $T_{mn}(x, y)$ as in the proof of Theorem I; replace the absolute difference in the integrand of (7) by $2\omega[(u^2+v^2)^{1/2}]\geqq\omega[(4u^2+4v^2)^{1/2}]$; split the resulting even integral up into

$$4h_{pq}\left\{\int_0^{1/q}\int_0^{1/p}+\int_{1/q}^{\pi/2}\int_{1/p}^{\pi/2}+\int_{1/q}^{\pi/2}\int_0^{1/p}+\int_0^{1/q}\int_{1/p}^{\pi/2}\right\},$$

and apply the properties (10), (11), (12), and (13) of $\omega(\delta)$ and the inequalities (8) and (9) to h_{pq} and to the integrals separately, noting particularly that $\omega[(u^2+v^2)^{1/2}]\leqq\omega(u+v)\leqq\omega(u)+\omega(v)$, and that for $u\geqq 1/p$, $2\omega(1/p)/(1/p)\geqq\omega(u)/u$, so that $\omega(u)\leqq 2pu\omega(1/p)$, a similar inequality holding for $\omega(v)$ when $v\geqq 1/q$.

THEOREM III. *Let $f(x, y)$ be a periodic function of period 2π in x and in y separately for which the partial derivatives $f^{i,k-i}(x, y)$, $i=0, 1, \cdots, k$, all exist and are everywhere continuous, and let p and q be two such integers that $2p-2\leqq m\leqq 2p$ and $2q-2\leqq n\leqq 2q$, where m and n are two given positive integers. If the partial derivatives of order k are such that the $k+1$ functions*

$$I_{i,k-i}(x, y) = h_{pq}\int_{-\pi/2}^{\pi/2}\int_{-\pi/2}^{\pi/2} f^{i,k-i}(x + 2u, y + 2v)F_{pq}(u, v)dudv$$

satisfy the inequalities

$$|f^{i,k-i}(x, y) - I_{i,k-i}(x, y)| \leqq \epsilon$$

everywhere, where ϵ is some finite positive constant or zero, then there exists a sum $T(x, y)$ of type (1) *of order at most m in x and n in y such that*

$$|f(x, y) - T(x, y)| \leqq K_1^k \epsilon(1/m + 1/n)^k$$

everywhere.

Let

$$t_1(x, y) = h_{pq} \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} f(x + 2u, y + 2v) F_{pq}(u, v) du dv.$$

Since conditions for differentiating under the integral sign are fulfilled it is immediately apparent that

$$t_1^{i,k-i}(x, y) = I_{i,k-i}(x, y) \qquad (i = 0, 1, \cdots, k).$$

Furthermore, $t_1(x, y)$ is a sum of type (1) of order not exceeding m in x and n in y. Form the function

$$R_1(x, y) = f(x, y) - t_1(x, y).$$

By hypothesis the partial derivatives of order k of R_1 satisfy the relations $|R_1^{i,k-i}(x, y)| \leqq \epsilon$ everywhere, whence by the law of the mean

$$|R_1^{i,j-i}(x_1, y_1) - R_1^{i,j-i}(x_2, y_2)| \leqq \epsilon(|x_1 - x_2| + |y_1 - y_2|),$$

$i=0, 1, \cdots, j=k-1$. Theorem I is now applicable to each of the j functions $R_1^{i,j-i}(x, y)$ with λ replaced by ϵ and the $T_{mn}(x, y)$ replaced by the functions

$$I_{i,j-i}(x, y) = h_{pq} \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} R_1^{i,j-i}(x + 2u, y + 2v) F_{pq}(u, v) du dv$$

corresponding to $R_1^{i,j-i}(x, y)$. The theorem yields the inequalities

$$|R_1^{i,j-1}(x, y) - I_{i,j-i}(x, y)| \leqq K_1 \epsilon(1/m + 1/n).$$

Now let

$$t_2(x, y) = h_{pq} \int_{-\pi/2}^{\pi/2} \int_{-\pi/2}^{\pi/2} R_1(x + 2u, y + 2v) F_{pq}(u, v) du dv$$

and repeat the process begun on $t_1(x, y)$ and $R_1(x, y)$, and thereby construct a sequence of sums $t_1(x, y), t_2(x, y), \cdots, t_k(x, y)$, all of type (1) and of order at most m in x and n in y, and a sequence of functions $R_1(x, y), R_2(x, y), \cdots, R_k(x, y)$ in which $R_k(x, y) = f(x, y) - t_1(x, y) - t_2(x, y) - \cdots - t_k(x, y)$ satisfies the inequality

$$|R_k(x_1, y_1) - R_k(x_2, y_2)| \leqq K_1^{k-1} \epsilon(1/m + 1/n)^{k-1}(|x_1 - x_2| + |y_1 - y_2|.$$

Consequently, by a final application of Theorem I, there exists a sum $t_{k+1}(x, y)$ of type (1) and of order at most m in x and n in y such that

$$\left| R_k(x, y) - t_{k+1}(x, y) \right| \leqq K_1^k \epsilon(1/m + 1/n)^k$$

everywhere. In this inequality let $t_1(x, y)+t_2(x, y)+ \cdots +t_{k+1}(x, y)$ be denoted by $T(x, y)$, a trigonometric sum of type (1) of order at most m in x and n in y, and the theorem is proved.

If, now, the kth-order partial derivatives of $f(x, y)$ are continuous with uniform modulus of continuity $\Omega(\delta)$, then by Theorem II each of these derivatives can be approximated by a sum of type (1) given precisely by the $I_{i,k-i}(x, y)$ of Theorem III, and ϵ will have the value $K_2\Omega(1/m+1/n)$; if these derivatives satisfy a Lipschitz condition with parameter Λ, ϵ will have the form $K\Lambda(1/m+1/n)$. Consequently the following theorem is true:

THEOREM IV. *If $f(x, y)$, periodic of period 2π in x and in y separately, is such that its kth-order partial derivatives are everywhere continuous with uniform modulus of continuity $\Omega(\delta)$, then corresponding to every pair of positive integers m and n there exists a trigonometric sum $T_{mn}(x, y)$ of type* (1) *such that*

$$\left| f(x, y) - T_{mn}(x, y) \right| \leqq K_2 K_1^k (1/m + 1/n)^k \Omega(1/m + 1/n)$$

everywhere, and if these derivatives satisfy a Lipschitz condition with parameter Λ the right-hand side of the inequality becomes $K_1{}^k K\Lambda(1/m+1/n)^{k+1}$.

5. **Degree of convergence of polynomial approximation.** Polynomial approximations to a function $f(x, y)$ are effected by obtaining trigonometric approximations to the transformed function

$$F(\theta, \phi) = f(\cos \theta, \cos \phi)$$

from the theorems of the preceding article, using the relations of Lemmas I and II and the conclusion of the following lemma.*

LEMMA V. *If $F(\theta, \phi)$ is an even function of θ and ϕ separately and if there exists a sum $T(\theta, \phi)$ of type* (1) *such that*

$$\left| F(\theta, \phi) - T(\theta, \phi) \right| \leqq \epsilon$$

everywhere, then there exists a sum $t(\theta, \phi)$ of the same type, of order not higher than that of $T(\theta, \phi)$, and devoid of sines of multiples of either θ or ϕ, such that

$$\left| F(\theta, \phi) - t(\theta, \phi) \right| \leqq \epsilon$$

everywhere.

On account of the evenness of $F(\theta, \phi)$ and the inequality in the hypothesis of the lemma,

* Cf. Jackson, *Colloquium*, p. 13.

$$\begin{aligned}
&|F(\theta,\phi) - \tfrac{1}{4}\{T(\theta,\phi) + T(\theta,-\phi) + T(-\theta,\phi) + T(-\theta,-\phi)\}| \\
&\quad \leqq \tfrac{1}{4}\{|F(\theta,\phi) - T(\theta,\phi)| + |F(\theta,-\phi) - T(\theta,-\phi)| \\
&\qquad + |F(-\theta,\phi) - T(-\theta,\phi)| + |F(-\theta,-\phi) - T(-\theta,-\phi)|\} \\
&\quad \leqq \tfrac{1}{4}4\epsilon = \epsilon.
\end{aligned}$$

But the sum in braces on the left is $4T(\theta,\phi)$ with all the terms containing sines removed; hence the lemma is true.

Proofs of theorems paralleling those of §4 all follow the same scheme. Of these theorems the one based on Theorem IV is the most general; it is sufficiently typical to warrant omitting the others.

THEOREM V. *Let $f(x, y)$ together with its partial derivatives of order $k \geqq 1$ and lower be continuous in the square region $-1 \leqq x, y \leqq 1$; let M be the maximum of the absolute values of these derivatives in the region; let $\Omega(\delta) \not\equiv 0$ be the uniform modulus of continuity of the derivatives of order k. Then corresponding to every pair of positive integers m and n there exists a polynomial $P_{mn}(x, y)$ of type (2) such that*

$$|f(x, y) - P_{mn}(x, y)| \leqq K_3 N(1/m + 1/n)^k \Omega(1/m + 1/n)$$

*throughout the region, where $K_3 = K_2 K_1^k k!(e^{e-1} - 1)$ and N is the larger of unity and $8M/\Omega(2^{3/2})$.**

Under the hypotheses of the theorem and on account of Lemma I, $F(\theta, \phi) = f(\cos\theta, \cos\phi)$ is a periodic function of period 2π in θ and in ϕ separately having kth-order partial derivatives everywhere continuous with a uniform modulus of continuity which does not exceed $Nk!(e^{e-1} - 1)\,\Omega(\delta)$. By Theorem IV there exists a sum of type (1) which everywhere approximates $F(\theta, \phi)$ within an error nowhere exceeding that assigned for $P_{mn}(x, y)$ in Theorem V. Since $F(\theta, \phi)$ is even in θ and in ϕ separately, there exists by Lemma V a sum $T_{mn}(\theta, \phi)$ of the same type containing no sines of either θ or ϕ and giving at least as good an approximation; this $T_{mn}(\theta, \phi)$ is a polynomial in $\cos\theta$ and $\cos\phi$ of degree not exceeding m in $\cos\theta$ and n in $\cos\phi$: $T_{mn}(\theta, \phi) = P_{mn}(\cos\theta, \cos\phi)$. Consequently, $f(x, y)$ is approximated by $P_{mn}(x, y)$ within an error not exceeding that permitted in the theorem. If $\Omega(\delta) \equiv 0$ then $f(x, y)$ is itself a polynomial of degree at most k, in which case the above theorem does not, and need not, apply.

6. **Degree of convergence of mixed approximation.** In case $f(x, y)$ is periodic in one variable only and satisfies conditions of continuity in an infinite strip of finite width, methods analogous to those of §5 lead to approximations

* For a theorem on polynomial approximation related to this one, but neither containing it nor contained in it, see P. Montel, *Sur les polynomes d'approximation*, Bulletin de la Société Mathématique de France, vol. 46 (1918), pp. 151–192; p. 191.

in the form of sums which are trigonometric in that variable and polynomial in the other: mixed sums of type (3). As in §5 one representative theorem will suffice.

THEOREM VI. *Let $f(x, y)$ be a periodic function of period 2π in x alone which together with its partial derivatives of order $k \geqq 1$ and lower is continuous in the region $-\infty < x < \infty$, $-1 \leqq y \leqq 1$; let M be the maximum of the absolute values of the partial derivatives in the region; let $\Omega(\delta)$ be the uniform modulus of continuity of the partial derivatives of order k; let $\Omega(d) \neq 0$, where $d^2 = \pi^2 + 4$. Then corresponding to every pair of positive integers m and n there exists a mixed sum $H_{mn}(x, y)$ of type* (3) *such that*

$$| f(x, y) - H_{mn}(x, y) | \leqq K_3 N''(1/m + 1/n)^k \Omega(1/m + 1/n)$$

throughout the region, where K_3 is the same as in Theorem V *and N'' is the larger of unity and $2^{3/2}Md/\Omega(d)$.*

Let $F(\theta, \phi) = f(\theta, \cos\phi)$. By Lemma III, $F(\theta, \phi)$ is a periodic function of period 2π in θ and in ϕ separately, which has everywhere continuous partial derivatives of order $k \geqq 1$ whose uniform modulus of continuity does not exceed $N''k!(e^{e-1} - 1)\,\Omega(\delta)$. Hence by Theorem IV there exists a sum of type (1) which approximates $F(\theta, \phi)$ within the error assigned in the above theorem. Since $F(\theta, \phi)$ is an even function in ϕ, this can be replaced by a sum $T_{mn}(\theta, \phi)$ containing no sines of ϕ and, consequently, is expressible as a polynomial of degree not exceeding n in $\cos\phi$: a mixed sum of type (3), $T_{mn}(\theta, \phi) = H_{mn}(\theta, \cos\phi)$. The proof is an appropriate simplification of that of Lemma V. Therefore, $f(x, y)$ is approximated by $H_{mn}(x, y)$ within the error given. If $\Omega(d) = 0$, $f(x, y)$ does not contain x and is a polynomial of degree at most k in y. As in §5 the above theorem does not, and need not, apply in this case.

Upon examination it will be noticed that, except for the absolute constants involved, the inclusive theorems on the approximation to a function having continuous partial derivatives of order k (Theorems IV, V, and VI) reduce to the simpler theorems for an ordinary continuous function by placing k equal to zero.

II. APPROXIMATION BY SUMS OF SURFACE SPHERICAL HARMONICS

7. **Degree of convergence of Laplace's series.** Let θ and ϕ be the co-latitude and longitude, respectively, of a point on the surface of a sphere of unit radius, and let $F(\theta, \phi)$ be a real, single-valued, integrable point function on the sphere. The partial sum of degree n of the expansion of $F(\theta, \phi)$ in Laplace's series* is

* See, e.g., Byerly, *Fourier's Series and Spherical Harmonics*, Boston, Ginn and Co., 1895, p. 211.

$$(17) \qquad S_n(\theta, \phi) = \sum_{i=0}^{n} \frac{2i+1}{4\pi} \int_0^{2\pi} \int_0^{\pi} F(\theta', \phi') P_i(\cos\gamma)(\sin\theta') d\theta' d\phi',$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')$ and P_i is Legendre's polynomial (the Legendrian) of degree i.

By means of an existence theorem (cf. §8 below) Gronwall* gave an elegant proof concerning the degree of convergence of $S_n(\theta, \phi)$ to $F(\theta, \phi)$. His method, however, does not seem to permit extension to functions having continuous derivatives beyond those of the first order. The attack below through the medium of the mean-value function used by Dirichlet†, Darboux‡, and others, leads readily to an extended theorem. The success of the attack is due largely to the two accompanying lemmas.

LEMMA VI. *Let* $\sigma_n(x) = (1/2)\sum_0^n (2i+1) P_i(x)$. *If* $g(x)$ *is an integrable function such that* $|g(x)| \leqq G$ *throughout the region* $-1 \leqq x \leqq 1$, *then*

$$\left| \int_{-1}^{1} g(x)\sigma_n(x)dx \right| \leqq c'Gn^{1/2}, \ n \geqq 1,$$

where c' *is an absolute constant.*

LEMMA VII. *If* $p_n(x)$ *is a polynomial of degree at most* n *in* x *then*

$$\int_{-1}^{1} p_n(x)\sigma_n(x)dx = p_n(1)$$

for all positive integral values of n.

The first of these lemmas is an adaptation of the fact that

$$\lim_{n\to\infty} \left\{ (1/n)^{1/2} \int_{-1}^{1} |\sigma_n(x)| dx \right\} = 2(2/\pi)^{1/2},$$

a fact proved by Gronwall.§ The second follows on substituting the identity¶

$$\sigma_n(x) = \tfrac{1}{2}(d/dx)[P_{n+1}(x) + P_n(x)]$$

in the integrand and integrating by parts. Thus

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* T. H. Gronwall, *On the degree of convergence of Laplace's series*, these Transactions, vol. 15 (1914), pp. 1–30; see pp. 14–23.

† Dirichlet, *Sur les séries dont le terme général dépend de deux angles* · · · , Journal für Mathematik, vol. 17 (1837), pp. 35–56.

‡ Darboux, same title as the preceding, Journal de Mathématiques, (2), vol. 19 (1874), pp. 1–18.

§ Gronwall, loc. cit., pp. 3–14; also *Über die Laplacesche Reihe*, Mathematische Annalen, vol. 74 (1913), pp. 213–270; see pp. 222–230.

¶ See, for example, Byerly, op. cit., p. 180.

$$\int_{-1}^{1} p_n(x)\sigma_n(x)dx = \tfrac{1}{2}[p_n(x)\{P_{n+1}(x) + P_n(x)\}]_{-1}^{1}$$
$$- \tfrac{1}{2}\int_{-1}^{1} p_n'(x)[P_{n+1}(x) + P_n(x)]dx.$$

The value of the first term on the right is $p_n(1)$. Since $p'_n(x)$ is of lower degree than either $P_{n+1}(x)$ or $P_n(x)$, it is orthogonal to each and, consequently, to their sum; hence the second term on the right is zero, and the lemma is proved.

Let the system of coördinates be rotated to place the pole at the point (θ, ϕ); let the principal meridian be any fixed great circle through this pole; let x and y be the new geographic coördinates. Then the pole (θ, ϕ) transforms into the point $(0, y)$; $\cos \gamma$ into $\cos x$; $S_n(\theta, \phi)$ into a constant, say $s_n(0)$; $F(\theta', \phi')$ into a new function $f(x, y)$; $F(\theta, \phi)$ into $f(0, y)$. Consequently, (17) becomes

$$S_n(\theta, \phi) = s_n(0) = \sum_0^n \left\{\frac{2i+1}{4\pi}\int_0^{2\pi}\int_0^{\pi} f(x, y)P_i(\cos x)(\sin x)dxdy\right\}$$
$$= \int_0^{\pi}\left[\sum_0^n \frac{2i+1}{2} P_i(\cos x)\right]\left[\frac{1}{2\pi}\int_0^{2\pi} f(x, y)dy\right](\sin x)dx$$
$$= \int_0^{\pi} \Phi(x; \theta, \phi)\sigma_n(\cos x)(\sin x)dx,$$

where $\Phi(x; \theta, \phi) = (1/(2\pi))\int_0^{2\pi} f(x, y)dy$ is the *mean value* of F on a circle of curved radius (polar distance) x with center (pole) at the point (θ, ϕ).

With the aid of the last two lemmas the following theorem is established.

THEOREM VII. *Let $F(\theta, \phi)$ be a real, single-valued, integrable point function on the unit sphere, and let $\Phi(x; \theta, \phi)$ be the mean value of F on a circle of curved radius x with center at (θ, ϕ). If $F(\theta, \phi)$ is of such a nature that corresponding to a positive constant ϵ_n there exists a polynomial $p_n(\cos x)$ of degree at most n in $\cos x$ satisfying the inequality*

$$| \Phi(x; \theta, \phi) - p_n(\cos x) | \leqq \epsilon_n$$

for all values of x, then

$$| S_n(\theta, \phi) - \Phi(0; \theta, \phi) | \leqq c\epsilon_n n^{1/2}$$

for all positive integral values of n, where c is an absolute constant.

Let $\Phi(x; \theta, \phi) - p_n(\cos x)$ be denoted by $g(x)$. By hypothesis $|g(x)| \leqq \epsilon_n$; therefore, by Lemma VI,

$$\left|\int_0^\pi g(x)\sigma_n(\cos x)(\sin x)dx\right| \leqq \epsilon_n \int_{-1}^1 |\sigma_n(x)|\, dx \leqq c'\epsilon_n n^{1/2}.$$

In other words,

$$\left|\int_0^\pi \Phi(x;\theta,\phi)\sigma_n(\cos x)(\sin x)dx - \int_0^\pi p_n(\cos x)\sigma_n(\cos x)\,(\sin x)dx\right|$$
$$\leqq c'\epsilon_n n^{1/2}.$$

Here the first integral is $s_n(0)$ and the second, by virtue of Lemma VII, is $p_n(\cos 0)$. The inequality, then, takes the form

$$|\, s_n(0) - p_n(\cos 0)\,| \leqq c'\epsilon_n n^{1/2}.$$

It was assumed in the hypothesis, however, that

$$|\,\Phi(0;\theta,\phi) - p_n(\cos 0)\,| \leqq \epsilon_n.$$

By combining these last two inequalities the following inequality is obtained:

$$|\,\Phi(0;\theta,\phi) - s_n(0)\,| \leqq \epsilon_n(1 + c'n^{1/2}) \leqq c\epsilon_n n^{1/2}.$$

But $s_n(0) = S_n(\theta, \phi)$, and the theorem is proved.

Suppose now that $F(\theta, \phi)$ is continuous on the surface of the sphere with modulus of continuity $\omega(\delta)$; i.e.,

$$|\,F(\theta_1,\phi_1) - F(\theta_2,\phi_2)\,| \leqq \omega(\delta)$$

for all points for which $\Gamma \leqq \delta$, where Γ is the shorter great-circle distance between the points. If $\omega(\delta)$ does not exceed $\lambda\delta$, where λ is a positive constant, $F(\theta, \phi)$ will be said to satisfy a Lipschitz condition with parameter λ. Since each point (x, y) on the sphere can be thought of as having infinitely many alternative pairs of coördinates, $(x+2\mu\pi,\ y+2\nu\pi)$, $(-x+2\mu\pi,\ y+(2\nu+1)\pi)$, where μ and ν are arbitrary integers, positive, negative, or zero, $f(x, y)$, considered as a point function on the sphere, is periodic of period 2π in x and y separately and $\Phi(x; \theta, \phi)$ is a periodic even function of x of period 2π having the same modulus of continuity, $\omega(\delta)$. For a fixed pole, then, it can be inferred from a well known theorem* and from the analogue of Lemma V for functions of a single variable that corresponding to every positive integer n there exists a trigonometric sum $T_n(x)$ *containing only cosine terms*, of order at most n in x, such that

$$|\,\Phi(x;\theta,\phi) - T_n(x)\,| \leqq K'\omega(2\pi/n) \tag{18}$$

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* See, e.g., Jackson, *Colloquium*, p. 7; *On the approximate representation of an indefinite integral* · · · , these Transactions, vol. 14 (1913), pp. 343–364; p. 350.

where K' is an absolute constant. (If $F(\theta, \phi)$ satisfies a Lipschitz condition the absolute error in (18) does not exceed $K''\lambda/n$, where K'' is an absolute constant.) Since $T_n(x)$ contains only cosine terms it is a polynomial $p_n(\cos x)$ of degree at most n in $\cos x$.

The selection of $T_n(x) = p_n(\cos x)$ depends on the choice of the pole, but no matter what point is chosen for pole the accompanying polynomial in $\cos x$ satisfies (18). The hypotheses of Theorem VII are fulfilled at every point on the sphere. On account of the continuity of $F(\theta, \phi)$, evidently $\Phi(0; \theta, \phi) = F(\theta, \phi)$. Hence the theorem stated below is true.

THEOREM VIII. *If $F(\theta, \phi)$ is continuous with modulus of continuity $\omega(\delta)$ on the surface of the sphere, then*

$$|F(\theta, \phi) - S_n(\theta, \phi)| \leqq cK'\omega(2\pi/n)n^{1/2}, \quad n > 0,$$

for all points on the sphere, where c and K' are absolute constants. If $F(\theta, \phi)$ satisfies a Lipschitz condition with parameter λ, the absolute error does not exceed $cK''\lambda/n^{1/2}$, where K'' is an absolute constant.

COROLLARY I. *If $F(\theta, \phi)$ has a modulus of continuity $\omega(\delta)$ such that $\lim_{\delta\to 0} \omega(\delta)/\delta^{1/2} = 0$, Laplace's series converges uniformly to $F(\theta, \phi)$ over the surface of the sphere.*

The theorem above, which is substantially the same as that of Gronwall, permits the following extension:

THEOREM IX. *Let $F(\theta, \phi)$ be continuous and such that the kth-order derivatives $(\partial^k/\partial s^k)F(\theta, \phi)$ with respect to arc-length exist on every great circle on the sphere with moduli of continuity not exceeding a common upper bound $\omega(\delta)$ such that $\lim_{\delta\to 0}\omega(\delta) = 0$. Then the partial sum of order n, $S_n(\theta, \phi)$, of Laplace's series for $F(\theta, \phi)$, satisfies the inequality*

$$|F(\theta, \phi) - S_n(\theta, \phi)| \leqq A\omega(2\pi/n)(1/n)^k n^{1/2}, \quad n > 0,$$

where A is an absolute constant.

Under the hypotheses it follows on differentiating under the integral sign that $(\partial^k/\partial x^k)\Phi(x; \theta, \phi)$ is a continuous function of x with modulus of continuity not exceeding $\omega(\delta)$. Under these circumstances $\Phi(x; \theta, \phi)$ satisfies the conditions of a theorem* which in substance states that there exists a polynomial $p_n(\cos x)$ which approximates Φ with an absolute error not exceeding $A'\omega(2\pi/n)\ (1/n)^k$, where A' is an absolute constant. Consequently, by the argument used in the proof of Theorem VIII, the present theorem holds.

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* Jackson, *Colloquium*, p. 12.

8. **An existence theorem.** The partial sum of Laplace's series is a special form of the general sum of degree n of surface spherical harmonics*:

$$(19)\qquad Y_n(\theta, \phi) = \sum_{i=0}^{n} \sum_{j=0}^{i} \{ [A_{ij} \cos j\phi + B_{ij} \sin j\phi] P_i^j(\cos\theta) \} ;$$

the A's and B's are real constants and

$$P_i^j(\cos\theta) = (\sin j\theta)\ (d/d \cos\theta)^j P_i(\cos\theta)$$

is the associated function of order j and degree i.

Gronwall† has shown that there exists a sum of the form (19) which under certain conditions approximates a given function more closely than does the partial sum of Laplace's series for the function. (Cf. (4), (5), and (6) of §2.) Let

$$(20)\qquad T_n(\theta, \phi) = h_p \int_0^{2\pi} \int_0^{\pi} F(\theta', \phi') g_p(\gamma)(\sin\theta') d\theta' d\phi',$$

where γ is the great-circle distance between (θ', ϕ') and (θ, ϕ).

$$g_p(\gamma) = [(\sin(p\gamma/2))/(\sin(\gamma/2))]^4,$$

and

$$1/h_p = \int_0^{2\pi} \int_0^{\pi} g_p(\gamma)(\sin\theta') d\theta' d\phi'.$$

That $T_n(\theta, \phi)$, thus defined by Gronwall, is of the form (19) is established from the following facts: $[(\sin(p\gamma/2))/(\sin(\gamma/2))]^2 = (1-\cos p\gamma)/(1-\cos\gamma)$ is a cosine sum of order $p-1$, so that $g_p(\gamma)$ is such a sum of order $2p-2$; g_p is therefore a polynomial of degree $2p-2$ in $\cos\gamma$ and, consequently, is expressible as a linear combination of Legendrians in $\cos\gamma$, $\sum_0^{2p-2} a_i P_i(\cos\gamma)$; since a Legendrian, $P_i(\cos\gamma)$, is a surface spherical harmonic of degree i, it follows that $T_n(\theta, \phi)$ is a surface spherical harmonic sum of degree not exceeding n when p is an integer such that $2p-2 \leqq n \leqq 2p$. Gronwall proved that if $F(\theta, \phi)$ has a modulus of continuity $\omega(\delta)$ on the sphere, $T_n(\theta, \phi)$ satisfies the inequality

$$| F(\theta, \phi) - T_n(\theta, \phi) | \leqq B'\omega(1/n)$$

for all points on the sphere, where B' is an absolute constant. This theorem can be extended to include a function having continuous directional derivatives.

* Byerly, op. cit., p. 197.

† Gronwall, these Transactions, loc. cit., pp. 14–23.

THEOREM X. *If $F(\theta, \phi)$ has at every point of the sphere continuous first-order directional derivatives as described in Theorem* IX *with moduli of continuity not exceeding $\omega(\delta)$, where $\lim_{\delta\to 0}\omega(\delta)=0$, then corresponding to every positive integer n there exists a sum $T_n(\theta, \phi)$ of form* (19), *of degree at most n, such that*

$$|F(\theta, \phi) - T_n(\theta, \phi)| \leqq B\omega(1/n)(1/n)$$

for all points on the sphere, where B is an absolute constant.

Let (θ, ϕ) be chosen for a new pole of coördinates; let $\Phi(x; \theta, \phi)$ be the mean value of F on a circle of curved radius x with center at (θ, ϕ). Let the definition of $T_n(\theta, \phi)$ given by (20) be subjected to the following modifications: let p be an integer such that $3p-3\leqq n\leqq 3p$, let

$$g_p(\gamma) = [(\sin\ (p\gamma/2))/(\sin\ (\gamma/2))]^6,$$

and let a factor 2π be divided out of the corresponding integral defining $1/h_p$. From the remarks accompanying the definition (20) it is apparent that $T_n(\theta, \phi)$ thus modified is also a surface spherical harmonic sum of degree not exceeding n. At the pole (θ, ϕ), then,

$$T_n(\theta, \phi) = h_p \int_0^\pi \Phi(x; \theta, \phi) g_p(x)(\sin x)dx,$$

with

$$1/h_p = \int_0^\pi g_p(x)(\sin x)dx.$$

From this last equation evidently

$$\Phi(0; \theta, \phi) = h_p \int_0^\pi \Phi(0; \theta, \phi) g_p(x)(\sin x)dx.$$

Consequently

$$T_n(\theta, \phi) - \Phi(0; \theta, \phi) = h_p \int_0^\pi [\Phi(x; \theta, \phi) - \Phi(0; \theta, \phi)] g_p(x)(\sin x)dx;$$

whence by the law of the mean, since $\Phi_x(x; \theta, \phi)$ is continuous in x,

$$T_n(\theta, \phi) - \Phi(0; \theta, \phi) = h_p \int_0^\pi x\Phi_x(qx; \theta, \phi) g_p(x)(\sin x)dx,$$

where $0<q<1$. Inasmuch as $\Phi(x; \theta, \phi)$ is an even function possessing a continuous derivative with respect to x it follows that $\Phi_x(0; \theta, \phi)=0$. Therefore

$$|\Phi_x(qx;\theta,\phi)| = |\Phi_x(qx;\theta,\phi) - \Phi_x(0;\theta,\phi)|$$
$$\leqq \omega(qx) \leqq \omega(x).$$

Since, also, $\Phi(0;\theta,\phi)=F(\theta,\phi)$,

$$|T_n(\theta,\phi) - F(\theta,\phi)| \leqq h_p \int_0^\pi x\omega(x)g_p(x)(\sin x)dx.$$

By a method analogous to that suggested in the outline of the proof of Theorem II in §4, the right-hand side of this inequality does not exceed $B \cdot\omega(1/n)\,(1/n)$. Since the hypotheses are assumed to hold at every point (θ,ϕ) of the sphere so also does the conclusion.

9. **A problem of closest approximation in terms of surface spherical harmonics.** As in other cases of approximation by means of orthogonal functions, it is easily demonstrated that the particular choice of the coefficients A_{ij} and B_{ij} in the general surface spherical harmonic sum $Y_n(\theta,\phi)$ which minimizes the integral

$$\int_0^{2\pi}\int_0^\pi [F(\theta,\phi) - Y_n(\theta,\phi)]^2(\sin\theta)d\theta d\phi,$$

where $F(\theta,\phi)$ is a given continuous function, is that choice which yields the partial sum $S_n(\theta,\phi)$ of Laplace's series for $F(\theta,\phi)$; there is one and only one choice of the coefficients which produces a minimum value of the integral.

As in the case of polynomial and of ordinary trigonometric approximation the above problem can be generalized into a problem of minimizing the integral of a power other than the square of the absolute discrepancy. Jackson* has given a general existence theorem which shows in particular that if $p_1(x), p_2(x), \cdots, p_k(x)$ is any set of k linearly independent continuous functions of x in an interval $a \leqq x \leqq b$ and $f(x)$ is continuous in this interval, then there exists one and only one choice of the coefficients in the linear combination $\sum_1^k c_i p_i(x)$ which minimizes the integral

$$\int_a^b \left| f(x) - \sum_1^k c_i p_i(x)\right|^m dx,$$

where m is any number greater than unity. By suitable adaptation the same method yields a proof of

* Jackson, *A generalized problem in weighted approximation*, these Transactions, vol. 26 (1924), pp. 133–154; see pp. 133–138.

THEOREM XI. *Let $F(\theta, \phi)$ be a continuous single-valued function on the unit sphere, and let $Y_n(\theta, \phi)$ be a general surface spherical harmonic sum of degree n. There exists one and only one choice of the coefficients A_{ij} and B_{ij} which will render the integral*

$$\int_0^{2\pi}\int_0^{\pi} |F(\theta, \phi) - Y_n(\theta, \phi)|^m(\sin\theta)d\theta d\phi$$

a minimum when $m>1$. Such a sum $Y_n(\theta, \phi)$ is called the approximating sum for $F(\theta, \phi)$ corresponding to exponent m.

The proof of existence, apart from the question of uniqueness, holds also for $0<m\leqq 1$.

10. **Convergence of the approximating sum.** To begin with, it is to be observed that $Y_n(\theta, \phi)$ is a trigonometric sum of order n of type (1), §1. Hence Bernstein's theorem is applicable: if $|Y_n(\theta, \phi)|\leqq L$ over the entire sphere, then also $|(\partial/\partial\theta)Y_n(\theta, \phi)|\leqq nL$ and $|(\partial/\partial\phi)Y_n(\theta, \phi)|\leqq nL$ over the entire sphere. Since by an arbitrary rotation of coördinates which places the pole at the point (θ, ϕ), Y_n is transformed into another sum of the same character, the statement $|(\partial/\partial\theta)Y_n|\leqq nL$ can be given the more general form $|(\partial/\partial s)Y_n|\leqq nL$, where s is along any great circle through the point (θ, ϕ). With this observation a device used in connection with other problems* becomes available for finding conditions on the function $F(\theta, \phi)$ sufficient to insure uniform convergence of its approximating sum.

Let $F(\theta, \phi)$ be continuous on the sphere; for fixed n let $Y_n(\theta, \phi)$ be its approximating sum corresponding to exponent m; let γ_n be the minimum attained by the integral

$$\int_0^{2\pi}\int_0^{\pi} |F(\theta, \phi) - Y_n(\theta, \phi)|^m(\sin\theta)d\theta d\phi. \tag{21}$$

Suppose that there exists another sum $y_n(\theta, \phi)$ of the same type (19) such that

$$|F(\theta, \phi) - y_n(\theta, \phi)| \leqq \epsilon_n$$

everywhere on the sphere, where ϵ_n depends only on n. Place

$$\rho_n(\theta, \phi) = Y_n(\theta, \phi) - y_n(\theta, \phi)$$

and

$$r_n(\theta, \phi) = F(\theta, \phi) - y_n(\theta, \phi).$$

* See, e.g., Jackson, *On the convergence of certain trigonometric and polynomial approximations*, these Transactions, vol. 22 (1921), pp. 158–166; *Colloquium*, Chapter III. For an application of the method to a problem of approximation in two variables, see E. Carlson, *On the convergence of trigonometric approximations for a function of two variables*, Bulletin of the American Mathematical Society, vol. 32 (1926), pp. 639–641.

Now, ρ_n is continuous; let μ_n be the maximum of its absolute value and (θ_0, ϕ_0) a point at which it is attained: $\mu_n = |\rho_n(\theta_0, \phi_0)|$. Let the point (θ, ϕ) be restricted to a circle R of curved radius $1/(2n)$ with center at (θ_0, ϕ_0). If (θ, ϕ) is distinct from (θ_0, ϕ_0), ρ_n is a continuous function of s with continuous derivatives with respect to s along the great circle joining the points. By the law of the mean,

$$|\rho_n(\theta, \phi) - \rho_n(\theta_0, \phi_0)| = |\partial\rho_n'/\partial s|\, s$$

where ρ_n' is a value of ρ_n at a point on the arc between the given points. Consequently, whether (θ, ϕ) is distinct from (θ_0, ϕ_0) or not,

$$|\rho_n(\theta, \phi) - \rho_n(\theta_0, \phi_0)| \leqq n\mu_n/(2n) = \mu_n/2.$$

Therefore, $|\rho_n(\theta, \phi)| \geqq \mu_n/2$ for all points in R.

For the moment let it be supposed that $\epsilon_n \leqq \mu_n/4$, the contrary case being considered presently. Then, since $|r_n(\theta, \phi)| \leqq \epsilon_n \leqq \mu_n/4$ everywhere and $|\rho_n(\theta, \phi)| \geqq \mu_n/2$ in R, it follows that $|r_n - \rho_n| \geqq \mu_n/4$ in R, and, consequently, that

$$\begin{aligned}\int_0^{2\pi}\int_0^{\pi} &|F(\theta, \phi) - Y_n(\theta, \phi)|^m(\sin\theta)d\theta d\phi \\ &= \int_0^{2\pi}\int_0^{\pi} |r_n - \rho_n|^m(\sin\theta)d\theta d\phi \\ &\geqq (\mu_n/4)^m \int_R(\sin\theta)d\theta d\phi = (\mu_n/4)^m 2\pi\int_0^{1/(2n)}(\sin\theta)d\theta \\ &= (\mu_n/4)^m 4\pi \sin^2[1/(4n)].\end{aligned}$$

Therefore, $\gamma_n \geqq 4\pi(\mu_n/4)^m \sin^2[1/(4n)]$. But, since $Y_n(\theta, \phi)$ minimizes the integral (21), $\gamma_n \leqq 4\pi\epsilon_n^m$, and it follows that

$$\mu_n \leqq 4\{\gamma_n/(4\pi \sin^2[1/(4n)])\}^{1/m} \leqq 4\{\sin^2[1/(4n)]\}^{-1/m}\epsilon_n.$$

In the contrary case $\epsilon_n > \mu_n/4$, certainly $\mu_n < 4\epsilon_n$, so that in either case

$$\mu_n \leqq 4\{\sin^2[1/(4n)]\}^{-1/m}\epsilon_n + 4\epsilon_n. \tag{22}$$

Since $\sin x > (2/\pi)x$ when $0 < x \leqq \pi/2$, $\sin^2[1/(4n)] \geqq n^{-2}/(4\pi^2)$. The upper bound (22) for μ_n then assumes the form

$$\mu_n \leqq Cn^{2/m}\epsilon_n + 4\epsilon_n,$$

where C depends only on m, a constant.

Finally, then,

$$|F(\theta, \phi) - Y_n(\theta, \phi)| \leqq |r_n| + |\rho_n| \leqq Cn^{2/m}\epsilon_n + 5\epsilon_n,$$

and one can state the following theorem:

THEOREM XII. *If $F(\theta, \phi)$ can be approximated by a surface spherical harmonic sum $y_n(\theta, \phi)$ of type* (19) *with maximum absolute error ϵ_n then the approximating sum $Y_n(\theta, \phi)$ corresponding to exponent $m>1$ represents $F(\theta, \phi)$ with maximum absolute error not exceeding*

$$Cn^{2/m}\epsilon_n + 5\epsilon_n$$

where C depends only on m.

Theorems on the convergence of $Y_n(\theta, \phi)$ to $F(\theta, \phi)$ can now be written as corollaries to Theorem XII. By Gronwall's theorem of §8, ϵ_n may be replaced by a constant multiple of $\omega(1/n)$ if $F(\theta, \phi)$ is continuous with modulus of continuity $\omega(\delta)$ on the sphere. The immediate consequence is

COROLLARY I. *If $m>2$ and if $F(\theta, \phi)$ is such that* $\lim_{\delta\to 0}\omega(\delta)/\delta^{2/m}=0$, *then $Y_n(\theta, \phi)$ converges uniformly to $F(\theta, \phi)$.*

If $F(\theta, \phi)$ satisfies the hypotheses of Theorem X, ϵ_n may be replaced by a constant multiple of $(1/n)\ \omega(1/n)$, and one can state

COROLLARY II. *If $m>1$ and if $F(\theta, \phi)$ has at all points of the sphere continuous first-order directional derivatives as in Theorem* X, *with modulus of continuity not exceeding $\omega(\delta)$, where* $\lim_{\delta\to 0}\omega(\delta)/\delta^{(2/m)-1}=0$, *then $Y_n(\theta, \phi)$ converges uniformly to $F(\theta, \phi)$.*

As in the case of trigonometric and polynomial approximation it is to be emphasized that the conditions imposed on $F(\theta, \phi)$ in the above corollaries are by no means necessary for the uniform convergence to $F(\theta, \phi)$ of the approximating sum corresponding to exponent m. The following observation will suffice to bring out this fact. In §9 it was pointed out that the partial sum of Laplace's series is the approximating sum corresponding to exponent $m=2$. It has already been shown in the corollary to Theorem VIII that a sufficient condition for the uniform convergence of this sum to $F(\theta, \phi)$ is that $F(\theta, \phi)$ have a modulus of continuity such that $\omega(\delta)/\delta^{1/2}\to 0$. This is a less restrictive condition on $F(\theta, \phi)$ than that afforded by Corollary II of Theorem XII. Also as in the case of other forms of approximating functions the problem treated here can be generalized by admitting a positive, continuous weight function in the integrand of (21).

Since methods are not available for finding ϵ_n by arbitrary sums other than those used in the above corollaries, the values of ϵ_n must be taken as the errors assigned in Theorems VIII and IX for representation by partial sums of Laplace's series. Conclusions arrived at by such considerations appear to be of secondary interest, and will not be included in this discussion. Fur-

thermore, since theorems on convergence for $m \leqq 1$ which employ such values of ϵ_n, and theorems on the degree of convergence for all positive values of m, are adaptations of the corresponding existing theorems for trigonometric approximation paralleling the adaptation herein given for the case $m > 1$, they, also, will not be included.

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A GEOMETRIC THEORY OF SOLUTION OF LINEAR INEQUALITIES*

BY

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1. **Introduction.** The general theory of solution of linear inequalities has been treated previously by at least two authors, and particular aspects of the subject have been discussed by others.

Minkowski's† treatment is built around the theorem that the general solution of a set of linear homogeneous inequalities of the form

$$(1.1)\qquad \sum_{i=1}^{n} \lambda^{i} x_{i}^{j} \geqq 0 \qquad (j = 1, \cdots, N),$$

in which the λ's are the unknowns, is a linear homogeneous combination of a fundamental set of particular solutions. Because of this, his theory has a marked analogy with the theory of linear equations. His treatment of (1.1) is theoretically complete, and his ideas are of such a fundamental nature that they will necessarily appear in some guise in any thorough discussion of the subject.

Dines‡ gives a necessary and sufficient condition that a system of the form

$$(1.2)\qquad \sum_{i=1}^{n} \lambda^{i} x_{i}^{j} > 0$$

have a solution, and a criterion for the degree of arbitrariness of the general solution. These are stated in terms of what he calls the I-rank of the system, this quantity being defined in terms of a sequence of matrices formed from

* Presented to the Society, April 3, 1931; received by the editors in March, 1931; presented, in substance, as a Doctor's dissertation at Duke University.

† H. Minkowski, *Geometrie der Zahlen*, Leipzig, 1910, pp. 40–45.

‡ L. L. Dines, *Systems of linear inequalities*, Annals of Mathematics, (2), vol. 20 (1918–19), pp. 191–199.

Other articles by this author on related subjects are:

Definite linear dependence, Annals of Mathematics, (2), vol. 27 (1926), pp. 57–64.

Note on certain associated systems of linear equalities and inequalities, Annals of Mathematics, (2), vol. 28 (1926–27), pp. 41–42.

On positive solutions of a system of linear equations, Annals of Mathematics, (2), vol. 28 (1926–27), pp. 386–392.

Linear inequalities and some related properties of functions, Bulletin of the American Mathematical Society, vol. 36 (1930), pp. 393–405.

the matrix of the coefficients. It is in this particular that his theory has its analogy (not so close as Minkowski's) with the theory of linear homogeneous equations. In addition, he gives a method of finding the general solution of (1.2), which appears in unattractive form due to its lack of symmetry.* He has also considered particular problems of related types such as the determination of a particular positive solution of a system of linear homogeneous equations.

Minkowski also considers briefly system (1.2). He gives a necessary and sufficient condition that it have a solution but does not give its general solution.

Carver† gives a necessary and sufficient condition for the non-existence of solutions of (1.2). He also considers questions of independence and equivalence.

The present paper is designed to obtain results combining the advantages of the work just cited, and extending it. The theory of systems (1.1) is developed in such a form that it readily gives a solution of any system formed by uniting a system (1.1), a system (1.2), and a system

$$\sum_{i=1}^{n} \lambda^{i} x_i^{j} = 0. \tag{1.3}$$

We thus have a theory which has three extreme cases: Minkowski's system, Dines' system, and a system of linear homogeneous equations. The present paper exhibits the general solution of (1.1) in Minkowski's form and the general solution of the combined problem in a similar form. It makes the numerical work of solution center about a *single* matrix formed from the matrix of coefficients.

The method of developing the necessary theory is geometric in the following sense. For three dimensions the theorems are capable of visualization and in fact are almost intuitively obvious. When proofs of them are formulated in terms of analytic geometry, their extension to n dimensions, together with the necessary proofs, is immediate. Although Minkowski's treatment occurs in a book on the geometry of numbers and occasionally employs geometric nomenclature, both his treatment and Dines' are essentially analytic.

The two- and three-dimensional cases are treated in detail at the start in order to exhibit the geometric character of the reasoning (§§4 and 5).

A geometric method distinct from that of the present paper is employed

* L. L. Dines, *Systems of linear inequalities*, Annals of Mathematics, (2), vol. 20 (1918–19), p. 199.

† W. B. Carver, *Systems of linear inequalities*, Annals of Mathematics, (2), vol. 23 (1921), pp. 212–220.

by Lovitt* in the solution of a particular problem in three unknowns.†

2. **Statement of the problem.** It is proposed to solve the system of inequalities (1.1) in which the coefficients x are known and the λ's are to be determined.

Every system (1.1) is satisfied by $\lambda^i=0$. This will accordingly be called the *trivial solution*. Any other solution will be by definition *non-trivial*.

In order to apply geometric methods to the solution of this problem, it is convenient to recall certain properties of euclidean space of n dimensions referred to a rectangular cartesian coördinate system, or to a system arising from a rectangular cartesian system by a linear homogeneous transformation.‡

A linear homogeneous expression in the coördinates x, at least one of whose coefficients is not zero, defines an *oriented* $(n-1)$*-flat* (passing through the origin§). Points whose coördinates make this expression positive lie on the *positive side* of the $(n-1)$-flat, those making it negative lie on the *negative side*, and those making it zero lie on it.

Two oriented $(n-1)$-flats are the same if and only if the expression defining one is a positive multiple of that defining the other.

The totality of points on an oriented $(n-1)$-flat forms an unoriented $(n-1)$-flat whose equation is obtained by equating to zero the expression defining the oriented $(n-1)$-flat.

An $(n-1)$-flat (oriented or unoriented) separates a pair of points if and only if the substitution of the coördinates of those points in the expression defining it gives opposite signs.

To each condition (1.1) there corresponds a point $(x_1^j, \cdots, x_n^j)$ in n-space. The given system is thus represented geometrically by a set of points which we shall denote by S_n.¶

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* W. V. Lovitt, *Preferential voting*, American Mathematical Monthly, vol. 23 (1916), pp. 363–66.

† In order to render the bibliography more nearly complete, we cite here two papers whose existence has been called to our attention, but which we have not seen. They are by M. Fujiwara and appear in the 1928 and 1930 volumes of the Proceedings of the Imperial Academy, Tokyo.

‡ We find it desirable to describe the admissible coördinate systems in this manner. Any given set of data is most conveniently represented by interpreting the coördinates as rectangular cartesian. But this interpretation of x, y, z being once chosen (we take $n=3$ for illustration), it is convenient at times to choose x, y as the coördinates in a plane through the origin oblique to the coördinate planes, and x, y are not rectangular cartesian coördinates for that plane.

§ For our purpose it suffices to consider only flats containing the origin, and we shall understand, even in the absence of express mention of the fact, that every geometric element of one or more dimensions contains the origin, except in §15.

¶ Dines, Annals of Mathematics, (2), vol. 27 (1926), pp. 58–59, mentions that the representation of each condition can be accomplished by means of a point or by means of a vector. He uses the latter representation to describe the condition for non-existence of a solution of (1.2). He does not represent the solution geometrically, however, and does not employ a geometric method of reasoning in obtain-

To any non-trivial solution λ of (1.1) there corresponds an oriented $(n-1)$-flat determined by the expression

$$\sum_{i=1}^{n}\lambda^i x_i, \tag{2.1}$$

such that none of the points of S_n is on the negative side of it.*

Conversely, the coefficients of any oriented $(n-1)$-flat having this property give a non-trivial solution of (1.1).

Two solutions of (1.1) will be called the same if the corresponding two $(n-1)$-flats are the same. Hence the solution of (1.1) can be accomplished by finding every oriented $(n-1)$-flat having the property that no point of S_n lies on the negative side of it.

A necessary condition on a solution is that the corresponding unoriented $(n-1)$-flat do not separate any two points of the set.

3. **Reduction of dimensionality.** If the rank of the matrix

$$\| x_i^j \| \tag{3.1}$$

is $r<n$, the given points S_n lie in an r-flat, Π_r. As coördinates in that r-flat we may take any r of the coördinates x such that the matrix formed from the corresponding columns of (3.1) is of rank r. For convenience of language we assume that $x_1, \cdots, x_r$ are the coördinates in Π_r. The remaining $n-r$ coördinates $x_{r+1}, \cdots, x_n$ are linear homogeneous functions of them.

If an unoriented $(n-1)$-flat contains Π_r, either oriented $(n-1)$-flat associated with it gives a solution. This solution, being of the particular type which makes every left member of (1.1) zero, will be called an *equality solution*.

If an oriented $(n-1)$-flat defined by an expression

$$\sigma(x) \tag{3.2}$$

does not contain Π_r, then it meets Π_r in an oriented $(r-1)$-flat whose defining expression, when it is regarded as an oriented hyperplane in the space of r dimensions, can be obtained by eliminating $x_{r+1}, \cdots, x_n$ from (3.2) by means of the equations defining Π_r. Hence (3.2) regarded as an $(n-1)$-flat in n-space separates two points of S_n, if and only if it separates them when it is

ing his results. Actually, when the coördinates are rectangular cartesian, the vector used by Dines in describing the condition for the non-existence of solutions of (1.2) is the directed normal to the $(n-1)$-flat which we employ to represent the solution.

* Following a well established practice in the application of geometric methods to analysis, we regard the solution, the corresponding $(n-1)$-flat, and its defining linear homogeneous expression as identical.

regarded as an $(r-1)$-flat in Π_r. Given an $(r-1)$-flat defining a solution in Π_r, any $(n-1)$-flat containing it, when given the proper orientation, defines a solution in n-dimensional space, and, conversely, every solution not containing Π_r in n-dimensional space is obtained in this manner.

Let σ be the general solution of the problem in Π_r. Let the expressions whose vanishing defines Π_r be $u_{r+1}, \cdots, u_n$. Then

$$\sigma + \sum_{i=r+1}^{n} a^i u_i, \tag{3.3}$$

where the a's are arbitrary real constants, is an $(n-1)$-flat whose trace on Π_r is σ. Hence (3.3) is a solution in n dimensions. Conversely, any solution in n dimensions is a linear homogeneous function of the x's which reduces to σ on Π_r and has, therefore, the form (3.3). Accordingly, (3.3) gives the general solution in n dimensions. The solutions containing Π_r are obtained by making $\sigma=0$ in (3.3).

Because of the above result, in the sequel we need only indicate how to solve (1.1) when its matrix is of rank n, i.e., when the set S_n (plus the origin) is actually n-dimensional.

A useful consequence of the above is

THEOREM 1. *The intersection of two solutions σ, σ' is an $(n-2)$-flat which is a solution of the problem for $n-1$ dimensions defined by the points of the set in σ, and also of the problem defined by the points of the set in σ'.*

Since σ is a solution for the whole set of points, it is a solution for the points contained in σ' and by the result given above must intersect σ' in a solution for the set of points in σ'.

4. Linear homogeneous inequalities in two variables. In accordance with the results developed in the last section, it is sufficient to discuss the case where not all the representative points lie on a single line through the origin.

A non-trivial solution is necessarily of one of two types: it may contain a point of the set other than the origin, in which case it will be called a *fundamental solution*; or it may contain no such point.

Suppose the system proposed has a solution σ of the latter type. If the line representing it be rotated without passing through a point of the set, it continues to represent a solution. If it be so rotated until it contains a point of the set, it becomes a fundamental solution. As the rotation can take place in either sense, two fundamental solutions are thus obtained. If these fundamental solutions coincided, the set of points would be collinear with the

origin, contrary to hypothesis. Hence there exist two fundamental solutions σ_1 and σ_2. Moreover, we can write

$$\sigma = a\sigma_1 + b\sigma_2 \tag{4.1}$$

where a and b are constants. Since σ and σ_1 are positive for any point of the set on σ_2, a must be positive. In similar fashion b is proved positive.

Conversely, if the constants in the right member of (4.1) are given arbitrary positive values, the expression σ is positive for any point of the set because σ_1 and σ_2 have non-negative values for any point of the set and vanish simultaneously at no point of the set.

THEOREM 2. *The system*

$$\lambda x^j + \mu y^j \geqq 0 \qquad (j = 1, 2, \cdots, N), \tag{4.2}$$

whose matrix is of rank two, has a solution which is not a fundamental solution if and only if it has two fundamental solutions σ_1 and σ_2. The general solution of (4.2) *which contains no point of the set is then expressible as*

$$a\sigma_1 + b\sigma_2,$$

where a and b are arbitrary positive constants.

A consequence of the above is that if (4.2) has only one fundamental solution, that fundamental solution is the only solution.

THEOREM 3. *The system*

$$\lambda x^j + \mu y^j \geqq 0 \qquad (j = 1, 2, \cdots, N), \tag{4.3}$$

whose matrix is of rank two, may have zero, one, or two fundamental solutions. Its general solution is a linear homogeneous combination of all the fundamental solutions, the coefficients being arbitrary non-negative constants.

5. **Linear homogeneous inequalities in three variables.** Here it is sufficient to discuss the case where not all the representative points lie on a single plane through the origin.

A non-trivial solution is necessarily of one of the three types: (i) it contains two points of the set that are not collinear with the origin, in which case it will be called a *fundamental solution*; (ii) it contains one point P of the set but no other point not collinear with P and the origin; or (iii) it contains no point of the set.

Suppose the system proposed has a solution σ of the second type, containing the point P of the set. A plane coinciding initially with σ can be rotated in either sense about the line OP without passing through a point of the set until it contains another point of the set and thus becomes a fundamental

solution. If the two fundamental solutions so obtained coincided, all the points would be coplanar with the origin, contrary to hypothesis. Hence there are two fundamental solutions σ_1 and σ_2, coaxial with σ, and σ can be expressed in the form

$$\sigma = a\sigma_1 + b\sigma_2, \tag{5.1}$$

where a and b are constants. By considering a point of the set in σ_2 and not in σ_1 we prove that a is positive. Similarly b is positive.

Suppose the system proposed has a solution σ containing no point of the set. We may rotate a plane initially coinciding with σ about any line in σ through the origin until the variable plane contains a point of the set. Hence there is a solution of type (i) or (ii) and consequently at least one fundamental solution. Let σ_1 be any fundamental solution. Rotate a plane initially coinciding with σ about the intersection of σ and σ_1 until it contains a point of the set. Let the final position be σ'. The rotation can be accomplished in either sense. The two solutions so obtained could coincide with σ_1 only if the given point set were coplanar with the origin. Hence we may assume that σ' is distinct from σ_1, and we have

$$\sigma = a\sigma_1 + b\sigma', \tag{5.2}$$

where a and b are constants. By substitution of the coördinates of a point of the set which is on σ' but not on σ_1, we prove that a is positive. Likewise b is positive. Now σ' is a solution of type (i) or (ii). If of type (i), σ' is a fundamental solution in (5.2). If of type (ii), formula (5.1) applies to σ', and we have on substituting in (5.2)

$$\sigma = a\sigma_1 + b\sigma_2 + c\sigma_3, \tag{5.3}$$

where a, b and c are positive constants.

Let $\sigma_1, \sigma_2, \cdots, \sigma_p$ be the *complete system of fundamental solutions*, i.e., include all the fundamental solutions there are. Then by the above discussion, any solution σ of (1.1) can be expressed in the form

$$\sigma = \sum_{i=1}^{p} a^i\sigma_i, \tag{5.4}$$

where the a's are non-negative constants. Conversely, it is seen that (5.4) is a solution for arbitrary non-negative values of the a's because at any point of the given set the right member of (5.4) is computed by addition and multiplication from non-negative numbers.

The number of fundamental solutions in three dimensions may have any value. If we take as the given points the vertices of a convex polygon of N

sides whose plane does not contain the origin, the system has N fundamental solutions, as is readily seen geometrically.

6. **Rotation of an $(n-1)$-flat in n-dimensional space.** If σ and τ are two distinct oriented $(n-1)$-flats, they intersect in an $(n-2)$-flat which is contained by the $(n-1)$-flat

$$\sigma \cos \alpha + \tau \sin \alpha, \tag{6.1}$$

where α is any constant.

For $\alpha = 0$, (6.1) is the $(n-1)$-flat σ. If α increases continuously from zero to any positive value α_1, expression (6.1) for any value on the interval $(0, \alpha_1)$ represents an $(n-1)$-flat through the intersection of σ and τ. The passage from σ to $\sigma \cos \alpha_1 + \tau \sin \alpha_1$ can be defined as a *rotation* in the positive sense*. In the same way, if α decreases from zero to any negative value α_2, the passage from σ to $\sigma \cos \alpha_2 + \tau \sin \alpha_2$ can be defined as a *rotation in the negative sense*. The intersection of σ and τ can be called the *axis of the rotation*. If the coördinates of any point P not on the axis are substituted in (6.1) and the result is placed equal to zero, an α can be found for which the $(n-1)$-flat (6.1) contains P. Moreover, there will be both a positive and a negative α, i.e., the rotation can be accomplished in either one of two senses. If a finite set of points is given, we can determine the least positive α corresponding to each of them. The point, or points, in the set having the smallest positive α, say α_0, is the first point reached in the rotation in the positive sense. The result of substituting the coördinates of any point of the set in (6.1) is a continuous function of α, say $f(\alpha)$, which does not vanish within the interval $(0, \alpha_0)$. Hence $f(\alpha)$ does not change sign on the closed interval $(0, \alpha_0)$, i.e., if σ is a solution of the inequalities for the given set of points, then the final position of (6.1), namely, $\sigma \cos \alpha_0 + \tau \sin \alpha_0$, is also a solution.

7. **Linear homogeneous inequalities in n variables.** The generalization to the problem in n variables follows readily from the preceding case of three variables. Here we shall discuss the case where not all the representative points lie on a single $(n-1)$-flat through the origin.

A given non-trivial solution is necessarily of one of the n types (i) it contains $n-1$ points of the set that are not on an $(n-2)$-flat, in which case it will be called a *fundamental solution*; (ii) it contains $n-2$ points of the set which determine an $(n-2)$-flat, but no points of the set not on that $(n-2)$-flat; $\cdots$; (n) it contains no points of the set.

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* It is not to be supposed that α is the measure of the angle through which the $(n-1)$-flat is rotated. The latter is a function of α, whose explicit expression in terms of α can be obtained when the expression for the linear element in the coördinates x is known. This expression is of no use in the present discussion.

Suppose the system proposed has a solution σ which contains l points of the set which determine an l-flat but no points of the set not on that l-flat.

If $l<n-1$, there is an $(n-1)$-flat τ_1 distinct from σ and containing the l points. σ and τ_1 intersect in an $(n-2)$-flat. Let a variable $(n-1)$-flat, initially coinciding with σ, be rotated without passing through a point of the set until it contains another point P of the set and thus becomes a solution τ_2 containing at least $l+1$ points of the set not on an l-flat. The rotation can be accomplished in either of two senses. If the two solutions so obtained coincided with τ_1, all the points would be in the same $(n-1)$-flat, namely τ_1, contrary to hypothesis. Hence we may assume $\tau_1 \neq \tau_2$. Since the three $(n-1)$-flats are coaxial,

$$\sigma = a\tau_1 + b\tau_2, \tag{7.1}$$

where a and b are constants. At a point of S_n in τ_2, but not in τ_1, the quantities σ and τ_1 are positive and τ_2 is zero. Hence a is positive.

If the points of S_n in τ_2 lie in a flat of less than $n-1$ dimensions, this process can be repeated. Finally, we obtain a solution which is the $(n-1)$-flat determined by $n-1$ points of the set, i.e., *a fundamental solution containing the l points which are in σ.*

Let σ_1 be any fundamental solution containing the l points. In the above argument it can replace τ_1, so that we have

$$\sigma = a\sigma_1 + b\tau_2, \tag{7.2}$$

where a is positive. It is in this case possible to prove that b is also positive. σ_1, being distinct from τ_2 and containing $n-1$ points of the set, contains a point of the set not in τ_2. The substitution of this point in (7.2) makes σ_1 zero, and σ and τ_2 positive. Hence b is positive.

The solution τ_2 contains at least $l+1$ points of the set not on the same l-flat. The above general result shows that there is a fundamental solution σ_2 containing the same points of the set as τ_2. If σ_1 were the same as σ_2, then σ would contain at least $l+1$ points of the set not on the same l-flat. Hence σ_1 and σ_2 are distinct.

Moreover, we have, by the above general argument,

$$\tau_2 = c\sigma_2 + d\tau_3,$$

where c and d are positive constants.

This process can be continued until a τ_p which contains $n-1$ points is obtained. The number of operations $p-1$ is at most equal to $n-1-l$. Hence

$$p + l \leqq n. \tag{7.3}$$

Since τ_p is a fundamental solution, we write it σ_p and we have

$$\sigma = a^1\sigma_1 + a^2\sigma_2 + \cdots + a^p\sigma_p, \tag{7.4}$$

where the a's are positive constants. The σ's are distinct fundamental solutions. We have already seen that any two consecutive σ's are distinct. σ_3 is distinct from σ_1 because it has a point in common with σ_2 which is not contained in σ_1. Similarly for any other non-consecutive pair.

The totality of fundamental solutions containing the l points is called the *complete system of fundamental solutions containing those points*. The number of such solutions is necessarily finite because only a finite number of $(n-1)$-flats are determined by a finite number of points.

If there is a fundamental solution σ_{p+1} containing the l points and distinct from $\sigma_1, \cdots, \sigma_p$, we may repeat the above process, getting

$$\sigma = a^{p+1}\sigma_{p+1} + \cdots + a^q\sigma_q, \tag{7.5}$$

where the a's are positive constants, and $\sigma_{p+1}, \cdots, \sigma_q$ are distinct fundamental solutions containing the l points. We do not know they are all distinct from $\sigma_1, \cdots, \sigma_p$. The addition of (7.4) and (7.5) gives an expression for σ in which $\sigma_1, \cdots, \sigma_{p+1}$ surely occur with positive coefficients. If there is a fundamental solution through the l points not appearing in the right hand member, we may repeat the process.

If a solution σ contains l points of the set and in addition other points of the set not on the l-flat determined by them, there is a set of l' points containing the l points, to which the above argument is applicable. Hence any solution through l points of the set can be expressed as a linear homogeneous combination of the complete system of fundamental solutions through those l points, the coefficients being non-negative. Conversely, any linear homogeneous combination of the complete system of fundamental solutions through the l points, the coefficients being arbitrary non-negative constants, obviously passes through the l points, and is a solution because its value at S_n is computed by multiplication and addition from non-negative numbers.

THEOREM 4. *If the representative points do not all lie in an $(n-1)$-flat through the origin, the general solution of*

$$\sum_{i=1}^{n} \lambda^i x_i^j \geqq 0 \qquad (j = 1, 2, \cdots, N),$$

containing a specified sub-set of the given set of points S_n, is a linear homogeneous combination of the complete system of fundamental solutions containing those points, the coefficients being arbitrary non-negative constants.

A corollary is Minkowski's theorem:

THEOREM 5. *The general solution of the system*

$$\sum_{i=1}^{n}\lambda^i x_i^j \geqq 0 \qquad (j = 1, 2, \cdots, N),$$

whose representative points do not all lie in an $(n-1)$-flat through the origin, is a linear homogeneous combination of the complete system of fundamental solutions, the coefficients being arbitrary non-negative constants.

If the complete system mentioned in either of the two foregoing theorems is vacuous, the general solution is the trivial solution.

Suppose a solution σ contains a set A of points and contains no point of another set B. By Theorem 4, we know that σ can be expressed in the form (7.4), where the a's are non-negative and the σ's are fundamental solutions through A. In order that σ do not contain a given point of B, at least one σ_i must not contain that point. Hence, *a necessary condition for the existence of σ is that corresponding to every point of B there be at least one fundamental solution through the points A not containing it.*

Any fundamental solution through A which does not contain all the points B can be employed as the σ_1 in (7.2). If we use every such fundamental solution in turn as the σ_1 in (7.2) and add the resulting (7.4)'s, we have σ expressed as a linear homogeneous expression in which every fundamental solution passing through A and not containing all points of B occurs with a positive coefficient, and we are sure that the other coefficients are non-negative.

Conversely, if the necessary condition of the theorem is fulfilled, a linear homogeneous combination of the complete system of fundamental solutions through A in which every fundamental solution not containing all points of B has an arbitrary *positive* coefficient and those containing all points of B have *non-negative* coefficients satisfies the system, obviously contains A, and contains at least one non-vanishing σ corresponding to every point of B. Hence we have the

FUNDAMENTAL THEOREM 6. *The system*

$$\sum_{i=1}^{n}\lambda^i x_i^j \geqq 0,$$

whose representative points do not all lie in an $(n-1)$-flat through the origin, has a solution containing every point of a specified sub-set A of the given set of points and containing no point of another specified sub-set B if and only if corresponding to every point of B there is at least one fundamental solution through A not containing it. When a solution exists, the general solution is a linear homogeneous combination of the complete system of fundamental solutions through A, the

coefficient of every fundamental solution not containing all points B being an arbitrary positive constant, and that of every fundamental solution containing all points B being an arbitrary non-negative constant.

Making the point set B contain all points of S_n not in the flat space determined by A, since no fundamental solution through A contains all points of B in this case, we get

THEOREM 7. *If the representative points do not all lie in an $(n-1)$-flat through the origin, the general solution of the system*

$$\sum_{i=1}^{n} \overset{i}{\lambda} \overset{j}{x_i} \geqq 0$$

which contains a subset A of the given set and contains no points of the given set not on the flat space determined by A, is a linear homogeneous combination of the complete system of fundamental solutions through A, the coefficients being arbitrary positive constants.

If the complete system of fundamental solutions referred to in the preceding theorem is vacuous, there is no solution to the problem.

If A contains no points, we have a new solution of the problem (1.2) considered by Dines:

THEOREM 8. *If the representative points do not all lie in an $(n-1)$-flat through the origin, the general solution of the system*

$$\sum_{i=1}^{n} \overset{i}{\lambda} \overset{j}{x_i} \geqq 0$$

which contains no points of the given set is a linear homogeneous combination of the complete system of fundamental solutions, the coefficients being arbitrary positive constants.

Again, if the complete system referred to is vacuous, there is no solution.

8. **Necessary and sufficient conditions for the existence of a non-trivial solution.** In the preceding section we proved that a system of inequalities (1.1) has a non-trivial solution only if it has a fundamental solution. This condition is obviously also sufficient. We propose in the present section to get the necessary and sufficient condition in a slightly different form.

Given a set of points S_n having a fundamental solution Π_{n-1} containing l points of the set which determine an l-flat. Consider the points of S_n contained in Π_{n-1}. They form a set which we call S_{n-1}. Let Π_{n-2} be a fundamental solution, through the same l points, for the set S_{n-1}. And so on until a Π_m is reached, which is a fundamental solution for the points S_{m+1} of S_n in

Π_{m+1}, and is such that the points S_m in Π_m have only the trivial solution. S_m will be called the *inconsistent set containing the given l points*. Clearly $l \leqq m \leqq n$.

THEOREM 9. *Every solution for the set S_n which passes through l given points must contain the corresponding inconsistent set S_m.*

Any solution σ for the set S_n which is distinct from Π_{n-1} will, by Theorem I, meet Π_{n-1} in a solution for S_{n-1}. The latter solution will meet Π_{n-2} in a solution for S_{n-2}, and so on until we reach a solution for set S_{m+1}. If that solution were distinct from Π_m, it would meet Π_m in a non-trivial solution for S_m. As such a solution does not exist, the solution for the set S_{m+1} finally reached must coincide with Π_m. Hence σ contains Π_m.

THEOREM 10. *A solution containing a given set of l points and the l-flat determined by them, but no other points of the set, exists, if and only if $l=m$, that is, if the inconsistent set for the l points is contained by the l-flat determined by them.*

THEOREM 11. *There are exactly $n-m$ linearly independent fundamental solutions through l points determining an l-flat, where m is the dimensionality of the inconsistent set containing the l points.*

In the space Π_{m+1} above there is one and only one fundamental solution Π_m for the set S_{m+1}. Let us assume that in Π_{m+a} there are a linearly independent fundamental solutions for S_{m+a}. In Π_{m+a+1} we know the existence of one fundamental solution, namely, Π_{m+a}. We can rotate an $(m+a)$-flat, initially coinciding with Π_{m+a}, about any one of the fundamental solutions in Π_{m+a} for S_{m+a}. This rotation can be carried out in either of two senses. Since by the result of §3 any $(m+a)$-flat which passes through a fundamental solution for S_{m+a} and which is contained by Π_{m+a+1} is a solution in Π_{m+a+1} for S_{m+a}, when properly oriented, we know that the points of Π_{m+a} which do not remain in the variable $(m+a)$-flat are all on the same side of that $(m+a)$-flat. Hence for one sense of rotation the variable $(m+a)$-flat will remain a solution for S_{m+a+1}, and as soon as it contains a point of S_{m+a+1} not on Π_{m+a}, will be fundamental for S_{m+a+1}. In this way we deduce a fundamental solutions for S_{m+a+1}, which together with Π_{m+a} constitute a set of $a+1$ fundamental solutions for S_{m+a+1}.

If these solutions were linearly dependent, they would have in common a point P not on Π_m. Since one of these solutions, namely Π_{m+a}, meets the others in fundamental solutions for the set S_{m+a}, this point P would be on the a fundamental solutions for S_{m+a}. But this contradicts their linear independence. Hence by induction we conclude that the set S_{m+a} proposed for solution in the space Π_{m+a} has a linearly independent solutions passing through

the l given points for all possible values of a. The theorem stated follows by making $a = n - m$.

9. **Solution of numerical problems.** In the preceding sections we have developed the theory which will enable us to solve any of the problems mentioned in the introduction. The actual solution requires the knowledge of the complete system of fundamental solutions of the system (1.1) corresponding to all the points representing the given conditions; for example, if the system proposed is

$$u_1 > 0, \quad u_2 = 0, \quad u_3 \geqq 0,$$

the system of which it is necessary to know the complete system of fundamental solutions is

$$u_1 \geqq 0, \quad u_2 \geqq 0, \quad u_3 \geqq 0.$$

If the general solution of only the problem (1.1) is desired, the simplest way to compute the fundamental solutions is as follows. Assuming unknowns have been omitted, if necessary, so that the rank of the system is equal to the number of remaining unknowns, write the equation of every $(n-1)$-flat determined by $n-1$ points of the set. Substitute successively in each the coördinates of all the points of the set. If for a given $(n-1)$-flat the substitution gives a variation in sign, the $(n-1)$-flat is not a fundamental solution. If the substitution gives only non-negative values, the left member of the equation of the $(n-1)$-flat defines an oriented $(n-1)$-flat which is a fundamental solution. If the substitution gives only non-positive values, the left member of the equation with its sign changed gives a fundamental solution.

If, however, the solution of one of the associated problems is desired, it is convenient to make use of a rectangular array whose formation we proceed to describe. Each column is headed by a combination of the indices of the points of the set taken $n-1$ at a time, each combination being written once and only once in a definite but arbitrary order. The rows are numbered with the indices of the points of the set. The entry made in any position is the sign of the determinant having as its rows the coördinates of the points involved, written in the order determined by the indices at the head of the column followed by the index indicating the row. A zero is used to denote absence of sign. Any column containing no variation in sign is headed by a combination of points which determine a fundamental solution, if they do not lie in an $(n-2)$-flat. A zero in the array indicates that the point corresponding to the row is in the $(n-1)$-flat determined by the indices at the top of the column.

The totality of fundamental solutions containing a given set of points is readily picked out by means of this array, and it is easy to tell whether any point of the set is in a given fundamental solution. It is also possible from the disposition of zeros in the columns involved to tell whether two fundamental solutions are distinct.

An examination of the array will show what points are common to all fundamental solutions through a given set of l points, that is, will determine the set S_m (§8) for those l points.

10. **Skeleton sets.** If in the sign matrix of the last section every column which is headed by a combination containing a given index has a variation in sign, the point corresponding to that index is in no fundamental solution. As the general solution of (1.1) involves only the fundamental solutions, any point in no fundamental solution can be omitted without altering the solution of (1.1), provided its omission leaves the set n-dimensional. The corresponding condition (1.1) is satisfied as a consequence of the others, and its left member is positive for all solutions.

By a *skeleton set* of (1.1) we mean *a minimum set of the given points which are necessary to determine the solution*. This set is determined in the following manner.

From the n-dimensional set S_n form the array described in the preceding section. Omit as many points which are in no fundamental solution as will leave the dimensionality n, using the test just given. Consider all the points of S_n which are in a particular fundamental solution Π_{n-1}. Call them S_{n-1}. Form a new array with the points of S_{n-1}. If S_{n-1} is consistent, it will have at least one fundamental solution. Consider a point P of S_{n-1} contained in no fundamental solution in Π_{n-1}. P is contained by no solution of the problem in Π_{n-1} defined by S_{n-1}. Any solution of the problem in n dimensions, other than Π_{n-1}, will intersect Π_{n-1} in a solution for S_{n-1} (Theorem 1). Hence P is on no solution of the original n-dimensional problem except Π_{n-1}, and if S_{n-1} remains $(n-1)$-dimensional upon its omission, P can be omitted from S_n. The condition (1.1) corresponding to P will be a consequence of the remaining conditions (1.1), and its left member will be positive for every solution of (1.1) other than Π_{n-1}, which is the only fundamental solution through P. Let Π_{n-2} be a fundamental solution for S_{n-1}, call the points of the set in it S_{n-2}, and repeat the above operation until finally a k-dimensional set S_k is reached which is inconsistent in Π_k, i.e., has no fundamental solution if proposed as a problem in k dimensions.

The sign matrix of S_k, as a set in k dimensions, has a variation in sign in every column which contains a non-zero element. There must be at least one such column because the set is k-dimensional. Any point whose omission

gives an array with at least one non-zero column and a variation in sign in every such column can be omitted from the set because the resulting set will be both k-dimensional and inconsistent.

Any column contains $k-1$ zeros. In order that a column contain a variation in sign, the number of points in a k-dimensional inconsistent set must be at least $k+1$. It is not always possible, conversely, to reduce an inconsistent set of k dimensions to a skeleton set of $k+1$ points: for example, the inconsistent set composed of the vertices of a square whose center is the origin cannot be reduced to $k+1=3$ points because the omission of any one of the points leaves a consistent set having the diagonal connecting the two adjacent vertices as a fundamental solution.

When all possible points have been omitted by considering all fundamental solutions and all fundamental solutions contained in them, the remnant of the set is a skeleton set.

A particular consequence of the above discussion is

THEOREM 12. *Any set of n or fewer inequalities* (1.1) *in n unknowns has a fundamental solution.*

To illustrate the formation of the skeleton set, consider the case of three dimensions. Draw the planes through the origin determined by every pair of points of the set. Omit any plane which separates two points of the set. The planes remaining after the omission will be fundamental solutions. The only points retained are those in fundamental solutions, with the exception that when there is only one fundamental solution, a single point not on it is retained in order to leave the set three-dimensional.

Next consider in a fundamental solution the lines joining the points of the set to the origin, and omit any one of them which separates two points of the set, together with all points of the set on it. Repeat for the other fundamental solutions, if any exist.

Next consider one of the lines which have not been omitted. If the origin separates two points of the set on it, we retain those points and omit the others. If the points on it are all on the same side of the origin, we retain any one of them and omit the others.

If the original set is inconsistent, i.e., has no fundamental solution, we examine each point in turn to see if it can be omitted without making the remaining set consistent. By trial, we thus find a three-dimensional inconsistent set, the omission of any one of whose points leaves a consistent set.

If the set of points in a fundamental solution is inconsistent, the process just outlined for inconsistent three-dimensional sets is to be used to reduce it.

The points of the original set which remain after the above omissions form a skeleton set for the given three-dimensional set.

If (1.1) has a solution not containing any point of the set, the geometric configuration defined by the skeleton set is a convex pyramidal space, having as its vertex the origin, and as faces all the planes giving fundamental solutions. The consecutive planes intersect along the lines joining the points of the skeleton set to the origin.

Minkowski has also given a criterion for finding when one of the inequalities is a consequence of the others.*

11. **Examples.** Solve the system:

$$\lambda - \mu + \nu \geqq 0, \qquad \nu \geqq 0, \qquad \mu \geqq 0,$$
$$2\lambda \geqq 0, \quad \lambda + \mu \geqq 0, \; \lambda + \mu + \nu \geqq 0.$$

The set of points defined by the system is

$$(1, -1, 1), \quad (0, 0, 1), \quad (0, 1, 0), \quad (2, 0, 0), \quad (1, 1, 0), \quad (1, 1, 1).$$

We plot these points. The figure shows that there are four fundamental solutions. The same result is obtained from the rectangular array:

	12	13	14	15	16	23	24	25	26	34	35	36	45	46	56
1	0	0	0	0	0	−	−	−	−	−	−	0	+	+	+
2	0	+	+	+	+	0	0	0	0	−	−	−	+	+	0
3	−	0	+	+	0	0	+	+	+	0	0	0	0	−	−
4	−	−	0	−	−	−	0	−	−	0	0	+	0	0	+
5	−	−	+	0	−	−	+	0	0	0	0	+	0	−	0
6	−	0	+	+	0	−	+	0	0	−	−	0	+	0	0

The complete set of fundamental solutions is x; z; $y+z$; and $x+y$.

The general solution is

$$ax + bz + c(y + z) + d(x + y),$$

where a, b, c, d are arbitrary non-negative constants.

A particular solution containing no points of the set is

$$\sigma = 2x + y + 3z.$$

Let us express this in terms of the fundamental solutions by the method of §7.

The family of planes passing through the intersection of σ and the fundamental solution $\sigma_1 = x$ is

$$(11.1) \qquad (2x + y + 3z) + ax.$$

* Loc. cit., pp. 43–45.

The a's corresponding to the other points of the set are -4, -2, -3, -6. We choose the a numerically smaller than any of the other a's and substitute in (11.1), whence

$$\tau_2 = y + 3z.$$

The family of planes through the intersection of τ_2 and the fundamental solution $\sigma_2 = z$ is

$$y + 3z + bz.$$

The b's corresponding to all the other points of the set are -2, -3, -3, -4. Hence

$$\sigma_3 = y + z.$$

Since σ_3 contains two points of the set, it is a fundamental solution, and

$$\sigma = 2x + 2z + (y + z).$$

Starting anew with the fundamental solution $\sigma_1 = x+y$, we get

$$\sigma = (x + y) + 3z + x.$$

The addition of the two expressions for σ gives

$$2\sigma = 3x + 5z + (y + z) + (x + y).$$

This is a linear homogeneous combination of the complete system of fundamental solutions, the coefficients being positive constants.

Example 2. Suppose the system proposed for solution is

$$(0, 0, 1), \quad (-1, 0, 0), \quad (0, 1, 0), \quad (1, 0, 0).$$

There are just two fundamental solutions and they both contain $(-1, 0, 0)$, $(1, 0, 0)$, which therefore constitute the inconsistent set. The general solution for the case $\geqq 0$ is $ay+bz$, where a and b are arbitrary non-negative constants.

Example 3. The points $(1, -1, -1)$, $(2, 0, 0)$, $(2, 2, 2)$ lie in the plane $y-z=0$. In that plane x and y can be taken as coördinates. The solution of system (1.1) corresponding to the given points $(1, -1)$, $(2, 0)$, $(2, 2)$ is readily found to be $(a+2b)x+(a-2b)y$. The general solution of the problem in three unknowns is given by the pencil of planes through this line and is obtained by adding $c(y-z)$ to the above. Thus the general solution is furnished by

$$(a + 2b)x + (a - 2b + c)y - cz,$$

and is

$$\lambda = a + 2b, \quad \mu = a - 2b + c, \quad \nu = - c,$$

where a and b are arbitrary non-negative constants and c is an arbitrary constant.

12. **An application.** The problem considered by Lovitt* is to find the conditions of compatibility of the system

$$\begin{aligned} &\lambda - \mu > 0, \quad \mu - \nu > 0, \\ (12.1) \qquad &(A_1 - B_1)\lambda + (A_2 - B_2)\mu + (A_3 - B_3)\nu > 0, \\ &(A_1 - C_1)\lambda + (A_2 - C_2)\mu + (A_3 - C_3)\nu > 0, \end{aligned}$$

where the A, B, C are given non-negative numbers satisfying the relations

$$(12.2) \qquad \sum A = \sum B = \sum C.$$

The matrix of the coefficients is

$$\left\| \begin{matrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ A_1 - B_1 & A_2 - B_2 & A_3 - B_3 \\ A_1 - C_1 & A_2 - C_2 & A_3 - C_3 \end{matrix} \right\|.$$

The sum of the elements on each row being zero by virtue of (12.2), the representative points all lie in the plane

$$x + y + z = 0.$$

The convex polygon defined by a set of coplanar points is the figure assumed by a stretched rubber band placed around pins fixed at the points. It can be obtained geometrically by drawing all segments determined by two points of the set, and erasing all those segments which, produced if necessary, separate two points of the set. It reduces to a straight line if the given points are collinear.

Hence a geometric form of the necessary and sufficient condition that (12.1) have a solution is as follows: *the origin must not be inside the convex polygon defined by the representative points nor on its boundary*. Since the property involved in this condition is unaltered by orthogonal projection on another plane through the origin, in applying the test for consistency the representative points can be replaced by their projections on one of the coordinate planes.

To obtain the condition in analytic form, we may take x, y as the coordinates in the plane $x+y+z=0$. The sign matrix is then

$$\begin{array}{ccccc} & 1 & 2 & 3 & 4 \\ 1 & 0 & - & A_3 - B_3 & A_3 - C_3 \\ 2 & + & 0 & A_1 - B_1 & A_1 - C_1 \\ 3 & B_3 - A_3 & B_1 - A_1 & 0 & P \\ 4 & C_3 - A_3 & C_1 - A_1 & -P & 0 \end{array},$$

* Loc. cit.

where

$$P = A_2(B_1 - C_1) + B_2(C_1 - A_1) + C_2(A_1 - B_1).$$

We desire the condition that the associated system with symbols $\geqq 0$ have a solution containing no point of the set. The fundamental theorem (§7) gives immediately the result, which can be stated as follows: *if any column, say the ith, is deleted from the matrix, there must remain a column containing no variation of sign and containing a non-zero element on the ith row.*

It is easy to deduce from the above the two italicized results given by Lovitt on page 365 of his article. The first situation is described by (we have interchanged the significance of Lovitt's A and B)

$$A_1 = B_1,\ A_2 < B_2,\ A_3 > B_3,\ B_i = C_i,$$

and the matrix is

$$\begin{array}{cccc} 0 & - & + & + \\ + & 0 & 0 & 0 \\ - & 0 & 0 & 0 \\ - & 0 & 0 & 0\,. \end{array}$$

There is no solution because after the deletion of the fourth column the only column with a non-zero element on the fourth row contains a variation of sign.

The second situation is

$$A_1 < B_1,\ A_2 = B_2,\ A_3 > B_3,$$

for which the matrix is

$$\begin{array}{cccc} 0 & - & + & A_3 - C_3 \\ + & 0 & - & A_1 - C_1 \\ - & + & 0 & P \\ C_3 - A_3 & C_1 - A_1 & -P & 0\ . \end{array}$$

When the last column is omitted, the remaining columns all have variations of sign. Hence there is no solution.

Strictly speaking, the system treated by Lovitt contains another condition,

$$\nu > 0,$$

which insures that λ, μ, ν are positive. For the sake of simplicity we have omitted this condition. It is easy to see, however, that the inclusion of the additional point (0, 0, 1) will not alter the result given above. For any plane passed through a line giving a solution in the plane $x+y+z=0$ obviously can

be rotated until all the points, including $(0, 0, 1)$, are on its positive side; and, conversely, the trace of any solution on the plane $x+y+z=0$ is a solution in that plane by Theorem 1.

13. **Non-homogeneous case.** With the system of non-homogeneous inequalities

$$x_0^j + \sum_{i=1}^{n} \lambda^i x_i^j \sim 0 \qquad (j = 1, \cdots, N), \tag{13.1}$$

there is associated a homogeneous system

$$\begin{gathered} \lambda^0 x_0^j + \sum_{i=1}^{n} \lambda^i x_i^j \sim 0, \\ \lambda^0 > 0, \end{gathered} \tag{13.2}$$

where the symbol $\sim$ represents any one of the signs $\geqq$, $>$, $=$.

If a set of λ's satisfy (13.1), then $1, \lambda^1, \lambda^2, \cdots, \lambda^n$ satisfy (13.2).

Conversely, if a solution of (13.2) is $\lambda^0, \lambda^1, \lambda^2, \cdots, \lambda^n$, then

$$x_0^j + \sum_{i=1}^{n} (\lambda^i/\lambda^0) x_i^j \sim 0,$$

that is, $\lambda^1/\lambda^0, \lambda^2/\lambda^0, \cdots, \lambda^n/\lambda^0$ is a solution of (13.1). Hence the solution of (13.1) can be determined from that of (13.2).

The methods of the preceding sections may, therefore, be applied to give solutions of systems of non-homogeneous inequalities of the general type.

The above reduction is employed both by Minkowski and Dines. The adjunction of $\lambda^0 > 0$ to Dines' system gives a system of the same sort, but its adjunction to Minkowski's system gives one of the intermediate types considered in the present paper. As Minkowski gives the general solution only of (1.1), his treatment of the non-homogeneous case is incomplete.

14. **Positive solutions of systems of homogeneous or non-homogeneous linear equations.** It suffices to discuss the case of homogeneous equations, because it is always possible to convert a non-homogeneous into a homogeneous system by the use of homogeneous coördinates, as in §13.

This case is covered completely by the fundamental theorem of §7. The method of solving is given by the following

Example. Find the positive solutions of the equation

$$\lambda^1 + 2\lambda^2 - \lambda^3 - \lambda^4 = 0.$$

To solve, we must find the fundamental solutions of

$$\lambda^1 + 2\lambda^2 - \lambda^3 - \lambda^4 \geqq 0,\ \lambda^1 \geqq 0,\ \lambda^2 \geqq 0,\ \lambda^3 \geqq 0,\ \lambda^4 \geqq 0.$$

For the set of points

$$(1, 2, -1, -1),\quad (1, 0, 0, 0),\quad (0, 1, 0, 0),\quad (0, 0, 1, 0),\quad (0, 0, 0, 1)$$

we form the array

	123	124	125	134	135	145	234	235	245	345
1	0	0	0	0	0	0	−	+	+	−
2	0	0	0	+	−	−	0	0	0	−
3	0	−	+	0	0	+	0	0	+	0
4	+	0	+	0	−	0	0	−	0	0
5	−	−	0	+	0	0	+	0	0	0.

The complete set of fundamental solutions containing the first point is

$$y + 2t;\ y + 2z;\ x + t;\ x + z.$$

Hence the general solution is

$$a(y + 2t) + b(y + 2z) + c(x + t) + d(x + z),$$

where a, b, c and d are arbitrary positive constants. Thus

$$\lambda^1 = c + d,\ \lambda^2 = a + b,\ \lambda^3 = 2b + d,\ \lambda^4 = 2a + c.$$

15. **Another form of the necessary and sufficient condition for a solution.** The geometric condition given in §12 can be extended, as we proceed to show. First we must generalize the notion of the convex polygon defined by a set of points.

We shall not suppose, as previously, that all the flats contain the origin.

Consider a set of points S which are in a k-flat $F_k(S)$ $(0 \leqq k \leqq n)$, but are not all in a $(k-1)$-flat. Imagine constructed the set of all $(k-1)$-flats determined by S. Omit from it any $(k-1)$-flat which separates two points of S, and denote the remaining $(k-1)$-flats by $F_{k-1}(S)$.

Consider next the set of all points of S in a member of F_{k-1}. Construct the $(k-2)$-flats and discard as before. Denote by $F_{k-2}(S)$ the totality of $(k-2)$-flats remaining after this process has been applied to all the members of F_{k-1}.

If this process is continued, we finally obtain a set of lines $F_1(S)$, none of which separates any two points of S coplanar with it. Each of these lines, being determined by points of S, contains at least two distinct points of S. The application of the above general process to each line leaves two points which are separated by every other point of S on their join. The points of S finally remaining are denoted by $F_0(S)$ and are said to determine the *convex figure associated with* S. The points in $F_0(S)$ are called its *vertices*.

From the method of the construction no $(k-1)$-flat of $F_{k-1}(S)$ separates two points of S. Consequently, as in §7, any $(k-1)$-flat of F_{k-1} can be rotated about the $(k-2)$-flat determined by $k-1$ points of $F_0(S)$ in it in such a sense that the points of S continue to be not separated by it. When it acquires another point of S, it again becomes a member of $F_{k-1}(S)$. If the initial and final positions coincided, all the points of S would lie in a $(k-1)$-flat, contrary to hypothesis. Hence $F_{k-1}(S)$ contains at least two $(k-1)$-flats, i.e., the set $F_0(S)$ is not contained by a single $(k-1)$-flat. Therefore we have

THEOREM 13. *The convex figure associated with a set of points has the same dimensionality as the set of points.*

By an induction entirely analogous to that used in proving Theorem 11, we prove

THEOREM 14. *There are p linearly independent $(p-1)$-flats in $F_{p-1}(S)$ which contain a given vertex of $F_0(S)$.*

A particular consequence is obtained by making $p=k$:

THEOREM 15. *There are k linearly independent $(k-1)$-flats of $F_{k-1}(S)$ through every vertex of $F_0(S)$.*

By an equation $F_0(S)=F_0(T)$ we mean that the two point sets involved are identical.

A point P of space for which

$$F_0(S+P) \neq F_0(S) \tag{15.1}$$

is said to be *exterior to the convex figure* $F_0(S)$. P is then necessarily a vertex of $F_0(S+P)$.

Relation (15.1) surely holds if P is not on the k-flat containing S.

Any point P which is on a $(k-1)$-flat of $F_{k-1}(S)$ and for which

$$F_0(S+P) = F_0(S) \tag{15.2}$$

holds is said to be *on the boundary of the convex figure* $F_0(S)$. The totality of points P in any $(k-1)$-flat of $F_{k-1}(S)$ and on the boundary of $F_0(S)$ constitute a *face* of the convex figure. By Theorem 15 there are at least k faces through each vertex. The regular octahedron furnishes an example of a case where there are more than this minimum.

A point P which is on no $(k-1)$-flat of $F_{k-1}(S)$ and for which (15.2) holds is said to be *interior to the convex figure* $F_0(S)$.

We are now in a position to prove

THEOREM 16. *The system*

$$\sum_{i=1}^{n} \lambda^i x_i^j > 0$$

has a solution if and only if the origin is exterior to the convex figure associated with the representative points.

The condition is necessary. Suppose the origin P is not exterior. Because of equation (15.2), either P is a vertex of $F_0(S)$ or it is collinear with two points of S which it separates. In the former case, every solution of the associated system (1.1) contains a point of the set S, namely, P. In the latter case, any flat through the origin contains the two points of the set or separates them. In either case, therefore, there is no solution.

The condition is sufficient. When it is fulfilled, the origin is a vertex of $F_0(S+P)$. Hence by Theorem 15 there are n linearly independent faces of $F_0(S)$ through P, i.e., n linearly independent fundamental solutions for the set S, where n is the dimensionality of the point set $S+P$ (either k or $k+1$). From Theorem 11 the dimensionality of the inconsistent set for a set of $l=0$ points is zero. Hence by Theorem 10 there is a solution.

In the same way we readily prove

THEOREM 17. *The system*

$$\sum_{i=1}^{n} \lambda^i x_i^j \geqq 0$$

has a solution other than an equality solution if and only if the origin is not interior to the convex figure associated with the representative points.

It is to be noted that the last two theorems are true whatever the rank of the matrix of the coefficients may be.

The writer wishes to acknowledge indebtedness to Professor J. M. Thomas under whose direction this work was done.

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ON CERTAIN TYPES OF PLANE CONTINUA*

BY

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Continua which are the sum of a set of continua mutually exclusive except for a point which all have in common play a rôle of importance in analysis situs. In particular they are likely to obtrude upon any study of unbounded point sets, since in connection with these it is so often of value to perform an inversion of space. An investigation of some properties of such continua is the task of this paper. In it will be derived some interrelations of the almost mutually exclusive continua out of which they are built.

1. **Introduction.** A property of upper semicontinuous collections of continua‡ presently to be found useful in several connections will be derived first.

LEMMA I. *If S is a euclidean plane, G is an upper semicontinuous collection of mutually exclusive bounded continua filling up S and none dividing S, K is a bounded continuous curve contained in S, and G_k is that maximal§ subcollection of G each of whose elements contains a point of K, then G_k is a bounded continuous curve of elements of G.*¶

LEMMA II. *If S is a euclidean plane, G and H are upper semicontinuous collections of mutually exclusive bounded continua filling up S and none dividing it, and every element of H is a subset of some element of G, then in the space W whose points are the elements of H the point sets corresponding to the elements of G are an upper semicontinuous collection of mutually exclusive bounded continua filling up W and none dividing W.*

THEOREM I.‖ *If S is a euclidean plane, G is an upper semicontinuous collection of mutually exclusive bounded continua filling up S and none dividing it, M_s and N_s are two simple closed curves of S whose points are elements of G, and*

* Presented to the Society, June 21, 1929; received by the editors July 28, 1930.

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‡ For definitions and development of fundamental properties of such collections, see R. L. Moore, *Concerning upper semi-continuous collections of continua*, these Transactions, vol. 27 (1925), pp. 416–428. Moore proves if the elements of an upper semicontinuous collection of mutually exclusive bounded continua filling a euclidean plane and none dividing it be considered the points of a space, that this space is itself topologically equivalent to a euclidean plane.

§ Maximal in the sense that it is a proper subset of no similarly defined subcollection of G.

¶ For proof of this see H. M. Gehman, *A special type of upper semicontinuous collection*, Proceedings of the National Academy of Sciences, vol. 16, No. 9, pp. 609–613, Theorem III.

‖ The writer is indebted to Professor H. M. Gehman for a suggestion which has given this theorem a more general form than that which the writer obtained originally.

*a_s^m, b_s^m, and c_s^m are distinct points of M_s while a_s^n, b_s^n, and c_s^n are distinct points of N_s; then a_s^m, b_s^m, and c_s^m have the same sense on M_s as a_s^n, b_s^n, and c_s^n have on N_s if and only if the corresponding points a_t^m, b_t^m, and c_t^m on the corresponding curve M_t in the space T whose points are the elements of G have the same sense as a_t^n, b_t^n, and c_t^n upon N_t.***

An outline of the proof of this theorem will be sufficient.

Suppose first that there exists a simple closed curve L_s which together with its interior is contained in the common exterior of M_s and N_s and which also is composed of points that are themselves elements of G.

Let a_s^l, b_s^l, and c_s^l and a_t^l, b_t^l, and c_t^l be corresponding sets of distinct points upon the corresponding curves L_s and L_t respectively. It will be considered that the assertion is established for the special case now being considered when it has been shown that if a_s^m, b_s^m, and c_s^m may be simply joined to a_s^l, b_s^l, and c_s^l, then a_t^m, b_t^m, and c_t^m may be simply joined to a_t^l, b_t^l, and c_t^l, and conversely.

Suppose that a_s^m, b_s^m, and c_s^m may be simply joined to a_s^l, b_s^l, and c_s^l. Let A_s be an arc joining a_s^m and a_s^l in the common exterior of L_s and M_s, and let G_a be the maximal subaggregate of G each element of which contains a point of A_s. Owing to Lemma I, G_a contains as a subset an arc of elements H_a from a_s^m to a_s^l which as a subset of S is a continuum K_a not dividing the plane and contained except for the points a_s^m and a_s^l in the common exterior of L_s and M_s. Clearly there may now be constructed in the common exterior of L_s and M_s and the complement of K_a an arc B_s with end points b_s^m and b_s^l. The arc B_s, like A_s above, identifies a subcollection G_b of G including an arc of elements H_b of G whose one end element is b_s^m and the other b_s^l, and whose sum is a continuum K_b contained except for the points b_s^m and b_s^l in the common exterior of L_s and M_s and the complement of K_a, and not dividing space. Let M and L be arcs with end points respectively a_s^m, b_s^m and a_s^l, b_s^l contained except for their ends in the interiors of the simple closed curves M_s and L_s. Then $L+M+A_s+B_s$ is a simple closed curve X and $L+M+K_a+K_b$ is a continuum Y which divides S into precisely two mutually separated connected sets. If Y were to separate c_s^m from c_s^l it would follow that X must do this also, a contradiction of hypothesis, so there exists in the common exterior of L_s and M_s and in the complement of Y an arc C_s with end points c_s^m and c_s^l. This arc identifies a subcollection G_c of G including an arc of elements H_c of G whose

* For proofs and discussion of the fundamental problems of sense underlying this and similar statements, see J. R. Kline, *A definition of sense on closed curves in non-metrical plane analysis situs*, Annals of Mathematics, (2), vol. 19 (1917–1918), pp. 185–200; also by the same author, *Concerning sense on closed curves in non-metrical analysis situs*, Annals of Mathematics, (2), vol. 21 (1919–1920), pp. 113–119.

one end element is c_s^m and the other c_s^l, and whose sum is a continuum K_c contained except for the points c_s^m and c_s^l in the common exterior of L_s and M_s and the complement of V. Thus H_a, H_b, and H_c correspond in T to the three arcs A_t, B_t, and C_t simply joining a_t^m, b_t^m, and c_t^m to a_t^l, b_t^l, and c_t^l.

Suppose conversely that A_t, B_t, and C_t are three arcs simply joining a_t^m, b_t^m, and c_t^m to a_t^l, b_t^l, and c_t^l. In S and corresponding to these are three continua K_a, K_b, and K_c mutually exclusive and contained, except for the respective pairs of points a_s^m and a_s^l, b_s^m and b_s^l, c_s^m and c_s^l, in the common exterior of M_s and L_s. Neighborhoods of K_a, K_b, and K_c exist in which it may readily be seen that three arcs A_s, B_s, and C_s may be drawn simply joining a_s^m, b_s^m, and c_s^m to a_s^l, b_s^l, and c_s^l.

Suppose now that there exists no simple closed curve L_s satisfying the conditions mentioned above. Let σ_s be a bounded connected and simply connected subdomain of elements of G containing M_s+N_s. Let ρ_s be a bounded connected and simply connected subdomain of elements of $G-\sigma_s$. Let H be the collection whose elements are the elements of G in $\bar{\rho}_s$ and all points of S not contained in any element of G belonging to $\bar{\rho}_s$. In W, the space whose points are the elements of H, let L_w be a simple closed curve contained in ρ_w. Now a_s^m, b_s^m, and c_s^m can be simply joined to a_s^n, b_s^n, and c_s^n if and only if a_w^m, b_w^m, and c_w^m can be simply joined to a_w^n, b_w^n, and c_w^n, for S and W within σ_s and σ_w are in continuous one to one reciprocal correspondence; and a_w^m, b_w^m, and c_w^m can be simply joined to a_w^l, b_w^l, and c_w^l if and only if a_t^m, b_t^m, and c_t^m can be simply joined to a_t^l, b_t^l, and c_t^l, by the first part of the argument. These two facts together with Lemma II permit the desired conclusion to be deduced easily.

COROLLARY I. *If S is a euclidean plane, G is an upper semicontinuous collection of mutually exclusive bounded continua filling up S and none dividing it, M_s and N_s are two simple closed curves of S and all elements of G containing points of M_s or N_s are respectively subsets of M_s or N_s, and A_s^m, B_s^m, and C_s^m are distinct elements of G contained by M_s while A_s^n, B_s^n, and C_s^n are distinct elements of G contained by N_s; then A_s^m, B_s^m, and C_s^m have the same sense on M_s as A_s^n, B_s^n, and C_s^n have on N_s if and only if the corresponding points a_t^m, b_t^m, and C_t^m on the corresponding curve M_t in the space T whose points are the elements of G have the same sense as a_t^n, b_t^n and c_t^n upon N_t.**

The corollary follows readily from Lemma II and Theorem I.

* Let a_s^m, b_s^m, c_s^m and a_s^n, b_s^n, c_s^n be points contained respectively by the continua A_s^m, B_s^m, C_s^m and A_s^n, B_s^n, C_s^n. Then the *sense* of A_s^m, B_s^m, C_s^m is said to be the same as that of A_s^n, B_s^n, C_s^n on M_s and N_s respectively, or different from it, if the sense of a_s^m, b_s^m, c_s^m is the same as that of a_s^n, b_s^n, c_s^n or different from it.

2. **Notation**. The letter Z will be used to designate a plane bounded point set composed of the continuum X and the set of continua $[X_\alpha]$, where each element of $[X_\alpha]$ contains a point of X, no two have any point in common which is not a point of X, no one is disconnected by the omission from it of its subset in X, and no one when added to X forms a point set any bounded component of whose complement contains any point whatever of an element of $[X_\alpha]$.

If C is a simple closed curve in S, let $i(C)$ represent its bounded complementary domain or interior, and $e(C)$ its unbounded complementary domain or exterior. If K is a point set in S let $c(K)$ be the subset of S complementary to K. Thus, in S, $ci(C)$ represents the complement of the interior of the simple closed curve C, that is, $C+e(C)$, and $ce(C)$ the complement of its exterior, that is, $C+i(C)$.

Remarks. Order may be assigned among the elements of $[X_\alpha]$. A description of one method for doing this is already in print.* Another may be outlined as follows. If C is any simple closed curve enclosing X, and X_a is any element of $[X_\alpha]$, let X_a^c be the sum of the components of $X_a \cdot i(C)$ each of which contains a point of X. Associated with each element X_a^c of $[X_\alpha]$ is a subarc C_a of C containing all the points of $\overline{X}_a^c \cdot C$ and containing no point whatever in common with the subarc of C similarly associated with any other element of $[X_\alpha]$. Suppose that both C and D are simple closed curves which enclose X and which intersect each of the three different elements X_l, X_m, and X_n of $[X_\alpha]$. Then the sense of C_l, C_m, and C_n upon C is the same as the sense of D_l, D_m, and D_n upon D. Accordingly no confusion can arise if the sense of those elements $[X_\alpha]_c$ of $[X_\alpha]$ having non-vacuous intersections with the simple closed curve C be defined as the same as the sense of the non-vacuous elements of $[C_\alpha]$. Of course the set $[X_\alpha]_c$ may be not identical with $[X_\alpha]$; however, it is clear that by selecting a simple closed curve properly the order relations of any particular element of $[X_\alpha]$ not included in $[X_\alpha]_c$ with the elements of $[X_\alpha]_c$ may be determined. It is evident now what will be meant by the separation of two elements of $[X_\alpha]$ by some other pair of them. With reference to some particular element X_r of $[X_\alpha]$ the expressions *X_a precedes X_b*, $[X_i]$ $(i=1, 2, 3, \cdots)$ is a *clockwise series* and $[X_i]$ is a *counterclockwise series*, all of course referring to the elements of $[X_\alpha]$, may be defined. When, for instance, X_i precedes X_{i+1} $(i=1, 2, 3, \cdots)$ in the series $[X_i]$, the series will be called clockwise. In what follows it is immaterial which sense about the simple closed curve C is referred to as clockwise, but a

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* R. L. Moore, *Concerning the sum of a countable number of continua in the plane*, Fundamenta Mathematicae, vol. 6, pp. 189–202.

definite one of the two is of course assumed to be such at the outset. It will be convenient to assume that $X_r - X$ contains a point which is the end of a ray R, all points of which except its end point belong to the complement of Z. It proves unnecessary to specify whether X_r precedes or follows other elements of $[X_\alpha]$. If ab is an arc with its ends, a and b respectively, in $X_a - X_a \cdot X$ and $X_b - X_b \cdot X$ and no other points whatever in $X_a + X_b + X_r + X + R$, then there exists a bounded complementary domain δ of $X_a + X_b + X + ab$ such that if T represents the sum of the components of $X_c \cdot c(ab+X)$ with limit points in X, then δ contains T or contains no points of T according as the element X_c of $[X_\alpha]$ is or is not between X_a and X_b.

LEMMA III. *Given* (1) *Z is the set defined above and X is a point*; (2) *a is a point of $Z - X$ and X_a is the element of $[X_\alpha]$ containing a*; (3) *$X_1, X_2, \cdots$ is a clockwise series of elements of $[X_\alpha]$ all following X_a and such that a is a limit point of $\sum X_i$*; (4) *V is a ray in $c(X_a)$ containing a point of X_1. Then V contains a point of at least one other element of the series.*

Let v be the end of V and v^* be the last point of V belonging to X_1. The lemma will be proved about V^*, that subray of V whose end is v^*. Suppose that the lemma is false and that for every subscript n, $n > 1$, $V^* \cdot X_n = 0$. Let $[C_i]$ be a set of circles having common center a, satisfying for $i = 1, 2, 3, \cdots$ the condition $i(C_i) \supset ce(C_{i+1})$, and having radii converging to zero. Let the radius of C_1 be less than $d(a, X + X_1 + X_r + R)$. As a is a limit point of $\sum X_i$ there is a first element X_1^* of $[X_i]$ following X_1 containing a point of $ce(C_1)$, and a first circle C_2^* of $[C_i]$ containing no point of X_1^*. As a is a limit point of $\sum X_i$ there is a first element X_2^* of $[X_i]$ containing a point of $ce(C_2^*)$, and a first circle C_3^* of $[C_i]$ containing no point of X_2^*. In general as a is a limit point of $\sum X_i$ there is a first element X_j^* of $[X_i]$ containing a point of $ce(C_j^*)$, and a first circle C_{j+1}^* of $[C_i]$ containing no point of X_j^*. The infinite sequences $[X_i^*]$ and $[C_i^*]$ satisfy all the hypotheses imposed by supposition upon $[X_i]$ and $[C_i]$. Obviously V^* has no point in any element of $[X_i^*]$. For $i = 1, 2, 3, \cdots$ let S_i be a straight line segment contained in $ce(C_i^*)$ with one end in X_a, the other in X_i^*, and no additional point whatever in $X_a + X_i^*$. As $S_i \cdot (X_r + R) = 0$, by reason of a remark made above, $S_i + X_a + X_i^*$ for each value of i bounds a bounded complementary domain δ_i which must contain all but the point X of X_1. The domain δ_i contains points of V^* as it contains v^*, but can not contain V^* as v^* is unbounded. Thus for each value of i the boundary Δ_i of δ_i contains a point of V^*. Since $\Delta_i \subset S_i + X_a + X_i^*$ and by assumption $V^* \cdot (X_i^* + X_a) = 0$, $V^* \cdot S_i \neq 0$ for each value of i. The set of points $\sum V^* \cdot S_i$ being contained in V^* has all its limit points in V^*. As it has a point within any circle whose center is a, it contains a or has a as limit point. In either case V^* contains a, a contradiction. The lemma is thus established.

THEOREM II. *Suppose that X is a point and Z a continuum. If a is any point of $Z-X$ and X_a is the element of $[X_\alpha]$ containing a, and $[X_i]$ $(i=1, 2, 3, \cdots)$ is a clockwise series of elements of $[X_\alpha]$ all following X_a, then a is not a limit point of $\sum X_i$.*

Let C be a circle whose interior contains Z. Let $[Y_\alpha]$ be the subset of continua of $[X_\alpha]$ consisting of those each of which contains a point different from X which may be joined to a point of C by an arc contained except for one end in the complement of Z. Let A_p be any such arc associated with Y_p.

Let the elements of $[X_\alpha]$ be well ordered.† This having been done the elements of $[Y_\alpha]$ are also well ordered. Let Y_1^1, Y_2^1, Y_3^1, and Y_4^1 be the first four elements of $[Y_\alpha]$, and let no two of the arcs A_1, A_2, A_3, and A_4 have a common point and no one have more than one point in C. Let the four sets Y_1^1, Y_2^1, Y_3^1, and Y_4^1 be grouped in pairs each of which separates the other in $[X_\alpha]$. Suppose that Y_i^1 and Y_j^1 separate Y_p^1 and Y_q^1, i, j, p, and q being 1, 2, 3, and 4 in some order. Then $i(C)\cdot(Y_i^1+A_i^1+Y_j^1+A_j^1)$ separates $i(C)$ into two domains each containing a subset of Z, these being respectively z_p^1 and z_q^1. Let $Z_p^1=z_p^1+Y_i^1+Y_j^1$ and $Z_q^1=z_q^1+Y_i^1+Y_j^1$. Both Z_p^1 and Z_q^1 are continua similar in structure to the continuum Z.

Now suppose that the theorem is not true. Then $[X_\alpha]$ contains an element X_a containing a point a different from X, and a series $[X_i]$ clockwise and including only elements of $[X_\alpha]$ which follow X_a, such that the point set $\sum X_i$ has a among its limits. From Lemma III it follows that the pair of elements Y_i^1 and Y_j^1 can not separate two elements of $[X_i]$, for if they could, a proper selection of one of them as X_r and the other as X_1 would contradict the lemma, as both, owing to the existence of A_i^1 and A_j^1, contain points different from X which are ends of rays contained except for their ends in the complement of Z. Accordingly the elements of $[X_i]$ can not be distributed between Z_p^1 and Z_q^1 but must occur all in one of the two, say in Z_p^1. Since Z_p^1 is a continuum and a is a limit of the set $\sum X_i$, Z_p^1 contains points of X_a and must therefore contain X_a. It thus appears that Z_p^1 like Z is a continuum about which the theorem is not true. Let Z_p^1 be Z^1, $[X_\alpha^1]$ be the subset of elements of $[X_\alpha]$ each of which contains a point different from X in Z^1 and so is entirely contained in Z^1, and $[Y_\alpha^1]$ be the subset of the elements of $[X_\alpha^1]$ each of which contains a point different from X arcwise accessible from C in the complement of Z^1. It is readily seen that both $[X_\alpha^1]$ and $[Y_\alpha^1]$ are well ordered and that $[Y_\alpha^1]\subset[Y_\alpha]$.

With $n=2, 3, 4, \cdots$, let Y_1^n, Y_2, Y_3^n, and Y_4^n be the first four elements of $[Y_\alpha^{n-1}]$. At least one of these follows all four of $[Y_k^{n-1}]$ $(k=1, 2, 3, 4)$ in the

† It is well known that this may be accomplished by means of the Zermelo postulate.

well ordered sequence $[Y_\alpha^{n-1}]$. The arcs A_1^n, A_2^n, A_3^n, and A_4^n exist as in the first case. Suppose that Y_i^n and Y_j^n separate Y_p^n and Y_q^n in $[X_\alpha^{n-1}]$. Then $i(C)\cdot(Y_i^n+A_i^n+Y_j^n+A^n)$ separates $i(C)$ into two domains each containing a subset of Z, z_p^n and z_q^n respectively. Let $Z_p^n=z_p^n+Y_i^n+Y_j^n$, and $Z_q^n=z_q^n+Y_i^n+Y_j^n$. As in the case for $n=1$, these are continua like Z. One of them, say Z_p^n, contains X_a and $[X_i]$, and consists of a well ordered subset $[X_\alpha^n]$ of the elements of $[X_\alpha]$. Evidently the subset $[Y_\alpha^n]$ may be defined as $[Y_\alpha^1]$ was defined, is then well ordered, and is a subset of $[Y_\alpha^{n-1}]$.

For each value of i ($i=1, 2, 3, \cdots$) there is thus determined a continuum Z^i, no two identical, which is of the same type as Z, and contains both X_a and the series $[X_i]$. Each of $[Z^i]$ is composed of a certain subset of the elements of $[X_\alpha]$. Taken together they form a well ordered sequence. Let

$$Z^\omega = \prod_{i=1}^{\infty} Z^i.$$

Then $Z^\omega \supset X$, and if $Z^\omega \supset p$ where p is a point not X of X_p, an element of $[X_\alpha]$, then $Z^\omega \supset X_p$. Therefore Z^ω is a continuum of the same kind as Z. Let it be made up of the elements $[X_\alpha^\omega]$ of $[X_\alpha]$ and let $[Y_\alpha^\omega]$ be the subset of these whose relation to C and the complement of Z^ω is like the relation of the elements of $[Y_\alpha]$ to C and the complement of Z. Clearly $[Y_\alpha^\omega]$ is a well ordered aggregate. It is not true in general that $[Y_\alpha^\omega]\subset[Y_\alpha^i]$, but it is true that

$$[Y_\alpha^\omega] \supset \prod_{i=1}^{\infty} [Y_\alpha^i.]$$

Evidently $Z^\omega \supset (X_a+\sum X_i)$.

It is now clear how to extend the well ordered sequence $[Z^\alpha]$. If at any stage this sequence has a last element Z^{f-1}, the next may be defined by selecting Y_1^f, Y_2^f, Y_3^f, and Y_4^f, the first four elements of $[Y_\alpha^{f-1}]$, choosing two that separate the other two, and thus obtaining a proper subcontinuum Z^f of Z^{f-1} composed of elements of $[X_\alpha]$ and including X_a and $[X_i]$. If at any stage the sequence has no last element, then the next element consists of the set of points common to all the elements of the sequence already defined, a continuum which clearly exists as it contains X_a and $[X_i]$ and just as clearly consists of a subset of the elements of $[X_\alpha]$.

Under what conditions will it be impossible to extend the sequence $[Z^\alpha]$? If this does become impossible clearly that must be in some case where the sequence up to that point defined has a last element, as when it has no last element the process being employed immediately determines another, namely the set of points common to all the elements already determined. Suppose then that Z^λ is the last element of $[Z^\alpha]$. Can Z^λ contain four or more ele-

ments of $[X_\alpha]$? Supposing that it can, various simple methods for proving that the subset $[Y_\alpha^{\lambda+1}]$ of $[X_\alpha]$ consists of at least four elements will soon present themselves. Such being the case it would be possible immediately to obtain an element following Z^λ, and so Z^λ, the last element of $[Z^\alpha]$, is composed of fewer than four of the elements of $[X^\alpha]$.

But the assumption that the theorem was not true of the continuum Z in the particular instance of X_a and $[X_i]$ has been seen to imply that each element of $[Z^\alpha]$ contains $X_a+\sum X_i$. Thus Z^λ must contain more than four elements of $[X_\alpha]$. The contradiction between these two inferences concerning Z^λ establishes the theorem.

COROLLARY II. *Suppose Z is a continuum. If a is a point of $Z-X$ and X_a is the element of $[X_\alpha]$ containing a, and $[X_i]$ $(i=1, 2, 3, \cdots)$ is a clockwise series of the elements of $[X_\alpha]$ all following X_a, then a is not a limit of $\sum X_i$.*

Let D be the sum of the bounded components of $c(X)$. $X+D$ is a continuum K. Let G be the upper semicontinuous collection of mutually exclusive bounded continua whose elements are respectively K and the points of $c(K)$. The elements of G are points of a space T which is topologically equivalent to S.† In T there corresponds to the continuum Z of S a continuum Z_t of the same type as Z except that the set X_t corresponding to X is a single point. Concerning Z_t, accordingly, Corollary II is identical with Theorem II and is therefore true. But if it were to be supposed that Corollary II is not true of Z, then owing to Corollary I and the nature of Z_t it would follow that Corollary II could not be true for Z_t either; a contradiction.

3. **Notation.** In the following paragraphs capital letters in script, as $\mathcal{P}$, will represent prime ends.‡

Remarks. It will be assumed in this paper that the prime ends discussed are defined by chains of cuts whose members are simple continuous arcs. Some facts which may easily be deduced from published researches upon prime ends and which will be used presently are as follows. Any connected and simply connected domain whose boundary is bounded, although the domain may itself be unbounded, may have its prime ends defined in the usual way, and these may then be shown to have cyclic arrangement.§

Suppose that X is a single point. If X_a is any element of $[X_\alpha]$, then X is arcwise accessible from one and but one prime end of the boundary of the

† R. L. Moore, *Concerning upper semicontinuous collections of continua*, loc. cit., p. 424, Theorem 21.

‡ For definitions and fundamental research upon prime ends, see C. Carathéodory, *Über die Begrenzung einfach zusammenhängender Gebiete*, Mathematische Annalen, vol. 73 (1912), pp. 323–370.

§ In the paper just referred to, Carathéodory considers only bounded domains.

unbounded component of $c(X_a)$. If X_a and X_b are distinct elements of $[X_\alpha]$ then X is arcwise accessible from two and only two prime ends of the boundary of the unbounded component of $c(X_a+X_b)$. Suppose that the two prime ends just identified are $\mathfrak{Q}$ and $\mathfrak{R}$. If x_r is a subset of X_r-X with X as limit point, one and only one prime end of the unbounded component of $c(X_a+X_b)$ is limit of x_r. Suppose that it is $\mathfrak{R}$; $\mathfrak{R}$ must contain X, must be limit of any subset of X_r-X with X as limit point, and must be arcwise accessible from the unbounded connected complementary domain of X_a+X_b. It will be convenient to say that $\mathfrak{R}$ is *limit* of X_r. It may then be proved that $\mathfrak{Q}$ and only $\mathfrak{Q}$ is limit of each element of $[X_\alpha]$ which is between X_a and X_b, while $\mathfrak{R}$ and only $\mathfrak{R}$ is limit of each element of $[X_\alpha]$ which is not between X_a and X_b, prime ends of course of the unbounded component of $c(X_a+X_b)$ alone being considered and *between* being understood with reference to X_r.

If X is a true continuum, not a point, conditions are somewhat more involved. X_a and X_b being any two different elements of $[X_\alpha]$, and $\mathfrak{C}$ being the simple closed curve of prime ends of the unbounded component of $c(X_a+X+X_b)$, then all the elements of $\mathfrak{C}$ except possibly exactly four, or exactly three, or exactly two are themselves prime ends of X_a, or of X, or of X_b. The prime ends of $\mathfrak{C}$ not belonging to X_a or X_b form two components $\mathfrak{Q}$ and $\mathfrak{R}$, and the three possibilities just enumerated respectively characterize the following cases, both $\mathfrak{Q}$ and $\mathfrak{R}$ are true arcs of prime ends of $\mathfrak{C}$, either $\mathfrak{Q}$ or $\mathfrak{R}$ is an arc but the other is a single prime end, and both $\mathfrak{Q}$ and $\mathfrak{R}$ are single prime ends. Irrespective of the case considered it can be proved that no element of $[X_\alpha]$ different from X_a and X_b can have both an end of $\mathfrak{Q}$ and an end of $\mathfrak{R}$ among its limits. Supposing that $\mathfrak{R}$ contains the limits of X_r, it can then be shown that $\mathfrak{R}$ contains all the limits of those elements of $[X_\alpha]$ which are not between X_a and X_b while $\mathfrak{Q}$ contains all the limits of those elements of $[X_\alpha]$ which are between X_a and X_b.

Lemma IV. *Given* (1) *Z is the set defined above and X is a point*; (2) *$\mathfrak{Q}$ is the prime end identified above of the unbounded component of $c(X_a+X_b)$*; (3) *X_1, $X_2, \cdots$ is a clockwise series of elements of $[X_\alpha]$ all following both X_a and X_b and having the prime end $\mathfrak{Q}$ as limit*; (4) *V is a ray in $c(X_a+X_r+X_b)$ containing a point of X_1. Then V contains a point of at least one other element of the series.*

Let V^* be determined as it was in the proof of Lemma III. Suppose that, for every subscript n greater than 1, $X_n\cdot V^*=0$, that is, suppose the lemma to be false. Let $[Q_i]$ be a chain of cuts defining $\mathfrak{Q}$ and $[\delta_i]$ be the corresponding chain of domains. Suppose that X is the only limit point of $\sum Q_i$, and that

$Q_1+\delta_1$ contains no point of X_1+X_r+R. Let Q_1 be Q_1^*. As $\mathfrak{Q}$ is a limit of $\sum X_i$, there is a first element X_1^* of $[X_i]$ following X_1, containing a point of Q_1^*, and as $\mathfrak{Q}$ is not a limit of X_1^* there is a first one Q_2^* of $[Q_i]$ following Q_1^* containing no point of X_1^*. As $\mathfrak{Q}$ is a limit of $\sum X_i$, there is a first element X_2^* of $[X_i]$ following X_1^* containing a point of Q_2^*, and as $\mathfrak{Q}$ is not a limit of X_2^*, also a first one Q_3^* of $[Q_i]$ following Q_2^* and containing no point of X_2^*. In general for $n=3, 4, 5, \cdots$ there is a first element X_n^* of $[X_i]$ following X_{n-1}^* containing a point of Q_n^*, and a first one Q_{n+1}^* of $[Q_i]$ following Q_n^* and containing no point of X_n^*. The two infinite series $[X_i^*]$ and $[Q_i^*]$ satisfy all the conditions imposed by supposition upon the series $[X_i]$ and $[Q_i]$. For $i=1, 2, 3, \cdots$, let S_i be an arc of Q_i^* whose non-end points include no points of $X_a+X_i^*$ and whose ends belong to X_a and X_i^* respectively. S_i together with the two elements of $[X_\alpha]$ in which its ends lie, for each value of i, determines a domain containing v^*, a point of V^*, but not containing V^*. Thus $V^*\cdot S_i\neq 0$. By the method used in the proof of Lemma III a contradiction similar to the one deduced in it can now be obtained.

THEOREM III. *Suppose that Z is a continuum and X a point. If X_a and X_b are distinct elements of $[X_\alpha]$ and $\mathfrak{Q}$ is the prime end of the unbounded connected complementary domain of their sum which contains X and is arcwise accessible from this domain and is not a limit of X_r, and if $[X_i]$ is a clockwise series of elements of $[X_\alpha]$ all following both X_a and X_b, then $\mathfrak{Q}$ is not a limit of $\sum X_i$.*

Suppose the theorem is not true; suppose in short that the prime end $\mathfrak{Q}$ is limit of $[X_i]$. It is clear that all the hypotheses of Lemma IV are now fulfilled or may easily be fulfilled. The process used in obtaining a contradiction to establish Theorem II is now available in this case also.

COROLLARY III. *Suppose Z is a continuum. If X_a and X_b are distinct elements of $[X_\alpha]$ and $\mathfrak{Q}$ is that maximal arc of prime ends of the unbounded component of $c(X_a+X+X_b)$ none of which is already a prime end of X_a or X_b and none of which contains a limit point of $X_r-X_r\cdot X$, and if $[X_i]$ is a clockwise series of elements of $[X_\alpha]$ all following both X_a and X_b, then no prime end in $\mathfrak{Q}$ contains a limit point of $\sum X_i$.*

Corollary III may be inferred from Theorem III in the way that Corollary II was inferred from Theorem II.

THEOREM IV. *If W is a plane bounded continuum composed of the point X and a set of continua $[X_\alpha]$ no two of which have any point in common except the point X which is common to all, then W separates the point p from the point q*

only if some element of $[X_\alpha]$ *separates* p *from* q.†

COROLLARY IV. *If* W *is a plane bounded continuum composed of the continuum* X *and a set of continua* $[X_\alpha]$ *each having a point in common with* X *and no two having in common any point which is not a point of* X, *and,* p *and* q *being two distinct points of the complement of* W *and* X_α *being any element at all of* $[X_\alpha]$, *if* $X+X_\alpha$ *does not separate* p *from* q, *then* W *does not separate* p *from* q.

As $X+X_\alpha$ does not separate p from q, X does not separate p from q and so owing to the availability of inversion it may as well be assumed that both p and q belongs to γ, the unbounded component of $c(X)$. In place of W and X, regard now the continuum X^* which is the sum of X and all its bounded connected complementary domains, and the continuum W^* which is the sum of X^* and the continua $[X_\alpha^*]$, that subset of $[X_\alpha]$ comprising all those elements of $[X_\alpha]$ with points in γ. Upper semicontinuity now provides a means of reducing the question to the one solved by Theorem IV.

† Since the submission of this paper to the editors a proof of Theorem IV has appeared in print. Accordingly, although the published demonstration is quite different from the one formerly contained in this paper, it having resembled in its principal details the argument for Theorem II, the proof of the theorem has been omitted. For the proof see J. H. Roberts, *Concerning collections of continua not all bounded*, American Journal of Mathematics, vol. 52 (1930), pp. 551–562, Theorem I on p. 553.

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THE k-FUNCTION, A PARTICULAR CASE OF THE CONFLUENT HYPERGEOMETRIC FUNCTION*

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In his well known paper† in which he defines the function $W_{k,m}(z)$ Professor E. T. Whittaker says: "There are other members of the family of functions $W_{k,m}(z)$ which have not hitherto been noticed, but which give promise of interesting properties. Among these may be mentioned the families of functions for which $m=0$ and those for which $m=\frac{1}{2}$." The functions considered here correspond to the case $m=\frac{1}{2}$. The associated differential equation has arisen recently in the theory of turbulence, particularly in the researches of W. Tollmien‡ and Th. von Kármán.§

1. **Definition of the functions.** The function $k_n(x)$ may be defined for real values of x and n by the definite integral

$$k_n(x) = \frac{2}{\pi}\int_0^{\pi/2} \cos(x \tan\theta - n\theta)d\theta. \tag{1.1}$$

When $n=0$ we have an integral which is easily evaluated, in fact

$$k_0(x) = e^{-|x|}.$$

For other values of n we have the interesting relations

$$k_{-n}(x) = k_n(-x), \tag{1.2}$$

$$k_n(0) = \frac{2}{n\pi}\sin\frac{n\pi}{2}, \tag{1.3}$$

which follow immediately from the definition. When n is an integer the definite integral may be evaluated in terms of known functions. In particular

$$k_2(x) = (x + |x|)e^{-|x|}. \tag{1.4}$$

This expression shows that $k_2(x)$ is zero when x is negative and it will be seen presently that if n is a positive integer $k_{2n}(x)$ is also zero when x is negative.

* Presented to the Society, September 9, 1931; received by the editors November 10, 1930. The k-notation has been adopted in honor of Dr. Th. von Kármán, who submitted the differential equation to the present author for investigation.

† Bulletin of the American Mathematical Society, vol. 10 (1903–04), p. 133.

‡ Göttinger Nachrichten, 1929, p. 21. Put $U=c+by$ in equation (2). [Noted by C. B. Millikan.]

§ International Congress of Applied Mechanics, Stockholm, 1930. Göttinger Nachrichten, 1930, p. 58. Put $\psi=\alpha e^{i\lambda x}Y(y)$ in equation (8) and neglect α^2.

Since $|\cos(x \tan \theta - n\theta)| \leqq 1$, we have the important inequality

$$(1.5) \qquad |k_n(x)| \leqq 1,$$

which holds for all real values of n and x. When n is an odd integer the function $k_n(x)$ may be expressed in terms of the Bessel functions $K_0(x)$ and $K_1(x)$. In particular, if $t = \tan \theta$,

$$\begin{aligned} k_1(x) &= \frac{2}{\pi}\int_0^\infty \frac{\cos (xt)dt}{(1+t^2)^{3/2}} + \frac{2}{\pi}\int_0^\infty \frac{\sin (xt)tdt}{(1+t^2)^{3/2}} \\ &= \frac{2}{\pi}\int_0^\infty \frac{\cos (xt)dt}{(1+t^2)^{3/2}} - \frac{2x}{\pi}\int_0^\infty \frac{\cos (xt)dt}{(1+t^2)^{1/2}} \\ &= \frac{2x}{\pi}[K_1(x) - K_0(x)], \quad x > 0, \\ &= -\frac{2x}{\pi}[K_1(-x) + K_0(-x)], \quad x < 0. \end{aligned}$$

2. **The generating function.** When n is an even integer the function $k_n(x)$ may be defined with the aid of the expansion

$$(2.1) \qquad e^{ix\tan(\theta+i\alpha)} = k_0(x) + k_2(x)e^{2i\theta-2\alpha} + k_4(x)e^{4i\theta-4\alpha} + \cdots, \qquad \alpha > 0, \quad x > 0.$$

If $s = e^{2i\theta-2\alpha}$, the expansion becomes

$$(2.2) \qquad e^{-x(1-s)/(1+s)} = k_0(x) + sk_2(x) + s^2k_4(x) + \cdots,$$

and it is readily seen that the power series in s is absolutely convergent when $|s| < 1$. The convergence on the circle of convergence will be studied later; it will suffice now to say that the series takes the form

$$(2.3) \qquad e^{ix\tan\theta} = k_0(x) + k_2(x)e^{2i\theta} + k_4(x)e^{4i\theta} + \cdots,$$

and is the Fourier series of the function $e^{ix\tan\theta}$ for the interval $(-\pi < \theta < \pi)$. Indeed,

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ix\tan\theta - 2ni\theta}d\theta &= \frac{1}{\pi}\int_0^\pi \cos\,[x\tan\theta - 2n\theta]d\theta \\ &= \frac{2}{\pi}\int_0^{\pi/2} \cos\,[x\tan\theta - 2n\theta]d\theta = k_{2n}(x). \end{aligned}$$

The fact that $k_{2n}(x) = 0$ when n is a negative integer accounts for the absence of terms of type $k_{-2m}(x)$ and enables us to write the expansion in the usual form,

$$e^{ix\tan\theta} = \sum_{m=-\infty}^{\infty} k_{2m}(x)e^{2mi\theta}. \tag{2.4}$$

The expansion (2.2) shows that the function $k_{2m}(x)$ is closely related to the generalized polynomial of Laguerre which is defined by Sonine's expansion*

$$(1-z)^{-1-\alpha}e^{-xz/(1-z)} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x)z^n. \tag{2.5}$$

Indeed, if $L_n(x)$ denotes the polynomial of Lagrange and Laguerre, we have Abel's expansion

$$(1-z)^{-1}e^{-xz/(1-z)} = \sum_{n=0}^{\infty} L_n(x)z^n, \tag{2.6}$$

and it is at once seen that when $x>0$,

$$k_{2m}(x) = (-1)^m e^{-x}[L_m(2x) - L_{m-1}(2x)]. \tag{2.7}$$

This is a particular case of the more general formula

$$\begin{aligned} k_{2m}(x) = (-1)^m e^{-x}\Big[& L_m^{(\alpha)}(2x) - \binom{\alpha+1}{1} L_{m-1}^{(\alpha)}(2x) \\ & + \binom{\alpha+1}{2} L_{m-2}^{(\alpha)}(2x) + \cdots + (-1)^m \binom{\alpha+1}{m} L_0^{(\alpha)}(2x)\Big], \end{aligned} \tag{2.8}$$

which is proved by equating the coefficients of z^m on the two sides of the equation

$$e^z[k_0(x) - zk_2(x) + z^2k_4(x) - \cdots] = (1-z)^{\alpha+1}\sum_{n=0}^{\infty} z^n L_n^{(\alpha)}(2x).$$

A reciprocal relation†

$$\begin{aligned} & (-1)^n e^{-x} L_n^{(\alpha)}(2x) \\ & \quad = k_{2n}(x) - \binom{\alpha+1}{1} k_{2n-2}(x) + \binom{\alpha+2}{2} k_{2n-4}(x) - \cdots \end{aligned} \tag{2.81}$$

is obtained by equating the coefficients of z^n on the two sides of the equation

$$\sum_{n=0}^{\infty} e^{-x} L_n^{(\alpha)}(2x)z^n = (1-z)^{-\alpha-1}[k_0(x) - zk_2(x) + z^2k_4(x) - \cdots].$$

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* N. Sonine, Mathematische Annalen, vol. 16 (1880), p. 1. In Sonine's notation

$$L_n^{\alpha}(x) = (-1)^n\Gamma(n+\alpha+1)T_\alpha^n(x).$$

The notation used here is the same as that used by Hille and Szegö.

† S. Namuri, Tôhoku Mathematical Journal, vol. 30 (1928–29), p. 58.

An important property of the function $k_{2m}(x)$ may be deduced directly from equation (2.7) with the aid of the orthogonal relation*

$$(2.9)\qquad \int_0^\infty e^{-u}L_m(u)L_n(u)du = 0,\ m \neq n,\\ = 1,\ m = n.$$

We have in fact

$$(2.91)\qquad \begin{aligned} \int_0^\infty [k_{2m}(x)]^2 dx &= 1,\ m > 0,\\ &= \tfrac{1}{2},\ m = 0;\\ \int_0^\infty k_{2m}(x)k_{2m+2s}(x)dx &= 0,\ s > 1,\\ &= \tfrac{1}{2},\ s = 1.\end{aligned}$$

A second generating function may be derived from the expansion†

$$e^{-u}(ux)^{-m/2}I_m(2(ux)^{1/2}) = \sum_{n=0}^{\infty} \frac{(-u)^n}{\Gamma(m+n+1)} L_n^{(m)}(x)$$

by putting $m=0$ and using (2.7). The result is

$$e^{-z/2}(vz)^{-1/2}I_1(2(vz)^{1/2}) = \frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n n!}\frac{e^v k_{2n+2}(v)}{v}.$$

The function $I_m(y)$ is the Bessel function with imaginary argument.

3. **The Lagrangian expansion.** The expansion of the generating function may be derived by Lagrange's theorem from the implicit relation

$$z = x - sz$$

which may be used to define z as a function of s. The expansion of the function

$$\frac{dz}{dx}\left(\frac{1}{z}e^{-2z}\right) = \frac{1}{x}e^{-2x/(1+s)}$$

is then

$$\frac{1}{x}e^{-2x/(1+s)} = \frac{1}{x}e^{-2x} + \sum_{n=1}^{\infty}(-1)^n\frac{s^n}{n!}\frac{d^n}{dx^n}[e^{-2x}x^{n-1}].$$

We thus have a representation of $k_{2n}(x)$ for $x>0$,

$$(3.1)\qquad k_{2n}(x) = \frac{(-1)^n x e^x}{n!}\frac{d^n}{dx^n}[e^{-2x}x^{n-1}],$$

which is analogous to Sonine's formula

* This relation was obtained by N. H. Abel, Oeuvres (Sylow and Lie), vol. II, p. 284.

† N. Sonine, loc. cit., p. 41

$$L_n^{(\alpha)}(x) = \frac{1}{n!} e^x x^{-\alpha} \frac{d^n}{dx^n} [x^{n+\alpha} e^{-x}], \tag{3.2}$$

for the generalized Laguerre polynomial.

4. **The difference equations and inequalities.** It is readily seen from (1.1) that

$$\begin{aligned}(n-2)[k_n(x) + k_{n-2}(x)] + (n+2)[k_n(x) + k_{n+2}(x)] - 4xk_n(x) \\ = -\frac{8}{\pi}\int_0^{\pi/2} \frac{d}{d\theta}[\cos^2\theta \sin(x\tan\theta - n\theta)]d\theta \\ = 0.\end{aligned} \tag{4.1}$$

It is also seen that

$$4xk_n'(x) = (n-2)k_{n-2}(x) - (n+2)k_{n+2}(x), \tag{4.2}$$

for we have the relation

$$\begin{aligned} k_n'(x) + k_{n+2}'(x) &= -\frac{2}{\pi}\int_0^{\pi/2}[\sin(x\tan\theta - n\theta) \\ &\quad + \sin(x\tan\theta - n\theta - 2\theta)]\tan\theta d\theta \\ &= -\frac{4}{\pi}\int_0^{\pi/2}\sin[x\tan\theta - n\theta - \theta]\sin\theta d\theta \\ &= +\frac{2}{\pi}\int_0^{\pi/2}[\cos(x\tan\theta - n\theta) - \cos(x\tan\theta - n\theta - 2\theta)]d\theta,\end{aligned}$$

in which we must be careful to form the expression for $k_n(x)+k_{n+2}(x)$ as a definite integral before we differentiate to form an expression for the quantity on the left hand side. When the ensuing relation

$$k_n'(x) + k_{n+2}'(x) = k_n(x) - k_{n+2}(x) \tag{4.3}$$

is combined with (4.1) it leads to (4.2). It should be remarked that when the difference equation (4.1) is used to calculate $k_n(x)$ for even negative values of n, using the known values of $k_n(x)$ for positive even values of n it is found that when $x>0$ we have $k_{-2m}(x)=0$ for all positive integral values of m. This is what was anticipated in §1. With the aid of (1.5) and (4.1) we obtain the inequality

$$|k_n(x)| \leqq \frac{|n|}{|x|} \quad (n > 2). \tag{4.4}$$

Similarly (4.2) gives the inequality

$$|k_n'(x)| \leqq \frac{|n|}{2|x|} \quad (n > 2). \tag{4.5}$$

These inequalities show that as $|x|\to\infty$, $k_n(x)\to 0$ and $k_n'(x)\to 0$. Another useful inequality is obtained by combining (4.4) with (4.1):

$$(4.6)\qquad |k_n(x)| < \frac{n^2+2}{|x|^2} \quad (n>2).$$

When s is a positive integer, $k_{2s}(x)$ is zero for $x=0$ and is finite for positive values of x; consequently we can find a positive number $\phi(s)$ such that

$$(4.7)\qquad |k_{2s}(x)| \leqq x\phi(s),\quad 0\leqq x\leqq 1.$$

This inequality will be used later in combination with (4.4) and (4.6).

5. **The differential equation.** The relation (4.2) gives

$$4xk_n''(x) + 4k_n'(x) = (n-2)k_{n-2}'(x) - (n+2)k_{n+2}'(x),$$

that is,

$$\begin{aligned} 4xk_n''(x) &= (n-2)[k_{n-2}'(x) + k_n'(x)] - (n+2)[k_{n+2}'(x) + k_n'(x)] \\ &= (n-2)[k_{n-2}(x) - k_n(x)] - (n+2)[k_n(x) - k_{n+2}(x)] \\ &= 4(x-n)k_n(x). \end{aligned}$$

Hence the function $k_n(x)$ satisfies the differential equation

$$(5.1)\qquad xk_n''(x) = (x-n)k_n(x).$$

This is an equation of Laplace's type and is a degenerate form of the canonical equation adopted by Whittaker* in his study of the confluent hypergeometric functions.

6. **The orthogonal relations.** The usual method of deriving orthogonal relations from a differential equation suggests that we should consider the value of the definite integral

$$(6.1)\qquad I_{m,n} = \int_{-\infty}^{\infty} k_{2m}(x)k_{2n}(x)\frac{dx}{x},$$

in which m and n are not simultaneously zero. When m and n are positive integers, a reduction formula

$$\begin{aligned} (n-1)I_{m,n-1} + (n+1)I_{m,n+1} + 2nI_{m,n} &= 0,\quad n>m+1,\\ &= 1,\quad n=m+1,\\ &= 2,\quad n=m,\\ &= 1,\quad n=m-1,\\ &= 0,\quad n<m-1, \end{aligned}$$

* Whittaker and Watson, *Modern Analysis*, chapter 16; see also H. A. Webb and J. R. Airey, Philosophical Magazine, (6), vol. 36 (1918), p. 129.

for $I_{m,n}$ is readily derived from the difference equation (4.1) and the relations (2.91). When $n=1$ the formula (2.91) also gives

$$I_{m,1} = 2\int_0^\infty e^{-x}k_{2m}(x)dx = 0, \quad m > 1,$$
$$= 1, \quad m = 1,$$

while (3.1) gives

$$I_{m,0} = \int_0^\infty e^{-x}k_{2m}(x)\frac{dx}{x}$$
$$= \frac{(-1)^m}{m!}\int_0^\infty \frac{d^m}{dx^m}[e^{-2x}x^{m-1}]dx \tag{6.2}$$
$$= (-1)^{m-1}\frac{1}{m}, \quad m > 0.$$

With the aid of these particular relations and the reduction formula the integral $I_{m,n}$ can be calculated step by step and is found to be zero when $m>1$ and $n<m$. On the other hand the reduction formula and the particular values give

$$I_{m,m} = \frac{1}{m} \quad (m > 0). \tag{6.3}$$

Hence, if $m>0$ the functions $k_{2m}(x)$ form an orthogonal set. When m and n have any real values the integral $I_{m,n}$ may be understood to have its principal value. To find this we note that the differential equation gives

$$\frac{d}{dx}[k_{2n}(x)k'_{2m}(x) - k_{2m}(x)k'_{2n}(x)] = \frac{2}{x}(n-m)k_{2n}(x)k_{2m}(x).$$

Therefore since $k_{2n}'(x)$ and $k_{2n}(x)\to 0$ as $|x|\to\infty$, we have

$$\int_\epsilon^\infty k_{2n}(x)k_{2m}(x)\frac{dx}{x} = \frac{1}{2(m-n)}[k_{2n}(\epsilon)k'_{2m}(\epsilon) - k_{2m}(\epsilon)k'_{2n}(\epsilon)],$$

$$\int_{-\infty}^{-\epsilon} k_{2n}(x)k_{2m}(x)\frac{dx}{x} = \frac{1}{2(m-n)}[k_{2m}(-\epsilon)k'_{2n}(-\epsilon) - k_{2n}(-\epsilon)k'_{2m}(-\epsilon)].$$

Now as $\epsilon\to 0$,

$$k_{2n}(\epsilon) \to k_{2n}(-\epsilon) = \frac{1}{n\pi}\sin(n\pi),$$

and the integral (1.1) gives

$$k'_{2n}(x) = -\frac{2}{\pi}\int_0^\infty \frac{\sin (xt - 2n\theta)tdt}{1+t^2},$$

where $t=\tan\theta$. When $2n$ is an odd integer $k_{2n}'(x)$ becomes infinite as $x\to 0$ but in any case

$$k'_{2n}(\epsilon) - k'_{2n}(-\epsilon) = -\frac{4}{\pi}\int_0^\infty \frac{[\sin(\epsilon t)][\cos 2n\theta]tdt}{1+t^2}$$

$$= -\frac{4}{\pi}\int_0^\infty \frac{[\sin u][\cos 2n\theta]udu}{\epsilon^2+u^2}, \quad \tan\theta = \frac{u}{\epsilon},$$

$$\to -\frac{4}{\pi}(\cos n\pi)\int_0^\infty \sin u du/u = -2\cos n\pi.$$

Hence

$$P\int_{-\infty}^{\infty} k_{2n}(x)k_{2m}(x)\frac{dx}{x} = \frac{1}{\pi(m-n)}\left[\frac{1}{m}\sin m\pi\cos n\pi - \frac{1}{n}\sin n\pi\cos m\pi\right]$$

$$= \frac{m+n}{2\pi mn}\left[\frac{\sin(m-n)\pi}{m-n} - \frac{\sin(m+n)\pi}{m+n}\right]. \tag{6.4}$$

This formula may be used with (4.1) to obtain the following generalization of (2.91):

$$\int_{-\infty}^{\infty} k_{2n}(x)k_{2m}(x)dx = \frac{\sin[(n-m)\pi]}{\pi(m-n+1)(m-n)(m-n-1)}. \tag{6.5}$$

When m and n are integers,

$$P\int_{-\infty}^{\infty} k_{2n+1}(x)k_{2m+1}(x)\frac{dx}{x} = 0, \quad m\neq n,$$

$$= \frac{2}{\pi(2n+1)}, \quad m = n. \tag{6.6}$$

7. **The interpolation formula.** It is useful to have an alternative definition of $k_n(x)$ from which its properties may be developed. Such a definition is obtained by making use of the cardinal function of interpolation theory, the properties of which have been developed by Professor E. T. Whittaker.* We thus write for all real values of n and x

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* Proceedings of the Royal Society of Edinburgh, vol. 35 (1915), p. 181. See also W. L. Ferrar, ibid., vol. 45 (1925), p. 269, vol. 46 (1926), p. 323, vol. 47 (1927), p. 230; J. M. Whittaker, Proceedings of the Edinburgh Mathematical Society, (2), vol. 1 (1927), pp. 41, 169; E. T. Copson, ibid., p. 129.

$$k_n(x) = \frac{2}{\pi} \sum_{m=-\infty}^{\infty} \frac{\sin (2m - n)\dfrac{\pi}{2}}{2m - n} k_{2m}(x). \tag{7.1}$$

The absolute convergence of this expansion may be established with the aid of Fejér's asymptotic formula* for $L_n(2x)$. Assuming $x>0$, we may write

$$L_n(2x) \sim \pi^{-1/2}e^x(2nx)^{-1/4} \cos\left[(8nx)^{1/2} - \frac{\pi}{4}\right] + O(n^{-1/2}), \tag{7.2}$$

and make use of the relation

$$k_{2m}(x) = (-1)^m e^{-x}[L_m(2x) - L_{m-1}(2x)]. \tag{2.7}$$

The two Laguerre series obtained by substituting the last expression in (7.1) are also absolutely convergent and so we may add them together in a manner different from that adopted in (7.1), and obtain a single Laguerre series

$$e^x k_n(x) = -\frac{4}{\pi} \sin\left(\frac{n\pi}{2}\right) \sum_{m=0}^{\infty} \frac{L_m(2x)}{(2m-n)(2m+2-n)}, \tag{7.3}$$

which is absolutely and uniformly convergent for all positive values of x including zero. Making use of the equation

$$\int_0^{2x} L_m(u)du = L_m(2x) - L_{m+1}(2x),$$

it is readily seen that

$$\begin{aligned} \int_0^x e^u k_n(u)du &= \frac{2}{\pi} \sin\left(\frac{n\pi}{2}\right) \sum_{m=0}^{\infty} \frac{L_{m+1}(2x) - L_m(2x)}{(2m-n)(2m+2-n)} \\ &= \frac{1}{2} e^x[k_{n+2}(x) + k_n(x)] - \frac{2}{\pi} \sin\frac{n\pi}{2} \frac{1}{n(n+2)}. \end{aligned} \tag{7.4}$$

Differentiating this equation with respect to x we obtain the relation

$$k_n(x) - k_{n+2}(x) = k'_{n+2}(x) + k'_n(x). \tag{4.3}$$

Again, if we make use of the well known relation

$$(n+1)L_{n+1}(u) - (2n+1-u)L_n(u) + nL_{n-1}(u) = 0,$$

it is seen from (7.3) that $k_n(x)$ satisfies the difference equation

$$(n-2)k_{n-2}(x) + (n+2)k_{n+2}(x) = (4x - 2n)k_n(x), \tag{4.1}$$

* Simple proofs of the theorem are given by Szegö, Mathematische Zeitschrift, vol. 1 (1918), p. 341, and O. Perron, Journal für die reine und angewandte Mathematik, vol. 151 (1920), p. 163.

and with the aid of (4.3) the relation

$$4xk_n'(x) = (n-2)k_{n-2}(x) - (n+2)k_{n+2}(x) \tag{4.2}$$

can be established.

The differential equation (5.1) may now be obtained as in §5 and with the aid of (7.1) we may obtain the further relations

$$k_{-n}(x) = k_n(-x),$$

$$k_n(0) = \frac{2}{n\pi}\sin\frac{n\pi}{2}\cdot$$

Thus all the principal properties of the function $k_n(x)$ have been obtained directly from the interpolation formula (7.1). The value of the integral

$$\int_0^\infty k_n(x)k_{2s}(x)\frac{dx}{x}$$

may be calculated by making use of the B-test* for the integration of an infinite series over an infinite range. In using this test we make use of Szegö's inequality†

$$e^{-x}\,|\,L_m(2x)\,| \leqq 1, \quad x \geqq 0, \tag{7.5}$$

and the inequality of §4

$$|\,k_{2s}(x)\,| < \frac{2s}{x}, \qquad x > 1,$$

$$< x\phi(s), \quad 0 < x < 1,$$

to prove that the series

$$\sum_{m=0}^{\infty}\frac{1}{(2m-n)(2m+2-n)}\int_0^\infty |\,k_{2s}(x)\,|\,e^{-x}\,|\,L_m(2x)\,|\,dx$$

converges.

The definite integral

$$\int_0^\infty k_{2s}(x)(dx/x)e^{-x}L_m(2x)$$

is calculated with the aid of (2.81) and is found to have the value

$$\frac{1}{s}, \quad m \geqq s;$$

$$0, \quad m < s.$$

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* Bromwich's *Infinite Series*, p. 453 (1st edition).

† Mathematische Zeitschrift, vol. 1 (1918), p. 341.

It is thus found that when s is a positive integer

$$(7.6)\qquad \int_0^\infty k_n(x)k_{2s}(x)\frac{dx}{x} = \frac{4}{\pi n}\,\frac{\sin\,(2s-n)\dfrac{\pi}{2}}{2s-n}\,.$$

Taking (6.2) into consideration we can regard (7.1) as the k-series* for the function $k_n(x)$ whether this function is defined by (1.1) or (7.1).

The B-test may also be used to calculate the integral

$$\int_0^\infty k_n(x)e^{-x}L_{m'}(2x)dx$$

by the integration of (7.3) term by term. In this case we make use of (2.81) and a combination of (4.6) and (4.7) to prove the convergence of the series

$$\sum_{m=0}^{\infty}\frac{1}{(2m-n)(2m+2-n)}\int_0^\infty e^{-2x}\,|\,L_{m'}(2x)\,|\,|\,L_m(2x)\,|\,dx.$$

In this way it may be shown that (7.3) is the Laguerre series for the function $k_n(x)e^x$ whichever definition is adopted for $k_n(x)$.

When $k_n(x)$ is defined with the aid of (7.3) the integral (6.5) may be calculated by using the Parseval theorem for the Laguerre functions.† The analysis leads to the interesting equation

$$(7.7)\ \cot\,(m\pi) - \cot\,(n\pi) = \frac{1}{2\pi}\sum_{s=-\infty}^{\infty}\frac{(n-m)(m-n+1)(n-m+1)}{(s-n)(s-m)(s+1-n)(s+1-m)},$$

which holds for both real and complex values of m and n, as may be seen by comparing the residues of the functions of m on the two sides of the equation.

The equivalence of the two definitions of $k_n(x)$ may be inferred from the fact that the two functions have the same Laguerre series or it may be proved by means of Parseval's theorem for Fourier series, the two functions $f(\theta)$, $g(\theta)$ in the integral

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)g(\theta)d\theta$$

being defined as follows:‡

* The k-series for a function $f(x)$ is of type $f(0)k_0(x)+c_1k_2(x)+\cdots$ where the coefficients c_n are calculated with the aid of the orthogonal relation of §6.

† For this theorem see S. Wigert, Arkiv för Matematik, Astronomi och Fysik, vol. 15 (1921); M. Riesz, Acta Litterarum ac Scientiarum regiae Universitatis Hungaricae Francisco-Josephinae, vol. 1 (1923), p. 209.

‡ It should be noticed that $f(\theta)$ is not of bounded variation in the interval $(-\pi, \pi)$ and is discontinuous at the points $\theta=\pm\pi/2$. The convergence of its Fourier series is discussed briefly in §9. A really elementary proof of the equivalence of the two definitions of $k_n(x)$ has not been obtained.

$$f(\theta) = e^{ix\tan\theta}, \quad -\pi < \theta < \pi;$$
$$g(\theta) = 2e^{-in\theta}, \quad -\pi < 2\theta < \pi,$$
$$= 0, \quad -2\pi < 2\theta < -\pi,$$
$$\text{or} \quad \pi < 2\theta < 2\pi;$$

the integral then becomes equal to the integral (1.1) defining $k_n(x)$ while the Parseval series becomes identical with (7.1). The use of an interpolation formula of type (7.1) for the representation of a function of n in terms of its values when n is an integer (or has even integral values) is not new. W. L. Ferrar has kindly informed me that the Legendre function $P_n(x)$ was expressed as a series of Legendre polynomials by J. Dougall* long ago. The interpolation formula was also used by de la Vallée Poussin† to approximate to the value of a function over a limited range.

8. **The exponential integral.** If a is positive and n is a positive integer,

$$
\begin{aligned}
\int_0^\infty e^{-ax}k_{2n}(x)dx &= \frac{(-1)^n}{n!}\int_0^\infty xe^{x(1-a)}\frac{d^n}{dx^n}[e^{-2x}x^{n-1}]dx \\
&= \frac{1}{n!}\int_0^\infty e^{-2x}x^{n-1}\frac{d^n}{dx^n}\{xe^{x(1-a)}\}dx \\
&= \frac{1}{n!}\int_0^\infty e^{-x(1+a)}\{x^n(1-a)^n + nx^{n-1}(1-a)^{n-1}\}dx \\
&= \frac{(1-a)^n}{(1+a)^{n+1}} + \frac{(1-a)^{n-1}}{(1+a)^n} = \frac{2}{(1+a)^2}\left(\frac{1-a}{1+a}\right)^{n-1}.
\end{aligned} \tag{8.1}
$$

On the other hand,

$$\int_0^\infty e^{-ax}k_0(x)dx = \frac{1}{1+a}\,. \tag{8.2}$$

A general formula valid for all real values of n may be obtained from (7.3) with the aid of Parseval's theorem for the Laguerre functions and the known formula

$$\int_0^\infty e^{-x(1+a)}L_m(2x)dx = (-1)^m\frac{(1-a)^m}{(1+a)^{m+1}}, \tag{8.3}$$

which gives the Laguerre constants for the function $e^{-x(a-1)}$. The formula may be written in the two forms

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* Proceedings of the Edinburgh Mathematical Society, vol. 18 (1900), p. 79. See also H. B. C. Darling, Quarterly Journal of Mathematics, vol. 49 (1923), p. 289.

† Bulletin de l'Académie Royale de Belgique (Classe de Sciences), 1908, p. 341. See also J. M. Whittaker, Proceedings of the Edinburgh Mathematical Society, (2), vol. 1 (1928), p. 169.

$$\int_0^\infty e^{-ax}k_{2n}(x)dx = \frac{1}{\pi}\sin n\pi\sum_{m=0}^{\infty}(-1)^{m+1}\frac{1}{(m-n)(m+1-n)}\frac{(1-a)^m}{(1+a)^{m+1}}$$

$$(8.4)\qquad = \frac{1}{n\pi}\sin n\pi\left[\frac{1}{1+a}-\frac{2n}{(1+a)^2}\left\{\frac{1}{n-1}-\frac{1-a}{1+a}\frac{1}{n-2}\right.\right.$$
$$\left.\left.+\left(\frac{1-a}{1+a}\right)^2\frac{1}{n-3}-\cdots\right\}\right].$$

When n is a positive integer the limiting form of the right hand side must be taken. When n is a negative integer the integral is zero as we should expect.

If

$$(8.5)\qquad y_n(a) = \int_0^\infty e^{-ax}k_{2n}(x)dx,$$

it is readily seen that $y_n(a)$ satisfies the difference equation

$$y_{n+1}(a) - \frac{1-a}{1+a}y_n(a) = \frac{\sin n\pi}{\pi}\,\frac{1}{n(n+1)(1+a)}.$$

This may be seen directly with the aid of (4.3).

9. **Some special series.** The relation (2.7) and Fejér's asymptotic formula (7.2) may be used to prove that when m is a positive integer $k_{2m}(x)\to 0$ as $m\to\infty$ and the same result may be derived from the following asymptotic formula which is derived from a result given by Perron (loc. cit.):

$$\pi^{1/2}k_{2m}(x) \sim (-1)^{m+1}(2x)(2mx)^{-3/4}\cos\left[(8mx)^{1/2}-3\pi/4\right].$$

Since $k_{2m}(x)\to 0$ it follows from Fatou's theorem* that the power series (2.2) converges at all regular points on the circle of convergence. The point $s=-1$ is the only irregular point and from the extension of Fatou's theorem given by Riesz† it may be concluded that the convergence is uniform on any arc which does not contain the point $s=-1$. The same result can be derived also from the theory of Fourier series.‡

To examine the convergence of the series at the point $s=-1$ we first note that the relation (4.3) gives

$$k_0'(x)+2k_2'(x)+2k_4'(x)+\cdots+2k_{2n-2}'(x)+k_{2n}'(x) = k_0(x)-k_{2n}(x).$$

* P. Fatou, Acta Mathematica, vol. 30 (1906), p. 335.

† M. Riesz, Journal für die reine und angewandte Mathematik, vol. 140 (1911), p. 89.

‡ A similar result for the series defining the generalised Laguerre polynomials is mentioned by E. Hille, Proceedings of the National Academy of Sciences, vol. 12 (1926), p. 261. See also G. Szegö, Mathematische Zeitschrift, vol. 25 (1926), p. 87.

The asymptotic formulas indicate that if $0 \leqq x \leqq a$ we can find a number m independent of x such that for $n > m$

$$| k_{2n}(x) | < \epsilon,$$

where ϵ is any preassigned small positive quantity. The series on the left can be regarded, then, as converging uniformly in x when we put $n = \infty$, and since $k_0(x) = e^{-x}$, $k_0'(x) = -e^{-x}$ it seems that the series converges uniformly to zero. Integrating it term by term we find that, when $0 \leqq x \leqq a$,

$$k_0(x) + k_2(x) + \cdots + k_{2n}(x) \to 1 \text{ uniformly as } n \to \infty .$$

Again, the relation (4.3) gives

$$\begin{aligned} k_0(x) - k_2(x) + k_4(x) - \cdots + k_{4n}(x) - k_{4n+2}(x) \\ = k_0'(x) + k_2'(x) + k_4'(x) + \cdots + k_{4n+2}'(x). \end{aligned}$$

As $n \to \infty$ the series on the right tends uniformly to zero hence the series on the left also tends uniformly to zero. This establishes the convergence of the power series (2.2) for $s = -1$ and of the series (2.1) for $\alpha = 0$, $\theta = \pi/2$.

Putting $\alpha = 0$, $\theta = \pi/4$, we find that

$$\begin{aligned} \cos x &= k_0(x) - k_4(x) + k_8(x) - \cdots, \\ \sin x &= k_2(x) - k_6(x) + k_{10}(x) - \cdots, \end{aligned}$$

and in general, if $2n\theta = \pi$,

$$e^{ix \tan\theta} = N_0(x) + e^{2i\theta}N_2(x) + \cdots + e^{2(n-1)i\theta}N_{2(n-1)}(x),$$

where

$$\begin{aligned} N_0(x) &= k_0(x) - k_{2n}(x) + k_{4n}(x) - \cdots, \\ N_2(x) &= k_2(x) - k_{2n+2}(x) + k_{4n+2}(x) - \cdots, \\ &\cdots\cdots\cdots\cdots\cdots\cdots . \end{aligned}$$

Furthermore, with this value of θ,

$$\begin{aligned} e^{ix \tan 3\theta} &= N_0(x) + e^{6i\theta}N_2(x) + \cdots + e^{6(n-1)i\theta}N_{2(n-1)}(x), \\ e^{ix \tan 5\theta} &= N_0(x) + e^{10i\theta}N_2(x) + \cdots . \end{aligned}$$

It can, indeed, be shown directly from (2.3) that, if $D \equiv d/dx$,

$$\begin{aligned} N_0(x) &= D[N_2(x) + N_4(x) + \cdots + N_{2(n-1)}(x)], \\ DN_0(x) + N_2(x) &= D[N_4(x) + N_6(x) + \cdots + N_{2(n-1)}(x)], \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ D[N_0(x) + N_2(x) + \cdots + N_{2n-4}(x)] &+ N_{2(n-1)}(x) = 0. \end{aligned}$$

If

$$f(D) = \begin{vmatrix} 1 & -D & -D \cdots -D \\ D & 1 & -D \cdots -D \\ \cdot & \cdot & \cdots \\ D & D & \cdots \quad 1 \end{vmatrix},$$

where the determinant has n rows and n columns, the differential equation $f(D)w=0$ has the n particular solutions

$$w_1 = e^{ix\tan\theta},\quad w_2 = e^{ix\tan 3\theta}, \cdots, \quad w_n = e^{ix\tan(2n-1)\theta}.$$

To see this we consider the set of n linear equations

$$\begin{aligned} x_0 &= i\tan\theta[x_1 + x_2 + \cdots + x_{n-1}], \\ ix_0\tan\theta + x_1 &= i\tan\theta[x_2 + x_3 + \cdots + x_{n-1}], \\ &\cdots\cdots\cdots \\ i(x_0 + x_1 + \cdots + x_{n-2})\tan\theta + x_{n-1} &= 0. \end{aligned}$$

Writing $\lambda = x_0+x_1+x_2+\cdots+x_{n-1}$, we find successively

$$\begin{aligned} x_0 &= i\lambda\sin\theta e^{-i\theta}, \qquad x_1 = i\lambda\sin\theta e^{-3i\theta}, \\ x_2 &= i\lambda\sin\theta e^{-5i\theta}, \cdots, \quad x_{n-1} = i\lambda\sin\theta e^{-(2n-1)i\theta}, \\ \lambda &= i\lambda\sin\theta e^{-i\theta}[1 + e^{-2i\theta} + \cdots + e^{-(2n-2)i\theta}] \\ &= i\lambda\sin\theta e^{-i\theta}\frac{1 - e^{-2ni\theta}}{1 - e^{-2i\theta}} = \frac{\lambda}{2}[1 - e^{-2ni\theta}]. \end{aligned}$$

Hence, if $\lambda \neq 0$ we must have $e^{2ni\theta} = -1$ or $2n\theta = s\pi$, where s is an odd integer. It is easily seen that

$$\begin{aligned} nN_0(x) &= w_1 + w_2 + \cdots + w_n, \\ nN_2(x) &= w_1e^{-2i\theta} + w_2e^{-6i\theta} + \cdots + w_ne^{-2i(2n-1)}\theta, \\ nN_4(x) &= w_1e^{-4i\theta} + w_2e^{-12i\theta} + \cdots + w_ne^{-4i(2n-1)i\theta}, \end{aligned}$$

and so on.

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ON THE CURVATURES OF A CURVE IN RIEMANN SPACE*†

BY

E. H. CUTLER

Introduction. The curvature and torsion of a curve in ordinary space have three properties which it is the purpose of this paper to attempt to extend to the curvatures of a curve in Riemann space. First, if the curvature vanishes identically the curve is a straight line; if the torsion vanishes identically the curve lies in a plane. Second, the distances of a point of the curve from the tangent line and the osculating plane at a nearby point are given approximately by formulas involving the curvature and torsion. Third, the curvature of a curve at a point is the curvature of its projection on the osculating plane at the point. In extending to Riemann space we take as the Riemannian analogue of the line or plane, a geodesic space generated by geodesics through a point. Such a space possesses the property of the line or plane of being determined by the proper number of directions given at a point, but it will not in general have the three properties given above. On the other hand, if we take as the analogue of line or plane only totally geodesic spaces, then, if such osculating "planes" exist, the three properties will hold.

Curves with a vanishing curvature. Given a curve C: $x^i = x^i(s)$, $i=1, \cdots, n$, in a Riemann space V_n with fundamental tensor g_{ij} (assumed definite). Following Blaschke‡ we write the Frenet formulas for the curve. The n associate vectors are given by

$$(1) \qquad \xi_1|^i = \frac{dx^i}{ds}, \quad \xi_r|^i{}_{,j}\frac{dx^j}{ds} = \xi'_{r+1}|^i \ (r = 1, \cdots, (n-1)).$$

In general these n vectors are independent and will determine an orthogonal n-uple, $\lambda_r|^i$, in terms of which we have the Frenet formulas

$$(2) \qquad \begin{aligned} &\lambda_1|^i = \xi_1|^i = \frac{dx^i}{ds}, \\ &\lambda_r|^i{}_{,j}\frac{dx^j}{ds} = -(1/\rho_{r-1})\lambda_{r-1}|^i + (1/\rho_r)\lambda_{r+1}|^i, \quad r = 1, \cdots, n, \ (1/\rho_n) = 0. \end{aligned}$$

* Presented to the Society, February 28, 1931; received by the editors February 10, 1931.

† The results of this and the following paper are part of a thesis submitted at Harvard, June 1930, and written under Professor H. W. Brinkmann.

‡ Blaschke, *Frenets Formeln für den Raum von Riemann*, Mathematische Zeitschrift, vol. 6, pp. 94–99. See also Eisenhart, *Riemannian Geometry*, §32.

If, however, there are, at a general point of the curve, only $k(k<n)$ independent associate vectors, then (2) will hold for $r=1, \cdots, k$ where $(1/\rho_k)=0$. This case will be described by saying $(1/\rho_k)=0$. In the general case the vector $\xi_r|^i$ is always dependent on $\lambda_1|^i, \cdots, \lambda_r|^i$, and in this case the associate vectors of all orders will be dependent on $\lambda_1|^i, \cdots, \lambda_k|^i$.

DEFINITION. *The linear vector space at P determined by the vectors $\lambda_r|^i$, $r=1, \cdots, q$, will be the qth osculating vector space of C at P. If $(1/\rho_k)=0$, the kth osculating vector space will be called the complete osculating vector space.*

We wish to discuss the relations of C to its complete osculating geodesic space G_k at P, where G_k is the locus of the ∞^{k-1} geodesics through P in directions of the complete osculating vector space. To discuss the question we take Riemannian coördinates at P. Then G_k is given by linear equations, and we wish to see whether the vector $[x^i(s)-x^i(P)]$ satisfies these, or, in other words, whether in this coördinate system it is linearly dependent on $\lambda_1|^i, \cdots, \lambda_k|^i$. Expanding $x^i(s)$ about P, the question reduces to that of the dependence of the ordinary derivatives of $x^i(s)$ of all orders on the $\lambda_q|^i$, $q=1, \cdots, k$, or on the $\xi_r|^i$, $r=1, 2, \cdots$, since these are so dependent. From (1) we have

$$\xi_1|^i = \lambda_1|^i = \frac{dx^i}{ds},$$

$$\xi_2|^i = \frac{d^2x^i}{ds^2} + \begin{Bmatrix} i \\ jk \end{Bmatrix} \frac{dx^j}{ds}\frac{dx^k}{ds},$$

$$\xi_3|^i = \frac{d^3x^i}{ds^3} + 3\begin{Bmatrix} i \\ jk \end{Bmatrix}\frac{d^2x^j}{ds^2}\frac{dx^k}{ds} + \left[\frac{\partial}{\partial x^h}\begin{Bmatrix} i \\ jk \end{Bmatrix} + \begin{Bmatrix} i \\ jl \end{Bmatrix}\begin{Bmatrix} l \\ kh \end{Bmatrix}\right]\frac{dx^h}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}, \tag{3}$$

$$\xi_4|^i = \frac{d^4x^i}{ds^4} + 5\frac{\partial}{\partial x^h}\begin{Bmatrix} i \\ jk \end{Bmatrix}\frac{d^2x^i}{ds^2}\frac{dx^k}{ds}\frac{dx^h}{ds} + \frac{\partial}{\partial x^h}\begin{Bmatrix} i \\ jk \end{Bmatrix}\frac{dx^j}{ds}\frac{dx^k}{ds}\frac{d^2x^h}{ds^2}$$

$$+ \frac{\partial^2}{\partial x^h\partial x^l}\begin{Bmatrix} i \\ jk \end{Bmatrix}\frac{dx^h}{ds}\cdots\frac{dx^k}{ds} + \text{terms in undifferentiated } \begin{Bmatrix} i \\ jk \end{Bmatrix}.$$

The equations (3) hold in any coördinate system. For a Riemannian system of coördinates we have, at the center P,

$$\begin{Bmatrix} i \\ jk \end{Bmatrix} = 0, \qquad S\left(\frac{\partial^u}{\partial x^h\cdots\partial x^l}\begin{Bmatrix} i \\ jk \end{Bmatrix}\right) = 0, \quad S = \text{symmetric part}. \tag{4}$$

From (3) and (4) it follows that the first three derivatives of $x^i(s)$ at P are actually equal to the corresponding $\xi_r|^i$, and that for $\xi_4|^i$ we will have at P

$$\xi_4|^i = \frac{d^4x^i}{ds^4} + 3\frac{\partial}{\partial x^h}\left\{ {i \atop jk} \right\}\frac{d^2x^j}{ds^2}\frac{dx^h}{ds}\frac{dx^k}{ds}.$$

Using (4) and the definition of R^i_{jkl}, we show that at P

$$(5)\qquad \begin{aligned} 3\frac{\partial}{\partial x^h}\left\{ {i \atop jl} \right\} &= R^i_{jhl} - R^i_{ljh}, \\ \xi_4|^i &= \frac{d^4x^i}{ds^4} - R^i_{jlh}\frac{d^2x^l}{ds^2}\frac{dx^j}{ds}\frac{dx^h}{ds}. \end{aligned}$$

The second term in general neither vanishes nor lies in any special vector space, as may be shown by examples. Hence the theorem we were seeking does not hold in general. A special case for which it does hold is that for which the G_k is totally geodesic in V_n. To show this we make use of a development for $(\partial^r/\partial x^l \cdots \partial x^h)\left\{ {i \atop jk} \right\}$, due to Veblen,* of which (5) is a special case. According to this, $(\partial^r/\partial x^l \cdots \partial x^h)\left\{ {i \atop jk} \right\}$ is a combination of terms in $R^i_{jkl,\cdots h}$, subscripts permuted in all possible ways, and of terms of inner products of similar factors of lower orders. Beside this we need a lemma.

LEMMA. *Let G_k be totally geodesic in V_n and let $\eta_1|^i \cdots \eta_s|^i$ be any vectors tangent to G_k. Then the vectors $R^i_{jkl}\eta_1|^j\eta_2|^k\eta_3|^l, \cdots, R^i_{jkl,\cdots h}\lambda_1|^j \cdots \lambda_s|^h$ lie in G_k.*

To prove this we recall that for a totally geodesic $G_k: x^i = x^i(u^1, \cdots, u^k)$,

$$(6)\qquad \Omega_\sigma|_{\alpha\beta} \equiv 0 \quad (\alpha, \beta = 1, \cdots, k; \sigma = 1, \cdots, (n-k)),$$

and hence by the Codazzi equation

$$(7)\qquad R_{ijkl}\xi_\sigma|^i\frac{\partial x^j}{\partial u^\alpha}\frac{\partial x^k}{\partial u^\beta}\frac{\partial x^l}{\partial u^\gamma} = 0 \qquad (\alpha, \beta, \gamma = 1, \cdots, k),$$

or $R^i_{jkl}\eta_1|^j\eta_2|^k\eta_3|^l$ lies in G_k. Differentiating (7) covariantly, and using (6) and (7), we have

$$(8)\qquad R_{ijkl,h}\xi_\sigma|^i\frac{\partial x^j}{\partial u^\alpha}\frac{\partial x^k}{\partial u^\beta}\frac{\partial x^l}{\partial u^\gamma}\frac{\partial x^h}{\partial u^\delta} = 0,$$

and similarly for continued differentiation. Hence the lemma.

Combining this lemma, which holds for any coördinate system, with the expression for the derivatives of the $\left\{ {i \atop jk} \right\}$ at the center of a Riemannian coördinate system in terms of the R^i_{jkl} and its derivatives, we have

* O. Veblen, Proceedings of the National Academy of Sciences, vol. 8, p. 196.

THEOREM 1. *If G_k is totally geodesic in V_n and the coordinates are Riemannian at P, and if $\eta_1|^i, \cdots, \eta_s|^i$ lie in G_k, then*

$$\frac{\partial^r}{\partial x^h \cdots \partial x^l}\left\{ {i \atop jk} \right\}\eta_1|^h \cdots \eta_s|^k \tag{9}$$

is a vector in G_k.

Now assume that G_k, the complete osculating geodesic space of the curve C at P, is totally geodesic in V_n. By (3) we see that the rth derivative of $x^i(s)$ depends on $\xi_r|^i$ and on terms of type (9), or products of factors of type (9) where the $\eta_s|^i$ are derivatives of $x^i(s)$ of order l, $l<r$. Hence by the theorem above, and by induction, we show that the derivatives of $x^i(s)$ of all orders lie in G_k. Hence the curve lies in G_k.

THEOREM 2. *If the kth curvature $(k<n)$ of a curve is identically zero, and if the complete osculating geodesic space G_k at a point P is totally geodesic, then the curve lies in G_k.*

COROLLARY. *If the kth curvature of a curve in a space of constant curvature vanishes identically, then the curve lies in the complete osculating geodesic space at any point.*

Distance from an osculating geodesic space. Given a curve in V_n which now is not assumed to have vanishing curvatures. We wish to obtain a formula for the distance of a point P' of the curve from the various osculating geodesic spaces G_h at a nearby point P. Let P be the center of a system of normal Riemannian coördinates so chosen that G_h is given by

$$x^i = 0 \qquad (i = (h+1), \cdots, n). \tag{10}$$

THEOREM 3. *Given a system of normal Riemannian coördinates with center at P: (0), and given a sequence of points P_r: (x^i) approaching P as a limit, then the principal part of the infinitesimal distance of P_r from the geodesic sub-space G_h: $x^i=0$, $i=(h+1), \cdots, n$, is given by*

$$d^2 = \sum_{i=h+1}^{n}(x^i)^2 + \cdots . \tag{11}$$

The proof of this consists first in showing that the true length of the geodesic from P_r orthogonal to G_k and the length this geodesic would have in terms of δ_{ij} as fundamental tensor are equivalent infinitesimals. The same is true for the lengths of the curves through P_r which would be the orthogonal geodesics if δ_{ij} were fundamental tensor. From these facts and the fact that a geodesic gives the shortest distance the theorem follows.

We now apply the theorem to find approximately the distance of a point P' of the curve from the osculating geodesic space at a nearby point P by expanding $x^i(s)$ about P.

THEOREM 4. *Given a system of normal Riemannian coördinates at* P: (0), *then the principal part of the distance of a nearby point* P' *of* C *from* $G_h: x^i=0$, $i=(h+1), \cdots, n$, *is given by*

$$d^2 = \left[\frac{(\Delta s)^u}{u!}\right]^2 \sum_{i=h+1}^{n} \left(\frac{d^u x^i}{ds^u}\right)^2 + \cdots, \tag{12}$$

where u *is the least number for which the uth derivative of* $x^i(s)$ *in this coördinate system does not lie in* G_k.

This means that d is approximately $(\Delta s)^u/u!$ times length of component of uth derivative normal to G_k. Taking G_h as the tangent geodesic $(h=1)$, we have $u=2$ and $d^2x^i/ds^2=\xi_2|^i$

$$d = \frac{1}{2\rho_1}(\Delta s)^2 + \cdots. \tag{13}$$

Equation (13) will hold in any coördinate system. Similarly, taking G_2 as determined by $\lambda_1|^i$ and $\lambda_2|^i$, we have $u=3$ and

$$\frac{d^3x^i}{ds^3} = \xi_3|^i + \text{vector in } G_2,$$

$$\xi_3|^i = -(1/\rho_1)^2\lambda_1|^i + \frac{d}{ds}(1/\rho_1)\lambda_2|^i + (1/(\rho_1\rho_2))\lambda_3|^i.$$

Hence the only component perpendicular to G_2 is $(1/(\rho_1\rho_2))\lambda_3|^i$ and

$$d = (\Delta s)^3/(6(\rho_1\rho_2)) + \cdots. \tag{14}$$

Higher than this we cannot go because of the impossibility of replacing the derivatives of $x^i(s)$ by the corresponding associate vectors. We note however that if G_3 is determined by $\lambda_1|^i$, $\lambda_2|^i$, $\lambda_3|^i$ the distance of P' from G_3 is an infinitesimal of at least the fourth order.

THEOREM 5. *The principal parts of the infinitesimal distances of* P' *from the osculating* G_1 *and* G_2 *at* P *are given by* (13) *and* (14), *and the distance from* G_3 *is an infinitesimal of at least the fourth order.*

Now assume that G_h, the hth osculating geodesic space of the curve at P, is totally geodesic. The vector $(d^rx^i/ds^r-\xi_r|^i)$ consists of vectors d^lx^i/ds^l, $l<r$, contracted into partial derivatives of $\left\{{i \atop jk}\right\}$, and by Theorem 1 we show by steps that d^rx^i/ds^r lies in G_h whenever $\xi_r|^i$ does; that is, for $r\leqq h$. Also the

vector $(d^{h+1}x^i/ds^{h+1}-\xi_{h+1}|^i)$ must lie in G_h and the component of $d^{h+1}x^i/ds^{h+1}$ normal to G_h is the same as that of $\xi_{h+1}|^i$. But by (2) we have

$$\xi_{h+1}|^i = \text{vector in } G_h + (1/(\rho_1\rho_2\cdots\rho_h))\lambda_{h+1}|^i,$$

and by Theorem 4,

$$d = (\Delta s)^{h+1}/[(h+1)!\,\rho_1\rho_2\cdots\rho_h] + \cdots. \tag{15}$$

THEOREM 6. *If the kth osculating geodesic space at P to C is totally geodesic, then the principal part of the distance of a nearby point P' from it is given by* (15).

Projection of curve on osculating space. Another property of ordinary space curves which we shall extend where possible to curves in a Riemann space is that certain curvatures of the curve C are equal to the corresponding curvatures of the projection of C on one of its osculating spaces. For the Riemannian case the osculating spaces are osculating geodesic spaces and projection is by means of the orthogonal geodesics.

As before we take normal Riemannian coördinates at P, so that the osculating space in question is given by $x^i=0$, $i>h$, C is given by $x^i=x^i(s)$, $i=1,\cdots,n$. If we let C' be given by $x^i=x^i(s)$, $i=1,\cdots,h$; $x^i=0$, $i>h$, then C', while not the projection of C on G_h by orthogonal geodesics, will in the cases considered be sufficiently close to C near P to be used for it.

Let G_h be the G_2 determined by $\lambda_1|^i$ and $\lambda_2|^i$. Then at P we have that dx^i/ds and d^2x^i/ds^2 are in G_2 and hence equal the corresponding quantities for C'. By formulas (3) the $\xi_1|^i$ and $\xi_2|^i$ are the same for C and C', and hence we have

$$(1/\rho_1) = (1/\rho_1'). \tag{16}$$

If G_h is G_3 determined by $\lambda_1|^i$, $\lambda_2|^i$ and $\lambda_3|^i$, it follows as before that dx^i/ds, d^2x^i/ds^2, d^3x^i/ds^3 are in G_3 and that they must then equal the corresponding quantities for C'. Hence it follows that $\xi_1|^i$, $\xi_2|^i$ and $\xi_3|^i$ are the same for the two curves, and that

$$(1/\rho_1) = (1/\rho_1'),\quad (1/\rho_2) = (1/\rho_2'). \tag{17}$$

THEOREM 7. *If a curve C is projected on its osculating G_2 or G_3 at P, then the curvatures of C and its projection at P are connected at P by* (16) *or* (17).

We now assume that G_h, the hth osculating geodesic space of C at P, is totally geodesic in V_n. Then d^rx^i/ds^r, $r\leqq h$, will lie in G_h, and, since the d^rx^i/ds^r for C' are exactly the G_h components of d^rx^i/ds^r, the two sets will be the same $r\leqq h$. As $\xi_r|^i$ and $\xi_r'|^i$ are formed in the same way from the same ordinary derivatives, we have

$$\xi_r|^i = \xi_r'|^i \text{ at } P \text{ for } r \leqq h.$$

It follows from the definition of the curvatures of a curve that if at P the first h associate vectors are known and independent, then the curvatures $(1/\rho_l)$, $l<h$, and their derivatives $(d^r/ds^r)(1/\rho_s)$, $r+s<h$, of C at P are determined at P. From this and the equality of the $\xi_r|^i$, $r \leqq h$, we have

THEOREM 8. *If the first $(h-2)$ curvatures of C at P do not vanish, and if the hth osculating geodesic space G_h at P is totally geodesic, then the curvatures $(1/\rho_l)$, $l<h$, and their derivatives $(d^r/ds^r)(1/\rho_s)$, $r+s<h$, of C at P are equal to the corresponding quantities of the projection of C on G_h.*

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FRENET FORMULAS FOR A GENERAL SUBSPACE OF A RIEMANN SPACE*

BY

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Introduction. The Frenet formulas for a curve in ordinary space have been extended by Blaschke† to a curve in a Riemann space V_m. In §§1–4 of the present paper it is shown that by utilizing a properly defined covariant differentiation similar formulas can be obtained for any subspace V_n of a V_m.‡ For a curve the curvatures are arbitrary functions of the parameter; for a general subspace the corresponding quantities are functions of the coordinates x^i which must satisfy certain integrability conditions, the Gauss, Codazzi, Ricci equations. In §5 a curve in the subspace is considered and Meusnier's Theorem extended, while in §6 certain relations of V_n to its osculating geodesic spaces are discussed.

1. **Complete tensors and complete derivatives.** Consider a Riemann space V_m with definite fundamental tensor $a_{\alpha\beta}$, and let V_n with fundamental tensor g_{ij} be a subspace given by

$$(1.1)\qquad y^\alpha = y^\alpha(x^1, \cdots, x^n) \qquad (\alpha = 1, \cdots, m),$$

where these, as all other functions, will be assumed analytic. Assume there is given at each point P of V_n a set of n_1 mutually perpendicular unit vectors $\zeta_{a_1}|^\alpha$ in V_m:

$$(1.2)\qquad a_{\alpha\beta}\zeta_{a_1}|^\alpha\zeta_{b_1}|^\beta = \delta_{a_1b_1} \qquad (a_1, b_1 = 1, \cdots, n_1).$$

These vectors determine at P an n_1-dimensional linear vector subspace of V_m, and any other set of n_1 mutually perpendicular unit vectors $\zeta_{a_1}'|^\alpha$ in this subspace is given by

$$(1.3)\qquad \zeta_{a_1}'|^\alpha = t_{a_1}^{b_1}\zeta_{b_1}|^\alpha,$$

where

$$(1.4)\qquad t_{a_1}^{b_1}t_{c_1}^{b_1} = \delta_{a_1c_1},\S$$

* Presented to the Society, February 28, 1931; received by the editors February 10, 1931.

† Mathematische Zeitschrift, vol. 6, pp. 94–99.

‡ Since submission of the present paper there has appeared another on the same subject: J. A. Schouten and E. R. van Kampen, *Eine Revision der Krummungstheorie*, Mathematische Annalen, vol. 105, p. 144. These results were presented by Professor Schouten to the Society at its April meeting. See also Tucker, *Generalized covariant differentiation*, Annals of Mathematics, vol. 32, p. 451.

§ Repeated indices are summed regardless of position.

and conversely. We shall seek properties of this vector subspace which are independent of the choice of the n_1 vectors in it.

DEFINITION 1. *A quantity* $T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,b_1,\cdots}$, *whose components are distinguished by any number of indices* $\alpha, \beta, \cdots = 1, \cdots, m$, *and of* $i, j, \cdots = 1, \cdots, n$, *and of* $a_1, b_1, \cdots = 1, \cdots, n_1$, *is a complete tensor if for a coördinate change in* V_m *or* V_n *it transforms as an ordinary tensor, and if for a change* (1.3) *in chosen system of the vectors* $\zeta_{a_1}|^{\alpha}$ *it transforms as*

$$T'^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,b_1,\cdots} = t^{c_1}_{a_1} t^{d_1}_{b_1} \cdots T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,c_1,d_1,\cdots}. \tag{1.5}$$

The sum or outer product of two complete tensors is a complete tensor, as is the contraction of a complete tensor contracted for a pair of indices of any of the three types.

Consider the covariant derivative defined by

$$\begin{aligned} T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,\cdots;k} = {} & \frac{\partial}{\partial x^k} T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,\cdots} + \left\{ \begin{matrix} i \\ k\,l \end{matrix} \right\}_g T^{\alpha,\cdots,l,\cdots}_{\beta,\cdots,j,\cdots,a_1,\cdots} \\ & + \cdots - \left\{ \begin{matrix} l \\ j\,k \end{matrix} \right\}_g T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,l,\cdots,a_1,\cdots} + \left\{ \begin{matrix} \alpha \\ \gamma\delta \end{matrix} \right\}_a y^{\gamma}_k T^{\delta,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,\cdots} \\ & + \cdots - \left\{ \begin{matrix} \gamma \\ \beta\delta \end{matrix} \right\}_a y^{\delta}_k T^{\alpha,\cdots,i,\cdots}_{\gamma,\cdots,j,\cdots,a_1,\cdots}, \end{aligned} \tag{1.6}$$

where y_i^{α} means $\partial y^{\alpha}/\partial x^i$. We verify that under a change of coördinates either in V_m or in V_n, $(T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,\cdots;k})$ transforms as a tensor, but under change (1.3) of $\zeta_{a_1}|^{\alpha}$ we find that

$$\begin{aligned} T'^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,\cdots;k} = {} & t^{c_1}_{a_1} \cdots T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,c_1,\cdots;k} \\ & + \left(\frac{\partial}{\partial x^k} t^{c_1}_{a_1}\right) t^{d_1}_{b_1} \cdots T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,d_1,\cdots} + \cdots. \end{aligned} \tag{1.7}$$

Hence this covariant derivative is not itself a complete tensor.

The system of vectors $\zeta_{a_1}|^{\alpha}$ is, by definition, a complete tensor; we consider its covariant derivative,

$$\zeta_{a_1}|^{\alpha}_{;i} = \frac{\partial}{\partial x^i} \zeta_{a_1}|^{\alpha} + \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}_a y^{\gamma}_i \zeta_{a_1}|^{\beta}. \tag{1.8}$$

In terms of this we define

$$\Gamma_{a_1 b_1}|_i = a_{\alpha\beta} \zeta_{a_1}|^{\alpha} \zeta_{b_1}|^{\beta}_{;i}. \tag{1.9}$$

By differentiating (1.2) we have

$$\Gamma_{a_1 b_1}|_i + \Gamma_{b_1 a_1}|_i = 0, \tag{1.10}$$

and under a change (1.3) of $\zeta_{a_1}|^\alpha$ we have

$$\begin{aligned} \zeta'_{b_1}|^\alpha_{;i} &= t^{a_1}_{b_1}\zeta_{a_1}|^\alpha_{;i} + \zeta_{a_1}|^\alpha \frac{\partial}{\partial x^i} t^{a_1}_{b_1}, \\ \Gamma'_{a_1 b_1}|_i &= t^{c_1}_{a_1} t^{d_1}_{b_1} \Gamma_{c_1 d_1}|_i + t^{c_1}_{a_1}\frac{\partial}{\partial x^i}(t^{c_1}_{b_1}); \end{aligned} \tag{1.11}$$

solving this for $(\partial/\partial x^i)(t^{c_1}_{b_1})$ and substituting in (1.7), we have

$$T'^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,\cdots(k} = t^{b_1}_{a_1}\cdots T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,b_1,\cdots(k}, \tag{1.12}$$

where

$$T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,\cdots(k} = T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,\cdots;k} - \Gamma_{c_1 a_1}|_k T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,\cdots} - \cdots. \tag{1.13}$$

DEFINITION 2. $T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,\cdots(k}$ *as defined by* (1.6) *and* (1.13) *is the complete covariant derivative of the complete tensor* $T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_1,\cdots}$.

THEOREM 1. *The complete covariant derivative of a complete tensor is a complete tensor.*

THEOREM 2. *Complete differentiation obeys the ordinary rules of differentiation.*

A complete tensor which we are especially interested in differentiating completely is $\zeta_{a_1}|^\alpha$. We have

$$\zeta_{a_1}|^\alpha_{(i} = \zeta_{a_1}|^\alpha_{;i} - \Gamma_{b_1 a_1}|_i \zeta_{b_1}|^\alpha. \tag{1.14}$$

Projecting this on $\zeta_{c_1}|^\alpha$ we have by (1.9) and (1.2) that

$$a_{\alpha\beta}\zeta_{a_1}|^\alpha_{(i}\zeta_{c_1}|^\beta = 0. \tag{1.15}$$

2. **Successive derived vector spaces.*** As before we are given at each point of V_n an n_1-dimensional vector space in V_m. In this we choose arbitrarily n_1 mutually perpendicular unit vectors $\zeta_{a_1}|^\alpha$, $a_1 = 1, \cdots, n_1$. Consider $\zeta_{a_1}|^\alpha{}_{(i}$ defined above. These are $n_1 n$ vectors in V_m which may or may not be independent. Since $\zeta_{a_1}|^\alpha$ and $\zeta_{a_1}|^\alpha{}_{(i}$ are complete tensors, we see that any vector η^α in V_m dependent on them for one system of coördinates (x^i) in V_n, and one system of vectors $\zeta_{a_1}|^\alpha$, will be dependent on the corresponding vectors for any other systems. Hence $\zeta_{a_1}|^\alpha$ and $\zeta_{a_1}|^\alpha{}_{(i}$ determine a unique vector space in V_m at each point of V_n. Let (n_1+n_2) be its dimensionality at

* See Struik, *Mehrdimensionale Differentialgeometrie*, p. 109.

a general point of V_n; at special points it may be less. At such a general point of V_n there is determined an n_2-dimensional vector space lying in the (n_1+n_2)-space and perpendicular to the original n_1-dimensional vector space. In this we choose n_2 mutually perpendicular unit vectors $\zeta_{a_2}|^\alpha$ and proceed as before with the vectors $\zeta_{a_2}|^\alpha$ defining $\zeta_{a_2}|^\alpha{}_{(i}$. These will with $\zeta_{a_1}|^\alpha$ and $\zeta_{a_2}|^\alpha$ determine a vector space of $(n_1+n_2+n_3)$ dimensions in which we choose n_3 vectors $\zeta_{a_3}|^\alpha$, etc. We thus obtain a series of derived vector spaces each perpendicular to all the preceding.

Let $\zeta_{a_q}|^\alpha$ be the last of these derived vector spaces. $\zeta_{a_q}|^\alpha$ is the first set for which $\zeta_{a_u}|^\alpha{}_{(i}$ are all dependent on $\zeta_{a_v}|^\alpha$ for $v \leqq u$. In general it will happen that these exhaust the independent vectors of V_m, and $m = \sum_{u=1}^{q}(n_u)$. However, it may happen that there are n_{q+1} further independent vectors at a general point of V_n. Choosing these as perpendicular to each other and to the preceding we write them $\zeta_{a_{q+1}}|^\alpha$. These last vectors will be spoken of as residual rather than derived.

The vectors $\zeta_{a_u}|^\alpha$ satisfy the relations

$$a_{\alpha\beta}\zeta_{a_u}|^\alpha \zeta_{b_u}|^\beta = \delta_{a_u b_u} \qquad (u = 1, \cdots, (q+1)), \tag{2.1}$$

$$a_{\alpha\beta}\zeta_{a_u}|^\alpha \zeta_{b_v}|^\beta = 0, \quad u \neq v. \tag{2.2}$$

Differentiating (2.2) completely,

$$a_{\alpha\beta}\zeta_{a_u}|^\alpha{}_{(i}\zeta_{b_v}|^\beta + a_{\alpha\beta}\zeta_{a_u}|^\alpha \zeta_{b_v}|^\beta{}_{(i} = 0. \tag{2.3}$$

By definition of the $(u+1)$st vector space, $\zeta_{a_{u+1}}|^\alpha$, it follows that $\zeta_{a_u}|^\alpha{}_{(i}$ is dependent on $\zeta_{a_v}|^\alpha$, $v = 1, \cdots, (u+1)$. Hence by (2.3) and (1.15) $\zeta_{a_u}|^\alpha{}_{(i}$ is dependent only on $\zeta_{a_{u-1}}|^\alpha$ and $\zeta_{a_{u+1}}|^\alpha$. Letting

$$\Omega_{a_{u+1}a_u}|_i = a_{\alpha\beta}\zeta_{a_u}|^\alpha{}_{(i}\zeta_{a_{u+1}}|^\beta \tag{2.4}$$

we have, by (2.3),

$$\zeta_{a_u}|^\alpha{}_{(i} = \Omega_{a_{u+1}a_u}|_i\zeta_{a_{u+1}}|^\alpha - \Omega_{a_u a_{u-1}}|_i\zeta_{a_{u-1}}|^\alpha, \tag{2.5}$$
$$u = 1, \cdots, (q+1), \text{ where } \Omega_{a_u a_{u-1}}|_i = 0, \ u = 1 \text{ or } (q+1).$$

3. **Integrability conditions.** We define $\Gamma_{a_u b_u}|_{ij}$ by the equation

$$\Gamma_{a_u b_u}|_{ij} = \Gamma_{a_u b_u}|_{j,i} - \Gamma_{a_u b_u}|_{i,j} + \Gamma_{c_u a_u}|_j\Gamma_{c_u b_u}|_i - \Gamma_{c_u a_u}|_i\Gamma_{c_u b_u}|_j. \tag{3.1}$$

From (1.11) we verify directly that $\Gamma_{a_u b_u}|_{ij}$ is a complete tensor unlike $\Gamma_{a_u b_u}|_i$ itself.

We now obtain the integrability conditions for complete differentiation.

Differentiating (1.13) and then interchanging the order of differentiation and subtracting, we have

$$
\begin{aligned}
&T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_u,\cdots(h(l} - T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_u,\cdots(l(h} = \Gamma_{b_u a_u}\big|_{hl} T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,b_u,\cdots} \\
(3.2)\qquad &+ \cdots + R^{k}{}_{jhl} T^{\alpha,\cdots,i,\cdots}_{\beta,\cdots,k,\cdots,a_u,\cdots} + \cdots - R^{i}{}_{khl} T^{\alpha,\cdots,k,\cdots}_{\beta,\cdots,j,\cdots,a_u,\cdots} - \cdots \\
&+ \overline{R}^{\epsilon}{}_{\beta\gamma\delta} \overset{\gamma}{y}_h \overset{\delta}{y}_l T^{\alpha,\cdots,i,\cdots}_{\epsilon,\cdots,j,\cdots,a_u,\cdots} + \cdots - \overline{R}^{\alpha}{}_{\epsilon\gamma\delta} \overset{\gamma}{y}_h \overset{\delta}{y}_l T^{\epsilon,\cdots,i,\cdots}_{\beta,\cdots,j,\cdots,a_u,\cdots} - \cdots .
\end{aligned}
$$

Applying this condition to $\zeta_{a_u}\big|^{\alpha}$ we have

$$
(3.3)\qquad \zeta_{a_u}\big|^{\alpha}{}_{(i(j} - \zeta_{a_u}\big|^{\alpha}{}_{(j(i} = -\overline{R}^{\alpha}{}_{\beta\gamma\delta}\zeta_{a_u}\big|^{\beta}\overset{\gamma}{y}_i\overset{\delta}{y}_j + \Gamma_{a_u b_u}\big|_{ij}\zeta_{b_u}\big|^{\alpha} .
$$

On the other hand using (2.5), we have

$$
\begin{aligned}
&\zeta_{a_u}\big|^{\alpha}{}_{(i(j} = \Omega_{c_{u-1}b_{u-2}}\big|_j \Omega_{a_u c_{u-1}}\big|_i \zeta_{b_{u-2}}\big|^{\alpha} - \Omega_{a_u c_{u-1}}\big|_{i(j}\zeta_{c_{u-1}}\big|^{\alpha} \\
(3.4)\qquad &\quad - [\Omega_{a_u b_{u-1}}\big|_i \Omega_{c_u b_{u-1}}\big|_j + \Omega_{b_{u+1}a_u}\big|_i \Omega_{b_{u+1}c_u}\big|_j]\zeta_{c_u}\big|^{\alpha} \\
&\quad + \Omega_{b_{u+1}a_u}\big|_{i(j}\zeta_{b_{u+1}}\big|^{\alpha} + \Omega_{c_{u+2}b_{u+1}}\big|_j \Omega_{b_{u+1}a_u}\big|_i \zeta_{c_{u+2}}\big|^{\alpha} .
\end{aligned}
$$

Substituting in (3.3) and projecting on the various $\zeta_{a_v}\big|^{\alpha}$,

$$
\begin{aligned}
(3.5)\quad &\Gamma_{a_u b_u}\big|_{ij} + (\Omega_{c_{u+1}b_u}\big|_i \Omega_{c_{u+1}a_u}\big|_j - \Omega_{c_{u+1}b_u}\big|_j \Omega_{c_{u+1}a_u}\big|_i) \\
&\qquad + (\Omega_{a_u c_{u-1}}\big|_i \Omega_{b_u c_{u-1}}\big|_j - \Omega_{a\ c_{u-1}}\big|_j \Omega_{b_u c_{u-1}}\big|_i) = \overline{R}_{\alpha\beta\gamma\delta}\zeta_{a_u}\big|^{\beta}\zeta_{b_u}\big|^{\alpha}\overset{\gamma}{y}_i\overset{\delta}{y}_j,
\end{aligned}
$$

$$
(3.6)\qquad \Omega_{a_u b_{u-1}}\big|_{i(j} - \Omega_{a_u b_{u-1}}\big|_{j(i} = \overline{R}_{\alpha\beta\gamma\delta}\zeta_{b_{u-1}}\big|^{\alpha}\zeta_{a_u}\big|^{\beta}\overset{\gamma}{y}_i\overset{\delta}{y}_j,
$$

$$
(3.7)\qquad \Omega_{a_u b_{u-1}}\big|_i \Omega_{b_{u-1}c_{u-2}}\big|_j - \Omega_{a_u b_{u-1}}\big|_j \Omega_{b_{u-1}c_{u-2}}\big|_i = \overline{R}_{\alpha\beta\gamma\delta}\zeta_{c_{u-2}}\big|^{\alpha}\zeta_{a_u}\big|^{\beta}\overset{\gamma}{y}_i\overset{\delta}{y}_j,
$$

$$
(3.8)\qquad \overline{R}_{\alpha\beta\gamma\delta}\zeta_{a_u}\big|^{\alpha}\zeta_{b_v}\big|^{\beta}\overset{\gamma}{y}_i\overset{\delta}{y}_j = 0, \quad |u - v| > 2.
$$

4. Frenet and Gauss formulas. We will now and for the rest of the paper take our original n_1-dimensional vector space as the tangent vector space to V_n in V_m. Then $n_1 = n$,

$$
(4.1)\qquad \zeta_{a_1}\big|^{\alpha} = \zeta_{a_1}\big|^{i}\overset{\alpha}{y}_i,
$$

$$
(4.2)\qquad a_{\alpha\beta}\zeta_{a_1}\big|^{\alpha}\zeta_{b_1}\big|^{\beta} = g_{ij}\zeta_{a_1}\big|^{i}\zeta_{b_1}\big|^{i} = \delta_{a_1 b_1}.
$$

The vectors $\zeta_{a_1}\big|^{i}$ are vectors of an orthogonal n-uple in V_n. The vectors $\zeta_{a_v}\big|^{\alpha}$ for $v > 1$ are all normal to V_n.

We verify that y_i^{α}, $\zeta_{a_1}\big|^{i}$ and $\zeta_{a_1}\big|_i$ are complete tensors and that they satisfy

$$\zeta_{a_1}|_i = a_{\alpha\beta} \overset{\alpha}{y}_i \zeta_{a_1}|^{\beta}, \tag{4.3}$$

$$\overset{\alpha}{y}_i = \zeta_{a_1}|^{\alpha} \zeta_{a_1}|_i. \tag{4.4}$$

Since y_i^{α} is a complete tensor we can differentiate completely

$$\overset{\alpha}{y}_{i(j} = \frac{\partial^2 y^{\alpha}}{\partial x^i \partial x^j} + \left\{ \begin{matrix} \alpha \\ \beta\gamma \end{matrix} \right\}_a \overset{\beta}{y}_i \overset{\gamma}{y}_j - \left\{ \begin{matrix} h \\ ij \end{matrix} \right\}_g \overset{\alpha}{y}_h = \overset{\alpha}{y}_{j(i}. \tag{4.5}$$

Differentiating completely the equation

$$g_{ij} = a_{\alpha\beta} \overset{\alpha}{y}_i \overset{\beta}{y}_j,$$

we have by a cyclic permutation of the indices and (4.5) that

$$a_{\alpha\beta} \overset{\alpha}{y}_i \overset{\beta}{y}_{j(h} = 0. \tag{4.6}$$

Differentiating (4.4) we have

$$\overset{\alpha}{y}_{i(j} = \zeta_{a_1}|^{\alpha}_{(j} \zeta_{a_1}|_i + \zeta_{a_1}|^{\alpha} \zeta_{a_1}|_{i(j}.$$

By (4.6) and (4.5) we have

$$\zeta_{a_1}|_{i(j} = 0 \text{ or } \overset{\alpha}{y}_{i(j} = \zeta_{a_1}|_i \zeta_{a_1}|^{\alpha}_{(j}, \tag{4.7}$$

$$\zeta_{a_1}|_i \zeta_{a_1}|^{\alpha}_{(j} - \zeta_{a_1}|_j \zeta_{a_1}|^{\alpha}_{(i} = 0. \tag{4.8}$$

This latter is a set of $\frac{1}{2}n(n-1)$ vector relations connecting the n^2 vectors $\zeta_{a_1}|^{\alpha}{}_{(i}$. The number of independent vectors is at most $\frac{1}{2}n(n+1)$, and we have $n_2 \leqq \frac{1}{2}n(n+1)$.

By (1.9) and (4.6) we have

$$\Gamma_{a_1 b_1}|_i = a_{\alpha\beta} \overset{\alpha}{y}_l \zeta_{a_1}|^l (\overset{\alpha}{y}_h \zeta_{b_1}|^h)_{;i} = g_{lh} \zeta_{a_1}|^l \zeta_{b_1}|^h_{;i} \tag{4.9}$$

$$\Gamma_{a_1 b_1}|_{ij} = g_{lh} \zeta_{a_1}|^l (\zeta_{b_1}|^h{}_{,j,i} - \zeta_{b_1}|^h{}_{,i,j}) = \zeta_{a_1}|^l \zeta_{b_1}|^h R_{lhij}. \tag{4.10}$$

Equation (3.5) thus reduces for $u=1$ to an equivalent of the Gauss equation for V_n in V_m, and the equations (3.5)–(3.8) are, as a set, equivalent to the ordinary Gauss, Codazzi, Ricci equations for V_n in V_m.* Whenever, as now, the $\zeta_{a_1}|^{\alpha}$ are the tangents to the V_n, equations (2.5) will be referred to as the Frenet formulas for V_n in V_m. In justification of this consider a curve V_1. We can choose the arc-length as the coördinate x^1 and then $g_{11}=0$ and $\left\{ \begin{smallmatrix} 1 \\ 11 \end{smallmatrix} \right\}=0$. There will be just one vector tangent to the curve and just one in each of the

* In case there is only one derived vector space the present work is equivalent to that of Weyl, Mathematische Zeitschrift, vol. 12, pp. 154–160, and that of R. Lagrange, Thesis, Paris, 1923, chapter 5.

derived vector spaces. Hence by (1.10) $\Gamma_{a_u b_u}|_i = 0$ and the complete derivative reduces to

$$\frac{d}{ds}\zeta_{a_u}|^\alpha + \left\{ {\alpha \atop \beta\gamma} \right\}_a \zeta_{a_u}|^\beta \frac{dy^\gamma}{ds},$$

and we can write

$$\frac{d}{ds}\zeta_u|^\alpha + \left\{ {\alpha \atop \beta\gamma} \right\}_a \zeta_u|^\beta \frac{dy^\gamma}{ds} = -\,\Omega_{u,u-1}\zeta_{u-1}|^\alpha + \Omega_{u+1,u}\zeta_{u+1}|^\alpha, \tag{4.11}$$

which is precisely the Frenet equation where $(1/\rho_u) = \Omega_{u,u-1}$.

5. **A curve V_1 in V_n in V_m.** Assume we have given a curve V_1 in V_n by

$$x^i = x^i(s). \tag{5.1}$$

Then by (1.1) we have V_1 given in V_m by

$$y^\alpha = y^\alpha(s). \tag{5.2}$$

Let $\xi_1|^\alpha, \cdots, \xi_m|^\alpha$ be the m associate vectors of (5.2) in V_m, and let $\phi_1|^i, \cdots, \phi_n|^i$ be the n associate vectors of (5.1) in V_n. Let $\lambda_u|^\alpha$ and $\eta_u|^i$ be the normalized vectors corresponding to $\xi_u|^\alpha$ and $\phi_u|^i$ respectively. Then we have

$$\xi_u|^\alpha_{,\beta}\frac{dy^\beta}{ds} = \xi_{u+1}|^\alpha, \quad \phi_u|^i_{,j}\frac{dx^j}{ds} = \phi_{u+1}|^i. \tag{5.3}$$

This covariant differentiation may be replaced by complete differentiation of $\xi_u|^\alpha$ or $\phi_u|^i$, it being understood that the subscript u is not an index but a part of the symbol. Then

$$\begin{aligned} \xi_u|^\alpha_{,\beta}\frac{dy^\beta}{ds} &= \xi_u|^\alpha_{,\beta}\, y^\beta_i \frac{dx^i}{ds} = \xi_u|^\alpha_{(i}\frac{dx^i}{ds}, \\ \phi_u|^i_{,j}\frac{dx_j}{ds} &= \phi_u|^i_{(j}\frac{dx^j}{ds}. \end{aligned} \tag{5.4}$$

Moreover we have

$$\xi_1|^\alpha = \frac{dy^\alpha}{ds} = y^\alpha_i\frac{dx^i}{ds}.$$

Differentiating this completely, using the Frenet formulas for V_n in V_m,

$$\begin{aligned} \xi_2|^\alpha &= \xi_1|^\alpha_{(j}\frac{dx^j}{ds} = (\zeta_{a1}|^\alpha \zeta_{a1}|_i\phi_1|^i)_{(j}\frac{dx^j}{ds} \\ &= \zeta_{a1}|^\alpha(\zeta_{a1}|_i\phi_2|^i) + \zeta_{a_2}|^\alpha(\Omega_{a_2a1}|_j\zeta_{a1}|_i\phi_1|^i\phi_1|^j), \\ \xi_3|^\alpha &= \zeta_{a1}|^\alpha(\Omega_{a_2a1}|_j\Omega_{a_2b1}|_i\zeta_{b1}|_k\phi_1|^i\phi_1|^j\phi_1|^k + \zeta_{a1}|_i\phi_3|^i) \\ &\quad + \zeta_{a_2}|^\alpha(\Omega_{a_2a1}|_{j(k}\zeta_{a1}|_i\phi_1|^i\phi_1|^j\phi_1|^k + 3\Omega_{a_2a1}|_i\zeta_{a1}|_j\phi_1|^i\phi_2|^j) \\ &\quad + \zeta_{a_3}|^\alpha(\Omega_{a_3a_2}|_i\Omega_{a_2a1}|_j\zeta_{a1}|_k\phi_1|^i\phi_1|^j\phi_1|^k), \end{aligned} \tag{5.5}$$

and in general we have that $\xi_u|^\alpha$ is dependent on $\zeta_{a_1}|^\alpha, \cdots, \zeta_{a_u}|^\alpha$.

THEOREM 3. *Given a V_1 in a V_n in a V_m, then the uth osculating vector space of V_1 in V_m is contained in the uth osculating vector space of V_n in V_m.*

$\xi_u|^\alpha$ may also be expressed by repeated application of the Frenet formulas for V_1 in V_m. Thus we have

$$\begin{aligned} &\xi_1|^\alpha = \lambda_1|^\alpha, \\ (5.6)\qquad &\xi_2|^\alpha = (1/\rho_1)\lambda_2|^\alpha, \\ &\xi_3|^\alpha = -(1/\rho_1)^2\lambda_1|^\alpha + \frac{d}{ds}(1/\rho_1)\lambda_2|^\alpha + (1/(\rho_1\rho_2))\lambda_3|^\alpha. \end{aligned}$$

If V_n in V_m is known and if the vectors $\lambda_v|^\alpha$, $v \leqq u$, of a curve are known then the quantities $(1/\rho_v)$ and $(d^r/ds^r)(1/\rho_s)$ entering into $\xi_v|^\alpha$, $v \leqq u$, are determined. This follows from a comparison of (5.5) and (5.6). The coefficient of $\lambda_1|^\alpha$ in $\xi_1|^\alpha$ is known, being 1. Assume the coefficients in the first v equations (5.6) known and also $\phi_r|^i$, $r < v$. Then $\phi_v|^i$ is determined since $\xi_v|^\alpha$ is given by (5.6), and in (5.5) $\phi_v|^i$ will be the only unknown in the formula for $\xi_v|^\alpha$. Then by (5.5) the projection of $\xi_{v+1}|^\alpha$ on $\zeta_{a_{v+1}}|^\alpha$ is known; the only term in (5.6) having such a projection is the last. Equating we have

$$\begin{aligned} (5.7)\quad (1/(\rho_1\rho_2 \cdots \rho_{v+1}))(a_{\alpha\beta}\lambda_{v+1}|^\alpha\zeta_{a_{v+1}}|^\beta) \\ = \Omega_{a_{v+1}a_v}|_i\Omega_{a_v a_{v-1}}|_j \cdots \Omega_{a_2a_1}|_k\zeta_{a_1}|_l\phi_1|^i \cdots \phi_1|^l. \end{aligned}$$

This determines $(1/\rho_{v+1})$ assuming that none of the preceding curvatures were zero. Next projecting on $\zeta_{a_v}|^\alpha$ we determine $(d/ds)(1/\rho_v)$, and so on. Only one previously undetermined quantity occurs in each projection and so can be determined. Proceeding by successive steps we show that $(d^r/ds^r)(1/\rho_s)$, $r \geqq 0$, $s > 0$, $r+s \leqq u$, and $\phi_v|^i$, $v \leqq u$, are determined by the $\lambda_v|^\alpha$ of a curve $v \leqq u$. The $\lambda_v|^\alpha$ must however be the actual $\lambda_v|^\alpha$ of some curve in V_n in order that the conditions be compatible. For instance the component of $\lambda_{v+1}|^\alpha$ normal to the vth osculating space of v_n is determined except for magnitude by (5.7). For this reason we state the resulting theorem in the form

THEOREM 4. *Through a point P given two curves lying in V_n in V_m, and given further that, for a number u,*

(a) *the osculating vector spaces $\zeta_{a_1}|^\alpha, \cdots, \zeta_{a_u}|^\alpha$ of V_n exist;*

(b) *the vectors $\lambda_1|^\alpha, \cdots, \lambda_u|^\alpha$ of the two curves as curves of V_m are the same at P;*

(c) *the common tangent to the two curves is such that*

$$\Omega_{a_u a_{u-1}}|_i\Omega_{a_{u-1}a_{u-2}}|_j \cdots \Omega_{a_2a_1}|_k\zeta_{a_1}|_l\phi_1|^l \cdots \phi_1|^i$$

is not zero for all a_u; then the curvatures $(1/\rho_v)$, $v \leqq u$, and their derivatives

$(d^r/ds^r)(1/\rho_s)$, $r+s\leqq u$, and the V_n associate vectors of the two curves are the same at P.

For $u=1$ this theorem reduces to an equivalent of Meusnier's theorem.

The case of V_1 in V_n in V_m is a special case of the more general problem of V_l in V_n in V_m. For this more general problem it follows, just as for a curve, that the first u osculating spaces of V_l in V_m are contained in the first u osculating vector spaces of V_n in V_m. A formula analogous to (5.7) holds for the more general problem, but it cannot always be solved for the curvature tensor which replaces the $(1/\rho_u)$. Hence we cannot proceed in this case to the extension of the theorem above.

6. **Residual normals.** The residual normals, $\zeta_{a_{q+1}}|^\alpha$, were defined as normals not in derived vector spaces of any order; they were characterized by the equation

$$\Omega_{a_{q+1}a_q}|_i = 0. \tag{6.1}$$

Hence by (2.5)

$$\zeta_{a_{q+1}}|^\alpha_{(i} = 0. \tag{6.2}$$

Equation (6.1) expresses a condition in terms of the $\Omega_{a_{q+1}a_q}|_i$ that the set of n_{q+1} normals $\zeta_{a_{q+1}}|^\alpha$ be residual; conditions that V_n in V_m possess n_{q+1} residual normals may also be expressed in terms of the ordinary theory of subspaces where the normals are not separated into successive sets.

Following the notation of Eisenhart* we denote a set of $(m-n)$ normals by $\xi_\sigma|^\alpha$, $\sigma=1, \cdots, (m-n)$, and define

$$\mu_{\tau\sigma}|_i = a_{\alpha\beta}\xi_\tau|^\alpha\left[\xi_\sigma|^\beta_{,i} + \begin{Bmatrix}\beta\\ \gamma\delta\end{Bmatrix}_a y^\gamma_i\xi_\sigma|^\delta\right], \tag{6.3}$$

$$\begin{aligned}\Omega_\sigma|_{ij} &= a_{\alpha\beta}\xi_\sigma|^\alpha\left[y^\beta_{i,j}+\begin{Bmatrix}\beta\\ \gamma\delta\end{Bmatrix}_a y^\gamma_i y^\delta_j\right]\\ &= -\,a_{\alpha\beta}y^\alpha_i\left[\xi_\sigma|^\beta_{,j} + \begin{Bmatrix}\beta\\ \gamma\delta\end{Bmatrix}_a y^\gamma_j\xi_\sigma|^\delta\right].\end{aligned} \tag{6.4}$$

We now choose as one particular set $\xi_\sigma|^\alpha$ the set $\zeta_{a_{q+1}}|^\alpha, \zeta_{a_q}|^\alpha, \cdots, \zeta_{a_1}|^\alpha$ in that order. Writing (6.2) in the form

$$\zeta_{a_{q+1}}|^\alpha_{,i} + \begin{Bmatrix}\alpha\\ \beta\gamma\end{Bmatrix}_a y^\beta_i\zeta_{a_{q+1}}|^\gamma - \Gamma_{b_{q+1}a_{q+1}}|_i\zeta_{b_{q+1}}|^\alpha = 0,$$

* *Riemannian Geometry*, chapter IV.

we see from (6.3) and (6.4) that for this set of normals

$$\mu_{\tau\sigma}\mid_i = 0 \qquad (\sigma = 1, \cdots, n_{q+1};\ \tau = n_{q+1}, \cdots, (m-n)), \tag{6.5}$$

$$\Omega_\sigma\mid_{ij} = 0 \qquad (\sigma = 1, \cdots, n_{q+1}). \tag{6.6}$$

Equations (6.5), (6.6) for some normal system are sufficient as well as necessary for the existence of n_{q+1} residual normals. For from (6.6) it follows that the first derived vector space is in space of $\xi_\tau\mid^\alpha$, $\tau > n_{q+1}$, and by (6.5) this follows for the further derived spaces.

If we change to another set of normals

$$\xi'_\sigma\mid^\alpha = t_\sigma{}^\tau \xi_\tau\mid^\alpha \qquad (\tau, \sigma = 1, \cdots, (m-n)),$$

$$\Omega'_\sigma\mid_{ij} = t_\sigma{}^\tau \Omega_\tau\mid_{ij},$$

and $\Omega_\sigma\mid_{ij}$ is normal covariant unlike $\mu_{\tau\sigma}\mid_i$. We define the complete derivative of a normal covariant quantity in the way indicated by the example

$$\Omega_\sigma\mid_{ij(k} = \Omega_\sigma\mid_{ij,k} - \mu_{\tau\sigma}\mid_k \Omega_\tau\mid_{ij} \qquad (\tau, \sigma = 1, \cdots, (m-n)). \tag{6.7}$$

The complete derivative is itself normal covariant and differentiation obeys the ordinary rules.*

If (6.6) holds for one normal system, then for any other there are n_{q+1} independent sets of solutions of

$$\eta_\sigma \Omega_\sigma\mid_{ij} = 0. \tag{6.8}$$

If η_σ is such a solution both η_σ and $\eta_{\sigma(k}$ are normal covariant, and for our special normal system we verify that $\eta_{\sigma(k} = 0$. Hence for any normal system we have

$$\eta_{\sigma(k} = 0. \tag{6.9}$$

Differentiating (6.8) we have, by (6.9),

$$\begin{aligned} \eta_\sigma \Omega_\sigma\mid_{ij(k} &= 0, \\ \eta_\sigma \Omega_\sigma\mid_{ij(k(l} &= 0, \\ &\cdots\cdots\cdot \end{aligned} \tag{6.10}$$

A necessary condition that V_n in V_m possess n_{q+1} residual normals is that equations (6.8) and (6.10) admit n_{q+1} independent solutions.

Conversely, assume that the first Q equations of (6.8) and (6.10) admit a complete set of n_{q+1} solutions which satisfy the $(Q+1)$st. Taking the normals corresponding to these solutions as the first n_{q+1} reference normals $\xi_\sigma\mid^\alpha$, $\sigma = 1, \cdots, n_{q+1}$, we can show that (6.5) and (6.6) are satisfied.

* See Weyl and Lagrange, loc. cit.

THEOREM 5. *A necessary and sufficient condition that V_n in V_m possess n_{q+1} residual normals is that* (6.8) *and the first Q equations of* (6.10) *admit a complete set of n_{q+1} solutions which also satisfy the* $(Q+1)$*st.*

The simplest example of a space V_n possessing residual normals is a totally geodesic space T_n in V_m. For such a subspace the $\Omega_\sigma|_{ij}$ vanish identically and conditions (6.8), (6.10) are satisfied by all normals, which are therefore all residual. Conversely, if a V_n in V_m possess $(m-n)$ residual normals, by (6.8) we see that $\Omega_\sigma|_{ij}=0$ and V_n is a T_n of V_m.

THEOREM 6. *A necessary and sufficient condition that a V_n in V_m possess $(m-n)$ residual normals is that it be a totally geodesic subspace of V_m.*

The sufficient part generalizes as follows:

THEOREM 7. *If V_n is any subspace of a totally geodesic subspace T_N of V_m, then the normals to T_N will be residual normals of V_n in V_m.*

This could be proved directly by choosing Riemannian coördinates and direct computation; it follows also from Theorem 3. The converse of this theorem is not true; that is, the existence of residual normals does not imply that V_n lies in a totally geodesic subspace of V_m. For example, it can be shown that there always exist curves with only one derived normal, the principal normal; and in fact that such curves exist through any given point with any given pair of perpendicular vectors as tangent and principal normal. But the point and pair of vectors may be such that the geodesic surface in V_m determined by them is not totally geodesic.

Another question concerning a V_n with residual normals which arises is that of the relation of the V_n to its complete osculating geodesic space. For curves the author has shown that if the complete osculating geodesic space is totally geodesic the curve lies in it. The proof by means of Riemannian coördinates and direct computation holds for the general subspace.

DEFINITION. *The uth osculating geodesic space of V_n in V_m at P is made up of V_m geodesics through P in directions dependent on $\zeta_{a_1}|^\alpha, \cdots, \zeta_{a_u}|^\alpha$. The complete osculating geodesic space at P is made up of geodesics through P perpendicular to all the residual normals at P.*

THEOREM 8. *If the complete osculating geodesic space of V_n in V_m at P is totally geodesic then V_n lies in it.*

THEOREM 9. *A necessary and sufficient condition that a V_n in a V_m of constant curvature possess $(m-N)$ independent residual normals is that it be a subspace of a geodesic subspace G_N of V_m.*

THEOREM 10. *A necessary and sufficient condition that a V_n in a V_m of constant curvature lie in an N-dimensional geodesic subspace is that* (6.8) *and* (6.10) *admit* $(m-N)$ *solutions in the usual sense.*

Other theorems on the relations of a curve to its osculating geodesic spaces may be extended to the present case. The proofs are the same as for the curve.

THEOREM 11. *Let P': $(x^i+\Delta x^i)$ be a point of V_n near P: (x^i). The principal parts of the infinitesimal distances of P' from the tangent G_n and the osculating G_{n+n_2} at P are given by*

$$d^2 = \frac{1}{4} \sum_{a_2} (\Omega_{a_2a_1} |_i \zeta_{a_1} |_j \Delta x^i \Delta x^j)^2 + \cdots,$$

$$d^2 = \frac{1}{36} \sum_{a_3} (\Omega_{a_3a_2} |_i \Omega_{a_2a_1} |_j \zeta_{a_1} |_k \Delta x^i \Delta x^j \Delta x^k)^2 + \cdots,$$

except where these expressions vanish.

THEOREM 12. *If the uth osculating geodesic space G_N of V_n in V_m at a general point P is totally geodesic, then the principal part of the distance of P': $(x^i+\Delta x^i)$ from it is in general*

$$d^2 = \sum_{a_{u+1}} (\Omega_{a_{u+1}a_u} |_i \Omega_{a_ua_{u-1}} |_j \cdots \Omega_{a_2a_1} |_k \zeta_{a_1} |_l \Delta x^i \cdots \Delta x^l)^2 / [(u+2)!]^2 + \cdots.$$

THEOREM 13. *If V_n' is the projection of V_n in V_m on its osculating geodesic space of $(n+n_2)$ or $(n+n_2+n_3)$ dimensions, then for properly chosen reference systems we have at P*

$$\Omega_{a_2a_1} |_i = \Omega'_{a_2a_1} |_i \qquad \textit{(either case)},$$

$$\Omega_{a_3a_2} |_i = \Omega'_{a_3a_2} |_i \qquad \textit{(second case)}.$$

THEOREM 14. *If V_N, the uth osculating geodesic space of V_n in V_m at $P(N=n+n_2+\cdots+n_u)$, is totally geodesic, and if V_n' is the projection of V_n on it, then for properly chosen reference systems*

$$\Omega_{a_va_{v-1}} |_i = \Omega'_{a_va_{v-1}} |_i \qquad (v = 2, \cdots, u).$$

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NECESSARY AND SUFFICIENT CONDITIONS FOR THE REPRESENTATION OF A FUNCTION AS A LAPLACE INTEGRAL*

BY

D. V. WIDDER

1. Introduction. In a previous paper† in these Transactions the author studied the singularities of functions defined by integrals of the form

$$f(x) = \int_0^\infty e^{-xt}d\alpha(t), \tag{1.1}$$

considering such an integral as a generalization of a Taylor series. All developments of that paper were on the assumption that $f(x)$ permitted of the integral representation (1.1). We wish to study here conditions on $f(x)$, both necessary and sufficient, for the validity of such representation. Following the analogy of Taylor's series we might at first be tempted to suppose that the analyticity of $f(x)$ in a half-plane, the region of convergence of an integral (1.1), would be the condition required. That this is not the case we see at once by recalling that such a function as $\sin x$, analytic in the entire plane, admits of no representation‡ of the form (1.1).

We are led, however, to a correct conjecture by considering our problem as the analogue of the moment problem of F. Hausdorff.§ This is the problem of determining a function $\chi(x)$ bounded and non-decreasing in the interval $0 \leqq x \leqq 1$ and such that

$$\mu_k = \int_0^1 x^k d\chi(x) \qquad (k = 0, 1, 2, \cdots).$$

Hausdorff has shown that the problem has a solution if and only if the sequence $\mu_0, \mu_1, \mu_2, \cdots$ is completely monotonic ("total monotone"). That is, the differences

$$(-1)^n\Delta^n\mu_m = \mu_m - \binom{n}{1}\mu_{m+1} + \binom{n}{2}\mu_{m+2} - \cdots + (-1)^n\mu_{m+n}$$

* Presented to the Society, December 30, 1930; received by the editors December 11, 1931.

† D. V. Widder, *A generalization of Dirichlet's series and of Laplace's integrals by means of a Stieltjes integral*, these Transactions, vol. 31 (1929), p. 694.

‡ This follows at once from a result of M. Lerch, *Sur un point de la théorie des fonctions génératrices d'Abel*, Acta Mathematica, vol. 27 (1903), p. 339.

§ Felix Hausdorff, *Momentprobleme für ein endliches Intervall*, Mathematische Zeitschrift, vol. 16 (1923), p. 220.

satisfy the inequalities

$$(-1)^n\Delta^n\mu_m \geqq 0 \qquad (n = 0, 1, 2, \cdots; m = 0, 1, 2, \cdots).$$

If we generalize this moment problem by allowing k to run through a continuous set of values, we are led to the integral equation

$$\mu(y) = \int_0^1 x^y d\chi(x)$$

for the determination of a non-decreasing function $\chi(x)$. If we set $x=e^{-t}$, this equation becomes

$$\mu(y) = \int_0^\infty e^{-yt} d\alpha(t),$$

where

$$\alpha(t) = -\chi(e^{-t}).$$

If $\alpha(t)$ is a non-decreasing function of t, then $\chi(x)$ will be a non-decreasing function of x, so that we are now required to solve an integral equation of type (1.1) for a non-decreasing function $\alpha(t)$. From Hausdorff's results we should be led to conjecture that the equation has a solution of the type desired if and only if $f(x)$ has derivatives of all orders satisfying the inequalities

$$(-1)^n \frac{d^n}{dx^n} f(x) \geqq 0 \qquad (n = 0, 1, 2, \cdots),$$

and this is in fact the case. This fact was first proved by S. Bernstein* in 1929. The present paper begins with a proof of this theorem following methods quite different from those of Bernstein. The more general problem of determining a solution of (1.1) which is merely of bounded variation is then attacked. A necessary and sufficient condition on $f(x)$ to guarantee the existence of a function $\alpha(t)$ of bounded variation and making the integral absolutely convergent is then obtained. The corresponding problem for an integral of the form

$$f(x) = \int_0^\infty e^{-xt}\phi(t)dt$$

is then treated. It is found that this equation has a solution $\phi(t)$ which is bounded and integrable if and only if

$$|f^{(n)}(x)| \leqq Kn!/(x-c)^{n+1} \qquad (x > c),$$

where K is some constant.

* Serge Bernstein, *Sur les fonctions absolument monotones*, Acta Mathematica, vol. 52 (1929), p. 1. The author had completed the proof of this theorem a few months after the publication of Bernstein's paper without being aware of its existence.

We are then able to solve a problem of considerable importance in the theory of Dirichlet's series. We obtain conditions on $f(x)$ in order that the integral equation (1.1) may have a step-function solution. We thus obtain a necessary and sufficient condition for the representation of $f(x)$ in a Dirichlet series.*

We then investigate the representation of a function $f(x)$ by an integral of the form

$$f(x) = \int_0^1 e^{xt} d\alpha(t) \tag{1.2}$$

with $\alpha(t)$ a bounded non-decreasing function, and find that a necessary and sufficient condition for such representation is that the sequence of derivatives of $f(x)$ at a point x_0,

$$f(x_0), f'(x_0), f''(x_0), \cdots,$$

should be completely monotonic. We then inquire what properties a sequence $a_0, a_1, a_2, \cdots$ must have in order that there may exist a completely monotonic function $f(x)$ satisfying the equations

$$f(n) = a_n \qquad (n = 0, 1, 2, \cdots),$$

and find that it is necessary for the sequence to be completely monotonic. A slight change in the condition makes it both necessary and sufficient. Combining this with the previous result we are led to infer that the generalized derivative of arbitrary order ρ of $f(x)$,

$$_{-\infty}D_x^\rho f(x) = \frac{1}{\Gamma(1-\nu)} \int_0^\infty t^{-\nu} f^{(m+1)}(x-t)dt, \; [\rho] = m, \; \rho = m + \nu,$$

is a completely monotonic function of ρ for every x if and only if $f(x)$ has the form (1.2). Here the generalized derivative is defined in a form slightly different from that given by Riemann, but it is shown that the form adopted is equally good as a generalization for the functions under consideration since it reduces to the ordinary derivative when ρ is an integer.

Throughout most of the paper functions of the real variable are considered. In the last section, however, it is shown that this is no essential restriction in the case of certain of the theorems, and in particular in the case of the theorem regarding Dirichlet's series. Slight modifications are made to make the theorem applicable to functions of the complex variable.

* References to earlier attempts to find such conditions will be found in Mémorial des Sciences Mathématiques, Fascicule XVII, *Théorie Générale des Séries de Dirichlet*, by M. G. Valiron, p. 30. The referee has called the author's attention to the following paper: Th. Kaluza, *Entwickelbarkeit von Funktionen in Dirichletsche Reihen*, Mathematische Zeitschrift, vol. 28 (1928), p. 203.

2. Completely monotonic functions and sequences. We begin with several definitions.

DEFINITION 1. *A function $f(x)$ is completely monotonic in the interval $c<x<\infty$ if it has derivatives of all orders in this interval and if the inequalities*

$$(-1)^n f^{(n)}(x) \geqq 0 \qquad (n = 0, 1, 2, \cdots)$$

are satisfied there.

DEFINITION 2. *A function $f(x)$ is completely monotonic in the interval $c\leqq x<\infty$ if it is completely monotonic in the interval $c<x<\infty$ and if $f(c+0) = f(c) \neq \infty$.*

DEFINITION 3. *The set of constants $\mu_0, \mu_1, \mu_2, \cdots$ form a completely monotonic sequence if*

$$(-1)^n \Delta^n \mu_m \geqq 0 \qquad (n = 0, 1, 2, \cdots; m = 0, 1, 2, \cdots),$$

where

$$(-1)^n \Delta^n \mu_m = \mu_m - \binom{n}{1}\mu_{m+1} + \binom{n}{2}\mu_{m+2} - \cdots + (-1)^n \mu_{m+n}.$$

We now prove

THEOREM 1. *If $f(x)$ is completely monotonic in the interval $c<x<\infty$, and if δ is any positive constant, then the set of constants*

$$f(a), f(a+\delta), f(a+2\delta), \cdots \qquad (c < a < \infty)$$

forms a completely monotonic sequence.

For, we have $\Delta^n f(a+m\delta) = f^{(n)}(\xi)\delta^n$ $(a+m\delta<\xi<a+(m+n)\delta)$ by a familiar result in the theory of finite differences. It follows that

$$(-1)^n \Delta^n f(a+m\delta) \geqq 0.$$

3. Hankel's determinants whose elements are the terms of a completely monotonic sequence. First we introduce the abbreviation

$$(3.1) \qquad [f(a), f(a+\delta), \cdots, f(a+2m\delta)]$$

$$= \begin{vmatrix} f(a) & f(a+\delta) & \cdots & f(a+m\delta) \\ f(a+\delta) & f(a+2\delta) & \cdots & f(a+(m+1)\delta) \\ \cdot & \cdot & \cdot & \cdot \\ f(a+m\delta) & f(a+(m+1)\delta) & \cdots & f(a+2m\delta) \end{vmatrix},$$

and then prove

THEOREM 2. *If $f(x)$ is completely monotonic in the interval $c<x<\infty$ and if δ is any positive constant, then the Hankel determinants*

$$[f(a), f(a+\delta), \cdots, f(a+2m\delta)] \qquad (c<a<\infty; m=0,1,2,\cdots)$$

are all positive or zero.

Since the sequence

$$f(a), f(a+\delta), f(a+2\delta), \cdots$$

is completely monotonic, there exists* a function $\chi(x)$ bounded and non-decreasing in the interval $0 \leqq x \leqq 1$ such that

$$f(a+n\delta) = \int_0^1 x^n d\chi(x) \qquad (n=0,1,2,\cdots).$$

Construct the quadratic form

$$\begin{aligned}\sum_{i=0}^{n}\sum_{j=0}^{n} f(a+(i+j)\delta)x_i x_j \\ &= \int_0^1 \sum_{i=0}^{n}\sum_{j=0}^{n} x^{i+j}x_i x_j d\chi(x) \\ &= \int_0^1 \left(\sum_{i=0}^{n} x^i x_i\right)^2 d\chi(x) \geqq 0.\end{aligned}$$

That this form is never negative follows since $\chi(x)$ is non-decreasing and since the integrand is non-negative. It is known that this implies that the determinants (3.1) are non-negative for $m=0, 1, 2, \cdots, n$.

THEOREM 3. *If $f(x)$ is completely monotonic in the interval $c<x<\infty$, then the determinants*

$$[f(a), f'(a), \cdots, f^{(2m)}(a)] \qquad (c<a<\infty; m=0,1,2,\cdots) \tag{3.2}$$

are positive or zero.

It is a familiar fact that the determinant (3.1) may be written as

$$[f(a), \Delta f(a), \Delta^2 f(a), \cdots, \Delta^{2m} f(a)].$$

Divide this determinant by $\delta^{m(m+1)}$. By Theorem 2 the quotient is non-negative for all positive δ. Let δ approach zero. The limit, which is the determinant (3.2), must also be positive or zero.

Clearly the result also holds if the constant c is replaced by a constant b

* F. Hausdorff, loc. cit., p. 226.

greater than c, since if $f(x)$ is completely monotonic in $c<x<\infty$ it is also completely monotonic in $b<x<\infty$.

4. Consequences of the vanishing of certain Hankel determinants. We begin by stating two Lemmas, the proofs of which may easily be supplied.

LEMMA 1. *If the quadratic form*

$$\sum_{i=0}^{n}\sum_{j=0}^{n} a_{ij}x_ix_j, \quad a_{ij} = a_{ji},$$

is non-negative for $x_n=1$ and for all values of the other variables, $x_0, x_1, \cdots, x_{n-1}$, then it is non-negative for all values of the variables $x_0, x_1, \cdots, x_n$.

LEMMA 2. *If the quadratic form*

$$\sum_{i=0}^{n}\sum_{j=0}^{n} a_{ij}x_ix_j, \quad a_{ij} = a_{ji},$$

is non-negative for all values of the variables $x_0, x_1, \cdots, x_n$, and if $a_{00}=0$, then $a_{01}=a_{02}=\cdots=a_{0n}=0$.

By use of these Lemmas we can prove

THEOREM 4. *If $f(x)$ is completely monotonic in the interval $c<x<\infty$, and if*

$$(4.1)\quad \begin{aligned}[][f(a), f'(a), \cdots, f^{(2m)}(a)] &> 0 \quad (m = 0, 1, 2, \cdots, k-1), c < a < \infty,\\ &= 0 \quad (m = k),\end{aligned}$$

then

$$[f(a), f'(a), \cdots, f^{(2m)}(a)] = 0 \qquad (m = k, k+1, k+2, \cdots).$$

Set $a_{ij}=f^{(i+j)}(a)$. The quadratic form

$$\sum_{i=0}^{n}\sum_{j=0}^{n} \frac{\Delta^{i+j}f(a)}{\delta^{i+j}}x_ix_j = \int_0^1\left(\sum_{i=0}^{n}(x-1)^i\frac{x_i}{\delta^i}\right)^2 d\chi(x)$$

is obviously non-negative for every positive value of δ. Allowing δ to approach zero we see that the quadratic form

$$\sum_{i=0}^{n}\sum_{j=0}^{n} a_{ij}x_ix_j \tag{4.2}$$

is also positive or zero.

Now consider the quadratic form in the variables $x_0, x_1, x_2, \cdots, x_{k-1}$, z $(k-1<n)$ whose determinant is

$$
(4.3)\qquad \begin{vmatrix}
a_{00} & a_{01} & \cdots & a_{0,k-1} & \sum_{j=k}^{n} a_{0j}x_j \\
a_{10} & a_{11} & \cdots & a_{1,k-1} & \sum_{j=k}^{n} a_{1j}x_j \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
a_{k-1,0} & a_{k-1,1} & \cdots & a_{k-1,k-1} & \sum_{j=k}^{n} a_{k-1,j}x_j \\
\sum_{i=k}^{n} a_{i0}x_i & \sum_{i=k}^{n} a_{i1}x_i & \cdots & \sum_{i=k}^{n} a_{i,k-1}x_i & \sum_{i=k}^{n}\sum_{j=k}^{n} a_{ij}x_ix_j
\end{vmatrix}.
$$

This form reduces to

$$\sum_{i=0}^{n}\sum_{j=0}^{n} a_{ij}x_ix_j$$

for $z=1$, so that it is non-negative, by Lemma 1, for all values of the variables $x_0, x_1, \cdots, x_{k-1}, z$. Hence the determinant (4.3) is positive or zero. Expanding it we obtain

$$
\sum_{i=k}^{n}\sum_{j=k}^{n} \begin{vmatrix}
a_{00} & a_{01} & \cdots & a_{0,k-1} & a_{0j} \\
a_{10} & a_{11} & \cdots & a_{1,k-1} & a_{1j} \\
\cdot & \cdot & \cdot & \cdot & \cdot \\
a_{k-1,0} & a_{k-1,1} & \cdots & a_{k-1,k-1} & a_{k-1,j} \\
a_{i0} & a_{i1} & \cdots & a_{i,k-1} & a_{ij}
\end{vmatrix} x_ix_j.
$$

This is itself a quadratic form in the variables $x_k, x_{k+1}, \cdots, x_n$ which is non-negative. Denote the coefficient of x_ix_j in this form by D_{ij}. Then $D_{kk}=0$ by hypothesis. Consequently, by Lemma 2,

$$D_{kk} = D_{k,k+1} = \cdots = D_{kn} = 0,$$

so that

$$
(4.4)\qquad \begin{vmatrix}
D_{kk} & D_{k,k+1} & \cdots & D_{km} \\
D_{k+1,k} & D_{k+1,k+1} & \cdots & D_{k+1,m} \\
\cdot & \cdot & \cdot & \cdot \\
D_{mk} & D_{m,k+1} & \cdots & D_{mm}
\end{vmatrix} = 0 \qquad (m = k, k+1, \cdots, n)
$$

since all the elements of the first row vanish. Now apply Sylvester's determinant theorem* to equation (4.4). It becomes

* See, for example, G. Kowalewski, *Einführung in die Determinantentheorie*, p. 86.

$$\begin{vmatrix} a_{00} & a_{01} & \cdots & a_{0,k-1} \\ a_{10} & a_{11} & \cdots & a_{1,k-1} \\ \cdot & \cdot & \cdot & \cdot \\ a_{k-1,0} & a_{k-1,1} & \cdots & a_{k-1,k-1} \end{vmatrix}^{m-k} \cdot \begin{vmatrix} a_{00} & a_{01} & \cdots & a_{0m} \\ a_{10} & a_{11} & \cdots & a_{1m} \\ \cdot & \cdot & \cdot & \cdot \\ a_{m0} & a_{m1} & \cdots & a_{mm} \end{vmatrix} = 0.$$

Since the first factor is different from zero by hypothesis, it follows that the second factor vanishes for $m=k, k+1, k+2, \cdots, n$. Since n is arbitrary the theorem is proved.

COROLLARY. *If* $f(a)=0$, *the determinants* (4.1) *vanish for all* m.

For, since (4.2) is non-negative, Lemma 1 shows us that $a_{00}=a_{01}=\cdots=a_{0n}$, from which the result follows at once.

The next result to be proved is

THEOREM 5. *Under the conditions of Theorem* 4, $f(x)$ *satisfies a linear differential equation of order* k *with constant coefficients.*

If $k=0$, then by the Corollary to Theorem 4 we have $f(a)=f'(a)=f''(a)=\cdots=0$. Since every completely monotonic function is analytic,* it follows that $f(x)\equiv 0$. If $k>0$, we have seen that

$$D_{km} = 0 \qquad (m = k, k+1, \cdots).$$

This shows that the rank of the matrix

$$\left\| \begin{matrix} f(a) & f'(a) & \cdots f^{(n)}(a) \\ f'(a) & f''(a) & \cdots f^{(n+1)}(a) \\ \cdot & \cdot & \cdot \\ f^{(k)}(a) & f^{(k+1)}(a) & \cdots f^{(n+k)}(a) \end{matrix} \right\|$$

is k for every value of $n\geqq k$.†

Hence there exist constants $K_0, K_1, \cdots, K_k$, not all zero, such that

$$K_0 f^{(m)}(a) + K_1 f^{(m+1)}(a) + \cdots + K_k f^{(m+k)}(a) = 0 \qquad (m = 0, 1, 2, \cdots).$$

That is, the analytic function

$$K_0 f(x) + K_1 f'(x) + \cdots + K_k f^{(k)}(x)$$

vanishes with all its derivatives at $x=a$, and is consequently identically zero. It remains only to show that $K_k\neq 0$. This follows from the hypothesis that (4.1) is different from zero when $m=k-1$. We observe that the differential equation which $f(x)$ satisfies may be put in the form

* Serge Bernstein, *Sur la définition et les propriétés des fonctions analytiques d'une variable réelle*, Mathematische Annalen, vol. 75 (1914), p. 449.

† G. Kowalewski, loc. cit., p. 53, Theorem 18.

$$
(4.5)\qquad \begin{vmatrix} f(x) & f'(x) & \cdots & f^{(k)}(x) \\ f(a) & f'(a) & \cdots & f^{(k)}(a) \\ f'(a) & f''(a) & \cdots & f^{(k+1)}(a) \\ \cdot & \cdot & \cdot & \cdot \\ f^{(k-1)}(a) & f^{(k)}(a) & \cdots & f^{(2k-1)}(a) \end{vmatrix} = 0.
$$

THEOREM 6. *Under the conditions of Theorem* 4,

$$
(4.6)\qquad (-1)^m[f'(a), f''(a), \cdots, f^{(2m-1)}(a)] > 0 \qquad (m = 1, 2, 3, \cdots, k-1).
$$

Since the function $-f'(x)$ is itself a completely monotonic function, the determinant (4.6) is clearly non-negative for all positive integers m by Theorem 3. Moreover, if (4.6) vanished for $m<k-1$, it would also vanish for $m=k-1$ by Theorem 4. Consequently, we have only to show that (4.6) is not zero for $m=k-1$. If it were zero, we should have, as in Theorem 5, that the rank of the matrix

$$
\begin{Vmatrix} f'(a) & f''(a) & \cdots & f^{(n)}(a) \\ f''(a) & f'''(a) & \cdots & f^{(n+1)}(a) \\ \cdot & \cdot & \cdot & \cdot \\ f^{(k-1)}(a) & f^{(k)}(a) & \cdots & f^{(n+k-2)}(a) \end{Vmatrix}
$$

would be at most $k-2$. That is, the determinant (4.1) would vanish for $m=k-1$ contrary to assumption. That (4.6) may vanish for $m=k$ is seen by taking $f(x)=1+e^{-x}$, $a=0$, $k=2$. That it need not vanish may be seen by taking $f(x)=e^{-x}+e^{-2x}$, $a=0$, $k=2$.

THEOREM 7. *Under the conditions of Theorem* 4

$$
(4.7)\qquad \begin{aligned} & f(x) = c_1e^{-\lambda_1 x} + c_2e^{-\lambda_2 x} + \cdots + c_ke^{-\lambda_k x}, \\ & 0 \leqq \lambda_1 < \lambda_2 < \cdots < \lambda_k,\ c_i > 0 \qquad (i = 1, 2, \cdots, k). \end{aligned}
$$

To prove that $f(x)$ has the form (4.7) we must show that the roots of the algebraic equation

$$
(4.8)\qquad \begin{vmatrix} 1 & z & \cdots & z^k \\ f(a) & f'(a) & \cdots & f^{(k)}(a) \\ \cdot & \cdot & \cdot & \cdot \\ f^{(k-1)}(a) & f^{(k)}(a) & \cdots & f^{(2k-1)}(a) \end{vmatrix} = 0
$$

associated with (4.5) are real, distinct and non-negative. To do this we appeal to the theory of continued fractions. If (4.6) is different from zero for $m=k$, then the left-hand side of (4.8) divided by the determinant (4.6) for $m=k$ is,

except for sign, the denominator of the reduced form of the continued fraction

$$\cfrac{1}{a_1 z + \cfrac{1}{a_2 + \cfrac{1}{a_3 z + \cfrac{1}{\ddots + \cfrac{1}{a_{2k-1}z + \cfrac{1}{a_{2k}}}}}}}$$

where

$$A_0 = 1,\ B_0 = 1,\ A_n = [f(a), f'(a), \cdots, f^{(2n-2)}(a)],$$

$$B_n = (-1)^n[f'(a), f''(a), \cdots, f^{(2n-1)}(a)],\quad a_{2n} = \frac{A_n^2}{B_n B_{n-1}},\quad a_{2n+1} = \frac{B_n^2}{A_n A_{n+1}}.$$

The rational function of z which this continued fraction represents is defined in the neighborhood of infinity by the series

$$u(z) = \frac{f(a)}{z} + \frac{f'(a)}{z^2} + \frac{f''(a)}{z^3} + \cdots.$$

By Theorem 6 we see that the B_n are all positive, and the A_n are all positive by hypothesis, so that the a_n are all positive. Under these conditions the roots of (4.8) are known to be distinct and positive.* The left-hand side of (4.8) is the function $Q_{2n}(z)$ of Stieltjes defined on page 426 of the article cited.

If (4.6) is zero for $m=k$, then the continued fraction development of $u(z)$ is the same as above except that it stops with the term $a_{2k-1}z$. The denominator of the expanded form is now

$$(4.9)\quad \begin{vmatrix} z & z^2 & \cdots z^k \\ f'(a) & f''(a) & \cdots f^{(k)}(a) \\ \cdot & \cdot & \cdot \\ f^{(k-1)}(a) & f^{(k)}(a) & \cdots f^{(2k-2)}(a) \end{vmatrix} \div \begin{vmatrix} f(a) & f'(a) & \cdots f^{(k-1)}(a) \\ f'(a) & f''(a) & \cdots f^{(k)}(a) \\ \cdot & \cdot & \cdot \\ f^{(k-1)}(a) & f^{(k)}(a) & \cdots f^{(2k-2)}(a) \end{vmatrix},$$

and this is also known to have distinct zeros which are all positive except one which is zero.† It is not difficult to identify the zeros of this function with the roots of equation (4.8). For, since we are assuming that (4.6) vanishes for

* T. J. Stieltjes, Collected Works, p. 411, and p. 426.

† T. J. Stieltjes, loc. cit., p. 411, and p. 427.

$m=k$ and not for $m=k-1$, we know that constants $L_1, L_2, \cdots, L_{k-1}$ exist such that

$$
(4.10)\qquad
\begin{aligned}
&L_1 f'(a) + L_2 f''(a) + \cdots + L_{k-1} f^{(k-1)}(a) + f^{(k)}(a) = 0,\\
&L_1 f''(a) + L_2 f'''(a) + \cdots + L_k f^{(k)}(a) + f^{(k+1)}(a) = 0,\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&L_1 f^{(k)}(a) + L_2 f^{(k+1)}(a) + \cdots + L_k f^{(2k-2)}(a) + f^{(2k-1)}(a) = 0.
\end{aligned}
$$

Expanding the determinant (4.8) according to the elements of the first column we have

$$
\sum_{j=1}^{k}(-1)^j f^{(j-1)}(a)
\begin{vmatrix}
z & z^2 & \cdots z^k \\
f'(a) & f''(a) & \cdots f^{(k)}(a) \\
\cdot & \cdot & \cdot \\
f^{(j-1)}(a) & f^{(j)}(a) & \cdots f^{(j+k-2)}(a) \\
f^{(j+1)}(a) & f^{(j+2)}(a) & \cdots f^{(j+k)}(a) \\
\cdot & \cdot & \cdot \\
f^{(k)}(a) & f^{(k+1)}(a) & \cdots f^{(2k-1)}(a)
\end{vmatrix}.
$$

Substitute the values of $f^{(k)}(a), f^{(k+1)}(a), \cdots, f^{(2k-1)}(a)$, obtained from equations (4.10), in the last row of this determinant. Equation (4.8) thus becomes

$$
(4.11)\qquad
\left[\sum_{j=1}^{k-1} L_j f^{(j-1)}(a) + f^{(k-1)}(a)\right]
\begin{vmatrix}
z & z^2 & \cdots z^k \\
f'(a) & f''(a) & \cdots f^{(k)}(a) \\
\cdot & \cdot & \cdot \\
f^{(k-1)}(a) & f^{(k)}(a) & \cdots f^{(2k-2)}(a)
\end{vmatrix}.
$$

The first factor can not vanish for its vanishing (together with equations (4.10)) would imply that (4.1) would vanish for $m=k-1$ contrary to assumption. Hence the roots of (4.9) are the same as those of (4.11).

It remains only to show that $c_i>0$. We have seen that

$$
f(x) = \int_0^\infty e^{-xt} d\alpha(t)
$$

where $\alpha(t)$ is a step-function with a finite number of jumps. We wish to show that these jumps are all positive. Let R be a number so large that the points of discontinuity of $\alpha(t)$ are all to the left of the point $t=R$. Then if z is a fixed number for which $|z|>R$ the series

$$
\frac{e^{-at}}{z+t} = e^{-at}\left[\frac{1}{z} - \frac{t}{z^2} + \frac{t^2}{z^3} - \cdots\right]
$$

is uniformly convergent in the interval $0 \leqq t \leqq R$, so that we have

$$\int_0^\infty \frac{e^{-at}}{z+t}d\alpha(t) = \frac{1}{z}\int_0^\infty e^{-at}d\alpha(t) - \frac{1}{z^2}\int_0^\infty e^{-at}t\,d\alpha(t) + \frac{1}{z^3}\int_0^\infty e^{-at}t^2 d\alpha(t) - \cdots$$
$$= \frac{f(a)}{z} + \frac{f'(a)}{z^2} + \frac{f''(a)}{z^3} + \cdots = u(z).$$

We thus have the partial fraction development of $u(z)$:

$$u(z) = \frac{c_1 e^{-ax_1}}{z+x_1} + \frac{c_2 e^{-ax_2}}{z+x_2} + \cdots + \frac{c_k e^{-ax_k}}{z+x_k}.$$

But the coefficients of this development are known to be positive,* so that the c_i are all positive. The theorem is thus completely established.

5. The function $\alpha(t)$ a monotonic function. We are now in a position to prove

THEOREM 8. *A necessary and sufficient condition that $f(x)$ should be completely monotonic in the interval $c<x<\infty$ is that*

$$f(x) = \int_0^\infty e^{-xt}d\alpha(t), \tag{5.1}$$

where $\alpha(t)$ is a non-decreasing function of such a nature that the integral converges for $x>c$.

The sufficiency of the condition is obvious since†

$$f^{(n)}(x) = (-1)^n \int_0^\infty e^{-xt}t^n d\alpha(t), \qquad x > c \qquad (n = 0, 1, 2, \cdots).$$

To prove the necessity of the condition we appeal to Theorem 7 and to a result of H. Hamburger.‡ If one of the determinants (4.1) is zero, then $f(x)$ has the form (5.1), $\alpha(t)$ being a step-function with a finite number of positive jumps. If none of these determinants vanishes, then the determinants (4.6) are positive for all m, and we are in a position to apply Hamburger's Theorem.§ The function $f(x)$ is thus seen to have the integral expression (5.1).

We note that if $\alpha(t)$ is to be a non-increasing function it is necessary and sufficient that $-f(x)$ should be completely monotonic.

* T. J. Stieltjes, loc. cit., p. 413.

† D. V. Widder, loc. cit., p. 702.

‡ H. Hamburger, *Bemerkungen zu einer Fragestellung des Herrn Pólya*, Mathematische Zeitschrift, vol. 7 (1920), p. 304.

§ We must actually apply the theorem to $f(-x)$, but the modifications necessary are obvious.

6. The function $\alpha(t)$ of bounded variation. We prove the following theorem:

THEOREM 9. *A necessary and sufficient condition that $f(x)$ can be expressed as*

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t), \tag{6.1}$$

with $\alpha(t)$ of bounded variation in every finite interval and the integral absolutely convergent for $x>c$, is that $f(x)$ should be the difference of two functions that are completely monotonic in the interval $c<x<\infty$.

We prove first the necessity of the condition. Suppose $f(x)$ has the form (6.1). We may suppose without loss of generality that $\alpha(0)=0$. Since $\alpha(t)$ is of bounded variation in the interval $0 \leqq t \leqq R$ there exist two non-decreasing functions $P(t)$ and $N(t)$ such that

$$\alpha(R) = P(R) - N(R),$$
$$u(R) = P(R) + N(R).$$

Here $u(R)$ is the total variation of $\alpha(t)$ in the interval $0 \leqq t \leqq R$. In this way we see that

$$f(x) = \lim_{R=\infty} \left[\int_0^R e^{-xt} dP(t) - \int_0^R e^{-xt} dN(t) \right].$$

Since the integral (6.1) is known to converge absolutely, the limit

$$\int_0^\infty e^{-xt} du(t) = \lim_{R=\infty} \left[\int_0^R e^{-xt} dP(t) + \int_0^R e^{-xt} dN(t) \right]$$

exists, so that the integrals

$$\int_0^\infty e^{-xt} dP(t), \quad \int_0^\infty e^{-xt} dN(t)$$

converge for $x>c$. Hence

$$f(x) = \int_0^\infty e^{-xt} dP(t) - \int_0^\infty e^{-xt} dN(t).$$

An application of Theorem 8 now establishes the necessity of the condition.

We turn now to the sufficiency. First suppose that $c \geqq 0$. By virtue of Theorem 8

$$f(x) = \int_0^\infty e^{-xt} dP(t) - \int_0^\infty e^{-xt} dN(t),$$

where $P(t)$ and $N(t)$ are non-decreasing functions, vanishing at the origin, of such a nature that the integrals converge for $x>c$. For any such value of x, constants K and ϵ exist such that*

$$(6.2) \qquad P(t) < Ke^{t(x-\epsilon)}, \quad N(t) < Ke^{t(x-\epsilon)} \qquad (0 \leqq t < \infty;\ c < x - \epsilon < x).$$

Consequently

$$\int_0^\infty e^{-xt}dP(t) = x\int_0^\infty e^{-xt}P(t)dt,$$

$$\int_0^\infty e^{-xt}dN(t) = x\int_0^\infty e^{-xt}N(t)dt,$$

the integrals on the right-hand side converging for $x>c$. Now if $\alpha(t)=P(t)-N(t)$, the integral $\int_0^\infty e^{-xt}d\alpha(t)$ converges absolutely if $x>c$. For, the total variation $u(t)$ of $\alpha(t)$ clearly satisfies the inequality

$$(6.3) \qquad u(t) \leqq P(t) + N(t).$$

This inequality shows that the integral $\int_0^\infty e^{-xt}u(t)dt$ converges, and hence that

$$(6.4) \qquad \int_0^\infty e^{-xt}du(t) = \lim_{R=\infty} u(R)e^{-xR} + x\int_0^\infty e^{-xt}u(t)dt.$$

By virtue of the inequalities (6.2) and (6.3) we see that the indicated limit in (6.4) exists and is zero, so that (6.1) converges absolutely for $x>c$.

The case in which $c<0$ may be reduced to the case just treated by the change of variable $x-c=y$.

We shall next seek to determine a more convenient condition to replace that of Theorem 9. First we shall obtain certain necessary conditions.

THEOREM 10. *If the integral*

$$f(x) = \int_0^\infty e^{-xt}d\alpha(t), \qquad \alpha(0) = 0,$$

converges absolutely for $x>c$, *then*

$$\lim_{x=\infty} f(x) = \alpha(0+),$$

$$\lim_{x=\infty} f^{(k)}(x) = 0 \qquad (k = 1, 2, 3, \cdots).$$

In consideration of Theorem 9 it is sufficient to suppose that $f(x)$ is completely monotonic for $x>c$. Since $f(x)$ is a positive decreasing function it

* D. V. Widder, loc. cit., p. 703, Lemma 2.

tends to a limit as x becomes infinite. The same is true of every derivative of $f(x)$ since $(-1)^k f^{(k)}(x)$ is itself a completely monotonic function. But the limit is zero if $k \geqq 1$. For, suppose

$$\lim_{x=\infty} (-1)^k f^{(k)}(x) = B > 0.$$

Then

$$\int_{c+\delta}^{R} (-1)^k f^{(k)}(x)dx > B(R - c - \delta), \qquad \delta > 0,$$

whence

$$\lim_{R=\infty} (-1)^k [f^{(k-1)}(R) - f^{(k-1)}(c + \delta)] = \infty,$$

contrary to the fact just proved that $f^{(k-1)}(x)$ approaches a finite limit as x becomes infinite.

If we define a function $\beta(t)$ by the equations

$$\beta(0) = 0,$$
$$\beta(t) = \alpha(0 +), \qquad t > 0,$$

it remains only to show that

$$\lim_{x=\infty} \int_0^\infty e^{-xt} d[\alpha(t) - \beta(t)] = 0.$$

The positive function $\gamma(t) = \alpha(t) - \beta(t)$ is continuous at $t=0$. If ϵ is an arbitrary positive constant, we can find a number δ so small that

$$\int_0^\delta e^{-xt} d\gamma(t) < \gamma(\delta) < \epsilon/2, \qquad x \geqq 0.$$

Then we can choose x so large that

$$\int_\delta^\infty e^{-xt} d\gamma(t) = e^{-x\delta} \int_0^\infty e^{-xt} d\gamma(t + \delta) < \epsilon/2.$$

The latter choice is clearly possible since the integral involving $\gamma(t+\delta)$ approaches a finite limit,* and $e^{-x\delta}$ approaches 0 as x becomes infinite.

THEOREM 11. *Under the conditions of Theorem* 10, *there exists a constant* M_δ *independent of* x *and of* n, *but dependent on* δ, *such that*

$$\int_{c+\delta}^{x} \frac{(t - c - \delta)^n}{n!} | f^{(n+1)}(t) | dt < M_\delta \quad (\delta > 0;\ x \geqq c + \delta,\ n = 0, 1, 2, \cdots).$$

---

* Since $\gamma(t+\delta)$ is monotonic, this follows from the first part of the proof of the present theorem.

By Theorem 9 we have

$$f(x) = \int_0^\infty e^{-xt}d\alpha(t) = \int_0^\infty e^{-xt}dP(t) - \int_0^\infty e^{-xt}dN(t),$$

whence

$$f^{(n+1)}(x) = (-1)^{n+1}\int_0^\infty e^{-xt}t^{n+1}dP(t) - (-1)^{n+1}\int_0^\infty e^{-xt}t^{n+1}dN(t)$$

$$|f^{(n+1)}(x)| \leqq \int_0^\infty e^{-xt}t^{n+1}d\gamma(t),$$

where

$$\gamma(t) = P(t) + N(t).$$

Then

$$\int_{c+\delta}^{x} \frac{(y-c-\delta)^n}{n!}|f^{(n+1)}(y)|\,dy \leqq \int_0^\infty t^{n+1}d\gamma(t)\int_{c+\delta}^{x} e^{-yt}\frac{(y-c-\delta)^n}{n!}dy.$$

The interchange of the order of integration which we have effected here is permissible since the integral $\int_0^\infty e^{-xt}t^{n+1}d\gamma(t)$ is uniformly convergent* in the interval $c+\delta \leqq x < \infty$.

The inequality is only strengthened if we replace the upper limit x of the last integral in the above inequality by ∞. Thus

$$\int_{c+\delta}^{x} \frac{(t-c-\delta)^n}{n!}|f^{(n+1)}(t)|\,dt < \int_0^\infty t^{n+1}d\gamma(t)\int_{c+\delta}^{\infty} \frac{e^{-xt}(x-c-\delta)^n}{n!}dx$$

$$= \int_0^\infty t^{n+1}d\gamma(t)e^{-t(c+\delta)}\int_0^\infty \frac{e^{-xt}x^n}{n!}\,dx = \int_0^\infty e^{-t(c+\delta)}d\gamma(t) = M_\delta.$$

This completes the proof.

We shall now show that the necessary condition established in Theorem 11 is also sufficient, and thus prove

THEOREM 12. *A necessary and sufficient condition that $f(x)$ can be expressed as*

$$(6.5) \qquad f(x) = \int_0^\infty e^{-xt}d\alpha(t)$$

with the integral absolutely convergent for $x>c$ and $\alpha(t)$ of bounded variation in every finite interval is that

* D. V. Widder, loc. cit., p. 701.

(a) *$f(x)$ has derivatives of all orders for $x>c$,*

(b) *a constant M_δ exists independent of x and of n but dependent on δ such that*

$$\int_{c+\delta}^{x} \frac{(t-c-\delta)^n}{n!} \left| f^{(n+1)}(t) \right| dt < M_\delta \quad (\delta > 0;\ x \geqq c+\delta;\ n = 0, 1, 2, \cdots).$$

In order to prove this theorem we shall make use of three lemmas.

LEMMA 1. *If the functions $\phi_0(x), \phi_1(x), \phi_2(x), \cdots$ are continuous with their first derivatives for $x \geqq x_0$ and if a constant K exists such that*

$$\phi_n(x) \leqq \phi_{n+1}(x) \leqq K \qquad (x \geqq x_0;\ n = 0, 1, 2, \cdots),$$

$$\phi'_{n+1}(x) \leqq \phi'_n(x),$$

then the given sequence converges uniformly for $x \geqq x_0$.

By hypothesis

$$\phi'_{n+p}(x) \leqq \phi'_n(x) \qquad (p = 1, 2, 3, \cdots),$$

so that

$$\int_{x_0}^{x} \phi'_{n+p}(t)dt \leqq \int_{x_0}^{x} \phi'_n(t)dt \qquad (x \geqq x_0),$$

$$\phi_{n+p}(x) - \phi_{n+p}(x_0) \leqq \phi_n(x) - \phi_n(x_0),$$

$$\phi_{n+p}(x) - \phi_n(x) \leqq \phi_{n+p}(x_0) - \phi_n(x_0).$$

But the sequence $\phi_0(x), \phi_1(x), \phi_2(x), \cdots$ clearly converges for each x since it is an increasing bounded sequence. Hence to an arbitrary positive ϵ there corresponds an integer m independent of x for $x \geqq x_0$ such that

$$\phi_{n+p}(x) - \phi_n(x) \leqq \epsilon \qquad (p = 0, 1, 2, \cdots)$$

when $n > m$. This proves the lemma.

LEMMA 2. *If the function $\psi(x)$ is continuous with its first $(k+1)$ derivatives for $x>c$, and if*

$$(-1)^n \psi^{(n)}(x) \geqq 0 \qquad (x > c;\ n = 0, 1, 2, \cdots, k+1),$$

then

$$\left| \psi^{(k)}(x) \right| \leqq \frac{\psi(c+\delta)k!}{(x-c-\delta)^k} \qquad (\delta > 0,\ x > c+\delta).$$

By Taylor's theorem we have

$$\psi(c+\delta) = \psi(x) + \psi'(x)(c+\delta-x) + \cdots + \psi^{(k)}(x)\frac{(c+\delta-x)^k}{k!}$$
$$+ \int_x^{c+\delta} \frac{(c+\delta-t)^k}{k!}\psi^{(k+1)}(t)dt \qquad (x \geqq c+\delta).$$

Every term on the right-hand side is positive or zero so that

$$\psi^{(k)}(x)(c+\delta-x)^k \leqq \psi(c+\delta)k!,$$

from which the desired inequality results immediately.

LEMMA 3. *Under the conditions of Theorem* 12 *a constant A exists such that*

$$\lim_{x=\infty} f(x) = A,$$
$$\lim_{x=\infty} f^{(n)}(x) = 0 \qquad (n = 1, 2, 3, \cdots).$$

For, integration by parts gives the equality

$$\int_{c+\delta}^{x} \frac{(t-c-\delta)^n}{n!} f^{(n+1)}(t)dt = f^{(n)}(x)\frac{(x-c-\delta)^n}{n!}$$
$$- \int_{c+\delta}^{x} \frac{(t-c-\delta)^{n-1}}{(n-1)!} f^{(n)}(t)dt \qquad (n = 1, 2, 3, \cdots).$$

Hence

$$|f^{(n)}(x)|\frac{(x-c-\delta)^n}{n!} \leqq \int_{c+\delta}^{x} \frac{(t-c-\delta)^n}{n!}|f^{(n+1)}(t)|dt$$
$$+ \int_{c+\delta}^{x} \frac{(t-c-\delta)^{n-1}}{(n-1)!}|f^{(n)}(t)|dt \quad (x \geqq c+\delta),$$

and by condition (b)

$$|f^{(n)}(x)| \leqq \frac{2M_\delta n!}{(x-c-\delta)^n} \qquad (n = 1, 2, 3, \cdots).$$

This inequality is sufficient to show that $f^{(n)}(x)$ approaches zero as x becomes infinite $(n=1, 2, 3, \cdots)$. But

$$\int_{c+\delta}^{x} f'(t)dt = f(x) - f(c+\delta).$$

Hence if x is allowed to become infinite the function $f(x)$ must approach a limit, since by condition (b) $(n=0)$ the integral

$$\int_{c+\delta}^{\infty} f'(x)dx$$

converges absolutely.

We turn now to the proof of the theorem. By Lemma 3 we have

$$f(x) - A = - \int_x^\infty f'(t)dt,$$

or

$$| f(x) - A | \leqq \int_x^\infty | f'(t) | dt$$

provided that the integral converges. We see that it does converge for $x>c$ by taking $n=0$ in (b). More generally,

$$\int_x^\infty \frac{(t-x)^{n-1}}{(n-1)!} | f^{(n)}(t) | dt \leqq \int_x^\infty \frac{(t-x)^n}{n!} | f^{(n+1)}(t) | dt. \tag{6.6}$$

Both integrals converge for $x>c$ as one sees by again referring to (b). The inequality is established by first noting that

$$| f^{(n)}(x) | \leqq \int_x^\infty | f^{(n+1)}(t) | dt \qquad (x > c)$$

and then that

$$\int_x^\infty \frac{(t-x)^{n-1}}{(n-1)!} | f^{(n)}(t) | dt \leqq \int_x^\infty \frac{(t-x)^{n-1}}{(n-1)!} dt \int_t^\infty | f^{(n+1)}(y) | dy.$$

If we interchange the order of integration on the right-hand side of this inequality, we obtain (6.6). This is permissible since the integrand is positive and since the resulting iterated integral is convergent.* If we set

$$\phi_n(x) = \int_x^\infty \frac{(t-x)^n}{n!} | f^{(n+1)}(t) | dt,$$

we may state our result as follows:

$$| f(x) - A | \leqq \phi_0(x) \leqq \phi_1(x) \leqq \cdots \leqq \phi_n(x) \leqq \cdots \leqq M.$$

The sequence of functions $\phi_0(x), \phi_1(x), \phi_2(x), \cdots$ has a limit for all $x>c$ which we shall denote by $\phi(x)$. Now the derivative of $\phi_n(x)$ is

$$\phi_n'(x) = - \int_x^\infty \frac{(t-x)^{n-1}}{(n-1)!} | f^{(n+1)}(t) | dt. \tag{6.7}$$

To justify the differentiation under the integral sign, set

* E. W. Hobson, *The Theory of Functions of a Real Variable and the Theory of Fourier's Series*, second edition, vol. 2, p. 346.

$$g(x, t) = \frac{(t-x)^n}{n!} | f^{(n+1)}(t) | \qquad (t > x),$$
$$= 0 \qquad (t \leqq x).$$

Then

$$\phi_n(x) = \int_{c+\delta}^{\infty} g(x, t) dt.$$

The functions $g(x, t)$ and $(\partial/\partial x)g(x, t)$ are continuous in the region $x \geqq c+\delta$, $t \geqq c+\delta$ if $n>1$, and the integral

$$\int_{c+\delta}^{\infty} \frac{\partial g(x, t)}{\partial x} dt \tag{6.8}$$

converges uniformly in the interval $x \geqq c+\delta$. For,

$$\left| \frac{\partial}{\partial x} g(x, t) \right| \leqq \frac{(t-c-\delta)^{n-1}}{(n-1)!} | f^{(n+1)}(t) | \qquad (x \geqq c+\delta)$$
$$\leqq \frac{(t-c-\delta)^n}{n!} | f^{(n+1)}(t) | \qquad (t \geqq n+c+\delta).$$

Since

$$\int_{c+\delta}^{\infty} \frac{(t-c-\delta)^n}{n!} | f^{(n+1)}(t) | dt$$

converges, the integral (6.8) is uniformly convergent for $x \geqq c+\delta$. Formula (6.7) also holds if $n=1$, as one may see directly by writing

$$\phi_n(x) = \int_x^{\infty} t | f''(t) | dt - x \int_x^{\infty} | f''(t) | dt$$

and differentiating.

In a similar way we have

$$\phi_n''(x) = \int_x^{\infty} \frac{(t-x)^{n-2}}{(n-2)!} | f^{(n+1)}(t) | dt \qquad (n \geqq 2),$$

and in general

$$\phi_n^{(k)}(x) = (-1)^k \int_x^{\infty} \frac{(t-x)^{n-k}}{(n-k)!} | f^{(n+1)}(t) | dt \qquad (k \leqq n).$$

It follows that

$$(-1)^k \phi_n^{(k)}(x) \geqq 0 \qquad (k \leqq n).$$

Now treating $f^{(m)}(x)$ as we did $f(x)$, we have

$$f^{(m)}(x) = -\int_x^\infty f^{(m+1)}(t)dt \qquad (m > 0),$$

$$(6.9)\quad |f^{(m)}(x)| \leqq (-1)^m \phi_m^{(m)}(x) \leqq (-1)^m \phi_{m+1}^{(m)}(x)$$
$$\leqq \cdots \leqq (-1)^m \phi_{m+n}^{(m)}(x) \leqq \cdots .$$

To show that this sequence has a limit for every $x > c$ we show that it has an upper limit. Let x_0 be an arbitrary point for which $x_0 > c + \delta (\delta > 0)$. By Lemma 2 we have

$$|\phi_{m+n}^{(m)}(x)| \leqq \frac{\phi(c+\delta)m!}{(x-c-\delta)^m}$$
$$\leqq \frac{\phi(c+\delta)m!}{(x_0-c-\delta)^m} \qquad (x \geqq x_0).$$

The right-hand side of this inequality, being independent of n, serves as an upper limit for the sequence (6.9) for all $x \geqq x_0$. The functions $\phi_{m+n}^{(m)}(x)$ of the sequence satisfy all the conditions of Lemma 1 at least for $n \geqq 1$ (as we see by replacing m by $m+1$ in the inequalities (6.9)). Consequently, the sequence $\phi_m^{(m)}(x)$, $\phi_{m+1}^{(m)}(x)$, $\phi_{m+2}^{(m)}(x)$, $\cdots$ converges uniformly, and its limit is the derivative of the limit of the sequence $\phi_m^{(m-1)}(x)$, $\phi_{m+1}^{(m-1)}(x)$, $\phi_{m+2}^{(m-1)}(x)$, $\cdots$. We see in this way that

$$\lim_{n=\infty} \phi_{m+n}^{(m)}(x) = \phi^{(m)}(x),$$

and that

$$(6.10)\quad \begin{aligned} &|f(x) - A| \leqq \phi(x), \\ &|f^{(n)}(x)| \leqq (-1)^n \phi^{(n)}(x) \qquad (n = 1, 2, 3, \cdots). \end{aligned}$$

We are now in a position to show that $f(x)$ is the difference of two completely monotonic functions,

$$f(x) - A = \frac{f(x) + \phi(x) - A}{2} - \frac{\phi(x) - f(x) + A}{2}.$$

The inequalities (6.10) lead at once to the following:

$$\begin{array}{ll} \dfrac{f(x)+\phi(x)-A}{2} \geqq 0, & \dfrac{\phi(x)-f(x)+A}{2} \geqq 0, \\ -\dfrac{f'(x)+\phi'(x)}{2} \geqq 0, & -\dfrac{\phi'(x)-f'(x)}{2} \geqq 0, \\ \vdots & \vdots \\ (-1)^n \dfrac{f^{(n)}(x)+\phi^{(n)}(x)}{2} \geqq 0, & (-1)^n \dfrac{\phi^{(n)}(x)-f^{(n)}(x)}{2} \geqq 0, \\ \vdots & \vdots \end{array}$$

If $A \geqq 0$ we write

$$f(x) = \left[A + \frac{f(x)+\phi(x)-A}{2}\right] - \left[\frac{\phi(x)-f(x)+A}{2}\right],$$

and if $A \leqq 0$,

$$f(x) = \left[\frac{f(x)+\phi(x)-A}{2}\right] - \left[\frac{\phi(x)-f(x)+A}{2} - A\right].$$

In either case $f(x)$ is obviously the difference of two completely monotonic functions. An appeal to Theorem 9 completes the proof of the theorem.

COROLLARY. *For $f(x)$ to have the form* (6.5) *it is sufficient that*

(a) *$f(x)$ should have derivatives of all orders,*

(b′) *M should exist independent of x and of n such that*

$$|f^{(n)}(x)| < \frac{M\Gamma(n+p)}{(x-c)^{n+p}} \quad (x > c;\, n = 0, 1, 2, \cdots)$$

for some positive constant p.

We have only to show that condition (b′) includes condition (b). We can do this as follows:

$$\int_{c+\delta}^{\infty} \frac{(x-c-\delta)^n}{n!} |f^{(n+1)}(x)|\, dx = \int_0^{\infty} \frac{x^n}{n!} |f^{(n+1)}(x+c+\delta)|\, dx$$
$$\leqq \frac{\Gamma(n+p+1)}{n!} M \int_0^{\infty} \frac{x^n}{(x+\delta)^{n+p+1}}\, dx = \frac{M\Gamma(p)}{\delta^p}.$$

As a simple example take $f(x) = 1/x$. It satisfies the conditions (a) and (b′) for $x>0$, and has the expression

$$\frac{1}{x} = \int_0^\infty e^{-xt}dt,$$

the integral converging for $x>0$.

That conditions (a) and (b′) are not necessary may be seen by noting that the function $f(x)=1$ does not satisfy them. Yet it may be expressed in the form (6.5).

7. **The function $\alpha(t)$ an integral.** Let us next investigate conditions under which $\alpha(t)$ is an integral, that is, under which $f(x)$ has the form

$$f(x) = \int_0^\infty e^{-xt}\phi(t)dt.$$

In this connection we prove

THEOREM 13. *A necessary and sufficient condition that $f(x)$ can be expressed in the form*

$$f(x) = \int_0^\infty e^{-xt}\phi(t)dt \tag{7.1}$$

with $\phi(t)$ integrable in every finite interval and satisfying an inequality of the form

$$|\phi(t)| \leqq Ke^{ct} \qquad (0 \leqq t) \tag{7.2}$$

is that

$$|f^{(n)}(x)| \leqq \frac{Kn!}{(x-c)^{n+1}} \quad (x > c;\ n = 0, 1, 2, \cdots). \tag{7.3}$$

Obviously the inequality (7.2) implies the absolute convergence of the integral (7.1) for $x>c$. The necessity of the condition is at once apparent. For, if $x>c$, we have

$$f^{(n)}(x) = (-1)^n \int_0^\infty e^{-xt}t^n\phi(t)dt,$$

$$|f^{(n)}(x)| \leqq \int_0^\infty e^{-xt}t^nKe^{ct}dt = \frac{Kn!}{(x-c)^{n+1}} \qquad (n = 0, 1, 2, \cdots).$$

Conversely if (7.3) is satisfied, then

$$-\frac{Kn!}{(x-c)^{n+1}} \leqq (-1)^n f^{(n)}(x) \leqq \frac{Kn!}{(x-c)^{n+1}},$$

or

$$(-1)^n\left[f^{(n)}(x) + (-1)^n \frac{Kn!}{(x-c)^{n+1}}\right] \geqq 0,$$

$$(-1)^n\left[-f^{(n)}(x) + (-1)^n \frac{Kn!}{(x-c)^{n+1}}\right] \geqq 0.$$

This shows that the functions $K(x-c)^{-1}+f(x)$ and $K(x-c)^{-1}-f(x)$ are both completely monotonic in the interval $c<x<\infty$. Hence by Theorem 8, there exists a non-decreasing function $\beta(t)$ such that

$$f(x) + \frac{K}{x-c} = \int_0^\infty e^{-xt}d\beta(t),$$

the integral converging for $x>c$. But

$$\frac{K}{x-c} = K\int_0^\infty e^{-xt}d\left(\frac{e^{ct}}{c}\right) \qquad (x > c),$$

so that

$$f(x) = \int_0^\infty e^{-xt}d\alpha(t),$$

where

$$\alpha(t) = \beta(t) - Ke^{ct}/c.$$

Since $\alpha(t)$ is the difference of two monotonic functions, it is a function of bounded variation. But

$$\frac{K}{x-c} - f(x) = \int_0^\infty e^{-xt}d\left[\frac{Ke^{ct}}{c} - \alpha(t)\right].$$

In this way we see that the functions $Ke^{ct}c^{-1}+\alpha(t)$ and $Ke^{ct}c^{-1}-\alpha(t)$ are both increasing functions. From this fact it follows that

$$-Ke^{c(t+\theta\delta)} \leqq \frac{\alpha(t+\delta)-\alpha(t)}{\delta} \leqq Ke^{c(t+\theta\delta)} \qquad (0<\delta;\, 0<\theta<1).$$

Allowing δ to approach zero we obtain

$$-Ke^{ct} \leqq D^+\alpha(t) \leqq Ke^{ct},$$

where $D^+\alpha(t)$ denotes the upper derivative of $\alpha(t)$ on the right. Now $D^+\alpha(t)$ is integrable* since $\alpha(t)$ is of bounded variation and $D^+\alpha(t)$ is finite in every finite interval. Consequently, we may write

* E. W. Hobson, loc. cit., vol. 1, p. 549.

$$\alpha(x) = \int_0^x \phi(t)dt$$

where

$$\phi(t) = D^+\alpha(t), \quad |\phi(t)| \leqq Ke^{ct} \qquad (0 \leqq t).$$

This completes the proof of the theorem. We point out that condition (7.3) implies the vanishing of $f(x)$ at infinity. This also follows indirectly from Theorem 10, since $\alpha(t)$, being an integral, is continuous, and $\alpha(0+)=0$. We further call attention to the fact that condition (7.3) implies condition (b) of Theorem 12. That this should be the case is seen by observing that if $f(x)$ has the form (7.1) it also has the form (6.5) with $\alpha(x)$ defined as

$$\alpha(x) = \int_0^x \phi(t)dt.$$

8. Examples. At this point we illustrate Theorem 13 by a few examples.

A. Take $f(x)=1/x$, $\alpha(t)=t$, $\phi(t)=1$, $c=0$, $K=1$. Condition (7.3) is clearly satisfied since

$$|f^{(n)}(x)| = \frac{n!}{x^{n+1}} \qquad (x > 0;\ n = 0, 1, 2, \cdots).$$

B. Take $f(x)=e^{-x}$, $\alpha(t)=0$ $(0\leqq t<1)$, $\alpha(t)=1$ $(t\geqq 1)$. In this case condition (7.3) should not be satisfied since $\alpha(t)$ is not an integral. We have

$$\frac{|f^{(n)}(x)|(x-c)^{n+1}}{n!} = \frac{e^{-x}(x-c)^{n+1}}{n!},$$

$$\max \frac{e^{-x}(x-c)^{n+1}}{n!} = \frac{e^{-(n+1)}(n+1)^{n+1}}{n!}.$$

This latter quantity becomes infinite with n so that no constant K exists for condition (7.3) no matter how c may be chosen.

C. Take $f(x)=1$, $\alpha(t)=1$ $(t>0)$, $\alpha(0)=0$. Here again $\alpha(t)$ is not an integral. Condition (7.3) is not satisfied since $(x-c)$ is not bounded.

D. Take $f(x)=e^{-x}/x$, $\phi(t)=0$ $(t<1)$, $\phi(t)=1$ $(t\geqq 1)$, $\alpha(t)=0$ $(t\leqq 1)$, $\alpha(t)=t-1$ $(t\geqq 1)$. Here condition (7.3) is satisfied with $c=0$, $K=1$. For

$$f^{(n)}(x) = \frac{e^{-x}}{x}\sum_{p=0}^{n}\frac{(-1)^{n-p}}{x^p}\binom{n}{p}p!,$$

$$\frac{f^{(n)}(x)x^{n+1}}{n!} = \frac{e^{-x}(-1)^n}{n!}\sum_{p=0}^{n}(-1)^p\binom{n}{p}p!(x)^{n-p},$$

$$\frac{|f^{(n)}(x)|x^{n+1}}{n!} \leqq e^{-x}\sum_{p=0}^{n}\frac{x^{n-p}}{(n-p)!}$$
$$\leqq e^{-x}\sum_{p=0}^{\infty}\frac{x^{p}}{p!} = 1.$$

E. Take $f(x)=1/x^2$, $\phi(t)=t$. Then

$$t \leqq \frac{e^{\epsilon t}}{e\epsilon} \qquad (t \geqq 0)$$

for all positive ϵ. For the function $te^{-\epsilon t}$ attains its maximum value $(e\epsilon)^{-1}$ at $t=1/\epsilon$. By Theorem 13 it should follow that

$$|f^{(n)}(x)| \leqq n!(e\epsilon)^{-1}(x-\epsilon)^{-n-1}$$

for all integers $n\geqq 0$, all positive numbers ϵ, and all $x>\epsilon$. Now

$$|f^{(n)}(x)| = (n+1)!x^{-n-2},$$

so that we should have

$$(n+1)x^{-n-2} \leqq (x-\epsilon)^{-n-1}(\epsilon e)^{-1},$$

or

$$(n+1)\epsilon x^{-1}(1-\epsilon x^{-1})^{-n-1} \leqq e^{-1}.$$

But the function on the left attains its maximum value $(n+1)^{n+2}(n+2)^{-n-2}$ at $x=\epsilon(n+2)$. As n becomes infinite this maximum value increases and approaches e^{-1} as its limit. The above inequality is thus established. This example serves to illustrate the fact that $|\phi(t)|$ may be equal to Ke^{ct} at certain points of $(0, \infty)$ and yet $|f^{(n)}(x)|$ may never be equal to $Kn!(x-c)^{-n-1}$ no matter how large n is taken.

9. Application to Dirichlet series. By use of Theorem 13 we are now able to obtain a condition that is both necessary and sufficient for the development of a function $f(x)$ in a convergent Dirichlet series. We restrict ourselves at first to the case in which the series converges for $x>0$.

THEOREM 14. *A necessary and sufficient condition that a real function $f(x)$ can be represented in a Dirichlet series convergent for $x>0$ is that a set of real constants $a_1, a_2, a_3, \cdots, \lambda_1, \lambda_2, \lambda_3, \cdots$,*

$$0 \leqq \lambda_1 < \lambda_2 < \lambda_3 < \cdots, \quad \lim_{k=\infty}\lambda_k = \infty,$$

*exist of such a nature that to every positive ϵ and every integer k there corresponds a number M independent of n and of x such that**

* In (9.1) it is to be understood that k shall also take on the value zero, the quantity in the brace then reducing to $f(x)$.

$$\left|\frac{d^n}{dx^n}\left[\left\{f(x) - \sum_{m=1}^{k} a_m e^{-\lambda_m x}\right\} x^{-1} e^{\lambda_{k+1} x}\right]\right| \leqq \frac{Mn!}{(x-\epsilon)^{n+1}} \tag{9.1}$$

$$(x > \epsilon;\ n = 0, 1, 2, \cdots).$$

We prove first the necessity of the condition. Let $f(x)$ be the sum of the Dirichlet series

$$f(x) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n x}, \tag{9.11}$$

convergent for $x > 0$. Then

$$f(x) = \int_0^{\infty} e^{-xt} d\alpha(t)$$

where

$$\alpha(t) = \begin{cases} 0 & (0 \leqq t < \lambda_1), \\ a_1 + a_2 + \cdots + a_k & (\lambda_k \leqq t < \lambda_{k+1};\ k = 1, 2, 3, \cdots). \end{cases}$$

Since the series and integral converge for $x = \epsilon > 0$, there exists* a constant M such that

$$|\alpha(t)| \leqq M e^{\epsilon t} \qquad (0 \leqq t < \infty). \tag{9.2}$$

Integrating by parts we obtain

$$f(x) = x \int_0^{\infty} e^{-xt} \alpha(t) dt,$$

the integrated term disappearing by virtue of (9.2). Now since $\alpha(t)$ is constantly zero in the interval $0 \leqq t < \lambda_1$, an obvious change of variable gives us

$$f(x) = x e^{-\lambda_1 x} \int_0^{\infty} e^{-xt} \alpha(t + \lambda_1) dt,$$

where

$$|\alpha(t + \lambda_1)| \leqq M e^{\epsilon\lambda_1} e^{\epsilon t} = M' e^{\epsilon t}.$$

Now applying Theorem 13 we have

$$\left|\frac{d^n}{dx^n}\{f(x) x^{-1} e^{\lambda_1 x}\}\right| \leqq \frac{Mn!}{(x-\epsilon)^{n+1}} \quad (x > \epsilon;\ n = 0, 1, 2 \cdots).$$

* D. V. Widder, loc. cit., p. 703, Lemma 2.

This is the first of the conditions (9.1) corresponding to the case $k=0$. Since

$$f(x) - \sum_{m=1}^{k} a_m e^{-\lambda_m x}$$

is itself a Dirichlet series, the first term of which is $a_{k+1}e^{-\lambda_{k+1}x}$, we have only to apply the result just obtained to the new series to obtain (9.1). The proof of the necessity is thus complete.

We turn now to the proof of the sufficiency of the condition. By Theorem 13, we see that condition (9.1) taken for $k=0$ implies the existence of a function $\alpha(t)$ such that

$$f(x) = xe^{-\lambda_1 x}\int_0^\infty e^{-xt}\alpha(t)dt, \tag{9.3}$$

where

$$|\alpha(t)| \leqq Me^{\epsilon t} \qquad (0 \leqq t < \infty).$$

By a linear change of variable, equation (9.3) becomes

$$f(x) = x\int_{\lambda_1}^\infty e^{-xt}\alpha(t-\lambda_1)dt = x\int_0^\infty e^{-xt}\beta(t)dt,$$

where

$$\beta(t) = \begin{cases} 0 & (0 \leqq t < \lambda_1), \\ \alpha(t-\lambda_1) & (\lambda_1 \leqq t < \infty), \end{cases}$$

$$|\beta(t)| \leqq Me^{-\epsilon\lambda_1}e^{\epsilon t} \leqq Me^{\epsilon t}. \tag{9.4}$$

We can now show that $\beta(t)$ is a step-function, or differs from such a function at a set of points of measure zero. Again applying Theorem 13, but now using (9.1) for an arbitrary k, we see that

$$f(x) = \sum_{m=1}^{k} a_m e^{-\lambda_m x} + xe^{-\lambda_{k+1}x}\int_0^\infty e^{-xt}\alpha_k(t)dt.$$

As before this may be transformed into

$$\begin{aligned} f(x) &= \sum_{m=1}^{k} a_m e^{-\lambda_m x} + x\int_{\lambda_{k+1}}^\infty e^{-xt}\alpha_k(t-\lambda_{k+1})dt \\ &= \sum_{m=1}^{k} a_m e^{-\lambda_m x} + x\int_0^\infty e^{-xt}\beta_k(t)dt, \end{aligned}$$

where

$$\beta_k(t) = \begin{cases} 0 & (0 \leqq t < \lambda_{k+1}), \\ \alpha_k(t - \lambda_{k+1}) & (\lambda_{k+1} \leqq t < \infty). \end{cases}$$

Clearly the summation in this expression may be rewritten as follows:

$$f(x) = \sum_{m=1}^{k} x \int_{\lambda_m}^{\infty} e^{-xt} a_m dt + x \int_0^{\infty} e^{-xt} \beta_k(t) dt,$$

whence

$$f(x) = x \int_0^{\infty} e^{-xt} \gamma_k(t) dt,$$

$$\gamma_k(t) = \begin{cases} 0 & (0 \leqq t < \lambda_1), \\ a_1 + a_2 + \cdots + a_\nu & (\lambda_\nu \leqq t < \lambda_{\nu+1};\ \nu = 1, 2, \cdots, k), \\ \beta_k(t) + \sum_{m=1}^{k} a_m & (\lambda_{k+1} \leqq t < \infty). \end{cases}$$

If we now make use of the uniqueness theorem,* we see that $\gamma(t)$ must coincide with $\beta(t)$ almost everywhere. By allowing k to become infinite, we see that $\beta(t)$ differs from a step-function $\gamma(t)$ at most at a set of points of measure zero. Since $\gamma(t)$ is a step-function it follows that the inequality (9.4) implies

$$(9.5) \qquad | \gamma(t) | \leqq Me^{\epsilon t} \qquad (0 \leqq t < \infty).$$

Hence, on integrating by parts, we obtain

$$f(x) = \int_0^{\infty} e^{-xt} d\gamma(t) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n x}.$$

The integral and series converge for $x > \epsilon$ by virtue of (9.5). But ϵ was an arbitrary positive quantity. The above argument repeated for any positive ϵ must always lead to the same Dirichlet series since expansion in such a series is unique. It follows that the series converges for $x > 0$, and the proof is complete.

We can now see that the restriction of convergence for $x > 0$ was not an essential one. For, if the series (9.11) converges for $x > c$, then the series

$$f(x + c) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n c} e^{-\lambda_n x}$$

converges for $x > 0$, and we can apply Theorem 14. The inequality (9.1) in the statement of that theorem must be replaced by the following one:

$$\left| \frac{d^n}{dx^n} \left\{ [f(x) - \sum_{m=1}^{k} a_m e^{-\lambda_m x}](x - c)^{-1} e^{\lambda_{k+1} x} \right\} \right| \leqq \frac{Mn!}{(x - c - \epsilon)^{n+1}} .$$

* D. V. Widder, loc. cit., p. 705.

If we restrict ourselves to Dirichlet series with positive coefficients a theorem that is much simpler in statement may be obtained by use of Theorem 9.

THEOREM 15. *A necessary and sufficient condition that $f(x)$ can be represented in a Dirichlet series with positive coefficients convergent for $x>c$ is that a set of constants*

$$a_1, a_2, a_3, \cdots, \lambda_1, \lambda_2, \lambda_3, \cdots,$$

$$0 \leqq \lambda_1 < \lambda_2 < \lambda_3 < \cdots, \quad \lim_{k=\infty} \lambda_k = \infty,$$

exist such that the function

$$\left(f(x) - \sum_{m=1}^{k} a_m e^{-\lambda_m x}\right) e^{\lambda_{k+1} x}$$

is a completely monotonic function in the interval $c<x<\infty$ for $k=0, 1, 2, \cdots$.

We omit the proof since it follows closely that of Theorem 14, making application of Theorem 9 instead of Theorem 13.

10. Relation between completely monotonic functions and completely monotonic sequences. We turn now to the discussion of the following problem. Given an infinite set of constants $a_0, a_1, a_2, \cdots$. Under what conditions is it possible to determine a completely monotonic function $f(x)$ such that $f(n)=a_n$ for $n=0, 1, 2, \cdots$? In order to simplify the statement of the solution of this problem we introduce a

DEFINITION. *A completely monotonic set of constants $a_0, a_1, a_2, \cdots$ is minimal if decreasing a_0 makes of it a set which is no longer completely monotonic.*

That there exist completely monotonic sets which are not minimal may be seen by noting that increasing the first element of a set which is completely monotonic leaves it so. By a theorem of Hausdorff* it is known that any completely monotonic set $a_0, a_1, a_2, \cdots$ can be represented in the form

$$a_n = \int_0^1 t^n d\phi(t) \qquad (n = 0, 1, 2, \cdots)$$

where $\phi(t)$ is a non-decreasing function. Moreover, the representation in this form is unique if "normalized" functions $\phi(t)$ only are admitted, that is, functions for which

$$\phi(0) = 0, \quad \phi(t) = \frac{\phi(t+0) + \phi(t-0)}{2} \qquad (0 < t < 1).$$

We first establish the following

* F. Hausdorff, loc. cit., p. 226.

LEMMA. *The completely monotonic set $a_0, a_1, a_2, \cdots$ is minimal if and only if the function $\phi(t)$ of its Hausdorff representation is continuous at $t=0$.*

We prove first the necessity of the condition, showing that if $\phi(0+)>0$ the set is not minimal. Define a function $\psi(t)$ continuous at $t=0$ by the equations

$$\psi(t) = \begin{cases} 0 & (t = 0), \\ \phi(t) - \phi(0+) & (0 < t \leqq 1). \end{cases}$$

Then

$$\int_0^1 t^n d\psi(t) = \int_0^1 t^n d\phi(t) = a_n \qquad (n = 1, 2, 3, \cdots)$$

$$= \int_0^1 d\phi(t) - \phi(0+) = a_0 - \phi(0+) \qquad (n = 0).$$

Since $\psi(t)$ is itself a non-decreasing function, the set $a_0-\phi(0+), a_1, a_2, \cdots,$ is itself completely monotonic, so that the given set can not have been minimal.

Conversely, if $\phi(0+)=0$, then the set $a_0, a_1, a_2, \cdots$ is minimal. If it were not so, a positive constant k would exist such that $a_0-k, a_1, a_2, \cdots$ would be a completely monotonic set. That is, a unique normalized non-decreasing function $\psi(t)$ would exist satisfying the equations

$$a_n = \int_0^1 t^n d\psi(t) \qquad (n = 1, 2, 3, \cdots),$$

$$a_0 - k = \int_0^1 d\psi(t).$$

But we clearly have

$$a_n = \int_0^1 t^n d\bar{\psi}(t) \qquad (n = 1, 2, 3, \cdots),$$

$$a_0 - k = \int_0^1 d\bar{\psi}(t),$$

where

$$\bar{\psi}(t) = \phi(t) - k \qquad (0 < t \leqq 1),$$

$$\bar{\psi}(0) = 0.$$

Hence $\bar{\psi}(t) \equiv \psi(t)$, for $\bar{\psi}(t)$ is clearly normalized if $\phi(t)$ is. But $\bar{\psi}(t)$ is not a

non-decreasing function since $\bar{\psi}(0+)-\bar{\psi}(0)=-k<0$. Since $\psi(t)$ is non-decreasing we have a contradiction, thus completing the proof of the lemma.

By use of this result we are able to prove

THEOREM 16. *A necessary and sufficient condition that there should exist a function $f(x)$ completely monotonic in the interval $0\leqq x<\infty$ such that $f(n)=a_n$ for $n=0, 1, 2, \cdots$ is that the set $a_0, a_1, a_2, \cdots$ should be a minimal completely monotonic set.*

We begin with the sufficiency of the condition. Suppose the set $a_0, a_1, a_2, \cdots$ to be of specified type. Then

$$a_n = \int_0^1 t^n d\phi(t) \qquad (n = 0, 1, 2, \cdots),$$

where $\phi(t)$ is a non-decreasing function vanishing at $t=0$ and continuous there. Since $\phi(t)$ is continuous at $t=0$ we have

$$a_n = \lim_{\epsilon=0} \int_\epsilon^1 t^n d\phi(t) \qquad (n = 0, 1, 2, \cdots).$$

Make the change of variable $t=e^{-y}$. Then

$$a_n = \lim_{\epsilon=0} \int_0^{-\log \epsilon} e^{-ny} d\alpha(y) = \int_0^\infty e^{-ny} d\alpha(y),$$

where

$$\alpha(y) = -\phi(e^{-y}).$$

The function $\alpha(y)$ is clearly non-decreasing. The function

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t)$$

is completely monotonic in the interval $0<x<\infty$ since the integral converges for $x>0$. The function $f(x)$ is continuous* in the interval $0\leqq x<\infty$ since the integral converges for $x=0$. Since $f(n)=a_n$, the proof of the sufficiency of the condition is complete.

Consider now the necessity of the condition. Suppose that a function $f(x)$, completely monotonic in the interval $0\leqq x<\infty$, exists such that $f(n)=a_n$ for $n=0, 1, 2, \cdots$. We show first that the set of constants $a_0, a_1, a_2, \cdots$ is completely monotonic. The point is not covered by Theorem 1 since it is not known that the point $x=0$ is an interior point of an interval in which $f(x)$ is completely monotonic. By Theorem 8 we have

* D. V. Widder, loc. cit., p. 701.

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t) \qquad (x > 0), \tag{10.1}$$

where $\alpha(t)$ is a non-decreasing function and the integral converges for $x>0$. We can show that the integral also converges for $x=0$. For suppose that it diverged. Since $\alpha(t)$ is monotonic we should then have $\alpha(\infty)=\infty$. In that case $\lim_{x=0} f(x)=\infty$. For, if x is a fixed positive quantity, we have

$$\begin{aligned} f(x) &= x\int_0^\infty e^{-xt}\alpha(t)dt \\ &= x\int_0^{1/x} e^{-xt}\alpha(t)dt + x\int_{1/x}^\infty e^{-xt}\alpha(t)dt. \end{aligned}$$

Since $\alpha(t)\geqq 0$ and $e^{-xt}>0$ we have

$$x\int_0^{1/x} e^{-xt}\alpha(t)dt \geqq 0,$$

whence

$$f(x) \geqq x\int_1^\infty e^{-xt}\alpha(t)dt.$$

But $\alpha(t)\geqq\alpha(1/x)$ in the interval $1/x\leqq t<\infty$. Hence

$$f(x) \geqq \alpha(1/x)x\int_{1/x}^\infty e^{-xt}dt = \alpha(1/x)/e.$$

Consequently

$$\lim_{x=0} f(x) = \alpha(\infty)/e = \infty .$$

But since $f(x)$ is assumed continuous at $x=0$,

$$\lim_{x=0} f(x) = f(0) = a_0.$$

The assumption that the integral (10.1) diverged for $x=0$ was false. The integral thus defined for $x=0$ must be equal to $f(0)=a_0$ since both the integral and the function $f(x)$ are continuous at $x=0$. Hence

$$a_n = \int_0^\infty e^{-nt}d\alpha(t) \qquad (n = 0, 1, 2, \cdots).$$

It follows that the set $a_0, a_1, a_2, \cdots$ is completely monotonic since

$$(-1)^k\Delta^k a_m = \int_0^\infty (1-e^{-t})^k e^{-mt}d\alpha(t) \geqq 0 \; (k = 0, 1, 2, \cdots; m = 0, 1, 2, \cdots).$$

It remains only to show that this set is minimal. We have

$$a_n = \lim_{R=\infty} \int_0^R e^{-nt} d\alpha(t) = \lim_{R=\infty} \int_{e^{-R}}^1 t^n d\beta(t),$$

where

$$\beta(t) = -\alpha\left(\log \frac{1}{t}\right) \qquad (0 < t \leqq 1).$$

The function $\beta(t)$ is undefined for $t=0$. If $\beta(0)$ is defined as $-\alpha(\infty)$, we have $\beta(0)=\beta(0+)$ and

$$a_n = \int_0^1 t^n d\beta(t).$$

Since $\beta(t)$ is continuous at $t=0$, the set $a_0, a_1, a_2, \cdots$ must be minimal, and the proof is complete.

11. The integral $\int_0^1 e^{xt} d\alpha(t)$. We turn now to the determination of conditions both necessary and sufficient for the representation of a function $f(x)$ in the form

$$f(x) = \int_0^1 e^{xt} d\alpha(t)$$

where $\alpha(t)$ is a non-decreasing function. First consider necessary conditions. If $f(x)$ has the above representation it is clearly an entire function. Let x_0 be any real value. Then

$$\begin{aligned} f^{(n)}(x_0) &= \int_0^1 e^{x_0 t} t^n d\alpha(t) \\ &= \int_0^1 t^n d\beta(t), \end{aligned}$$

where

$$\beta(t) = \int_0^t e^{x_0 y} d\alpha(y) \qquad (0 < t \leqq 1),$$

$$\beta(0) = 0.$$

Since $\beta(t)$ is a non-decreasing function, the sequence $f(x_0), f'(x_0), f''(x_0), \cdots$ is completely monotonic. In particular if $\alpha(t)$ is continuous at $t=0$, $\beta(t)$ is also continuous there, and the above sequence is minimal. We can now show that these necessary conditions are sufficient. Let $f(x)$ be a function with derivatives of all orders at $x=x_0$ and such that the sequence of its successive derivatives there is completely monotonic. That is,

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)(x - x_0)^n}{n!}.$$

This function is entire. For

$$0 \leqq f^{(n)}(x_0) \leqq f(x_0) \qquad (n = 0, 1, 2, \cdots),$$

and

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)(x - x_0)^n}{n!} \ll f(x_0) \sum_{n=0}^{\infty} \frac{|x - x_0|^n}{n!} = f(x_0)e^{|x-x_0|}$$

$$(-\infty < x < \infty).$$

Now by Hausdorff's theorem, it is possible to determine a non-decreasing bounded function $\beta(t)$ such that

$$f^{(n)}(x_0) = \int_0^1 t^n d\beta(t),$$

whence

$$f(x) = \sum_{n=0}^{\infty} \frac{(x - x_0)^n}{n!} \int_0^1 t^n d\beta(t).$$

For any fixed value of x the series

$$e^{(x-x_0)t} = \sum_{n=0}^{\infty} \frac{(x - x_0)^n t^n}{n!}$$

is uniformly convergent in the interval $0 \leqq t \leqq 1$, since

$$\sum_{n=0}^{\infty} \frac{(x - x_0)^n t^n}{n!} \ll \sum_{n=0}^{\infty} \frac{|x - x_0|^n}{n!} = e^{|x-x_0|}.$$

Hence it may be integrated term by term with respect to the monotonic function $\beta(t)$. That is,

$$\int_0^1 e^{(x-x_0)t} d\beta(t) = \sum_{n=0}^{\infty} \frac{(x - x_0)^n}{n!} \int_0^1 t^n d\beta(t) = f(x),$$

so that

$$f(x) = \int_0^1 e^{xt} d\alpha(t),$$

where

$$\alpha(t) = \int_0^t e^{-x_0 y} d\beta(y) \qquad (0 < t \leqq 1),$$

$$\alpha(0) = 0.$$

Again we see that $\alpha(t)$ is non-decreasing, is continuous at $t=0$ if $\beta(t)$ is continuous there. We have seen that $\beta(t)$ has this property if and only if the sequence $f(x_0), f'(x_0), \cdots$ is minimal. We have thus established

THEOREM 17. *A necessary and sufficient condition that a function $f(x)$ can be represented in the form*

$$f(x) = \int_0^1 e^{xt} d\alpha(t)$$

with $\alpha(t)$ a non-decreasing bounded function (continuous at $t=0$) is that $f(x)$ should have derivatives of all orders at a point $x=x_0$ which form a (minimal) completely monotonic set, $f(x_0), f'(x_0), f''(x_0), \cdots$.

12. Generalized derivatives. Let us now combine the results of Theorems 16 and 17. If $\alpha(t)$ is non-decreasing and continuous at $t=0$, the set $f(x_0)$, $f'(x_0), f''(x_0), \cdots$ is a minimal completely monotonic set, so that there exists a function $\phi(x)$ that is completely monotonic in $0 \leqq x < \infty$ such that

$$\phi(n) = f^{(n)}(x_0) \qquad (n = 0, 1, 2, \cdots).$$

It is natural to inquire if there is not some sense in which this equation holds for non-integral values of n. We shall show that if $f^{(n)}(x_0)$ is replaced by the generalized derivative of Riemann (slightly modified* to meet our need) then the above equation holds for all $n > 0$. We define the generalized derivative of positive order ρ of a function $f(x)$ as

$$_{-\infty}D_x^{\rho} f(x) = \int_0^{\infty} \frac{t^{-\nu} f^{(m+1)}(x-t)}{\Gamma(1-\nu)} dt \qquad (0 \leqq \nu < 1),$$

where $m + \nu = \rho$ and m is the largest integer contained in ρ. Clearly if ρ is an integer m and if $f^{(m)}(x)$ is a function which vanishes for $x = -\infty$, then

$$_{-\infty}D_x^{m} f(x) = f^{(m)}(x).$$

With this definition at hand we can now prove

THEOREM 18. *The generalized derivative $_{-\infty}D_x^{\rho} f(x)$ is a completely monotonic function of ρ in the interval $\rho \geqq 0$ for every x if and only if*

$$f(x) = \int_0^1 e^{xt} d\alpha(t),$$

where $\alpha(t)$ is a bounded non-decreasing function that is continuous at $t=0$.

To prove the sufficiency of the condition we show first that the integral

* This generalized derivative, $_{-\infty}D_x^{\rho} f(x)$, is ordinarily defined for positive values of ρ through the medium of its values for negative ρ. For the functions under consideration, however, it need not exist for negative ρ. The definition we give is legitimate since for positive integral values of ρ the generalized derivative reduces to the ordinary derivative.

$$(12.1)\qquad {}_{-\infty}D_x^{\nu}f(x) = \frac{1}{\Gamma(1-\nu)}\int_0^\infty t^{-\nu}dt\int_0^1 e^{xy-ty}y^{m+1}d\alpha(y)$$

converges. The integral is improper both on account of the infinite upper limit of the integral and because the integrand becomes infinite at $t=0$ i $\nu\neq 0$. For every x the integral

$$\int_0^1 e^{y(x-t)}y^{m+1}d\alpha(y)$$

is a continuous function of t, so that the integrand of (12.1) is $O(t^{-\nu})$ as t approaches zero. Since $\nu<1$ we are assured of the convergence of the integral if the upper limit ∞ is replaced by any positive finite limit. We must now investigate the behavior of the integrand as t becomes infinite. We show that for any fixed x and for $m\geqq 0$

$$\int_0^1 e^{y(x-t)}y^{m+1}d\alpha(y) = o(t^{-1}).$$

For

$$(12.2)\qquad \int_0^1 e^{y(x-t)}y^{m+1}d\alpha(y) = \alpha(1)e^{x-t} - \int_0^1 \alpha(y)\{(m+1)y^m e^{y(x-t)} + (x-t)e^{y(x-t)}y^{m+1}\}dy.$$

The first term on the right-hand side of this equation is clearly $o(t^{-1})$. The integral on the right-hand side may be set equal to I_1+I_2 where

$$I_1 = \int_0^{t^{-1/2}} \alpha(y)\{(m+1)y^m e^{y(x-t)} + (x-t)e^{y(x-t)}y^{m+1}\}dy,$$

$$I_2 = \int_{t^{-1/2}}^1 \alpha(y)\{(m+1)y^m e^{y(x-t)} + (x-t)e^{y(x-t)}y^{m+1}\}dy.$$

Applying the second law of the mean to I_1 we obtain

$$I_1 = \alpha(t^{-1/2})\int_\xi^{t^{-1/2}}\{(m+1)y^m e^{y(x-t)} + (x-t)e^{y(x-t)}y^{m+1}\}dy \quad (0\leqq\xi<t^{-1/2}),$$

$$= \alpha(t^{-1/2})\left\{(m+1)\eta^m\int_\xi^{t^{-1/2}} e^{y(x-t)}dy + (x-t)\eta^m\int_\xi^{t^{-1/2}} ye^{y(x-t)}dy\right\} \quad (\xi<\eta<t^{-1/2}).$$

Hence if $t>x$ we have

$$| I_1 | < \alpha(t^{-1/2})(m+1)(t^{-m/2})\frac{1 - e^{xt^{-1/2} - t^{1/2}}}{t - x}$$

$$+ \alpha(t^{-1/2})(t - x)t^{-m/2}\left\{\frac{1 + e^{xt^{-1/2} - t^{1/2}}[1 + (t - x)t^{-1/2}]}{(t - x)^2}\right\}.$$

Since $\alpha(t^{-1/2})$ approaches zero as t becomes infinite, it is clear that $I_1 = o(t^{-1})$ if $m \geqq 0$. For I_2 we have the following equation and inequalities:

$$I_2 = \alpha(\xi)\{(m+1)\xi^m + (x - t)\xi^{m+1}\}\int_{t^{-1/2}}^{1} e^{y(x-t)}dt \qquad (t^{-1/2} < \xi < 1),$$

$$| I_2 | \leqq \alpha(1)\{(m+1) + (t - x)\}\int_{t^{-1/2}}^{1} e^{y(x-t)}dt,$$

$$| I_2 | \leqq \alpha(1)\{(m+1) + (t - x)\}\{e^{x-t} - e^{xt^{-1/2} - t^{1/2}}\}(t - x)^{-1},$$

from which we see that I_2 is also $o(1/t)$. The integrand of (12.1) when multiplied by $t^{1+\nu}$ approaches zero as t becomes infinite. By the usual limit test for convergence we infer therefore that the integral converges if $\nu > 0$. If $\nu = 0$ the integral may be integrated in finite form, and we are assured of convergence since

$$\lim_{t=\infty} f^{(m)}(x - t) = 0 \qquad (m = 0, 1, 2, \cdots)$$

by virtue of Theorem 10.

We show next that it is permissible to interchange the order of integration in (12.1). We rewrite that integral as

$$(12.3) \qquad {}_{-\infty}D_x^{\rho} f(x) = \frac{1}{\Gamma(1 - \nu)}\int_0^{\infty} t^{-\nu}\alpha(1)e^{x-t}dt$$

$$+ \frac{1}{\Gamma(1 - \nu)}\int_0^{\infty} t^{-\nu+1}dt \int_0^1 \alpha(y)e^{y(x-t)}y^{m+1}dy$$

$$- \frac{x}{\Gamma(1 - \nu)}\int_0^{\infty} t^{-\nu}dt \int_0^1 \alpha(y)e^{y(x-t)}y^{m+1}dy$$

$$- \frac{1}{\Gamma(1 - \nu)}\int_0^{\infty} t^{-\nu}dt \int_0^1 e^{y(x-t)}y^m(m+1)\alpha(y)dy$$

and apply a familiar theorem* to each of the iterated integrals.

We must show

(a) that the two repeated integrals in opposite orders over the domain (0, 0; 1, R) exist and have equal values for every positive R,

* E. W. Hobson, loc. cit., vol. 2, p. 398.

(b) that the iterated integrals of (12.3) converge,

(c) that the integrands are non-negative.

The last of these conditions is obvious. Since each term on the right-hand side of (12.2) has been shown to be $o(t^{-1})$, (b) follows at once. To prove (a) we have only to note again that the integrands are non-negative and apply a known theorem.* Consequently

$$
\begin{aligned}
{}_{-\infty}D_x^{\rho} f(x) = \alpha(1)e^x - x\int_0^1 \alpha(y)e^{xy}y^{m+1}y^{\nu-1}dy + (1-\nu)\int_0^1 \alpha(y)e^{xy}y^{m+1}y^{\nu-2}dy \\
- \int_0^1 \alpha(y)e^{yx}(m+1)y^m y^{\nu-1}dy,
\end{aligned}
$$

since

$$\int_0^\infty t^{-\nu}e^{-ty}dt = y^{\nu-1}\Gamma(1-\nu) \qquad (\nu < 1).$$

But

$$
\begin{aligned}
\int_0^1 e^{xy}y^{m+\nu}d\alpha(y) = \alpha(1)e^x - x\int_0^1 \alpha(y)e^{xy}y^{m+\nu}dy \\
- (m+\nu)\int_0^1 \alpha(y)e^{xy}y^{m+\nu-1}dy.
\end{aligned}
$$

Consequently,

$$ {}_{-\infty}D_x^{\rho} f(x) = \int_0^1 e^{xy}y^{m+\nu}d\alpha(y) = \int_0^1 e^{xy}y^{\rho}d\alpha(y) \qquad (\rho \geqq 0).$$

To show that this is a completely monotonic function of ρ set $y=e^{-u}$. Since $\alpha(0+)=0$, we have

$$
\begin{aligned}
\int_0^1 e^{xy}y^{\rho}d\alpha(y) &= \lim_{\epsilon=0}\int_\epsilon^1 e^{xy}y^{\rho}d\alpha(y) = \lim_{\epsilon=0}\int_0^{-\log\epsilon} e^{xe^{-u}}e^{-\rho u}d[-\alpha(e^{-u})] \\
&= \int_0^\infty e^{-\rho u}d\beta(u),
\end{aligned}
$$

where

$$
\begin{aligned}
\beta(u) &= \int_0^u e^{xe^{-t}}d[-\alpha(e^{-t})] \qquad (u > 0), \\
\beta(0) &= 0.
\end{aligned}
$$

Since the function $-\alpha(e^{-u})$ is a non-decreasing function of u, it follows that $\beta(u)$ is also non-decreasing, and an appeal to Theorem 8 gives the desired result.

It only remains to prove the necessity of the condition. Assume then that

* E. W. Hobson, loc. cit., vol. 2, the first theorem on p. 340.

${}_{-\infty}D_x^{\rho} f(x)$ is a completely monotonic function of ρ in the interval $0 \leqq \rho < \infty$ for each x. Then by Theorem 16 the sequence $f(x_0), f'(x_0), f''(x_0), \cdots$ is a completely monotonic minimal set. Consequently, by Theorem 17,

$$f(x) = \int_0^1 e^{xt} d\alpha(t),$$

where $\alpha(t)$ is a non-decreasing bounded function continuous at $t=0$. This completes the proof of Theorem 18.

13. The complex case. We have been dealing thus far with real functions of the real variable. Although certain of our theorems, such as Theorem 8, are in their very nature real function theorems, others are easily extended to include complex functions. Since it is usually desirable to consider Dirichlet series in the complex plane, it is important to make such an extension in the case of Theorem 14. We begin by making a similar extension of Theorem 13. We prove

THEOREM 19. *A necessary and sufficient condition that the function $f(x+iy)$ can be expressed in the form*

$$f(x + iy) = \int_0^\infty e^{-t(x+iy)} \phi(t) dt,$$

where $\phi(t)$ is a complex function of the real variable t which is $O(e^{ct})$ as t becomes infinite, is that a real constant K should exist such that

$$| f^{(n)}(x) | \leqq \frac{Kn!}{(x - c)^{n+1}} \quad (x > c; n = 0, 1, 2, \cdots).$$

The necessity of the condition is obvious from the inequalities

$$| f^{(n)}(x) | \leqq \int_0^\infty e^{-xt} t^n | \phi(t) | dt \leqq K \int_0^\infty e^{-xt} t^n e^{ct} dt,$$

$$| f^{(n)}(x) | \leqq \frac{Kn!}{(x - c)^{n+1}}.$$

To prove the converse, set $f(x) = u(x) + iv(x)$. Then

$$([u^{(n)}(x)]^2 + [v^{(n)}(x)]^2)^{1/2} \leqq \frac{Kn!}{(x - c)^{n+1}},$$

whence

$$| u^{(n)}(x) | \leqq \frac{Kn!}{(x - c)^{n+1}},$$

$$| v^{(n)}(x) | \leqq \frac{Kn!}{(x - c)^{n+1}}.$$

Hence, by Theorem 13, functions $\beta(t)$ and $\gamma(t)$ exist such that

$$u(x) = \int_0^\infty e^{-xt}\beta(t)dt, \qquad |\beta(t)| \leqq Ke^{ct} \qquad (t \geqq 0),$$

$$v(x) = \int_0^\infty e^{-xt}\gamma(t)dt, \qquad |\gamma(t)| \leqq Ke^{ct} \qquad (t \geqq 0).$$

Consequently,

$$f(x) = \int_0^\infty e^{-xt}(\beta(t) + i\gamma(t))dt,$$

$$f(x+iy) = \int_0^\infty e^{-(x+iy)t}\phi(t)dt,$$

where

$$\phi(t) = \beta(t) + i\gamma(t) = O(e^{ct}).$$

The theorem is thus established.

THEOREM 20. *A necessary and sufficient condition that the complex function $f(x+iy)$ can be represented by a Dirichlet series convergent in the half-plane $x>0$ is that a set of complex constants $a_1, a_2, a_3, \cdots$ and a set of real constants $\lambda_1, \lambda_2, \lambda_3, \cdots$,*

$$0 \leqq \lambda_1 < \lambda_2 < \cdots, \quad \lim_{k=\infty} \lambda_k = \infty,$$

exist of such a nature that to every positive number ϵ and to every integer k there corresponds a number M independent of n and of x such that

$$\left|\frac{d^n}{dx^n}\left\{\left[f(x) - \sum_{m=1}^{k} a_m e^{-\lambda_m x}\right]x^{-1}e^{\lambda_{k+1}x}\right\}\right| \leqq \frac{Mn!}{(x-\epsilon)^{n+1}} \tag{13.1}$$

$$(x > \epsilon; n = 0, 1, 2, \cdots).$$

To prove this set $f(x) = u(x) + iv(x)$ and $a_n = \alpha_n' + i\alpha_n''$. Assume first that $f(x+iy)$ may be expanded in a Dirichlet series

$$f(x+iy) = \sum_{m=1}^\infty a_m e^{-\lambda_m(x+iy)}.$$

Then

$$u(x) = \sum_{m=1}^\infty \alpha_m' e^{-\lambda_m x},$$

$$v(x) = \sum_{m=1}^\infty \alpha_m'' e^{-\lambda_m x},$$

and both series converge for $x>0$. Then by Theorem 14 we have

$$(13.2)\qquad \left|\frac{d^n}{dx^n}\left\{\left[u(x) - \sum_{m=1}^{k} \alpha_m' e^{-\lambda_m x}\right] x^{-1} e^{\lambda_{k+1} x}\right\}\right| \leqq \frac{M'n!}{(x-\epsilon)^{n+1}},$$

$$(13.3)\qquad \left|\frac{d^n}{dx^n}\left\{\left[v(x) - \sum_{m=1}^{k} \alpha_m'' e^{-\lambda_m x}\right] x^{-1} e^{\lambda_{k+1} x}\right\}\right| \leqq \frac{M''n!}{(x-\epsilon)^{n+1}}.$$

Combining these two inequalities and setting $M=M'+M''$ we have (13.1).

Conversely if (13.1) holds, then the inequalities (13.2) and (13.3) hold if $M'=M''=M$. It follows by Theorem 14 that $u(x)$ and $v(x)$ are expressible in real Dirichlet series convergent for $x>0$,

$$u(x) = \sum_{m=1}^{\infty} \alpha_m' e^{-\lambda_m x},$$

$$v(x) = \sum_{m=1}^{\infty} \alpha_m'' e^{-\lambda_m x}.$$

That is, the series

$$f(x+iy) = \sum_{m=1}^{\infty} a_m e^{-\lambda_m (x+iy)}$$

converges in the half-plane $x>0$. This completes the proof of the theorem. The statement of the theorem could easily be altered so as to deal with an arbitrary half-plane of convergence.

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ON A THEOREM OF S. BERNSTEIN-WIDDER*

BY

J. D. TAMARKIN

The present note is merely a comment to the preceding paper by D. V. Widder, and was suggested by the reading of its manuscript. It gives a simplified proof of the following important theorem discovered recently by S. Bernstein, and subsequently, but independently, by Widder, whose proof is based upon entirely different principles.

THEOREM. *A necessary and sufficient condition that the function $f(x)$ should be completely monotonic in the interval $c<x<\infty$ is that*

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t),$$

where $\alpha(t)$ is a non-decreasing function of such a nature that the integral converges for $x>c$.

The sufficiency of the condition is obvious since

$$f^{(n)}(x) = (-1)^n \int_0^\infty e^{-xt} t^n d\alpha(t), \quad x > c \qquad (n = 0, 1, 2, \cdots).$$

Conversely let $f(x)$ be completely monotonic in the interval $c<x<\infty$. Let a be an arbitrary constant greater than c and set $c_i=f^{(i)}(a)$. It follows from the monotonic character of $f(x)$ that the quadratic form

$$Q_n(x) = \sum_{i=0}^{n} \sum_{j=0}^{n} c_{i+j} x_i x_j \qquad (n = 0, 1, 2, \cdots)$$

is non-negative. This fact is sufficient to ensure the existence of at least one non-decreasing function $\rho(t)$ such that†

$$c_i = \int_{-\infty}^\infty t^i d\rho(t) \qquad (i = 0, 1, 2, \cdots).$$

We now distinguish two cases:

CASE I. The function $\rho(t)$ is a step-function with a finite number of jumps.

CASE II. The function $\rho(t)$ is any other non-decreasing function.

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* Presented to the Society, September 9, 1931; received by the editors June 13, 1931.

† See, for example, Marcel Riesz, *Sur le problème des moments*, Arkiv för Matematik, Astronomi och Fysik, vol. 17, no. 16 (1923).

CASE I. If $\rho(t)$ is a step-function with p positive jumps at the points $-\lambda_1, -\lambda_2, \cdots, -\lambda_p$ we have

$$c_m = \sum_{k=1}^{p} \sigma_k(-\lambda_k)^m, \quad \sigma_k > 0.$$

From the Taylor development of $f(x)$ we obtain

$$\begin{aligned} f(x) &= \sum_{i=0}^{\infty} \frac{c_i(x-a)^i}{i!} = \sum_{i=0}^{\infty} \sum_{k=1}^{p} \sigma_k(-\lambda_k)^i(x-a)^i/i! \\ &= \sum_{k=1}^{p} \sigma_k e^{\lambda_k a} e^{-\lambda_k x}. \end{aligned} \tag{1}$$

We can now show that all the λ_k are positive or zero. It is only a matter of notation to suppose that $\lambda_1 < \lambda_2 < \cdots < \lambda_p$. Suppose that λ_1 were negative. We should have

$$f'(x) = \sum_{k=1}^{p} \sigma_k e^{\lambda_k a}(-\lambda_k) e^{-\lambda_k x}$$

$$f'(x)e^{\lambda_1 x} = -\lambda_1 \sigma_1 e^{\lambda_1 a} + \sum_{k=2}^{p} \sigma_k e^{\lambda_k a}(-\lambda_k) e^{-x(\lambda_k - \lambda_1)}.$$

From the latter equation it is clear that $f'(x)e^{\lambda_1 x}$ tends to a limit as x becomes infinite, in fact to the positive limit $-\lambda_1\sigma_1 e^{\lambda_1 a}$. But since $f(x)$ is completely monotonic for $x > c$ we have $f'(x) \leqq 0$ and

$$\lim_{x=\infty} f'(x)e^{\lambda_1 x} \leqq 0.$$

The contradiction shows that λ_1 must be positive or zero.* But equation (1) may be written in the form

$$f(x) = \int_0^{\infty} e^{-xt} d\alpha(t),$$

where $\alpha(t)$ is a non-decreasing function. Hence the theorem is established in Case I.

CASE II. If $\rho(t)$ is not a step-function then the quadratic form is not only non-negative but is a positive *definite* form. For,

$$Q_n(x) = \int_{-\infty}^{\infty} \left(\sum_{i=0}^{n} t^i x_i \right)^2 d\rho(t) \qquad (n = 0, 1, 2, \cdots),$$

* That it may be zero is seen by the example $f(x) = 1 + e^{-x}$, which is certainly completely monotonic for all x.

and this is clearly positive unless $x_0=x_1=\cdots=x_n=0$. It follows that

$$(2)\qquad \begin{vmatrix} c_0 & c_1 & \cdots & c_n \\ c_1 & c_2 & \cdots & c_{n+1} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ c_n & c_{n+1} & \cdots & c_{2n} \end{vmatrix} > 0 \qquad (n = 0, 1, 2, \cdots).$$

We may also show in this case that

$$(3)\qquad \begin{vmatrix} c_1 & c_2 & \cdots & c_{n+1} \\ c_2 & c_3 & \cdots & c_{n+2} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ c_{n+1} & c_{n+2} & \cdots & c_{2n+1} \end{vmatrix} > 0 \qquad (n = 1, 2, 3, \cdots).$$

For, since $-f'(x)$ is itself a completely monotonic function, the two cases applicable to $f(x)$ are also applicable to $-f'(x)$. In the second of these cases we have (3) (which is merely (2) with all subscripts increased by unity). In the first of these cases we are led to a contradiction. For we should have

$$(4)\qquad -f'(x) = \sigma_0' + \sum_{k=1}^{p} \sigma_k' e^{\lambda_k' a} e^{-\lambda_k' x},$$

$$0 < \lambda_1' < \lambda_2' < \cdots < \lambda_p'; \ \sigma_0' \geqq 0, \ \sigma_k' > 0 \qquad (k = 1, 2, \cdots, p).$$

Integrating equation (4) we should obtain

$$(5)\qquad f(x) = -\sigma_0' x + \sum_{k=1}^{p} \sigma_k' e^{\lambda_k' a} e^{-\lambda_k' x}/\lambda_k' + C,$$

where C is a constant of integration. But σ_0' must be zero, for otherwise

$$\lim_{x=\infty} f(x) = -\infty.$$

This is impossible since $f(x) \geqq 0$. But if $f(x)$ has the form (5) it is clear that the functions $f(x), f'(x), f''(x), \cdots, f^{(p+1)}(x)$ are linearly dependent. Hence the Wronskian determinant of these functions must vanish identically. But this determinant reduces to (2) for $x=a$, $n=p+1$. We thus reach a contradiction. It follows that both (2) and (3) must hold in Case II. Hence we are in a position to apply a theorem of Hamburger* and obtain

* H. Hamburger, *Bemerkungen zu einer Fragestellung des Herrn Pólya*, Mathematische Zeitschrift, vol. 7 (1920), p. 304.

$$f(x) = \int_0^\infty e^{-xt} d\alpha(t),$$

where $\alpha(t)$ is a non-decreasing function. The theorem is thus established in all cases.

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ARITHMETICAL COMPOSITION AND INVERSION OF FUNCTIONS OVER CLASSES*

BY

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I. Ovoids and ova

1. If one or more of the postulates of a field be modified (§13), contradicted (§12), or suppressed (§5), and if the altered set of postulates be self-consistent, the set defines what we shall call a *variety*, which is *abstract* or *special* according as the marks in the postulates are arbitrary to the extent permitted by the postulates, or have specific interpretations (§64). A special variety is an *instance* of the abstract variety from which it is obtained by specific interpretation, and the existence of an abstract variety is said to be demonstrated by exhibiting an instance.

2. The varieties in the present theory are among the most rudimentary possible; I have called them ovoids and ova. For certain of the classic postulate systems of a field the complete sets of conceivable varieties (obtained from the truth tables of the systems) exist; for others, not all varieties exist. For the moment it is sufficient to observe that a ring, not necessarily commutative, is an instance of the ovoids and ova defined presently. Hence all exist. On the other hand, none of these varieties is an instance of a ring, ray, or module, commutative or not, and hence all would appear to be new. They were not constructed ad hoc, but arose by necessity in the general algebra of numerical functions, of which they are the simple, abstract structure. It is noteworthy that a comprehensive theory, among whose instances are several extensive theories of numerical functions, many of which have themselves a high degree of generality, should be comprised in such narrow compass.

The principal definitions and postulates are in §§3–6, 9, 15–21, 33, 42, 44, 45, 58, 60, and the main conclusions in §§31, 34–41, 47–50, 56–60, 61–64. The purpose of the theory is indicated in §§7, 23, 34, 42.

3. The notation $K|x, y, \cdots$ shall signify that the class K, different from the null class, contains the *elements* $x, y, \cdots$. To avoid separation into cases it is not assumed, unless so stated (as in §§15, 16), that the elements of a class are all distinct. Hence, unless the elements are stated to be all distinct and finite in number, K will contain any finite number of elements.

* Presented to the Society, September 11, 1931; received by the editors May 4, 1931.

A class will be called *countable* if it contains either only a finite number or a denumerable infinity of distinct elements.

4. Abstract equality, indicated by $=$, has its usual significance as a reflexive, symmetric, transitive relation, with the postulate (or theorem, as the case may be in a given context) that, if $K|x, y$, where K is any non-null class, and if $x=y$, then either of x, y may replace the other in any relation concerning x or y or both. The relation $\sim$ of §21 is an instance of $=$, and is such, not by postulation, but by proof. All classes in the sequel are postulated to be such that equality is significant for the elements of any given class. The sign $.\supset.$ of implication will be used where convenient.

5. Capital Greek letters shall denote operations. K is any non-null class, t is an arbitrary constant integer >0.

A capital Latin superscript $O_t, C_t, P_t, O, C, P, B, A, D, \cdots$, as in $\Phi^B K$, connotes a property of Φ with respect to some (or all) one-row matrices whose elements are in K, and a formula (or compound symbol) of the type $\Phi^{I_1 \cdots I_r}K$, where $I_1, \cdots, I_r$ denote properties, is an assertion. The like applies later to matrices of any numbers of rows and columns. The assertion $\Phi^{I_1 \cdots I_r}K$ is the simultaneous assertion of $\Phi^{I_j}K$ $(j=1, \cdots, r)$. If $\Phi^{I_1 \cdots I_r}K$, we call K a Φ-*void* of *character* $I_1 \cdots I_r$. Formal definitions follow.

$\Phi^{O_t}K$: asserts that the result $(x_1, \cdots, x_t)^\Phi$ of operating with Φ on the one-row matrix $(x_1, \cdots, x_t)$ is uniquely known whenever $K|x_1, \cdots, x_t$ and $(x_1, \cdots, x_t)$ is given. If $t=1$, by convention* $(x)^\Phi = x^\Phi = x$ whenever $K|x$.

$\Phi^{C_t}K$: asserts $\Phi^{O_t}K$ *and* $K|(x, \cdots, x)^\Phi$ whenever $K|x_1, \cdots, x_t$ and the matrix $(x_1, \cdots, x_t)$ is given.

$\Phi^{P_t}K$: asserts that $(x_1, \cdots, x_t)^\Phi$ is a symmetric function of $x_1, \cdots, x_t$ whenever $\Phi^{O_t}K$ and $(x_1, \cdots, x_t)^\Phi$ is defined.

$\Phi^O K$: asserts $\Phi^{O_t}K$ for all finite integers $t>0$;

$\Phi^C K$: asserts $\Phi^{C_t}K$ for all finite integers $t>0$;

$\Phi^P K$: asserts $\Phi^{P_t}K$ for all finite integers $t>0$;

$\Phi^B K$: asserts $\Phi^C K$ *and that*

$$(x_1, \cdots, x_r, x_{r+1})^\Phi = ((x_1, \cdots, x_r)^\Phi, x_{r+1})^\Phi$$

for all integers $r>0$, whenever $x_1, \cdots, x_r, x_{r+1}$ are in K and $(x_1, \cdots, x_r, x_{r+1})^\Phi$, $(x_1, \cdots, x_r)^\Phi$ are defined. *By convention*, if $\Phi^B K$, then $(x)^\Phi = x^\Phi = x$ whenever $K|x$. This convention as a separate statement is unnecessary, by the previous convention, but is included on account of its importance.

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* The restriction on Φ imposed by this convention is only apparent, and is a mere convenience of notation. The important case where the convention need not hold is provided for in the discussion of functions in §21, for which a different notation is used.

$\Phi^A K$: asserts $\Phi^B K$ *and that*

$$((x, y)^\Phi, z)^\Phi = (x, (y, z)^\Phi)^\Phi$$

whenever $K|x, y, z$ and (x, y, z) is given.

$(\Psi, \Phi)^D K$: asserts $\Psi^B K$ *and* $\Phi^B K$, *and that*

$$((x, y)^\Psi, z)^\Phi = ((x, z)^\Phi, (y, z)^\Phi)^\Psi$$

whenever $K|x, y, z$ and (x, y, z) is given.

The compound assertion $\Phi^A K$ *and* $K|x, \cdots$ will be written $\Phi^A K|x, \cdots$, and likewise in all similar situations.

6. Several modifications of the preceding definitions lead to effectively the same conclusions. For example, in defining $\Phi^B K$, it is sufficient to postulate $\Phi^O K$, or even $\Phi^{O_2} K$, instead of the much stronger $\Phi^C K$, as the clause following *and* implies $\Phi^C K$ when $\Phi^O K$. The foregoing however are in their most convenient form for our purpose.

If desirable to describe the processes and properties just defined, the following will be adopted: O, *over* or *open*, signifying that closure C of K under Φ is not postulated; C, *closed*; P, *totally permutable*, or *totally commutative*; P_2, *commutative*; B, *binary*; A, *associative*; D, (Ψ, Φ)-*distributive*; a suffix t on a property adds the restriction *of order* t; if $\Phi^C K$ is postulated we say that Φ is *on* K; if $\Phi^O K$, Φ is *over* K.

If $(x_1, \cdots, x_r)^\Phi$ is defined, we call it the Φ-*composite* of $(x_1, \cdots, x_r)$, and say that $(x_1, \cdots, x_r)^\Phi$ has been derived from $(x_1, \cdots, x_r)$ by Φ-*composition*; the Φ-*components* of $(x_1, \cdots, x_r)^\Phi$, in this order, are $x_1, \cdots, x_r$.

When two operations are connected by a property for whose definition both are necessary, as in the definition of D, the class K is called an *ovum* with respect to the two operations in a prescribed order.*

7. One of the problems with which we shall be concerned can now be stated. *Given ovoids and ova of prescribed characters, to construct from them further operations and classes of values of functions such that with respect to the new operations the new classes shall be varieties abstractly identical with those occurring in the algebra of numerical functions.* The sense in which inversion is used in this theory is explained in §23.

The solution obtained here is extremely general. It includes all known algebras of numerical functions, unifies them by demonstrating their abstract identity, exhibits them as simple instances of the abstract theory constructed, and provides the means for obtaining an indefinite number of further in-

* I introduced the term ovum in a previous paper, American Mathematical Monthly, vol. 37 (1930), p. 400. A note in the same volume, p. 484, may be glanced at in connection with the present paper. See also §62.

stances. A point of particular interest is the formal identity of the abstract theory, for functions of any finite number of general variables, with the classical algebra of either power series or Dirichlet series of a single numerically valued variable or with the like for r variables. The principal question propounded in the second paper cited in the footnote is therefore answered in the negative, in spite of the very different appearances which particular solutions present. Further generalizations will be indicated as we proceed, by suggesting weakened hypotheses which lead to similar but more general conclusions.

Whenever, as in parts of §8, an assertion is an immediate consequence of the definitions or of what has preceded, proof will be omitted without comment. For example, $\Phi^C K$ does not imply $\Phi^B K$, but $\Phi^B K. \supset .\Phi^C K$.

8. If $\Phi^{I_1 \cdots I_r} K$ is significant, so also are $\Phi^{I_i} K (i=1, \cdots, r)$, and $\Phi^{I_1 \cdots I_r} K. \supset .\Phi^{I_i} K (i=1, \cdots, r)$. By this remark the following columns may be extended. We collect for easy reference the simplest consequences of the definitions in §5 that will be most frequently used, generally without further reference. By $\Phi^{I_1 + \cdots + I_r} K$ we assert $\Phi^{I_1} K$ *or* $\Phi^{I_2} K, \cdots,$ *or* $\Phi^{I_r} K$, and the statement that $\Phi^I K. \supset .\Phi^{I_1 + \cdots + I_r} K$ is false is the assertion that all of $\Phi^I K. \supset .\Phi^{I_i} K (i=1, \cdots, r)$ are false.

TRUE	FALSE
$\Phi^A K. \supset . \Phi^B K.$	$\Phi^O K. \supset . \Phi^{A+B+C+P} K.$
$\Phi^B K. \supset . \Phi^C K.$	$\Phi^C K. \supset . \Phi^{A+B+P} K.$
$\Phi^C K. \supset . \Phi^O K.$	$\Phi^B K. \supset . \Phi^{A+P} K.$
$\Phi^P K. \supset . \Phi^O K.$	$\Phi^A K. \supset . \Phi^P K$, if K contains at at least 2 distinct elements.
$\Phi^{C_t} K. \supset . \Phi^{O_t} K.$	$\Phi^{O_t} K. \supset . \Phi^{C_t} K (t > 1).$
$\Phi^{P_t} K. \supset . \Phi^{O_t} K.$	$\Phi^{O_t} K. \supset . \Phi^{P_t} K (t > 1).$
$\Phi^{BP_2P_3} K. \supset . \Phi^{AP} K.$	$\Phi^{OP_2P_3} K. \supset .\Phi^{A+P} K$, if K contains at least 4 distinct elements.
$\Phi^P K. \supset .\Phi^{P_j} K$ (all j).	

We prove $\Phi^{BP_2P_3} K. \supset .\Phi^{AP} K$. It will suffice to prove the A part, as the rest follows, precisely as may be proved in a field, from $\Phi^{BAP_2} K. \supset .\Phi^P K$.

Let $\Phi K \mid (x, y, z)^\Phi$. Then

$$\Phi^B K. \supset .(x, y, z)^\Phi = ((x, y)^\Phi, z)^\Phi;$$

$$\Phi^{BP_3} K. \supset .(x, y, z)^\Phi = (y, z, x)^\Phi = ((y, z)^\Phi, x)^\Phi;$$

$$\Phi^{BP_2} K. \supset .((y, z)^\Phi, x)^\Phi = (x, (y, z)^\Phi)^\Phi.$$

Hence

$$\Phi^{BP_2P_3} K \mid (x, y, z)^\Phi. \supset .((x, y)^\Phi, z)^\Phi = (x, (y, z)^\Phi)^\Phi,$$

which completes the proof.

A useful property implied by $\Phi^A K$ is the following. If

$$\Phi^A K \mid x_1, \cdots, x_r, y_1, \cdots, y_s, \cdots, z_1, \cdots, z_t$$

then

$$((x_1, \cdots, x_r)^\Phi, (y_1, \cdots, y_s)^\Phi, \cdots, (z_1, \cdots, z_t)^\Phi)^\Phi$$
$$= (x_1, \cdots, x_r, y_1, \cdots, y_s, \cdots, z_1, \cdots, z_t)^\Phi.$$

If $\Phi^A K \mid x$, we shall write $(x, \cdots, x)^\Phi = x^{r\Phi}$ (precisely r x's in first). Hence

$$\Phi^A K \mid x . \supset . (x^{r\Phi}, x^{s\Phi})^\Phi = x^{(r+s)\Phi}.$$

9. Of the many possible kinds of (Ψ, Φ)-ova, defined by D and specific properties of Ψ, Φ, from which we may construct compositions of functions, we shall consider only one. If

$$(\Psi, \Phi)^D K, \textit{ and } \Psi^{CAP} K, \textit{ and } \Phi^{CAP} K,$$

we shall call K a *double* (Ψ, Φ)-ovum, and write $(\Psi, \Phi)^{CAPD} K$. By §8 this definition contains several redundancies. But as it is in the form most often applied, we shall not restate the hypotheses on Ψ, Φ in terms of their weakest equivalents.

If K is a double (Ψ, Φ)-ovum, and if $K \mid x, y, z$, then K contains all of the following, which are equal:

$$((x, y)^\Psi, z)^\Phi, ((x, z)^\Phi, (y, z)^\Phi)^\Psi,$$

and the elements obtained from these by interchanging the symbols within $(\ )^\Psi$, $(\ )^\Phi$ respectively in all possible ways.

10. A commutative ring is an instance of a double (Ψ, Φ)-ovum, but not conversely, as Ψ is not assumed to have an inverse, and neither Ψ nor Φ is postulated to have a modulus. For the same reason a double (Ψ, Φ)-ovum is not a commutative group with respect to either Ψ or Φ, nor is it a Ψ or Φ commutative semigroup, since there is no postulate of cancellation.

The following device is useful in supplying details of proofs in complicated situations concerning double ova. By what has just been remarked, if K is a double (Ψ, Φ)-ovum, (Ψ, Φ) may be replaced by the $(+, \times)$ of a commutative ring, *provided that* neither the existence of an inverse to Ψ nor that of moduluses with respect to either Ψ or Φ be assumed.

A similar device, with the obvious greater restrictions necessary, applies to ovoids of prescribed characters, say A, P, CP, AP, compared with modules, rings, rays, groups and semigroups. By the indicated changes of notation, classical proofs can be transferred directly to ovoids and ova, and there is no need to reproduce them, after having verified that the above proviso, or the necessary equivalent for a particular character, is not violated.

11. Applying §10, we readily see the following, which will be required later. Let K be a double (Ψ, Φ)-ovum, and let K contain all of

$$a_{\lambda 1}, \cdots, a_{\lambda r}; b_{\mu 1}, \cdots, b_{\mu s}; \cdots; c_{\nu 1}, \cdots, c_{\nu t}$$
$$(\lambda = 1, \cdots, l); (\mu = 1, \cdots, m); \cdots; (\nu = 1, \cdots, n).$$

Then the Φ-composite

$$\begin{aligned}[&\{(a_{11}, \cdots, a_{1r})^{\Phi}, \cdots, (a_{l1}, \cdots, a_{lr})^{\Phi}\}^{\Psi},\\ &\{(b_{11}, \cdots, b_{1s})^{\Phi}, \cdots, (b_{m1}, \cdots, b_{ms})^{\Phi}\}^{\Psi},\\ &\cdots \qquad \cdots \qquad \cdots\\ &\{(c_{11}, \cdots, c_{1t})^{\Phi}, \cdots, (c_{n1}, \cdots, c_{nt})^{\Phi}\}^{\Psi}]^{\Phi}\end{aligned}$$

is equal to the Ψ-composite of the $lm \cdots n$ Φ-composites

$$(a_{\lambda 1}, \cdots, a_{\lambda r}, b_{\mu 1}, \cdots, b_{\mu s}, \cdots, c_{\nu 1}, \cdots, c_{\nu t})^{\Phi}$$
$$(\lambda = 1, \cdots, l; \mu = 1, \cdots, m; \cdots; \nu = 1, \cdots, n).$$

12. The distinction between suppressing a postulate and contradicting it leads to the consideration of disjunctive varieties. Let R' denote the negation of the property R; for example, $A' \equiv$non-associative. Let $R, Q, \cdots$ be any properties significant for Φ when $\Phi^{O}K$. Then if at least one but not all of $R', Q, \cdots$ is a negation, and $\Phi^{R'Q\cdots}K$, we call K a Φ-ovoid of *disjunctive character* $R'Q \cdots$. The extension to more than one operation is obvious and need not be formally stated.

There are at least 32 disjunctive ovoids in the literature of postulate systems for groups and fields, and many more exist. These will be referred to in §64. For some of the classical postulate systems for a field, the complete sets of disjunctive ovoids characterized by the truth tables of the systems exist; for others of these systems there is only partial existence.

From an ovoid with a given disjunctive character, a theory parallel to that developed in the sequel can be constructed. According to the specific characters, these theories will be either more or less general than that constructed here, which is based on ovoids and double ova; none of the possible theories has precisely the same degree of generality as this one. I have worked out several, but a complete discussion is beyond any reasonable limits of a single paper.

13. The sense in which a postulate is modified, as stated in §1, is as follows. If a postulate is a compound assertion, the compound is to be replaced by the simple predications of which it is composed. For example, "addition has a unique modulus," is to be replaced by "addition has a modulus" and "the modulus is unique." When so decomposed, the postulates are then to be

contradicted or suppressed in all possible ways. There thus appear to be not more than $2^{12}-1$ conceivable varieties, of which all but a possible maximum of 1152 are non-existent.*

Some of the less well known extant varieties, for example an ovoid with character $A'C$, give rise to striking algebras of numerical functions.

The theory based on ovoids and double ova outlined here is closer than any of the others to the algebra of numerical functions in rational arithmetic. As it includes all of the known algebras of numerical functions and produces an infinity more, it is sufficient for the present.

II. Ω-NUMBERS AND FUNCTIONS

14. As the Ω-numbers now introduced are a new species of number, and as they are basic for the sequel, we shall describe their nature in detail. It is to be noticed that as first defined in §15, the only property of Ω assumed is that its component operations shall be respectively over r arbitrary classes according to the general definition of $\Phi^O K$ in §5. If the operations be specialized by the imposition of further properties, so that the elements of the Ω-number are restricted to be in varieties of prescribed characters, for example, if the elements are positive rational integers, we obtain the numbers necessary for composition over varieties having determinate characters. Thus the procedure to be followed in constructing any of the possible theories mentioned in §§7, 12, 13 will be evident from the present ones, which have the common character A. The numbers defined in §15 are generalized in §§50, 52, 53, 58.

15. An *Ω-number* Z of *order* r, *degree* h, and *index* (r, h) is a matrix of r rows and h columns, whose *rh elements* belong to any r countable classes $K_1, \cdots, K_r$, of the following kind: the h elements in the jth row of Z belong to K_j, and it is postulated that there exist r operations Ω_j such that $\Omega_j{}^O K_j (j=1, \cdots, r)$.

The Ω from which the numbers take their name is the one-column matrix

$$\Omega \equiv \begin{Vmatrix} \Omega_1 \\ \cdot \\ \cdot \\ \cdot \\ \Omega_r \end{Vmatrix},$$

interpreted as an operation over the class J of Ω-numbers of order r. The result Z^Ω of operating with Ω on Z is defined to be the one-column matrix of r rows in which the element in the jth row is the Ω_j-composite (§6) of the h

* For these calculations I am indebted to Mr. C. R. Worth, whose investigation, when complete, will exhibit the totality of varieties that exist. See also §64.

elements in the jth row of $Z(j=1, \cdots, r)$. Thus, if Z is defined by the first of the following matrices, the second defines Z^Ω:

$$\left\| \begin{matrix} z_{11}, \cdots, z_{1h} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ z_{r1}, \cdots, z_{rh} \end{matrix} \right\|, \qquad \left\| \begin{matrix} (z_{11}, \cdots, z_{1h})^{\Omega_1} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ (z_{r1}, \cdots, z_{rh})^{\Omega_r} \end{matrix} \right\|.$$

If, as above, J denotes the class of all Ω-numbers, and if Q is a character such that $\Omega_j{}^Q K(j=1, \cdots, r)$, then $\Omega^Q J$.

Since Ω-numbers of index (r, h) are matrices of r rows and h columns, *equal* and *distinct* Ω-numbers of index (r, h) are automatically defined, and similarly for the Ω-decompositions introduced in a moment.

If each of $\Omega_j(j=1, \cdots, r)$ has the character $QR \cdots S$, so that

$$\Omega_j{}^{QR\cdots S}K_j \qquad (j = 1, \cdots, r),$$

we say that Ω has the *character* $QR \cdots S$, with respect to $(K_1, \cdots, K_r)$ if necessary. The only character of Ω so far is O.

With the notation as above for Z, Z^Ω, let Ω now have the character C (§5). Then Z^Ω is an Ω-number, say

$$Z^\Omega = \left\| \begin{matrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{matrix} \right\|, \quad K_j \mid z_j \qquad (j = 1, \cdots, r).$$

If, conversely, Ω has the character C, and if an Ω-number X of index $(r, 1)$ be given, say

$$X = \left\| \begin{matrix} x_1 \\ \cdot \\ \cdot \\ \cdot \\ x_r \end{matrix} \right\|, \quad \Omega_j{}^C K_j \mid x_j \qquad (j = 1, \cdots, r),$$

and if further there exists an Ω-number $X_{(h)}$ of index (r, h),

$$X_{(h)} \equiv \left\| \begin{matrix} x_{11}, \cdots, x_{rh} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_{r1}, \cdots, x_{rh} \end{matrix} \right\|, \quad \Omega_j{}^C K_j \mid x_{ji} \quad (j = 1, \cdots, r;\ i = 1, \cdots, h),$$

such that $X_{(h)}{}^\Omega = X$, we call $X_{(h)}$ an *Ω-decomposition* of *index* (r, h) of X.

The class of all distinct Ω-decompositions of index (r, h) of X is called the *total Ω-decomposition of index* (r, h) *of* X. Since the order r will be fixed in a given context unless otherwise noted (§59), we may refer only to the degree h, and speak of total Ω-decompositions of *degree h*. These definitions have con-

tent if and only if Ω has the character C. The possible generalization where C is not postulated for Ω is not considered in this paper; it presents no difficulty. Since $A\,.\supset.\,C$, the definitions are significant if Ω has the character A.

16. We shall assume henceforth the following

POSTULATE. The total Ω-decomposition of index (r, h) of an Ω-number of index $(r, 1)$ contains only a finite number of elements (distinct Ω-decompositions of index (r, h)).

It is on account of this postulate that the present theory is said to be arithmetical. The reasons for our assumption will appear as we proceed. In composition of functions over classes, Ω-decompositions of degree h play a part abstractly identical with that of the resolution in all distinct ways (order relevant) of a positive integer into a product of h positive integers. For this isomorphism to have a meaning, the postulate is necessary.*

17. Returning to §15, we define $(K_1, \cdots, K_r) \equiv (K^r)$, r superscript, to be the class of all one-row matrices $(z_1, \cdots, z_r)$ generated as z_i ranges over all elements of $K_i (i=1, \cdots, r)$. If z_i' is a particular element of K_i, and z_i a variable element of $K_i(i=1, \cdots, r)$, we call $(z_1', \cdots, z_r')$ a *value* of the *matric variable* $(z_1, \cdots, z_r)$ of *order* r *over* (K^r). Thus (K^r) is the class of all values of the matric variable $(z_1, \cdots, z_r)$.

When, as in §15, $\Omega_i{}^O K_i (i=1, \cdots, r)$, we shall write† $(\Omega^O K^r)$. Similarly, if $\Omega_i{}^Q K_i (i=1, \cdots, r)$, we write $(\Omega^Q K^r)$, and, if necessary to emphasize the character Q, we shall refer to Ω^Q*-numbers* and Ω^Q*-decompositions*.

If for the moment L denotes the class of all matric variables of order r, and Q is any character, then

$$(\Omega^Q K^r)\,.\supset.\,\Omega^Q L.$$

18. The notation being as in §15, we write for the moment

$$(x_{1j}, \cdots, x_{rj}) \equiv X_j' \qquad (j = 1, \cdots, h).$$

The one-row matrix $(X_1', \cdots, X_h')$, whose h elements are the one-row matrices $X_1', \cdots, X_h'$ in this order, namely the transposes of the columns of $X_{(h)}$ in their order from left to right, is called the *transfer* of the Ω-number $X_{(h)}$ of index (r, h). If $h=1$, transfer and transpose coincide; if $h>1$, there is no correspondent of the transfer in the classical algebra of matrices.

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* If the postulate is not assumed, the corresponding theory applies to the field of all algebraic numbers, among others, but appears to be of only artificial interest.

† By §5, it would be meaningless to write $\Omega^O(K^r)$, as Ω has not been defined as an operation over elements of (K^r), but as an operation on Ω-numbers. An operation Ω' is defined in §41 such that $\Omega'^O(K^r)$. When and only when $r=1$, $\Omega'^O(K^r)$ and $(\Omega^O K^r)$ have the same content.

The transfer of the X_r in §15 is the matric variable $(x_1, \cdots, x_r)$.

The relation of transference is reciprocal in the sense of the following definition. We say that, conversely, $X_{(h)}$ as in §15 is the *transfer* of $(X_1', \cdots, X_h')$. Hence X as in §15 is the transfer of $(x_1, \cdots, x_r)$.

19. The following definitions are fundamental in the theory of composition. In the notation of §18, $(X_1', \cdots, X_h')$ is called an *Ω-decomposition* of *index* (r, h) *of the matric variable* $(x_1, \cdots, x_r)$.

The *total Ω-decomposition* of *index* (r, h) *of the matric variable* $(x_1, \cdots, x_r)$ is the class whose elements are the transfers of the elements in the total Ω-decomposition of index (r, h) of the transfer of $(x_1, \cdots, x_r)$.

20. The *conjoint** $\|X, \cdots, Z\|$ of any number of Ω-numbers $X, \cdots, Z$ of the same degree r,

$$X \equiv \begin{Vmatrix} x_{11}, \cdots, x_{1a} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_{r1}, \cdots, x_{ra} \end{Vmatrix}, \cdots, Z \equiv \begin{Vmatrix} z_{11}, \cdots, z_{1c} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ z_{r1}, \cdots, z_{rc} \end{Vmatrix},$$

in the order $X, \cdots, Z$, is the Ω-number W, where

$$W \equiv \begin{Vmatrix} x_{11}, \cdots, x_{1a}, \cdots, z_{11}, \cdots, z_{1c} \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ x_{r1}, \cdots, x_{ra}, \cdots, z_{r1}, \cdots, z_{rc} \end{Vmatrix}.$$

If $X', \cdots, Z', W'$ are the transfers of $X, \cdots, Z, W$ above, the *conjoint* $(X', \cdots, Z')$ of $X', \cdots, Z'$, in this order, is W'.

From the last of §8 it follows that if $(\Omega^A K^r)$, and $W \equiv \|X, \cdots, Z\|$, where $X, \cdots, Z$ are Ω^A-numbers, then

$$W^\Omega \equiv \|X, \cdots, Z\|^\Omega = \|X^\Omega, \cdots, Z^\Omega\|^\Omega,$$

and W^Ω is of degree 1. If $(\Omega^A K^r)$ be weakened to $(\Omega^B K^r)$, or modified to $(\Omega^{A'B} K)$, A' as in §12, the above equality can not be asserted.

The following obvious remarks are frequently useful. They will be applied where necessary without further reference. If $(\Omega^O K^r)$, and if $Z_1, \cdots, Z_h$ are the Ω-numbers of degree 1 formed from the columns of the Ω-number Z of degree h in the order in which they occur from left to right, $Z = \|Z_1, \cdots, Z_h\|$. Conversely, if Z is of degree h, and this equality is given, Z_i is the Ω-number of degree 1 formed from the ith column of $Z (i = 1, \cdots, h)$. Again, if $X, \cdots, Z$ are any Ω^Q-numbers, where Q is either of P, AP_2, then $\|X, \cdots, Z\|^\Omega$ is a symmetric function of $X, \cdots, Z$.

* The first instances of conjunction were given in earlier papers, these Transactions, vol. 22 (1921), p. 1.

21. Composition as required for the purpose outlined in §7 is concerned with functions of matric variables. We next define such functions and the relation of equivalence for certain classes of their values.

If whenever the mark ξ and a value $(z_1', \cdots, z_r')$ of the matric variable $(z_1, \cdots, z_r)$ over (K^r), as defined in §17, are assigned, there is a unique determination $\xi(z_1', \cdots, z_r')$ of the mark $\xi(z_1, \cdots, z_r)$, we call $\xi(z_1', \cdots, z_r')$ a *value* of the *ξ-function* $\xi(z_1, \cdots, z_r)$ of the *argument* $(z_1, \cdots, z_r)$, or briefly, we say that $\xi(z_1, \cdots, z_r)$ is a function of $(z_1, \cdots, z_r)$.

Three classes, $F(K^r)$, $F_r(K^r)$, F_r will be required.

$F(K^r) \equiv$ the class of all functions of the argument $(z_1, \cdots, z_r)$.

$F_r(K^r) \equiv$ the class of all values of all the functions in $F(K^r)$ generated as $(z_1, \cdots, z_r)$ ranges over all its values.

$\xi \equiv$ the class of all values of $\xi(z_1, \cdots, z_r)$ in $F_r(K^r)$.

$F_r \equiv$ the class of all such classes ξ.

The notation $F(K^r)$, $F_r(K^r)$, F_r is henceforth fixed.

If $F_r \mid \xi, \zeta$, we say that ξ, ζ are *equivalent*, $\xi \sim \zeta$, if and only if $\xi(z_1, \cdots, z_r) = \zeta(z_1, \cdots, z_r)$ for all values of $(z_1, \cdots, z_r)$. It is obvious that equivalence is an instance of abstract equality as defined in §4. Hence we may replace $\xi \sim \zeta$ by the usual notation $\xi = \zeta$, bearing in mind, however, that $\xi = \zeta$ is not equality of classes in the sense of mathematical logic. Equivalence implies equality as in mathematical logic, but not conversely.

22. Given now that an operation Λ is over $F(K^r)$, we shall define $\Lambda^O F_r$. Let

$$\Lambda^O F(K^r) \mid \xi(z_1, \cdots, z_r), \xi_i(z_1, \cdots, z_r) \qquad (i = 1, \cdots, s),$$

$$\xi(z_1, \cdots, z_r) = (\xi_1(z_1, \cdots, z_r), \cdots, \xi_s(z_1, \cdots, z_r))^\Lambda.$$

As $(z_1, \cdots, z_r)$ ranges over all its values, the left of the preceding equation generates ξ, and the right generates the class of all values of the Λ-composite of $(\xi_1(z_1, \cdots, z_r), \cdots, \xi_s(z_1, \cdots, z_r))$; this class will be denoted by $(\xi_1, \cdots, \xi_s)^\Lambda$. Evidently we have

$$\xi \sim (\xi_1, \cdots, \xi_s)^\Lambda;$$

and therefore by §21,

$$\xi = (\xi_1, \cdots, \xi_s)^\Lambda; \; F_r \mid \xi_1, \cdots, \xi_s . \supset . F_r \mid (\xi_1, \cdots, \xi_s)^\Lambda.$$

From the second of these we may assert $\Lambda^O F$, where Λ as an operation over elements of F_r has the meaning just defined for the first. This defines the Λ-composite $(\xi_1, \cdots, \xi_s)^\Lambda$.

23. The nature of the first stage of the theory of composition and inversion for functions of Ω-numbers can now be briefly indicated. The theory is extended in §50.

Given that there exist certain operations under which the class $F_r(K^r)$, as in §21, is a variety of a prescribed character (§§5, 12, 13), and given the like for $F(K^r)$ (§21), we are to construct from some or all of these data operations under which F_r is a variety (§§1, 21).

Suppose that in this way we reach $\Theta^Q F_r$, where Q is a given character. The problem of inversion then is as follows. Given that $\Theta^Q F_r \mid \alpha, \beta$, we are to construct ξ such that $\Theta^Q F_r \mid \xi$ and either $(\alpha, \xi)^\Theta = \beta$ or $(\xi, \alpha)^\Theta = \beta$.

In constructing Θ we shall need certain simple properties of total Ω-decompositions. Here we make a remark which need not be repeated, and which obviates a possible duplication of proofs in precisely similar situations. Theorems concerning total Ω-decompositions of the matric variable $(z_1, \cdots, z_r)$ over (K^r) can be written down at once by transference from the corresponding theorems for Ω-numbers; see §§18, 19. Again, by transference, any definition relating to Ω-numbers has a unique correspondent for matric variables, and conversely. Hence the theory may be developed with respect either to Ω-numbers or matric variables. The latter is the final form desired, as it is in consonance with properties of functions of r independent variables as usually presented and particularly as customarily written. But when stated in terms of Ω-numbers, the definitions and theorems may be apprehended at a glance, while the transfers are frequently less obvious in appearance. Accordingly we shall state the first forms of all definitions and theorems in terms of Ω-numbers, and later omit such preliminary statements.

24. Let $(\Omega^A K^r)$ (§17). If the Ω^A-number Z of degree 1 has Ω^A-decompositions of degree $h>1$, the total Ω^A-decomposition of degree $h+s-1$, $s>0$, of Z, if it exists, may be obtained as follows. Let (§20)

$$\|Z_{1i}, \cdots, Z_{h-1i}, Z_{hi}\| \qquad (i = 1, \cdots, p)$$

be all those Ω^A-decompositions of degree h of Z which are such that the Ω^A-numbers Z_{hi} of degree 1 have Ω^A-decompositions of degree s. Let the total Ω^A-decomposition of degree s of Z_{hi} be

$$\|Z_{hi1p_i}, \cdots, Z_{hisp_i}\| \qquad (p_i = 1, \cdots, q_i).$$

Then, the total Ω^A-decomposition of degree $h+s-1$ of Z, if it exists, is

$$\|Z_{1i}, \cdots, Z_{h-1i} Z_{hi1p_i}, \cdots, Z_{hisp_i}\| \qquad (p_i = 1, \cdots, q_i;\ i = 1, \cdots, p).$$

For, the Ω^A-numbers just written are all distinct, by the definition of a total decomposition in §15. The set is exhaustive; otherwise, there would exist an i such that

$$X \equiv \|Z_{1i}, \cdots, Z_{h-1i}\|, \quad Y \equiv \|Z_{hi1p_i}, \cdots, Z_{hisp_i}\|,$$
$$Y^{\Omega} \equiv W, \quad Z \equiv \|X, W\|, \quad W \neq Z_{hi};$$

which is a contradiction, since the decompositions of degree h are total.

If $(\Omega^A K^r)$ be weakened to $(\Omega^B K^r)$, the weaker hypothesis is insufficient for the above conclusion.

25. If the Ω^A-number Z has Ω^A-decompositions of degree $h>1$, it has Ω^A-decompositions of degree t, $0<t<h$. This follows from §§8, 20.

26. As in §§24, 25, we may prove the following. Let

$$\|Z_{1i}, \cdots, Z_{hi}\| \qquad (i = 1, \cdots, p)$$

be the total Ω^A-decomposition of degree h of the Ω^A-number Z of degree 1, and let the total Ω^A-decomposition of degree n_j of Z_{ji}, if it exists, be

$$\|Z_{ji1m_j}, \cdots, Z_{jin_jm_j}\| \qquad (m_j = 1, \cdots, t_j).$$

Then, if it exists, the total Ω^A-decomposition of degree $n_1 + \cdots + n_h$ of Z is

$$\|Z_{1i1m_1}, \cdots, Z_{1in_1m_1}, \cdots, Z_{hi1m_h}, \cdots, Z_{hin_hm_h}\|$$
$$(i = 1, \cdots, p;\ m_j = 1, \cdots, t_j;\ j = 1, \cdots, h).$$

27. In this and the following sections we introduce certain simple tactical considerations indispensable for proofs later. Let $(\Omega^{AP_2}K^r)$, and refer to §8.

Let $\|A_1, \cdots, A_s\|$ be any element of the total Ω^{AP_2}-decomposition S of degree s of the Ω^{AP_2}-number Z of degree 1. Then, if precisely s_j of $A_1, \cdots, A_s$ are each equal to A_j' $(j=1, \cdots, t)$, we have $s_1 + \cdots + s_t = s$, and $\|A_1, \cdots, A_s\|$ contributes to S precisely $s!/(s_1! \cdots s_t!)$ elements, obtained by permuting $A_1, \cdots, A_s$ in all possible ways.

If in what precedes $(\Omega^{AP_2}K^r)$ be replaced by either of the weaker hypotheses $(\Omega^A K^r)$, $(\Omega^{P_2}K^r)$, the conclusion does not follow.

28. If now S contains further elements, let $\|B_1, \cdots, B_s\|$ be one such, and proceed in the same way with this to obtain its total contribution to S. Continue this process till S is exhausted (§16), and write all the decompositions under one another, omitting the $\|\quad\|$ from $\|A_1, \cdots, A_s\|$, and similarly for the others. In this way we obtain the first of the following arrays or matrices,

$$\begin{array}{ccc} & A_1, \cdots, A_s & (\xi_1, A_1), \cdots, (\xi_s, A_s) \\ & *, \cdots, * & *, \cdots, * \\ (28) & B_1, \cdots, B_s & (\xi_1, B_1), \cdots, (\xi_s, B_s) \\ & *, \cdots, * & *, \cdots, * \\ & \cdots & \cdots \end{array}$$

in which the first row of stars indicates all the distinct permutations of

$A_1, \cdots, A_s$, except this one, and similarly for the second row of stars and $B_1, \cdots, B_s$, and so on. The second array is formed from the first by replacing any element H_i of the first by (ξ_i, H_i), where $\xi_1, \cdots, \xi_s$ are arbitrary marks. From the construction and the meanings of total decomposition and $(\Omega^{AP_2}K^r)$, we have the conclusion next stated.

29. If two columns of the first array in (28) be interchanged, the effect upon the array is at most a permutation of entire rows.

In the second array (28) permute $(\xi_1, \cdots, \xi_s)$ or, what is equivalent, perform the same given substitution upon the suffixes $1, \cdots, s$ of $\xi_1, \cdots, \xi_s$ in every row of the array. Then, since every permutation of $A_1, \cdots, A_s$, of $B_1, \cdots, B_s, \cdots$ occurs in the first array (28), and since no permutation of $A_1, \cdots, A_s$ is a permutation of $B_1, \cdots, B_s, \cdots$, and similarly for all, we have the second conclusion, stated next.

30. The effect of a permutation of $\xi_1, \cdots, \xi_s$ in the second array (28) is equivalent to at most a permutation of entire columns of the array followed by a permutation of entire rows.

From this we have the third conclusion, which is the particular one required in composition of functions over classes.

31. Let $(\Omega^{AP_2}K^r)$, and let the total Ω^{AP_2}-decomposition of degree s of Z be $\|Z_{1i}, \cdots, Z_{si}\| (i=1, \cdots, h)$. Let $\sigma(y_1, \cdots, y_s)$ be any symmetric function of the marks $y_1, \cdots, y_s$. Then a permutation of $\xi_1, \cdots, \xi_s$ at most permutes the h functions

$$\sigma((\xi_i, Z_{1i}), \cdots, (\xi_s, Z_{si})) \qquad (i = 1, \cdots, h)$$

among themselves.

32. In none of what follows is it necessary to postulate that if $(\Omega^A K^r) | (z_1, \cdots, z_r)$, then the matric variable $(z_1, \cdots, z_r)$ has Ω^A-decompositions of all finite degrees. To take care of the case where decompositions of all finite degrees exist (which, incidentally, is that of all extant theories of numerical functions, except that of Carlitz for finite fields*), we shall state sufficient definitions and conditions for this to be so.

33. An Ω-number U which is such that $\|X, U\|^\Omega = X$ for all Ω-numbers X, is called an *Ω-right modulus* of (K^r), and (K^r) is said to be *Ω-right modular* when U exists. Similarly for left, instead of right, from $\|U, X\|^\Omega = X$. If $\|X, U\|^\Omega = \|U, X\|^\Omega = X$ for all Ω-numbers X, (K^r) is said to be *Ω-modular*, and we write $(\Omega^M K^r)$. If the modulus U is unique, we write $(\Omega^{M_1} K^r)$.

If $(\Omega^{AP_2}K^r)$ and (K^r) is either right or left modular, then $(\Omega^{AP_2M}K^r)$.

If $(\Omega^A K^r)$, and if the Ω^A-number X has Ω^A-decompositions of all finite

* L. Carlitz, American Journal of Mathematics, 1931.

degrees, X is said to be Ω^A*-normal.* If all Ω^A-numbers over (K^r) are Ω^A-normal, (K^r) is said to be Ω^A-normal, and we write $(\Omega^{AN}K^r)$.

Sufficient conditions that $(\Omega^{AN}K^r)$ are that $(\Omega^A K^r)$ and (K^r) have an Ω-right modulus of degree 1. This condition is not necessary.

Apply the remarks in §23 to what precedes.

III. Composition of functions

34. We now construct an operation Θ as described in §23. Precisely: from the hypotheses

$$(\Omega^{AP_2N}K^r),\quad \Phi^O F_r(K^r),\quad \Psi^P F_r(K^r),$$

in which Φ, Ψ are arbitrary beyond the properties O, P with respect to $F_r(K^r)$ as indicated, we shall produce an interpretation of the matrix (Ψ, Φ) as an operation Θ such that $\Theta^O F_r$. For the notation, we refer to §§5, 15, 17, 21, 33, and for equivalent statements of the hypotheses to the general results in §8. The whole of §§34–49 is generalized in §§50–59.

Let $K_s \,|\, z_s (s=1, \cdots, r)$, and let the total Ω^{AP_2}-decomposition of degree h (and necessarily of order 1) of z_s be

$$(z_{s1i_s}, \cdots, z_{shi_s}) \qquad (i_s = 1, \cdots, p_s).$$

Write

$$Z \equiv \left\| \begin{matrix} z_1 \\ \cdot \\ \cdot \\ \cdot \\ z_r \end{matrix} \right\|.$$

Then the total Ω^{AP_2}-decomposition of degree h of Z is

$$\left\| \begin{matrix} z_{11i_1}, & z_{12i_1}, & \cdots, & z_{1hi_1} \\ z_{21i_2}, & z_{22i_2}, & \cdots, & z_{2hi_2} \\ \cdot & \cdot & \cdot & \cdot \\ z_{r1i_r}, & z_{r2i_r}, & \cdots, & z_{rhi_r} \end{matrix} \right\| \qquad (i_s = 1, \cdots, p_s;\ s = 1, \cdots, r);$$

and the total Ω^{AP_2}-decomposition of the matric variable $(z_1, \cdots, z_r)$ is

$$((z_{11i_1}, \cdots, z_{r1i_r}), \cdots, (z_{1hi_1}, \cdots, z_{rhi_r})),$$

with the same ranges of $i_1, \cdots, i_r$. Let

$$F_r \,|\, f_t \quad (t = 1, \cdots, h);\ p \equiv p_1 \cdots p_r.$$

Form the p Φ-composites, as permitted by the second hypothesis,

$$(f_1(z_{11i_1}, \cdots, z_{r1i_r}), \cdots, f_r(z_{1hi_1}, \cdots, z_{rhi_r}))^\Phi.$$

The third hypothesis implies that if we now form the Ψ-composite of these p Φ-composites, the result is independent of the order in which the p Φ-composites are Ψ-composed. Hence this Ψ-composite is uniquely known as an element of $F_r(K^r)$ when $\Theta\equiv(\Psi, \Phi)$ and the matrix

$$(f_1(z_1, \cdots, z_r), \cdots, f_h(z_1, \cdots, z_r))$$

are assigned, Φ, Ψ being as in the second and third hypotheses. Hence we have the situation of §22 with Λ there replaced by Θ, and we may assert $\Theta^O F_r$.

If $F_r | f_1, \cdots, f_h$, we call $(f_1, \cdots, f_h)^\Theta$, as just constructed, in accordance with the general definitions in §§6, 22, the *Θ-composite* of $(f_1, \cdots, f_h)$, ***with respect to*** $(\Omega^{AP_2N}K^r)$, if necessary to specify the classes $K_1, \cdots, K_r$ and the properties AP_2N of the Ω-numbers constructed from these classes.

We have not used all the implications of the hypothesis $(\Omega^{AP_2N}K^r)$ in the construction of Θ. The full restrictions on Ω are stated from the beginning, as they are required for the derivation of properties of Θ when Φ, Ψ are postulated to have further properties. The restriction N is later removed; the postulation of N as above ensures the existence of Θ-composites of all finite degrees h.

35. To prove theorems on Θ-composition we refer to the definition of equality (equivalence) in F_r, stated in §21, and proceed as next outlined.

Equality being the only relation which has been defined for elements of F_r, any theorem concerning Θ-composition must be of the type

$$\Theta F_r | f_1, \cdots, f_h, g_1, \cdots, g_k . \supset . (f_1, \cdots, f_h)^\Theta = (g_1, \cdots, g_k)^\Theta$$

which is trivially true if the matrices $(f_1, \cdots, f_h)$, $(g_1, \cdots, g_k)$ are equal. Suppose then that these matrices are unequal. Form the two sets of Φ-composites

$$(f_1(z_{11i_1}, \cdots, z_{r1i_r}), \cdots, f_h(z_{1hi_1}, \cdots, z_{rhi_r}))^\Phi \qquad (i_s = 1, \cdots, p_s;\ s = 1, \cdots, r);$$

$$(g_1(w_{11j_1}, \cdots, w_{r1j_r}), \cdots, g_k(w_{1kj_1}, \cdots, w_{rkj_r}))^\Phi \qquad (j_s = 1, \cdots, q_s;\ s = 1, \cdots, r);$$

the arguments of $f_1, \cdots, f_h$ being given as in §34 from the total Ω^{AP_2}-decomposition of degree h of $(z_1, \cdots, z_r)$, and those of $g_1, \cdots, g_k$ from the total Ω^{AP_2}-decomposition of degree k of $(z_1, \cdots, z_r)$. Under the hypothesis $\Psi^P F_r(K^r)$, which will be assumed in the theorems considered, the respective Ψ-composites of these sets of Φ-composites are significant and the order of Ψ-compositions is immaterial. They are uniquely known when Θ and the matrices

$$(f_1(z_1, \cdots, z_r), \cdots, f_h(z_1, \cdots, z_r)), \quad (g_1(z_1, \cdots, z_r), \cdots, g_k(z_1, \cdots, z_r))$$

are given. If the Ψ-composites are equal for all values of $(z_1, \cdots, z_r)$, the equality of $(f_1, \cdots, f_h)^\Theta$, $(g_1, \cdots, g_k)^\Theta$ is proved. The only question requiring attention is the extent to which postulated properties of Φ, Ψ carry over to Θ.

Detailed manipulative proofs usually involve complicated multiple suffixes. To avoid such unnecessary complications, we developed the preliminaries in detail, so that the properties of Θ-composition for Φ, Ψ with assigned properties are immediate consequences of the definitions and §§27–31. If proofs by manipulation are desired, in order to see more clearly the content of the theorems, they may be written out in full by the device of §10.

36. If the hypotheses in §34 be replaced by

$$(\Omega^{AP_2N}K^r), \quad \Phi^PF_r(K^r), \quad \Psi^PF_r(K^r),$$

then Θ^PF_r.

This follows from §31 applied to the construction of Θ in §34.

37. If the hypotheses in §34 be replaced by

$$(\Omega^{AP_2N}K^r), \quad \Phi^{CP}F_r(K^r), \quad \Psi^{CP}F_r(K^r),$$

then $\Theta^{CP}F_r$.

38. If the hypotheses in §34 be replaced by

$$(\Omega^{AP_2N}K^r), \quad \Phi^CF_r(K^r), \quad \Psi^{CP}F_r(K^r),$$

then Θ^CF_r.

39. We wish now to impose sufficient properties upon Φ, Ψ in §34 to ensure A for Θ. It is interesting to observe that D as postulated next is necessary. See §9 for the notation.

If the hypotheses in §34 be replaced by

$$(\Omega^{AP_2N}K^r), \quad (\Psi, \Phi)^{CAPD}F_r(K^r),$$

then $\Theta^{CAP}F_r$.

By §§8, 9, the second hypothesis may be replaced by

$$(\Psi, \Phi)^DF_r(K^r), \textit{ and } \Psi^{CP}F_r(K^r), \textit{ and } \Phi^{CP}F_r(K^r),$$

or by

$$(\Psi, \Phi)^DF_r(K^r), \textit{ and } \Psi^{CAP_3}F_r(K^r), \textit{ and } \Phi^{CAP_3}F_r(K^r);$$

the conclusion also is equivalent to either of

$$\Theta^{CP_2P_3}F_r, \quad \Theta^{CAP_3}F_r.$$

The theorem follows from §§37, 11, but a proof by manipulation will bring

out more clearly the nature of the conclusion. By §37 it is sufficient to prove $\Theta^A F$. Let

$$\Theta F_r \mid \xi_p, \xi_{pv_p} \qquad (v_p = 1, \cdots, n_p);$$
$$\xi_p = (\xi_{p1}, \cdots, \xi_{pn_p})^\Theta, \quad \xi = (\xi_1, \cdots, \xi_h)^\Theta,$$

where h is an arbitrary constant integer >0. Although it is sufficient to take $h=3$, we shall not impose this restriction. The desired conclusion of the type in §8 is here

$$\begin{aligned}\xi &= ((\xi_{11}, \cdots, \xi_{1n_1})^\Theta, \cdots, (\xi_{h1}, \cdots, \xi_{hn_h})^\Theta)^\Theta \\ &= (\xi_{11}, \cdots, \xi_{1n_1}, \cdots, \xi_{h1}, \cdots, \xi_{hn_h})^\Theta.\end{aligned}$$

Refer to §26, and indicate the transfers of Ω-numbers of degree 1 by accents; thus Z_{1i}' is the transfer of Z_{1i}, etc., $Z' \equiv (z_1, \cdots, z_r)$ is the transfer of the Z in §26. We have

$$\xi = (\xi_1, \cdots, \xi_h)^\Theta,$$

and therefore, by §§26, 34,

$$\xi(Z') = \{(\xi_1(Z_{11}'), \cdots, \xi_h(Z_{h1}'))^\Phi, \cdots, (\xi_1(Z_{1s}'), \cdots, \xi_h(Z_{hs}'))^\Phi\}^\Psi.$$

Similarly, $\xi_p(Z_{pi}')$ is the Ψ-composite of the t_p Φ-composites

$$(\xi_{pi}(Z_{pi1m_p}'), \cdots, \xi_{pn_p}(Z_{pin_pm_p}'))^\Phi \qquad (m_p = 1, \cdots, t_p).$$

On making the indicated substitutions and applying the general theorem of §11 to our present hypotheses, we see that $\xi(Z')$ is the Ψ-composite of the $st_1t_2 \cdots t_h$ Φ-composites

$$\{\xi_{11}(Z_{1i1m_1}'), \cdots, \xi_{1n_1}(Z_{1in_1m_1}'), \cdots, \xi_{h1}(Z_{hi1m_h}'), \cdots, \xi_{hn_h}(Z_{hin_hm_h}')\}^\Phi.$$

But this is precisely the form of $\xi(Z')$ that would have been obtained had we proceeded directly from

$$(\xi_{11}, \cdots, \xi_{1n_1}, \cdots, \xi_{h1}, \cdots, \xi_{hn_h})^\Theta,$$

calculated for the same argument Z'. This completes the proof.

40. The chain of theorems in §§34–39 is completed by the production of a double (Ψ, Φ)-ovum (§9). Later (§§60–63) we shall extend the chain, by postulating appropriate moduluses and inverses, to include rings of composition. The following theorem can easily be generalized by weakening the hypotheses, but in the form stated it is closest to extant algebras of numerical functions and includes them. For the notation $F(K^r)$, see §21.

If the hypotheses in §34 be replaced by

$$(\Omega^{AP_2N}K^r), \quad (\Psi, \Phi)^{CAPD}F_r(K^r), \quad \Psi^{AP_2}F(K^r),$$

then $(\Psi, \Theta)^{CAPD}F_r$.

With what has already been proved it is sufficient to show that

$$\Theta F_r \mid \xi, \zeta, \theta . \supset . (\xi, (\zeta, \theta)^\Psi)^\Theta = ((\xi, \zeta)^\Theta, (\xi, \theta)^\Theta)^\Psi,$$

under the stated hypotheses.

Let $(K^r) \mid z, x_i, y_i (i=1, \cdots, s)$, and let the total Ω^{AP_2}-decomposition of degree 2 of the matric variable z of order r be $(x_i, y_i)(i=1, \cdots, s)$. The new detail in the proof enters from the third hypothesis. Attending to the definition of $F(K^r)$, we have $(\xi(z), (\xi(z), \theta(z))^\Psi)^\Theta$ = the Ψ-composite of the s Φ-composites

$$\{\xi(x_i), (\zeta(y_i), \theta(y_i))^\Psi\}^\Phi \qquad (i = 1, \cdots, s),$$

by §34. But this Ψ-composite is equal to the Ψ-composite of the two Ψ-composites

$$\{(\xi(x_1), \zeta(y_1))^\Phi, \cdots, (\xi(x_s), \zeta(y_s))^\Phi\}^\Psi,$$
$$\{(\xi(x_1), \theta(y_1))^\Phi, \cdots, (\xi(x_s), \theta(y_s))^\Phi\}^\Psi,$$

by the second hypothesis. Hence, since $(w^\Psi)^\Psi = w$, the foregoing Ψ-composite is the Ψ-composite of the $2s$ Φ-composites

$$(\xi(x_i), \zeta(y_i))^\Phi, (\xi(x_i), \theta(y_i))^\Phi \qquad (i = 1, \cdots, s).$$

With a similar reduction of

$$((\xi(z), \zeta(z))^\Theta, (\xi(z), \theta(z))^\Theta)^\Psi,$$

the proof is completed.

41. An immediate extension of §§34–40 to the more general situation in which the first hypothesis of §34 is replaced by either of

$$(\Omega^{AP_2}K^r), \quad (\Omega^{AP_2N'}K^r),$$

is now evident. An integer $h>0$ exists such that there exist Ω-numbers having Ω-decompositions of degree h (we may take $h=1$ if necessary). For a particular h let S_h denote the set of all Ω^{AP_2}-numbers having Ω^{AP_2}-decompositions of degree h. Replace N in hypothesis 1 by N_h and K^r by S_h, where N_h indicates that elements of S_h have Ω^{AB_2}-decompositions of degree h. By the general property of A in §8, if $S_h \mid Z$, then Z has Ω^{AP_2}-decompositions of degree t, $0<t\leqq h$ (§25). Replace $F_r(K^r)$ wherever it occurs in the hypotheses by $F_h(S_h)$, the class of all values of all functions whose arguments range over S_h; replace $F(K^r)$ by $F(S_h)$, the class of all functions of Z', where Z' is the transfer of the general element of S. Replace F_r by F_r', defined for $F(S_h)$ precisely as F_r was for $F(K^r)$. Replace P wherever it occurs by P_h, and A, P_2 by A_h, P_h which are defined as associativity and commutativity for every t elements $(0<t\leqq h)$.

Finally replace Θ by Θ_h, which is Θ-composition defined only for classes of functions whose arguments are in S_h.

What precedes disposes incidentally of the case where at least one of $\Omega_i^{AP_2}K_i(i=1, \cdots, r)$ has elements having no $\Omega_i^{AP_2}$-decomposition of degree $>h_i$, while K_i for some i contains elements having $\Omega_i^{AP_2}$-decompositions of degree h_i. If $\Omega_i^{AP_2}K_i(i=j_1, \cdots, j_p)$ are all such, and if h is the least integer such that $h \geqq h_u(u=1, \cdots, p)$, Θ_h-composition as above replaces Θ-composition.

42. The object of §§42–48 is to reduce §§34–41 to abstract identity with an algebra of numerically valued functions of a single positive integral variable, and to prepare for a complete solution in §§60–63 of the problem of inversion (§23). The algebra in question is isomorphic with that of the ring properties of either ordinary Dirichlet series of one variable or ordinary power series of one variable, up to the proviso noted in §10. As indicated in §40, the proviso will be removed later. Let

$$\Omega_j^C K_j \mid z_j, z_{j1}, \cdots, z_{jh}; (z_{j1}, \cdots, z_{jh})^{\Omega_j} = z_j \quad (j = 1, \cdots, r).$$

Then, as in §17, $(\Omega^C K^r)$, and by the convention in §5,

$$z_j^{\Omega_j} = z_j, \quad z_{ji}^{\Omega_j} = z_{ji} \quad (j = 1, \cdots, r; \, i = 1, \cdots, h);$$

$$(\Omega^C K^r) \mid (z_1, \cdots, z_r), \; (z_{1i}, \cdots, z_{ri}) \qquad (i = 1, \cdots, h),$$

$$\left\| \begin{matrix} z_{11}, \cdots, z_{1h} \\ \cdots\cdots\cdots \\ z_{r1}, \cdots, z_{rh} \end{matrix} \right\|^{\Omega} = \left\| \begin{matrix} z_1 \\ \vdots \\ z_r \end{matrix} \right\|.$$

We now define $\Omega' \equiv (\Omega_1, \cdots, \Omega_r)$ as an operation such that Ω' is on (K^r), namely $\Omega'^C(K^r)$, as follows. By definition we put

$$((z_{11}, \cdots, z_{r1}), \cdots, (z_{1h}, \cdots, z_{rh}))^{\Omega'} \equiv ((z_{11}, \cdots, z_{1h})^{\Omega_1}, \cdots, (z_{r1}, \cdots, z_{rh})^{\Omega_r})$$

the right of which is known. Hence the left is defined and therefore also Ω', $\Omega'^O(K^r)$. We have now

$$((z_{11}, \cdots, z_{r1}), \cdots, (z_{1h}, \cdots, z_{rh}))^{\Omega'} = (z_1, \cdots, z_r),$$

$$(z_1, \cdots, z_r)^{\Omega'} = (z_1^{\Omega_1}, \cdots, z_r^{\Omega_r}) = (z_1, \cdots, z_r).$$

Hence $(\Omega^C K^r) . \supset . \Omega'^C(K^r)$, and obviously $(\Omega^Q K^r) . \supset . \Omega'^Q(K^r)$ $(Q=P, A, AP_2)$.

For simplicity in printing we shall denote matric variables and their values over (K^r) by small accented Latin letters, with or without suffixes. Thus, if $K_i | z_i (i=1, \cdots, r)$, we write $z' = (z_1, \cdots, z_r)$.

Let the total Ω^C-decomposition of degree h of z' (§19) be $(z_{1i}', \cdots, z_{hi}')$ $(i=1, \cdots, t)$, so that

$$z' = z'^{\Omega'} = (z_{1i}', \cdots, z_{hi}')^{\Omega'} \qquad (i = 1, \cdots, t).$$

Then, if z' is fixed, the complete solution $(x_1', \cdots, x_h')$ of

$$z'^{\Omega'} = (x_1', \cdots, x_h')^{\Omega'}$$

is $(x_1', \cdots, x_h') = (z_{1i}', \cdots, z_{hi}')(i=1, \cdots, t)$. As definitions of the expressions on the right of the following we now put

$$(x_1', \cdots, x_h') \equiv x_1' \cdots x_h', \quad (x_1', \cdots, x_h')^{\Omega'} \equiv x_1'^{\Omega'} \cdots x_h'^{\Omega'},$$

so that $z' = x_1' \cdots x_h'$ and $z'^{\Omega'} = (x_1', \cdots, x_h')^{\Omega'}$ are equivalent statements, and call $x_1' \cdots x_h'$ the *Ω'-product* of $x_1', \cdots, x_h'$ in this order. Neither A nor P is yet postulated for Ω'.

If $(\Omega^{AP_2}K^r)$, then $\Omega'^{AP_2}(K^r)$, and the *Ω'-multiplication* just defined for $x_1' \cdots x_h'$ is both A and P_2.

43. *Scalar multiplication* of Ω'^A-products *over* $F(K^r)$, $F_r(K^r)$ of §21 is defined to have all the formal properties of scalar multiplication in a linear associative algebra. For example, if $F_r(K^r)\,|\,p$, the scalar products $px_1' \cdots x_h'$, $x_1' \cdots x_h' p$ are equal. *Addition* of scalar products, indicated by $+$, is commutative and associative, precisely as in a ring, with the omission of the postulate concerning a modulus and an inverse. Thus the *sum* of the scalar products $px_1' \cdots x_h'$, $qy_1' \cdots y_k'$ is $p'x_1' \cdots x_h' + qy_1' \cdots y_k'$; $px_1' \cdots x_h' + qx_1' \cdots x_h' = (p+q)x_1' \cdots x_h'$.

Scalar processes have as yet merely a formal or abstract significance. The next sections, on scalar functions, relate scalar processes to F_r and Θ.

44. It will be sufficient to develop what follows under the hypotheses of §40, as the discussion under weaker hypotheses is precisely similar with the obvious modifications necessary on the Γ-processes next introduced. These modifications may be inclusively formulated: if Ψ has the character Q, then generators (as defined presently) can be combined only so as to preserve the character Q for addition; if Φ has the character R, generators can be combined only so as to preserve the character R for multiplication.

Let z' be an arbitrary element (matric variable of order r) of (K^r). The *generator* f_j^Γ of f_j, where $F_r\,|\,f_j$, is defined by the following scalar sum of scalar products:

$$f_j^\Gamma \equiv \sum f_j(z_i')z_i'^{\Omega'} \equiv f_j(z_1')z_1'^{\Omega'} + \cdots + f_j(z_n')z_n'^{\Omega'} + \cdots,$$

where the scalar summation $\sum$ refers to all z_i' such that $(K^r)\,|\,z_i'$. By §16, the z_i' are countable, as implied in the above.

The generators f^Γ, g^Γ are said to be *equal*, $f^\Gamma = g^\Gamma$, when and only when $f=g$ as in §21. Equality of generators is an instance of abstract equality in §4. If $f^\Gamma = g^\Gamma$, the coefficients of $z_n'^{\Omega'}(n=1, \cdots)$ in the two generators are equal, and conversely.

The class of all generators f^Γ as f ranges over all elements of F_r is denoted by F_r'.

45. Under the hypotheses of §40 we next define Γ-composition so that $\Gamma^C F_r'$, and show how $(f_1, \cdots, f_h)^\Gamma$ may be calculated precisely as is the general coefficient in the formal multiplication of h Dirichlet series in one variable.

Let $f=(f_1, \cdots, f_h)^\Theta$, as in §34. The Γ-composite $(f_1, \cdots, f_h)$ of $(f_1, \cdots, f_h)$ is defined to be the generator f^Γ. Hence

$$(f_1, \cdots, f_h)^\Gamma = f^\Gamma = \sum f(z')z'^{\Omega'}; \quad \Gamma^C F_r'.$$

Under the stated hypotheses we have also $\Gamma^{AP_2}F_r'$. Further, if Σ' denotes scalar addition in F', defined by

$$f^\Gamma + g^\Gamma \equiv \sum (f(z')g(z'))z'^{\Omega'},$$

then $(\Sigma', \Gamma)^{CAPD}F_r'$.

46. For the moment only, let the z' denote integers >0 and Ω' a real or complex variable. Let $\prod$, $\sum$ be as in ordinary analysis. Then

$$\prod_{j=1}^{h}\left[\sum f_j(z')z'^{\Omega'}\right] = \sum f(z')z'^{\Omega'},$$

where the f_j are any numerical functions, the summations refer to all z', and $f(z')$ is defined by

$$f(z') = \sum f_1(x_{1i}') \cdots f_h(x_{hi}'), \tag{46}$$

the sum extending to all matrices $(x_{1i}', \cdots, x_{hi}')$ of integers $x_{1i}', \cdots, x_{hi}'$ each >0 such that $z'=x_{1i}' \cdots x_{hi}'$. The statement that (46) holds for all integers $z'>0$ is written $f=f_1 \cdots f_h$, and the indicated (symbolic) multiplication in $f_1 \cdots f_h$ is associative, commutative and distributive over addition, $f+g$, where $f+g$ is defined by the assertion that $k=f+g$ and $k(z')=f(z')+g(z')$ for all integers $z'>0$ are identical statements. For references, see §56, footnote.

47. Returning to Ω-numbers, we have now as the equivalent of §46 the following abstractly identical situation. Write

$$\prod_{j=1}^{h} f_j^\Gamma = \prod_{j=1}^{h}\left[\sum f_j(z')z'^{\Omega'}\right] = \sum f(z')z'^{\Omega'},$$

$$\prod_{j=1}^{h}\left[\sum f_j(z')z'^{\Omega'}\right] = \sum\left[\sum f_1(z_{1i}') \cdots f_h(z_{hi}')\right]z'^{\Omega'},$$

where the inner $\sum$ in the last has the following interpretation, which automatically interprets the $\prod$'s. For each solution $(z_{1i}', \cdots, z_{hi}')$ as in §42 of $z' = x_1' \cdots x_h'$ as there, we form the Φ-composite $(f_1(z_{1i}), \cdots, f_h(z_{hi}))^{\Phi}$ and denote the composite by $f_1(z_{1i}) \cdots f_h(z_{hi})$. The Ψ-composite

$$(f_1(z_{11}) \cdots f_h(z_{h1}), \cdots, f_1(z_{1t}) \cdots f_h(z_{ht}))^{\Psi},$$

where t is as in §42, is now formed and is denoted by $\sum_{i=1}^{t} f_1(z_{1i}') \cdots f_h(z_{hi}')$. With these interpretations of multiplication (Φ-composition) and addition (Ψ-composition), indicated by the notations customary in a ring or a field, we have

$$f(z') = \sum_{i=1}^{t} f_1(z_{1i}') \cdots f_h(z_{hi}').$$

Hence, up to the proviso noted in §10, there is abstract identity between Γ-composition of generators and formal multiplication and addition of Dirichlet series of one variable.

48. It follows that theorems concerning Θ-composition are equivalent to identities obtained by equating coefficients of $z'^{\Omega'}$ in identities between generators combined according to Γ-composition (§45). The processes of Γ-composition can be carried out formally by operating on Dirichlet series as in §46 by addition and multiplication, the final results being reinterpreted in terms of Θ-composition for classes of functions of Ω-numbers, as in §47.

49. In this section we establish a second isomorphism between Θ-composition and the processes of Dirichlet series. It will be shown now that Θ-processes are abstractly identical with those originating in identities obtained by addition and multiplication from Dirichlet series in r independent variables. After the detailed discussion for the case of Dirichlet series in 1 variable, it will be sufficient to present an outline.

For the moment only let $\Omega_1, \cdots, \Omega_r$ denote independent real or complex variables and consider the following product of h r-fold Dirichlet series,

$$\sum_{j=1}^{h} \left[\sum f_j(z_{1j}, \cdots, z_{rj}) z_{1j}^{\Omega_1} \cdots z_{rj}^{\Omega_r} \right],$$

where the $\sum$ refers to all values of $z_{1j}, \cdots, z_{rj}$ ranging independently over all integers >0. On reducing this to the form

$$f^{\Gamma_r} \equiv \sum f(z_1, \cdots, z_r) z_1^{\Omega_1} \cdots z_r^{\Omega_r},$$

where the left is to be read Γ_r-*generator of* f, we have

$$f(z_1, \cdots, z_r) = \sum f(z_{11i_1}, \cdots, z_{1hi_1}) \cdots f(z_{r1i_r}, \cdots, z_{rhi_r}),$$

where the $\sum$ refers to all matrices

$$(z_{s1i_s}, \cdots, z_{shi_s}) \qquad (s = 1, \cdots, r)$$

of integers >0 such that $z_s = z_{s1i_s} \cdots z_{shi_s}$. Compare this with the matrix of r rows and h columns in §34.

By an obvious reinterpretation of the notation, as in passing from §46 to §47, we see that the processes of Γ_r-composition of Γ_r generators, whose formal definition is obvious from what precedes, can be carried out formally by operating on Dirichlet series of r independent variables by addition and multiplication, the final results being translated at once into terms of Θ-composition for classes of functions of Ω-numbers. Further, Γ_r may replace Γ in all conclusions of §§42–48.

The modifications necessary if the hypotheses of §40 are weakened as in §41 are obvious. The abstract identity with power series follows as in Euler algebra (see §56) by a reinterpretation of the notation.

50. Returning to §15, we take advantage of the generality of the definition of Ω-numbers to obtain an indefinite number of generalizations, or iterations, of the theory in §§34–49. In §15 replace Ω by ${}_i\Omega$, where i is a prefix to distinguish different species of numbers in the sense of §15. The class of all ${}_i\Omega$-numbers of given order s_i will be denoted by ${}_iK$, and we shall write $({}^rK) \equiv ({}_1K, \cdots, {}_rK)$. If, for example,

$${}_i\Omega \equiv \left\| \begin{matrix} {}_i\Omega_1 \\ \cdot \\ \cdot \\ {}_i\Omega_{s_i} \end{matrix} \right\|, \quad {}_i\Omega_j{}^O{}_iK_j \qquad (j = 1, \cdots, s_i),$$

we have merely repeated the definitions in §15 with r, Ω_j, K_j there replaced by s_i, ${}_i\Omega_j$, ${}_iK_j$ here.

We now postulate ${}_i\Omega^O{}_iK (i = 1, \cdots, r)$. This postulate is satisfied, for example, by conjunction as previously defined. We then define $\bar{\Omega}$ by

$$\bar{\Omega} \equiv \left\| \begin{matrix} {}_1\Omega \\ \cdot \\ \cdot \\ {}_r\Omega \end{matrix} \right\|.$$

Beginning with §15, we now replace Ω by $\bar{\Omega}$, K_j by ${}_jK$, Ω_j by ${}_j\Omega (j = 1, \cdots, r)$, and (K^r) by $({}^rK)$, up to and including §49. In §21 we may change the notation F_r to ${}_rF$ wherever F_r occurs, and to avoid a possible confusion, transpose all suffixes of elements to prefixes.

The process just described may be repeated indefinitely. Thus, having just defined $\bar{\Omega}$-numbers, we may replace ${}_i\Omega$ by ${}_i\bar{\Omega}$, ${}_iK$ by ${}_i\bar{K}$, where the last is the class of all ${}_i\bar{\Omega}$-numbers, and proceed in the same way to define $\bar{\bar{\Omega}}$-numbers where, for example,

$$\bar{\bar{\Omega}} \equiv \left\| \begin{matrix} {}_1\bar{\Omega} \\ \cdot \\ \cdot \\ \cdot \\ {}_t\bar{\Omega} \end{matrix} \right\|.$$

Thus we see that the theory of composition up to §49 is indefinitely iterable, and that a given iteration includes all those that precede it, but is not included in any of its predecessors.

51. Each of these iterates in §50 is abstractly identical with the algebra of Dirichlet series in one or in r variables, and hence also with the like for power series.

Conversely, it will be seen in §58, footnote, that the algebras of Dirichlet series and of power series, and the algebras of numerical functions constructed from them, are instances of the algebra of Θ-composition up to §49.

52. There is another generalization, to functions of $t(t>1)$ independent matric variables over any t classes. We shall merely indicate this, as the detailed development is isomorphic to what precedes and can be obtained by simple changes of notation, or by a reinterpretation of the notation as it is.

In §50 let $s_i = r(i=1, \cdots, t)$, and interpret the conjoint $\Omega^\dagger$,

$$\Omega^\dagger \equiv \|{}_1\Omega, \cdots, {}_t\Omega\|,$$

as an operator on generalized Ω-numbers $Z^\dagger$, where

$$Z^\dagger \equiv \|{}_1Z, \cdots, {}_tZ\|,$$

and ${}_iZ$ is an ${}_i\Omega$-number of index (r, h_i), by the equation of definition,

$$Z^{\dagger\Omega^\dagger} = \|{}_1Z^{{}_1\Omega}, \cdots, {}_tZ^{{}_t\Omega}\|.$$

The restatement of the abstract content of §§15–52 in terms of $\Omega^\dagger$-numbers is obvious and will be omitted.

53. Finally, there is the formal generalization of all that precedes to numbers of infinite order. In §52, t need not be finite. It is assumed in the infinite case that the processes are significant. If the classes concerned are modular with respect to the operations, decompositions of infinite degree can also be considered. For instance, see §58, footnote.

54. The Θ-compositions in what precedes are said to be *outer* compositions, to distinguish them from *inner* compositions. In outer composition no law of unique decomposition is postulated, while in inner composition such a

law, or the abstract equivalent of the fundamental theorem of arithmetic, enters at some stage.

55. The general theory of composition under the stated hypotheses on Ω, (K^r), $F_r(K^r)$, $F(K^r)$ is contained in what precedes. Instances of the theory are constructed in accordance with §1 by imposing further postulates on one or more of the following sets: Ψ, Φ, Θ; Ω, (K^r); $F(K^r)$, $F_r(K^r)$, F_r. For example, in the third set of possibilities, only the functions in a given subclass of $F(K^r)$ may be Θ-composed, the subclass being such that it is closed under Θ-composition; in the second set, only a subclass of all Ω-numbers need be considered; in the first set, moduluses and inverses for Ψ or Φ both may be postulated. Finally, examples of any instances so constructed are exhibited by specifying the classes (K^r), $F_r(K^r)$. Thus (K^r) may be the class of all one-row matrices of r integers >0, and $F_r(K^r)$ the class of all functions of r independent variables which are uniform and finite for finite integer values >0 of the r variables. As the possibilities are obviously unlimited, we shall outline the development only of that one which includes all types of inner composition, of which there are several instances in the literature.

56. For simplicity we shall state the requisite definitions and postulates in forms which are unnecessarily strong. We shall postulate the hypotheses of §39, and we shall use the accent notation of §42 for Ω', z', $\cdots$ with the meanings there explained. By transference (§§18, 19), the first hypothesis is now $\Omega'^{AP_2N}(K^r)$. We postulate further that (K^r) is Ω'-modular (§33), with the unique modulus u'. We postulate also that if f, g are any elements of F_r, then $f(u')=g(u')$ and $f(u')$ (hence also $g(u')$, $\cdots$) is the unique Φ-modulus for $F_r(K^r)$.

With respect to (K^r) we now postulate the following: (K^r) contains the subclasses $I(K^r)$, $P(K^r)$; $I(K^r)\,|P(K^r)$; and $I(K^r)$, $P(K^r)$ are the maximal classes such that no element of $P(K^r)$ is the Ω'-composite of two elements of $P(K^r)$, while each element other than u' of $I(K^r)$ not in $P(K^r)$ is, apart from permutations of the components, the Ω'-composite of elements of $P(K^r)$ in precisely one way. We have therefore postulated outright the law of unique Ω'-decomposition of elements of $I(K^r)$. An element of $I(K^r)$ is called an *Ω'-integer*, an element of $P(K^r)$ an *Ω'-prime*. Two Ω'-integers, neither u', having u' as their unique common Ω'-component, are said to be *Ω'-coprime*.

If in the Ω'-decomposition $z'=x_1'\cdots x_h'$ of the Ω'-integer z' into Ω'-primes $x_1', \cdots, x_h'$ (see §42), $x_1'=\cdots=x_h'$, we write $z'=x_1'^h$. Hence, as in rational arithmetic, any Ω'-integer w' has a unique Ω'-decomposition $x_1'^{h_1}\cdots x_t'^{h_t}$, where $x_1', \cdots, x_t'$ are distinct Ω'-primes and $h_1\cdots h_t\neq 0$. By convention, we define $x_1'^{k_1}\cdots x_t'^{k_t}$, where $k_1=\cdots=k_t=0$, to be u'.

At this stage we may refer to the complete development of the theory of factorable numerical functions of rational integers in previous publications.*

Proceeding as outlined in §48, we may reinterpret the whole of the papers referred to in terms of (Ω', Φ)-factorable functions, which are defined as follows. If $f((x', y')^{\Omega'}) = (f(x'), f(y'))^{\Phi}$ whenever x', y' are Ω'-coprime, we say that f is *(Ω', Φ)-factorable.* In particular, by a mere reinterpretation of a theorem of the previous theory, or independently from the definitions and an application of §10, we have the following fundamental theorem for (Ω', Φ)-factorable functions.

Let f, g be (Ω', Φ)-factorable, and let $(f, g)^{\Theta} = k$. Then k is (Ω', Φ)-factorable, and if $z' = z_1'^{\alpha_1} \cdots z_r'^{\alpha_r}$ is the Ω'-decomposition of the Ω'-integer z' into powers of distinct Ω'-primes $z_1', \cdots, z_r'$, we have

$$k(z') = (k(z_1'^{\alpha_1}), \cdots, k(z_r'^{\alpha_r}))^{\Phi}.$$

From the last follows at once for (Ω', Φ)-factorable functions the whole theory of the Euler product and Euler multiplication, as developed in the publications cited, and the restatement of the previous theory in terms of the present can be made without computations by a simple re-reading of the notation as described in §48. In particular, the whole theory of generators, as previously developed, comes over unchanged to the present theory, with the exception (removed later) that reciprocals have (as yet) no existence. Further development in this direction is a matter of unnecessary detail. To the rational unit 1 in the generators of the previous theory (the leading term of the generators there) corresponds here the Φ-modulus as above defined.

57. The means for developing the theory of (Ω', Φ)-factorable functions have just been indicated. In this section and the next we sketch two methods for the construction of further compositions from a given one. By §48 we may state the processes in terms of the generators of Euler algebra (§7 of paper cited in §56), and obtain the corresponding final results here by re-reading the conclusions. A generator in Euler algebra is a power series in the indeterminate t with leading term 1, in which the coefficients of the several powers of t are arbitrary one-valued functions of the arbitrary constant rational prime ξ (paper cited, p. 148), say $F(t, \xi) \equiv \sum_0^\infty t^n f_n(\xi)$. But this generator is equivalent in Euler composition to the one-row matrix $(f_0(\xi), \cdots, f_n(\xi), \cdots)$, $f_0(\xi) \equiv 1$, two such matrices being combined according to Cauchy

* A list of which is given in the Journal of the Indian Mathematical Society, vol. 17 (1928), p. 260. I cite particularly three: University of Washington Publications in Science and Mathematics, vol. 1, No. 1, 1915; *Euler algebra*, these Transactions, vol. 25 (1923), pp. 135–154; *Algebraic Arithmetic*, Colloquium Publications of the American Mathematical Society, vol. 7, 1927. These contain much more in the present connection than the theory of factorable functions.

multiplication, namely, the Cauchy product (or composite) of $(f_0(\xi), \cdots, f_n(\xi), \cdots)$ and $(g_0(\xi), \cdots, g_n(\xi), \cdots)$ is $(h_0(\xi), \cdots, h_n(\xi), \cdots)$, where

$$h_n(\xi) = \sum_{j=0}^{n} f_j(\xi) g_{n-j}(\xi) \qquad (n = 0, 1, \cdots).$$

The process just recalled has the characteristic property that it leaves the set of all generators invariant. The function generated by the Cauchy composite as above of two generators is the E-composite of the functions generated by the respective generators. In order to secure the properties A, P_2 for E-composition, it is sufficient to compound the generators according to any operation Δ which leaves the value of the leading term of the compounded generator invariant and has with respect to generators the properties A, P_2. For example, instead of Cauchy composition as above we may take that instance of Δ-composition which is defined by

$$(f_0(\xi)g_0(\xi), \cdots, f_n(\xi)g_n(\xi), \cdots).$$

For any such Δ the E-composite k of f, g is the function generated by the Δ-compound of the generators of f, g. To be precise, we therefore refer to the (Δ, E)-composite of f, g.

Transferring this to the present theory, we replace E by Θ, and get (Δ, Θ)-composition. The particular Δ above becomes here

$$((f_0(x'), g_0(x'))^{\Phi}, \cdots, (f_n(x'), g_n(x'))^{\Phi}, \cdots),$$

where each of $f_0(x'), g_0(x')$ denotes the Φ-modulus (by translating a convention of Euler algebra), and x' is an arbitrary constant Ω'-prime.

The abstract properties of Θ-composition and (Δ, Θ)-composition of Ω'-factorable functions are identical.

58. Let K, E be any classes, and Π, Σ operations such that $\Pi^O K, \Sigma^O E$. Let $X, Y, Z, U, \cdots$ denote elements of K, and $E(X), E(Y), E(Z), E(U), \cdots$ elements of E. The $E(X), \cdots$ are as yet mere marks. If K, E now are such that: whenever $K|X$, there is uniquely determined an element, denoted by $E(X)$, of E; whenever $E|E(X)$, there is uniquely determined an element, denoted by $E^{-1}(E(X))$ or X, of K;

$$(X, Y)^{\Pi} . \supset . (E(X), E(Y))^{\Sigma}$$

whenever $K|X, Y$, and

$$(E(X), E(Y))^{\Sigma} . \supset . (X, Y)^{\Pi}$$

whenever $E|E(X), E(Y)$, we say that (K, E) has (Π, Σ)-*correspondence*, and call $E(X)$ the (Π, Σ)-*exponent* of X.

We shall discuss only the case of (Π, Σ)-correspondence of (K, E) in which (§33 for M_1) $\Sigma^{CAP_2M_1}E$. Hence, by the above definition, $\Pi^{CAP_2M_1}K$, and if U is the unique Σ-modulus of E, then $E^{-1}(E(U))$ is the unique Π modulus of K.

We choose now for K the class J of all Ω-numbers of fixed order r (§15), and take $\Pi\equiv\Omega$. By transference as in §23, what follows concerning (Ω, Ξ)-correspondence goes over at once to matric variables over (K^r) (§17) and their exponents, the latter being defined by transference from what is next developed.

The (Ω, Ξ)-exponent of the arbitrary Ω-number Z is now defined for a particular Ξ of great generality. Let

$$\Xi_{ij}{}^{CAP_2M_1}K_{ij} \quad (i = 1, \cdots, r; j = 1, 2, \cdots),$$

where the range of j is all finite integers >0. A matrix of r rows and an infinity of columns in which the element in row i, column j, is a_{ij}, will be denoted by $\|a_{ij}\|$, and we need not indicate the ranges $i=1, \cdots, r; j=1, \cdots,$ where these are obvious. The Ξ_{ij} modulus of K_{ij} is denoted by u_{ij}, and it is postulated that u_{ij} is Ξ_{ij}-indecomposable into two elements of K_{ij} distinct from u_{ij}. The matrix $\|z_{ij}\|$, where $K_{ij}|z_{ij}$, is called a *Ξ-number*, where $\Xi\equiv\|\Xi_{ij}\|$. A Ξ-number $\|z_{ij}\|$ which is such that $z_{ij}=u_{ij}$ for all $j>n$, where n is a finite integer >0, is said to be *finite*. Composition of Ξ-numbers is defined by

$$(\|x_{ij}\|, \|y_{ij}\|)^{\Xi} = \|(x_{ij}, y_{ij})^{\Xi_{ij}}\|.$$

Hence, by the hypotheses on Ξ_{ij}, the Ξ-composite of any finite number of finite Ξ-numbers is a finite Ξ-number, and if E denotes the class of all finite Ξ-numbers,

$$\Xi^{CAP_2M_1}E,$$

the unique Ξ-modulus of E being $U\equiv\|u_{ij}\|$.

Precisely as for Ω-numbers we can now define Ξ-decomposition. The number of distinct solutions

$$(\|x_{ij}\|, \|y_{ij}\|) \text{ of } (\|x_{ij}\|, \|y_{ij}\|)^{\Xi} = \|z_{ij}\|,$$

for $\|z_{ij}\|$ given, is postulated to be finite (compare §16). This is equivalent to the postulate in §16 for each of $\Xi_{ij}{}^{O}K_{ij}$.

With J now the class of all Ω-numbers of fixed order r as stated above, and E the class of all finite Ξ-numbers, we postulate that each element of J has a finite (Ω, Ξ)-exponent, and we further postulate (Ω, Ξ)-correspondence for (J, E). If $J|Z$, the finite Ξ-number corresponding to Z is $E(Z)$, and the total Ω-decomposition of degree h of Z is

$$(E^{-1}(E(Z_{1i})), \cdots, E^{-1}(E(Z_{hi}))) \qquad (i = 1, \cdots, t),$$

where

$$(E(Z_{1i}), \cdots, E(Z_{hi})) \qquad (i = 1, \cdots, t)$$

is the total Ξ-decomposition of degree h of $E(Z)$. Thus the discussion of any question concerning total Ω-decompositions is referred directly to a similar one concerning total Ξ-decompositions.

In applying the last to Θ-composition as in §34, some care is necessary regarding functions of the transfer U' of the Ξ-modulus U. If f is any element of F_r(§21), we postulate now that $f(E^{-1}(E(U')))$ shall be the Φ-modulus in the Θ-compositions. Or, otherwise: we may first define $f(E^{-1}(E(Z')))$ to be in $F_r(K^r)$ (§21) whenever the transfer Z' of Z is in (K^r), and $Z \neq U$, $F|f$, and then define $f(E^{-1}(E(U')))$ independently as above.*

It is now clear that the whole of §§34–57 may be re-read with Ξ in place of Ω. There being no restrictions on Ξ_{ij}, K_{ij} beyond those implied in the hypotheses $\Xi_{ij}{}^{CAPM_1}K_{ij}$ and the finiteness of Ξ_{ij}-decompositions, the possibilities are again unlimited. The hypotheses can be lightened, as in §41. Primality, coprimality and factorability can be defined for Ξ-numbers in an obvious manner, following §§56–57, and then be transferred directly to the corresponding Ω-numbers. Further developments in this direction belong rather to special theories than to the abstract, and we shall not pursue them here, as the postulates for primality are easily realizable in many ways when all the elements of an exponent are either rational numbers or positive rational integers.

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* In the instance of numerically valued functions $f(n_1, \cdots, n_r)$ of positive rational integers $n_1, \cdots, n_r$, let, for example,

$$n_1 = p_1^{z_{11}} p_2^{z_{12}} \cdots p_a^{z_{1a}}, \cdots, n_r = p_1^{z_{r1}} p_2^{z_{r2}} \cdots p_c^{z_{rc}},$$

where the z's are integers $\geqq 0$, $p_1, p_2, \cdots$ the primes $2, 3, \cdots$ in ascending order, be the prime factor decompositions of $n_1, \cdots, n_r$. As one definition of the exponent of the transfer of the matric variable $(n_1, \cdots, n_r)$ we may take

$$\left\| \begin{matrix} z_{11}, \cdots, z_{1a}, 1, 1, \cdots \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \\ z_{r1}, \cdots, z_{rc}, 1, 1, \cdots \end{matrix} \right\|$$

and for Ξ_{ij} the operation of *multiplication* as in rational arithmetic. The Ξ-modulus is then the matrix of r rows and an infinity of columns in which every element is 1. The postulate provides against infinite values of functions whose arguments are the product of all the positive rational primes. In Dirichlet multiplication the 1's in the above exponent (after $z_{1a}, \cdots, z_{rc}$) are replaced by 0's; the $\Xi_{ij} \equiv$ *addition* as in rational arithmetic; the Ξ-modulus is the matrix of r rows and an infinity of columns, all of whose elements are zero. This instance was first developed in my paper, Bulletin of the American Mathematical Society, vol. 32 (1926), pp. 341–345. The L. C. M.-composition of von Sterneck (Monatshefte für Mathematik, vol. 5 (1894), pp. 255–266) and D. H. Lehmer (American Journal of Mathematics, 1931) has, for $r=1$, $\Xi_{ij} \equiv$ the operation of replacing a pair of integers by the greater of them. For further instances, see my papers cited in §§6, 56, also Annals of Mathematics, vol. 27 (1926), pp. 511–536; American Journal of Mathematics, vol. 49 (1927), pp. 489–510.

59. When the degree r of the Ω-numbers (§15) in a given context is variable, numerous special properties of Θ-composition arise. As the possibilities are unlimited, we shall consider only one of the simplest: in §15, r is now variable, the classes $K_1, \cdots, K_r, \cdots$ are taken to be identical, and the operations $\Omega_1, \cdots, \Omega_r, \cdots$ are the same. To avoid possible confusion we modify the notation; (K^r) of §17 now becomes (K).

(K) shall denote the class of all values of all matric variables of all finite orders whose elements are in the class K.

For example, $K|x_1, \cdots, x_r. \supset .(K)|(x_1, \cdots, x_r)$, for all finite integers r. Elements of K will be denoted by small Latin unaccented letters, with or without suffixes, elements of (K) by the like accented. Thus $x'=(x_1, \cdots, x_r)$, etc.

Instead of $F(K^r)$, $F_r(K^r)$ of §21 we now have $F(K)$, the class of all functions of all matric variables of finite order, and $V(K)$, the class of all values of all functions in $F(K)$. For any fixed r, the class ξ in §21 is as there defined, and the definition of equality, $\xi=\zeta$, is retained. The class of all ξ is denoted by F, which replaces F_r of §21.

Let Λ, Ξ be any operations over F, and let $F|f, g$ (the Ξ here has no connection with that in §58). The result of operating on (f, g) with Λ will be written $(f, g)^{\Lambda}$, as before, and the result of operating on $(f, g)^{\Lambda}$ (which is in F) with Ξ will be written $((f, g)^{\Lambda})^{\Xi}$, or $(f, g)^{\Lambda\Xi}$. If for all f, g such that $F|f, g$, Λ, Ξ satisfy $(f, g)^{\Lambda\Xi}=(f, g)^{\Xi\Lambda}$, we say that Λ, Ξ are *permutable over* F, and write $\Lambda\Xi=\Xi\Lambda$.

There is an extensive theory of operations permutable over F, but as it belongs to the details of the abstract theory, we shall merely exhibit one permutable pair Θ, Υ, where Θ is the special case of the operation denoted by the same letter in §34 which is applicable to the present situation.

We postulate the hypotheses

$$\Delta^{AP_2N}K, \quad (\Psi, \Phi)^{CAPD}V(K),$$

from which we shall construct $\Theta\equiv(\Psi, \Phi)$, Υ such that

$$\Theta^{CAPF}, \quad \Upsilon^{O}V(K), \quad \Theta\Upsilon = \Upsilon\Theta.$$

The Δ above has no connection with that in §57.

As a detail of notation to make what follows clear, if $(f, g)^{\Theta}=h$, we shall indicate the value of h for the argument z' by writing $(f(z'), g(z'))^{\Theta}=h(z')$, which accords with §22.

It is further postulated, as the equivalent here of §16, that if $K|x$, the total number of matrices $(x_1, \cdots, x_r)$ such that $x=(x_1, \cdots, x_r)^{\Delta}$, $K|x_i(i=1, \cdots, r)$, for each finite integer r, is finite. The set of all such

$(x_1, \cdots, x_r)$ is the *total Δ-decomposition* of *degree* r of x. Order and degree may be used interchangeably in this degenerate case.

The *conjoint* (x', y') of $x' = (x_1, \cdots, x_n)$, $y' = (y_1, \cdots, y_m)$ is defined as in §20 to be $(x_1, \cdots, x_n, y_1, \cdots, y_m)$. In particular the conjoint of x and y' is (x, y'), where, by the notation already explained, x is an element of K and hence, if preferred, the matrix (x) of one element. If $F \mid f$, the function f for the argument (x', y') is written $f(x', y')$ as before.

With x', y' as just above, let

$$(f(x', y'), g(x', y'))^{\Theta} = h(x', y')$$

for all (x', y') such that $(K) \mid (x', y')$. The order (or degree) of the conjoint (x', y') is here $n+m$. To be explicit, then, the preceding equation may be written

$$(f(x', y'), g(x', y'))^{\Theta(n,m)} = h(x', y'),$$

where the suffix (n, m) refers to (x', y') in an obvious manner. The assertion that this equation holds for all (x', y') such that $(K) \mid (x', y')$, where the respective orders of x', y' are n, m, will be written

$$(f, g)^{\Theta(n,m)} = h;$$

and finally, the assertion that the last holds for all (n, m) such that $nm > 0$ and $n+m$ is finite, will be written $(f, g)^{\Theta} = h$. These details are necessary, as we must consider operations which change the orders of the matric variables in the functions.

If $\Theta_{(n,m)}$ has the character Q with respect to the subclass F_{n+m} of F which is generated by all functions of matric variables of order $n+m$ the values of whose elements are in K, we write $\Theta^{Q}_{(n,m)} F$. The assertion $\Theta^{Q}_{(n,m)} F$ for all (n, m) as defined is written $\Theta^{Q} F$.

From what precedes and the hypotheses, $\Theta^{CAP} F$.

What follows will be clearer if we describe the structure of $h(x', y')$ as above in some detail. Write

$$(x', y') \equiv z' \equiv (z_1, \cdots, z_{n+m}).$$

The total Δ-decomposition of degree 2 of z' is the class of conjoints of the form (u', v'), a typical pair u', v' being

$$u' = (u_1, \cdots, u_{n+m}), \quad v' = (v_1, \cdots, v_{n+m}),$$

in which a particular pair (u_j, v_j) is determined as a solution of $(u_j, v_j)^{\Delta}$

$=z_j(j=1, \cdots, n+m)$. For each conjoint (u', v') thus determined the Φ-composite $(f(u'), g(u'))^{\Phi}$ is formed; the Ψ-composite of all these Φ-composites, for all (u', v'), is then formed. The result is $h(x', y')$.

It is important to observe that an alternative construction is possible by §26. This is to be emphasized, as the permutability of Θ, Υ, established presently, may seem rather abstruse at first sight; at bottom it is precisely the obvious result in §26. Applied here, §26 enables us to find the total Δ-decomposition of degree 2 of z', assumed of order $n+m$, as follows. First write z' as the conjoint (x', y'), where x' is of order n and y' of order m. Form the total Δ-decomposition of degree 2 of x', and let (x_i'', x_i''') be a typical element of this decomposition. Similarly for y' and (y_j'', y_j'''). Denote the conjoints (x_i'', y_j''), (x_i''', y_j''') by z_{ij}'', z_{ij}''' respectively. Then the total Δ-decomposition of degree 2 of z' is the class of all (z_{ij}'', z_{ij}''').

To define Υ we need the intermediary operator $\Upsilon_x^{x',n}$, which operates on any $h(x', y')$ in which x' is of fixed order n, x is any given element of K, and y' any given element of (K). The result of operating on $h(x', y')$ with $\Upsilon_x^{x',n}$ is denoted by $h^{(n)}(x, y')$, and is obtained as follows: for x' is substituted in $h(x', y')$ in turn each element of the total Δ-decomposition of degree n of x, giving, say, the class $h(x_i', y')(i=1, \cdots, t)$; $h^{(n)}(x, y')$ is the Ψ-composite of all $h(x_i', y')(i=1, \cdots, t)$. By the hypotheses on Ψ, the order of Ψ-composition is immaterial. If the order of y' is m, that of (x', y') is $n+m$, and the result $f^{(n)}(x, y')$ of operating with $\Upsilon_x^{x',n}$ on $f(x', y')$ is a function whose argument is of order $1+m$. Hence $\Upsilon_z^{z',r}$ operating on $f(z')$, where z' is of order r, produces $f^{(r)}(z)$, where the argument is of order 1, namely z is in K. We call $\Upsilon_x^{x',n}$ the *contraction* of x' with respect to x. This contraction has obvious analogies with that of tensor algebra.

By the above remarks on §26 it follows immediately that if x', y' are of the respective orders n, m, and f, g, h are such that

$$(f, g)^{\Theta} = h,$$

then

$$(f, g)^{\Theta\Upsilon} = (f, g)^{\Upsilon\Theta},$$

with the following interpretation:

$$h^{(n)}(x, y') = (f^{(n)}(x, y'), g^{(n)}(x, y'))^{\Phi(1,m)}$$

for all y', n, m as defined and all $\Upsilon_x^{x',n}$ for all x in K. In full, the last equation is

$$[(f(x', y'), g(x', y'))^{\Theta(n,m)}]^{\Upsilon_x^{x',n}} = ([f(x', y')]^{\Upsilon_x^{x',n}}, [g(x', y')]^{\Upsilon_x^{x',n}})^{\Theta(1,m)}.$$

Instances of Υ occur in the case of positive integral arguments and the

special Θ-composition as in the instance by *addition* in §58, footnote, in the papers of Ramanujan and Vaidyanathaswamy.*

IV. Inversion of functions

60. The hypotheses of §39 are postulated in §§60–63, in which we obtain a complete solution of the problem of Θ-inversion as stated in §23. It is obvious that a solution without further postulates on Ω, Φ, Ψ, (K^r) is impossible. To the postulates carried over from §39 we now add the following, in which M_1 is as in §33,

$$\Omega'^{M_1}(K),\ \Psi^{M_1}F_r(K^r),\ \Phi^{M_1}F_r(K^r),$$

where Ω' is as in §42. The first hypothesis of §39 is equivalent to $\Omega'^{AP_2N}(K)$, by the remark in §23. Small Latin accented letters shall be as defined in §42.

The unique Ω'-modulus postulated above for (K^r) will be denoted by u', the unique Φ-modulus for $F_r(K^r)$ by ϕ, and the unique Ψ-modulus for $F_r(K^r)$ by ψ.

An Ω'-decomposition of z' of degree h in which each of the h Ω'-components is different from u', will be called ***proper***; the set of all proper Ω'-decompositions of degree h of z' will be called the ***total proper*** Ω'-decomposition of degree h of z'.

By §§16, 23, the total proper Ω'-decomposition of degree h of z', if it exists, contains only a finite number of elements. For simplicity, although it is not necessary, we now impose the following postulate.†

Postulate. If $(K^r)\,|z'$, then there exists a finite integer $\lambda(z')$, such that z' has at least one proper Ω'-decomposition of degree $\lambda(z')$, and no proper Ω'-decomposition of degree $>\lambda(z')$.

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* S. Ramanujan, Transactions of the Cambridge Philosophical Society, vol. 22(1918), p. 260 (=Collected Papers, p. 180); R. Vaidyanathaswamy, Atti del Congresso internazionale dei Matematici (VI), 1928, vol. 2, pp. 105–12; these Transactions, vol. 33 (1931), pp. 579–662. In the last, the instance cited is called convolution, and the author describes the permutability property as a distributivity. The other processes in the paper are as in the references cited in §56. In the University of Washington publication, 1915, I also described a process which I called "ideal addition," as a complement to "ideal multiplication." Neither, of course, is of the same character as in a field, for in a field the "addition" and "multiplication" must satisfy *all* the postulates. It seems improper, therefore, to call composition "multiplication," although it satisfies the postulates of multiplication as in a ring, *in the fully developed form* of the theory. For the same reason, it would seem to be advisable not to speak of the composition in *any* ring or group as "multiplication." In the book on *Algebraic Arithmetic*, and in earlier papers, I abandoned the process of "ideal addition" in favor of *addition*, which has the required distributive properties with respect to "multiplication." Finally, in the present paper, in the interests of complete accuracy, and to avoid any possible misunderstanding, I have dropped the terms "addition," "multiplication" entirely, and have referred to compositions as defined by their respective postulate systems, for example, *Φ-composition*, *Ψ-composition*, *Θ-composition*.

† In the instance of rational integers, this amounts to excluding infinity.

If $z' \neq u'$, z' has at least one proper Ω'-decomposition, namely z' , $=z'$.

Postulate. The Ω'-modulus u' has no proper Ω'-decomposition.*

Two further postulates will be required. The first imposes an additional property on the Ψ-modulus ψ; the second gives Ψ a unique inverse over $F_r(K^r)$.

Postulate. $F_r(K^r)\,|f(z') . \supset . (f(z'), \psi)^{\Phi} = \psi$.

Postulate. If $F_r(K^r)\,|f(z')$, then there exists a unique element $\tilde{f}(z')$ of $F(K)$, called the *Ψ-inverse* of $f(z')$, such that $(f(z'), \tilde{f}(z'))^{\Psi} = \psi$, and the following conditions with respect to Φ are satisfied. If

$$F_r(K^r)\mid g(x'),\ g(y') \text{ and } (f(x'), g(y'))^{\Phi} = h(z')$$

(the second of which merely states that the Φ-composite of $(f(x'), g(y'))$ is necessarily some element of $F_r(K^r)$), then

$$(f(x'), \tilde{g}(y'))^{\Phi} = \tilde{h}(z'),\ (\tilde{f}(x'), \tilde{g}(y'))^{\Phi} = h(z).$$

Hence, if $(f_1(x_1'), \cdots, f_n(x_n'))^{\Phi} = f(z')$, where $f_i(x_i')(i=1, \cdots, n)$ are any elements of $F_r(K^r)$, then $(\tilde{f}_1(x_1'), \cdots, \tilde{f}_n(x_n'))^{\Phi}$ is the element $f(z')$ or the element $\tilde{f}(z')$ of $F(K)$ according as n is even or odd.†

In the case of functions of rational integers, a distinction between regular and irregular numerical functions is a prerequisite to inversion. The corresponding situation here is provided for by the following definition.

If $F_r\,|f$, we say that f is *regular* if and only if there exists a unique element, which will be denoted by $\bar{f}(u')$, of $F_r(K^r)$, such that $(f(u'), \bar{f}(u'))^{\Phi} = \phi$.

In the numerical case of Dirichlet multiplication of functions of one integer variable, $\bar{f}(u')$ is $1/f(1)$.

To reach Θ-inversion it is necessary to produce a Θ-modulus θ. If $F_r(K^r)\,|\theta(z')$ whenever $(K^r)\,|z'$, we define θ by

$$\theta(u') = \phi,\ \theta(z') = \psi\ (z' \neq u').$$

From the construction of Θ in §34 and the hypotheses on Θ assumed at the beginning of this section, it follows that $(f, \theta)^{\Theta} = f$ whenever $F_r\,|f$. If there is a second Θ-modulus θ', we have $(\theta, \theta')^{\Theta} = \theta$, since θ' is a modulus, and $(\theta, \theta')^{\Theta} = \theta'$, since θ is a modulus. But $(\theta, \theta')^{\Theta}$ is a uniquely determined element of F_r. Hence $\theta' = \theta$, and the Θ-modulus is unique.

* In the instance of rational numbers, this amounts to restricting the arguments of the functions in the Θ-inverse to be integers.

† No attempt has been made to state an independent set of postulates equivalent to the above. If linear order be postulated for the elements of (K), Θ-inversion can be reached by a shorter route. The way followed here however has the advantage of being immediately applicable to a Boolean algebra or to an abelian group, for the first of which order is not significant and for the second at best artificial. For applications to Boolean algebra, I proceed from my paper on the *Arithmetic of logic*, these Transactions, vol. 29 (1927), pp. 597–611, in an obvious way.

61. Let f be any element of F_r. If an f' exists such that $F_r|f'$ and $(f,f')^\Theta=\theta$, the equation $(f,g)^\Theta=h$, where h is any given element of F_r, has a unique solution g such that $F_r|g$. For,

$$(f,g)^\Theta = h. \supset .((f,g)^\Theta, f')^\Theta = (h,f')^\Theta;$$
$$((f,g)^\Theta, f')^\Theta = (g,(f,f')^\Theta)^\Theta = (g,\theta)^\Theta = g;$$
$$g = (h,f')^\Theta, \quad F_r \mid g.$$

If f' is unique for f given, g is unique, as is seen by a contradiction. If f' is not unique, let

$$(f,f')^\Theta = \theta = (f,f'')^\Theta, \quad f' \neq f''.$$
$$(f'',\theta)^\Theta = (f'',(f,f')^\Theta)^\Theta = ((f,f'')^\Theta,f')^\Theta = (\theta,f')^\Theta;$$
$$(f'',\theta)^\Theta = (\theta,f')^\Theta; \quad f'' = f',$$

a contradiction. Hence f' is unique, and therefore if f' exists the problem of Θ-inversion is uniquely solvable. We prove next that f' exists when and only when f is regular. The proof consists in exhibiting the explicit form of f' as a Θ-composite.

62. Let z' be any element $\neq u'$ of (K^r), and let the total proper Ω'-decomposition of degree k of z' be $(z_{1i}', \cdots, z_{ki}')(i=1,\cdots,p_k)$, where $k\leqq\lambda(z')$. Let f be any regular element of F_r. Denote by $[f(z')]_k$ the following Ψ-composite of p_k Φ-composites,

$$((\bar{f}(z_{11}'),\cdots,\bar{f}(z_{k1}'))^\Phi,\cdots,(\bar{f}(z_{1p_k}'),\cdots,\bar{f}(z_{kp_k}'))^\Phi)^\Psi;$$

form the Φ-composite of $[f(z')]_k$ and k components $\bar{f}(u')$, namely

$$([f(z')]_k, \ (\bar{f}(u'))^{k\Phi})^\Phi,$$

and denote the result by $\{f(z')\}_k$. Write $\lambda(z')\equiv n$,

$$f'(z') \equiv (\{f(z')\}_1, \cdots, \{f(z')\}_n)^\Psi.$$

Define $f'(u')$ by $f'(u')\equiv\bar{f}(u')$. Then f' is the required Θ-inverse of f, namely $(f,f')^\Theta=\theta$. Regularity of f is necessary in order that f' shall exist; otherwise, $f'(u')$ does not exist.

That f' is indeed the required inverse can be easily verified by direct calculation of the value of $(f,f')^\Theta$ for the argument z'. Such a calculation is however unnecessary, as we see that $(f,f')^\Theta=\theta$ directly by the method of §48 applied to the theorem of inversion for numerical functions of positive rational integers proved in a former paper.* The inversion in that paper and the Θ-inversion here are abstractly identical.

* Tôhoku Mathematical Journal, vol. 17 (1920), pp. 221–225.

63. We can now remove the exception in §56 regarding reciprocals. It is sufficient to combine the hypotheses of §§56, 60 in order to establish complete simple isomorphism between the present theory and that of the papers cited in §56. By means of this isomorphism the entire theory previously constructed can be read in terms of the present by a simple reinterpretation of the notation.

64. It was repeatedly remarked in my papers already cited that the elements and operations were *abstract*, that is, *any* marks satisfying the postulates. The present paper is, from one point of view, merely an elaboration of that remark. Abstractly, the present theory is identical with my former theory in its simplest form (Euler algebra, as I finally called it, for functions of one variable). The variable there was general, unrestricted beyond the postulates explicitly stated. The abstract point of view has the obvious advantage that inessential details, due to a particular instance, do not obscure the elementary simplicity of the processes involved. It has the disadvantage that an instance may at first sight appear to be a generalization. For example, my extension to functions of r variables* is not a generalization of Euler algebra as developed in the papers cited, but is an *instance* of that algebra in which the *general* variable is *restricted* to be matric of order r. The content of the present theory is precisely that of Euler algebra. An instance of this theory can be read as an instance of Euler algebra, by a mere reinterpretation of the notation. If factorability be left out of account, a yet more elementary conclusion emerges: the content of the theory of outer composition (§54) is identical with that of the formal addition and multiplication of either power series or Dirichlet series in one variable. If factorability in any form be included, the theory (inner composition, §54) is what it was plus the fundamental theorem of arithmetic.

To generalize, and so reach theories which *include* Euler algebra or its sub-varieties (algebras C, D of previous papers), it is *necessary and sufficient* to replace the double (Ψ, Φ)-ovum in §34, from which Θ was constructed, by *any* variety of which a double (Ψ, Φ)-ovum is an instance. As remarked in §13, many such are already known to exist. Those in the literature may be found in the numerous papers of B. A. Bernstein, L. E. Dickson, E. V. Huntington and W. A. Hurwitz on postulate systems, that have appeared in American mathematical periodicals of the past 30 years.

* Bulletin of the American Mathematical Society, cited §58, footnote.

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ON STIELTJES POLYNOMIALS*

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INTRODUCTION

1. According to a theorem of Heine,† there exist at most

$$\frac{(n+1)(n+2)\cdots(n+p-2)}{1\cdot 2\cdots(p-2)}$$

polynomials $\Phi(z)$ of degree $p-2$ such that the differential equation‡

$$\text{(E)} \qquad \frac{d^2w}{dz^2}+\left(\sum_1^p \frac{\alpha_j}{z-a_j}\right)\frac{dw}{dz}+\frac{\Phi(z)}{\prod_1^p(z-a_j)}w=0,$$

where

$$-\gamma \leqq \arg \alpha_j \leqq \gamma < \frac{\pi}{2}, \qquad \text{all } j,$$

has as solution a polynomial of the nth degree. Let us call each of these determinations of $\Phi(z)$ a *characteristic polynomial* and the corresponding solution a *Stieltjes polynomial.*

If

$$S_n(z) = (z-z_1)(z-z_2)\cdots(z-z_n)$$

is a Stieltjes polynomial, it follows from (E) that

$$S_n''(z_k)+\left(\sum_{j=1}^p \frac{\alpha_j}{z_k-a_j}\right)S_n'(z_k)=0 \qquad (k=1,2,\cdots,n).$$

Should $S_n'(z_k)=0$, the z_k would coincide with an a_j, because, otherwise, $S_n''(z_k)=0$ and therefore $S_n^{(i)}(z_k)=0$ all i, and $S_n(z)\equiv 0$. Should $S_n'(z_k)\neq 0$,

$$S_n(z)=(z-z_k)T_{n-1}(z), \qquad T_{n-1}(z_k)\neq 0,$$

and, hence,

$$\frac{S_n''(z_k)}{S_n'(z_k)}=\frac{2T_{n-1}'(z_k)}{T_{n-1}(z_k)}=\sum_{j=1,\,j\neq k}^{n}\frac{2}{z_k-z_j}.$$

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* Presented to the Society, September 12, 1930, and September 9, 1931; received by the editors March 27 and June 3, 1931.

† Heine, *Kugelfunktionen*, Berlin, 1878, pp. 472–479.

‡ A generalization of the Lamé equation ($\alpha_j=1/2$, all j) and of the hypergeometric equation ($p=3$).

Hence, the zeros of $S_n(z)$ are either points a_j or solutions of the system of equations whose left members are the linear partial fractions

$$\text{(S)} \qquad \sum_{j=1}^{p} \frac{\alpha_j}{z_k - a_j} + \sum_{j=1, j\neq k}^{n} \frac{2}{z_k - z_j} = 0 \qquad (k = 1, 2, \cdots, n).$$

When $n=1$, system (S) reduces to a single equation whose left member is a linear partial fraction of the kind investigated in our previous papers.*

Likewise, if t_k is a zero of the characteristic polynomial for which $S_n(z)$ is a solution of (E),

$$S_n''(t_k) + \left(\sum_{j=1}^{p} \frac{\alpha_j}{t_k - a_j}\right) S_n'(t_k) = 0.$$

Should $S_n'(t_k)=0$, the t_k would coincide with an a_j, for, otherwise, $S_n''(t_k)=0$ and therefore $S_n^{(i)}(t_k)=0$, all i, and $S_n(z)\equiv$constant. Should $S'(t_k)\neq 0$, let us write

$$S_n'(z) \equiv n(z - z_1')(z - z_2') \cdots (z - z_{n-1}').$$

As

$$\frac{S_n''(z)}{S_n'(z)} = \sum_{1}^{n-1} \frac{1}{z - z_j'},$$

the zeros of the characteristic polynomials are either points a_j or zeros of the linear partial fraction

$$\text{(F)} \qquad \sum_{j=1}^{p} \frac{\alpha_j}{t_k - a_j} + \sum_{j=1}^{n-1} \frac{1}{t_k - z_j'} = 0.$$

Thus, if we knew the exact positions of the points a_i, we could, by solving equations (S) and (F), locate exactly the points z_j and t_k. *Given, however, only that the points a_i lie in a given convex region K, can we find a second convex region K' in which will lie the points z_j and t_k for all values of p and n?* This is the question which in the present paper we propose to discuss.

CASE $\gamma=0$

2. For the case that $\gamma=0$, this question has already been partially answered, as follows.

THEOREM 1a. *If all the points a_j lie on the segment σ of the real axis, the zeros of every Stieltjes polynomial will also lie on σ.*†

* M. Marden, these Transactions, vol. 32 (1930), pp. 658–668.

† Proved first by Stieltjes (Acta Mathematica, vols. 6–7 (1885–86), pp. 321–326) as a problem in the equilibrium of particles; later by Klein (*Ueber lineare Differentialgleichungen der zweiten Ordnung*, Göttingen, 1894, pp. 211–218) by a method of conformal mapping; still later by Bôcher (Bulletin of the American Mathematical Society, vol. 4 (1897), pp. 256–258) by means of simple function-theoretic considerations.

THEOREM 1b. *Under the hypothesis of Theorem* 1a, *the zeros of every characteristic polynomial will also lie on σ.*[*]

THEOREM 2a. *Any convex polygon which contains all the points a_i will also contain the zeros of every Stieltjes polynomial.*[†]

To the last theorem we add

THEOREM 2b. *Any convex polygon which contains all the points a_i also contains the zeros of every characteristic polynomial.*

For suppose t_k, a zero of a characteristic polynomial, were to lie outside of this polygon K. Not being a point a_j, the point t_k would have to be a root of equation (F), i.e., a root of minus the conjugate imaginary of (F):

$$\sum_{j=1}^{p} \frac{\alpha_j}{\bar{a}_j - \bar{t}_k} + \sum_{j=1}^{n-1} \frac{1}{\bar{z}_j' - \bar{t}_k} = 0. \tag{2.1}$$

By the Gauss-Lucas theorem,[‡] the points z_j' as zeros of $S_n'(z)$ lie in the smallest convex polygon enclosing the points z_j, the zeros of $S_n(z)$. Since the latter points, by Theorem 2a, lie in K, the points z_j' also lie in K. The quantities $(z_j' - t_k)$ and $(a_j - t_k)$ are thus vectors from t_k to points in K and, therefore, lie in the angle ϕ subtended at t_k by K. The vectors $(\bar{z}_j' - \bar{t}_k)^{-1}$ and $\alpha_j(\bar{a}_j - t_k)^{-1}$ are also drawn within the angle ϕ. As t_k is by hypothesis outside of K, $\phi < 180°$ and hence the left-hand side of expression (2.1) cannot sum to zero. This result contradicts the assumption that t_k is a zero of equation (F).

An immediate generalization of Theorems 2a and 2b is

THEOREM 3. *Any convex region K containing all the points a_i also contains the zeros of every Stieltjes and every characteristic polynomial.*

In particular, if K is a circle with its center at the origin, this theorem leads to the following corollary:

If $|a_i| \leqq A$, *all i, then also* $|z_j| \leqq A$ *and* $|t_k| \leqq A$, *all j and k.*

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* E. B. Van Vleck, Bulletin of the American Mathematical Society, vol. 4 (1898), p. 438.

† The theorem was first proved in the case $p=3$ by Bôcher (*Ueber die Reihenentwickelungen der Potentialtheorie*, Leipzig, 1894, pp. 215–218) as a problem in the equilibrium of particles, a method which carries over at once to the general case. Klein states the general theorem (*Differentialgleichungen*, p. 208) crediting it to Bôcher. The theorem was discovered independently by Pólya (Comptes Rendus, 1912, p. 767). See also J. L. Walsh, Tôhoku Mathematical Journal, vol. 23 (1924), pp. 312–317.

‡ Gauss, Werke, vol. 3, p. 112; Lucas, Comptes Rendus, 1868, and Journal de l'École Polytechnique, vol. 46 (1879), p. 8.

CASE $\gamma \neq 0$

3. We shall begin by proving

THEOREM 4. *If all the points a_j lie in a circle C of radius r, the zeros of every Stieltjes polynomial and of every characteristic polynomial lie in the concentric circle C' of radius $r' = r$ sec γ.*

In order to prove the first part of this theorem, let us suppose that z_1 is the z_j farthest away from the center of circle C. Then Γ, the circle through z_1 and concentric with C, will contain in or on its circumference all of the points z_j. Let us denote by T the tangent to Γ at z_1. If z_1 were to lie outside of C', the circle C would subtend at z_1 an angle $\phi < \pi - 2\gamma$. The vector $(\bar{a}_k - \bar{z}_1)^{-1}$ would lie in this angle ϕ. The vector $\bar{\alpha}_k(\bar{a}_k - \bar{z}_1)^{-1}$ would therefore lie in the angle got through enlarging ϕ by γ on both its sides, that is to say, the vector $\bar{\alpha}_k(\bar{a}_k - \bar{z}_1)^{-1}$ would lie on the same side of the line T as circle C. The last statement, being also true of the vector $2(\bar{z}_j - \bar{z}_1)^{-1}$, would show that

$$\sum_{k=1}^{p} \frac{\bar{\alpha}_j}{\bar{a}_j - \bar{z}_1} + \sum_{j=2}^{n} \frac{2}{\bar{z}_j - \bar{z}_1} \neq 0,$$

in contradiction to the hypothesis that z_1 is a zero of (S).

Similarly, if t_1, a zero of a characteristic polynomial, were to lie outside of C', we should draw a circle Γ through t_1 concentric with C and denote by T the tangent to Γ at t_1. By the above reasoning, the vector $\bar{\alpha}_j(\bar{a}_j - \bar{t}_1)^{-1}$ would lie on the same side of T as the circle C. The same would hold of the vectors $(\bar{z}_j' - \bar{t}_1)^{-1}$. For, by the Gauss-Lucas theorem, the points z_j' are situated in any convex region enclosing all the z_j, which, according to the first part of our theorem, lie in the circle C'. Thus, t_1 cannot lie outside of C', if it is to be a zero of (F).

By specializing the circle C to have its center at the origin, we deduce that, *if* $|a_i| \leqq A$, *all* i, *then* $|z_j| \leqq A$ sec γ *and* $|t_k| \leqq A$ sec γ, *all* j *and* k.

The circle C' of Theorem 4 gives us the smallest convex region which, for all n and p, will enclose the zeros of every Stieltjes and every characteristic polynomial. For, when $n = 1$ the theorem coincides with our previous results, which were best approximations.*

4. Let us now consider how we may extend Theorem 4 to an arbitrary convex region K.

By the *covering function* of such a region let us mean a function $k(\lambda)$ such that the inequality

$$|z - \lambda| \leqq k(\lambda) \tag{4.1}$$

is satisfied for all values of λ by and only by the points z of K.

* M. Marden, these Transactions, vol. 32 (1930), p. 658–660.

One covering function $k(\lambda)$ of a given convex region K may be always chosen as the maximum value of $|z-\lambda|$ for points z in K. For, clearly through this choice of the function $k(\lambda)$ inequality (4.1) will be satisfied by all points z in K for each value of λ, and by no point Z outside of K. For, otherwise, there would exist a line which would separate the point Z from the region K, and hence there would exist a circle with center at some point Λ and radius κ which would contain K but not the point Z. As $k(\Lambda)\leqq\kappa$,

$$|Z-\Lambda| > k(\Lambda);$$

i.e., for $\lambda=\Lambda$, Z would not satisfy inequality (4.1).

Chosen in this way, the covering function of the circle with center at α and radius ρ, for example, is

$$k(\lambda) = |\lambda-\alpha| + \rho.$$

A given convex region has in general, however, more than one covering function. For example, another covering function for the circle with center at α and with radius ρ is

$$\begin{aligned} k(\lambda) &= \rho \quad \text{for } \lambda = \alpha, \\ k(\lambda) &= \infty \quad \text{for } \lambda \neq \alpha. \end{aligned} \tag{4.2}$$

Clearly, no covering function of a convex region may ever be negative. Conversely, if $k(\lambda)$ is any real, non-negative function, the totality of points z which for all λ satisfy the inequality

$$|z-\lambda| \leqq k(\lambda), \tag{4.3}$$

if any such points z exist, form a convex region K of which $k(\lambda)$ would be a covering function. For, region K would consist of the common points of the circles (4.3), and the common points of any number of convex regions also form a convex region.

A further important property of covering functions is that, if $k_1(\lambda)\geqq k_2(\lambda)$, the region K_1 with $k_1(\lambda)$ as covering function contains the region K_2 with $k_2(\lambda)$ as covering function. This is because any z which satisfies the inequality

$$|z-\lambda| \leqq k_2(\lambda)$$

for all values of λ must also satisfy the inequality

$$|z-\lambda| \leqq k_1(\lambda)$$

for all values of λ.

To return to our problem, let us suppose the points a_j all to lie in the convex region K with $k(\lambda)$ as covering function. For each value of λ the points a_j will therefore lie in the circle

$$|z-\lambda| \leqq k(\lambda)$$

and hence the points z_i and t_k will lie in the circle

$$|z-\lambda| \leqq k(\lambda) \sec \gamma.$$

The last is equivalent to saying that the points z_i and t_k will be situated in a convex region K' which has $k'(\lambda)=k(\lambda) \sec \gamma$ as a covering function and which, since $k'(\lambda) \geqq k(\lambda)$, will enclose K.

Thus we are led to a generalization of Theorems 3 and 4.

THEOREM 5. *If all the points a_j lie in a convex region K with $k(\lambda)$ as covering function, the zeros of every Stieltjes polynomial and of every characteristic polynomial will lie in a convex region K' which will contain K and which will have as a covering function* $k'(\lambda)=k(\lambda) \sec \gamma$.

5. What is this region K' in the case that K is an ellipse, rectangle or straight-line segment?

If K is an ellipse with center, vertices, and foci at the points $(0, 0)$, $(\pm a, 0)$, and $(\pm c, 0)$ respectively, it may be regarded as the envelope of the circle with center at λ and radius $a(1+c^{-2}\lambda^2)^{1/2}$ as λ describes the y-axis. One covering function of K is therefore

$$(5.1)\qquad \begin{aligned} k(\lambda) &= a(1+c^{-2}\lambda^2)^{1/2} && \text{for } \mathfrak{R}(\lambda) = 0,\\ k(\lambda) &= \infty && \text{for } \mathfrak{R}(\lambda) \neq 0. \end{aligned}$$

Consequently

$$\begin{aligned} k'(\lambda) &= (a \sec \gamma)\ (1+c^{-2}\lambda^2)^{1/2} && \text{for } \mathfrak{R}(\lambda) = 0,\\ k'(\lambda) &= \infty && \text{for } \mathfrak{R}(\lambda) \neq 0, \end{aligned}$$

from which follows that K' is a confocal ellipse with major axis $2a \sec \gamma$. Results concerning the zeros of equations (S) and (F) being independent of the choice of coördinate axes, we may state the following generalization of Theorem 4.

THEOREM 6a. *If all the points a_j lie in or on a given ellipse with a major axis of $2a$, the zeros of every Stieltjes polynomial and of every characteristic polynomial lie in or on the confocal ellipse with the major axis* $2a \sec \gamma$.

A line segment being a limiting case of an ellipse, we may obtain from the preceding theorem a generalization of Theorems 1a and 1b.

THEOREM 6b. *If all the points a_j lie on the line segment AB, the zeros of every Stieltjes and every characteristic polynomial lie in or on the ellipse which has the points A and B as foci and which has a major axis of length* $\overline{AB} \sec \gamma$.

In particular, the choice of A and B as the points $z=\pm a$ of the real axis gives the ellipse of Theorem 6b axes of length $2a \sec\gamma$ and $2a\tan\gamma$. Consequently, *if the a_i are real and $|a_i|\leqq a$, all i, then for all j and k*

$$|\mathfrak{R}(z_j)| \leqq a\sec\gamma; \quad |\mathfrak{I}(z_j)| \leqq a\tan\gamma; \quad |\mathfrak{R}(t_k)| \leqq a\sec\gamma; \quad |\mathfrak{I}(t_k)| \leqq a\tan\gamma.$$

This corollary reduces to Theorems 1a and 1b on setting $\gamma=0$.

Let us next study the case that K is the rectangle $ABCD$, where the side AB joins the points $(\pm a, b)$, and the side BC, the points $(a, \pm b)$. Such a rectangle could be considered as composed of the common points of the four regions K_1, K_2, K_3 and K_4 with the covering functions $k_1(\lambda)$, $k_2(\lambda)$, $k_3(\lambda)$ and $k_4(\lambda)$ respectively, where

$$\begin{aligned}
k_1(\lambda) &= [(\lambda+a)^2+b^2]^{1/2} && \text{for points } \lambda \text{ on the positive real axis,}\\
k_1(\lambda) &= \infty && \text{for all other } \lambda;\\[1ex]
k_2(\lambda) &= [(\lambda-a)^2+b^2]^{1/2} && \text{for points } \lambda \text{ on the negative real axis,}\\
k_2(\lambda) &= \infty && \text{for all other } \lambda;\\[1ex]
k_3(\lambda) &= [(\lambda+b)^2+a^2]^{1/2} && \text{for points } \lambda \text{ on the positive imaginary axis,}\\
k_3(\lambda) &= \infty && \text{for all other } \lambda;\\[1ex]
k_4(\lambda) &= [(\lambda-b)^2+a^2]^{1/2} && \text{for points } \lambda \text{ on the negative imaginary axis,}\\
k_4(\lambda) &= \infty && \text{for all other } \lambda.
\end{aligned}$$

The region K_1, for instance, consists of the interior of the circumscribing circle Γ (of radius $c=(a^2+b^2)^{1/2}$) from which has been removed the sector $AEDFA$.

The region K' will hence be composed of the common points of the regions K_1', K_2', K_3' and K_4' whose covering functions are respectively $k_1'(\lambda)$, $k_2'(\lambda)$, $k_3'(\lambda)$ and $k_4'(\lambda)$, where $k_j'(\lambda)=k_j(\lambda)\sec\gamma$.

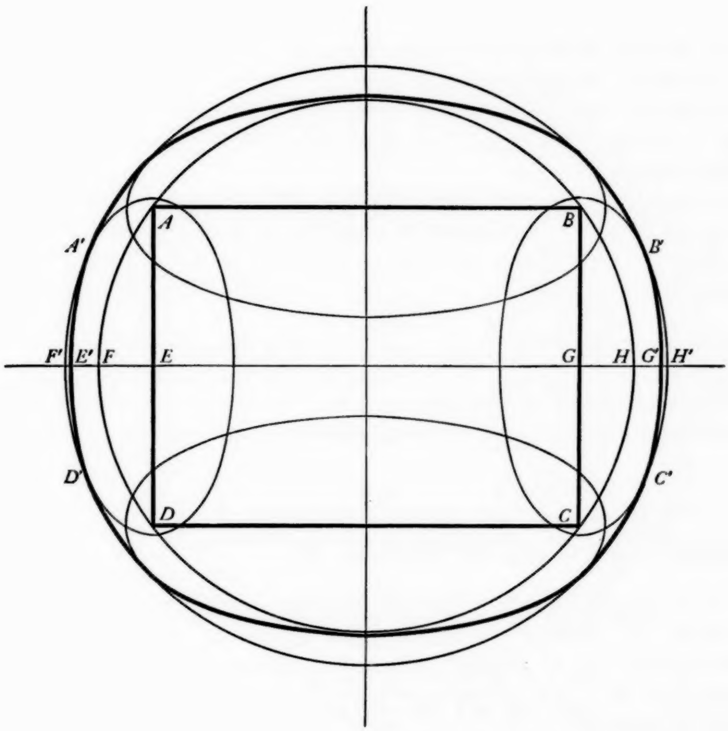
Compared with function (5.1), the functions $k_j'(\lambda)$ suggest how, for instance, the region K_1' may be constructed. The ellipse which has the points A and D as foci and the length $2b\sec\gamma$ as major axis is, at the points A' and D', tangent to the circle Γ' which is concentric with Γ and is of radius $c\sec\gamma$. The region K_1' will then consist of the points remaining in the circle Γ' after the sector $A'E'D'F'A'$ has been removed. Since the regions K_2', K_3' and K_4' are similar to K_1' in structure, the common points of the four form as K' the region bounded by the outer heavy line in the accompanying figure.

THEOREM 6c. *If all points a_j lie in a given rectangle $ABCD$, the zeros of every Stieltjes and every characteristic polynomial lie in the region K' enclosed by the outer heavy line in the accompanying figure.*

On calculating the axes of the four ellipses, we may deduce from the above theorem, that *if* $|\Re(a_i)| \leqq a$ and $|\Im(a_i)| \leqq b$ *for all* i, *then for all* j *and* k

$$|\Re(z_j)| \leqq A, \quad |\Im(z_j)| \leqq B;$$
$$|\Re(t_k)| \leqq A, \quad |\Im(t_k)| \leqq B,$$

where A *is the larger of the quantities* $a \sec \gamma$ *and* $a+b \tan \gamma$ *and* B *is the larger of the quantities* $b \sec \gamma$ *and* $b+a \tan \gamma$.



These results suggest the nature of the region K' in the case that K is an arbitrary convex polygon.

6. In addition to the hypotheses of §1, let us now assume the coefficient of dw/dz in equation (E) to represent the ratio of two real polynomials. Under these conditions a smaller zero-containing region is possible than that afforded by Theorem 5. This smaller region is described in the following theorem.

THEOREM 7. *If the a_j and the corresponding α_j are real or appear in conjugate imaginary pairs, the zeros of every Stieltjes and of every characteristic polynomial will lie in the smallest convex region which encloses all of the ellipses having a_j and $\bar{a}_j$ as foci and* $|a_j-\bar{a}_j|$ sec (arg α_j) *as major axis.**

For the purpose of establishing this theorem, we shall first prove that *the ray from any point w through the point*

$$\frac{e^{-i\gamma}}{1-\bar{w}}+\frac{e^{i\gamma}}{-1-\bar{w}}$$

always meets the ellipse $x^2\cos^2\gamma+y^2\cot^2\gamma=1$.

This lemma generalizes the intuitively evident fact that, if R is the resultant of the two forces at w due to unit particles at $z=\pm 1$ attracting according to the inverse distance law, then R produced must intersect the segment joining the points $z=\pm 1$.

Let α and β be the angles made with the positive real axis by the lines joining $z=-1$ and $z=1$ respectively to point $w=u+iv$. Then $M=e^{-i\gamma}(1-\bar{w})^{-1}$ and $N=e^{i\gamma}(-1-\bar{w})^{-1}$ may be represented by vectors drawn from point w to the points

$$m = w+\frac{1}{r_1}e^{(\pi+\alpha-\gamma)i}, \qquad n = w+\frac{1}{r^2}e^{(\pi+\beta+\gamma)i}$$

where r_1 and r_2 are the distances of w from the points $z=-1$ and $z=1$ respectively. As the mid-point of the segment joining m and n is

$$s = w-\frac{1}{2r_1r_2}(r_2e^{(\alpha-\gamma)i}+r_1e^{(\beta+\gamma)i}),$$

the slope of the line ws is tan arg $(s-w)$, or

$$\begin{aligned}\lambda &= \frac{r_2\sin(\alpha-\gamma)+r_1\sin(\beta+\gamma)}{r_2\cos(\alpha-\gamma)+r_1\cos(\beta+\gamma)}\\ &= \frac{[r_1^2+r_2^2]v\cos\gamma+[(r_1^2-r_2^2)u-(r_1^2+r_2^2)]\sin\gamma}{[(r_1^2+r_2^2)u+(r_2^2-r_1^2)]\cos\gamma+[r_2^2-r_1^2]v\sin\gamma}\\ &= \frac{v(u^2+v^2+1)\cos\gamma+(u^2-v^2-1)\sin\gamma}{u(u^2+v^2-1)\cos\gamma-2uv\sin\gamma}.\end{aligned}$$

From this it follows that

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* Insofar as it concerns the zeros of Stieltjes polynomials, the theorem was first discovered by Charles Vuille in his doctoral thesis, Ecole Polytechnique Fédéral de Zurich, 1916. His proof is based, however, upon a long computation covering pp. 62–75 of his thesis.

$$\lambda^2 + 1 = \frac{r_1^2 r_2^2 [u^2 \cos^2 \gamma + (v \cos \gamma - \sin \gamma)^2]}{[u(u^2 + v^2 - 1) \cos \gamma - 2uv \sin \gamma]^2}$$

and

$$(v - \lambda u)^2 + 1 = \frac{r_1^2 r_2^2 u^2}{[u(u^2 + v^2 - 1) \cos \gamma - 2uv \sin \gamma]^2}.$$

Now, the abscissas of points of intersection of the line $y - v = \lambda(x - u)$ with the ellipse are the roots of the equation

$$x^2(\cos^2 \gamma + \lambda^2 \cot^2 \gamma) + 2\lambda x(v - \lambda u) + (v - \lambda u)^2 \cot^2 \gamma = 1,$$

whose discriminant is

$$\begin{aligned} \Delta &= \left\{ [\lambda^2 + 1] - [(v - \lambda u)^2 + 1] \cos^2 \gamma \right\} \cot^2 \gamma \\ &= \frac{r_1^2 r_2^2 (v \cos \gamma - \sin \gamma)^2 \cot^2 \gamma}{[u(u^2 + v^2 - 1) \cos \gamma - 2uv \sin \gamma]^2}. \end{aligned}$$

This discriminant being non-negative, the ray ws always meets the given ellipse. ∎

The lemma just proved may be stated in a more general form; namely, that *the ray from any point w to the point*

$$\frac{\bar{\alpha}_j}{\bar{a}_j - \bar{w}} + \frac{\alpha_j}{a_j - \bar{w}}$$

always meets the ellipse E_j having a_j and $\bar{a}_j$ as foci and $|a_j - \bar{a}_j| \sec (\arg \alpha_j)$ *as major axis*. In this form, as we shall now see, the lemma leads almost immediately to a proof of Theorem 7.

If z_1, a zero of some Stieltjes polynomial, were to lie outside of K, the smallest convex region enclosing all the ellipses E_j, a line L could be drawn through it which would not cut K. We could suppose this line L to have on it, or on the same side of it as K, all of the remaining z_j. For, were this not the case, we could move L parallel to itself away from K until it met the last z_j, which we would rename as z_1. The vectors $2(\bar{z}_j - \bar{z}_1)^{-1}$ and $[\bar{\alpha}_k(\bar{a}_k - \bar{z}_1)^{-1} + \alpha_k(a_k - \bar{z}_1)^{-1}]$ would then lie on the same side of L as K and hence could not sum up to zero. This result contradicts the hypothesis that z_1 is a zero of system (S).

Similarly, if t_1, a zero of some characteristic polynomial, were to lie outside of K, a line L could be drawn through it not cutting K. According to the Gauss-Lucas theorem the z_j' lie in the same convex region as the z_j and

therefore lie in K. Consequently, the vectors $(\bar{z}_j' - \bar{t}_1)^{-1}$ and $[\bar{\alpha}_k(\bar{a}_k - \bar{t}_1)^{-1} + \alpha_k(a_k - \bar{t}_1)^{-1}]$ lie on the same side of L and cannot sum up to zero. This is contrary to the supposition that t_1 is a zero of partial fraction (F).

The above theorem is a generalization of Theorems 3 and 6b and to some extent an analogue to Jensen's theorem on the distribution of the non-real zeros of a real polynomial.

7. Theorem 5 may also be applied to the study of the mapping properties of the quotient of two linearly independent solutions w_1, w_2 of equation (E), when the second solution is a Stieltjes polynomial $S_n(z)$.

Let $\eta = w_1/w_2$. Then, since

$$\eta' = \frac{1}{w_2^2}(w_1' w_2 - w_1 w_2') = \frac{1}{w_2^2}\left[\exp\left(-\int \sum \frac{\alpha_j}{z - a_j} dz\right)\right],$$

$$\eta = \int \frac{(z - a_1)^{\alpha_1}(z - a_2)^{\alpha_2} \cdots (z - a_p)^{\alpha_p}}{S_n(z)^2} dz.$$

The function η becomes infinite at the zeros of $S_n(z)$ and possibly at some of the points a_j. Agreeing not to count the latter points, we may determine the number of zeros of $S_n(z)$ in a given region S as follows. The function η maps the region S upon a region Σ, which will, in general, be spread over several sheets of a Riemann surface. If the point at infinity on each sheet be considered as distinct from that on any other sheet, the number of zeros of $S_n(z)$ in S will be equal to the number of points at infinity in Σ.*

These considerations lead us to our final theorem.

THEOREM 8. *Let K be a convex region with $k(\lambda)$ as covering function, and S any finite region outside of the convex region K' which has $k'(\lambda) = k(\lambda)$ sec γ as covering function. Let w_2 be an arbitrary Stieltjes polynomial, and w_1 any solution of equation* (E) *which is linearly independent of w_2. Then, if all the singular points a_j lie in K, the region S is mapped by the function $\eta = w_1/w_2$ upon a region which does not contain any point at infinity.*

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* Cf. Klein, *Differentialgleichungen*, p. 213.

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ARITHMETIC OF DOUBLE SERIES*

BY

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Introduction. Two theories of numerical functions have received much attention. The first has for basis the Cauchy multiplication of power series and is appropriately used in considering functions sensitive to additive properties of integers. The second theory is based on the multiplication of Dirichlet series and is applied to multiplicative functions. Both theories may be developed‡ without reference to the analysis of infinite series, and relations thus obtained between numerical functions remain valid when other functions are substituted for which the corresponding infinite series fails to converge, or even when the integer arguments of the numerical functions are replaced by suitably defined elements.

Properties of integers other than additive and multiplicative can be studied by constructing the appropriate theory without regard to the corresponding infinite series§ (if it exists). These other theories whose existence has been doubted|| do have infinite series as we show in §16, and it is to the development of their common properties that this paper is devoted. The class of all these theories we call the *arithmetic of double series*. It is not, we repeat, a theory of infinite series, but rather a theory of composition of two numerical functions, each function being considered as a one-rowed matrix of its values. However we confine ourselves for simplicity to the infinite series aspect only.

1. The grouping function. Let us assume that the double series

$$\sum_{r,s=0}^{\infty} a_r b_s, \tag{1}$$

representing the product of the series $\sum a_n$ and $\sum b_n$, may be rearranged to form a simple series $\sum c_n$ by any manner of grouping the terms of (1). The grouping may be expressed by means of a function $\psi(x, y)$ for which the equation $\psi(x, y) = n$ has the solutions (i, j) corresponding to $c_n = \sum a_i b_j$ and no others. Conversely every single-valued function $\psi(x, y)$ which is an integer for integral arguments determines a method of grouping the terms of the double series (1). Familiar methods are characterized by

* Presented to the Society, April 11, 1931; received by the editors in December, 1930.

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‡ Compare E. T. Bell, these Transactions, vol. 25, pp. 135–144.

§ Such a theory based on the L.C.M. operation has been constructed in a recent paper.

|| American Mathematical Monthly, vol. 37 (1930), p. 484.

(2a)	$\psi(x, y) = x,$	(2b)	$\psi(x, y) = y,$
(3a)	$\psi(x, y) = \max(x, y),$	(3b)	$\psi(x, y) = \min(x, y),$
(4a)	$\psi(x, y) = x + y,$	(4b)	$\psi(x, y) = xy.$

(2a) and (2b) sum (1) by rows or by columns, (3a) and (3b) sum by borders, (4a) and (4b) correspond separately to Cauchy and Dirichlet multiplication of $\sum a_n$ and $\sum b_n$. In (3a), (4a), and (if $a_0 = b_0 = 0$) (4b) each sum c_n contains a finite number of elements; in the other cases c_n is an infinite series.

2. The ψ-calculus of numerical functions. The coefficients of an infinite series are merely values of a numerical function. Associated with each choice of $\psi(x, y)$ there is a calculus of numerical functions whose fundamental operation is

$$\sum f(i)g(j) = h(n) \tag{5}$$

in which f and g are arbitrary numerical functions and the sum extends over all integers (i, j) for which $\psi(i, j) = n$.

3. ψ-multiplication. The operation (5) is called ψ-multiplication and the function h is the ψ-product of f and g. For simplicity we write (5) in the form

$$f \bigcirc g = h$$

when emphasis on ψ is unnecessary.

A random way of rearranging the product of two series would enable us to say very little about the corresponding ψ-product of two arbitrary functions f and g. It is desirable therefore to restrict ourselves to ψ-functions which satisfy the following postulates.

POSTULATE I. *For each $n > 0$, $\psi(x, y) = n$ has a finite number of solutions (x, y).*

POSTULATE II. $\psi(x, y) = \psi(y, x)$.

POSTULATE III. $\psi(x, \psi(y, z)) = \psi(\psi(x, y), z)$.

DEFINITION. $\psi(x, y, z) = \psi(\psi(x, y), z)$.

THEOREM 1. *ψ-multiplication is commutative and associative.*

Commutativity is an obvious consequence of Postulate II. Associativity follows from Postulate III. In fact, if f_1, f_2, f_3 are any functions, then the expressions $f_1 \bigcirc (f_2 \bigcirc f_3)$ and $(f_1 \bigcirc f_2) \bigcirc f_3$ may be written $\sum f_1(i_1) f_2(i_2) f_3(i_3)$, the sum extending over all solutions (i_1, i_2, i_3) of $\psi(x, y, z) = n$.

In what follows we consider only integers $n > 0$. If necessary we renumber the terms of our infinite series or we set $a_0 = 0$.

4. The unit function $\eta(n)$. We next introduce

POSTULATE IV. *If n is any integer, $\psi(x, 1) = n$ implies $x = n$.*

THEOREM 2. *If*

$$\eta(n) = \left[\frac{1}{n}\right] = \begin{cases} 1, & n = 1, \\ 0, & n > 1, \end{cases}$$

then for every function f we have $f\bigcirc\eta = f$, and η is the only function enjoying this property.

The only terms different from zero in the sum

$$\sum f(i)\eta(j)$$

are those for which $j=1$, and by Postulate IV there is but one such term and in it $i=n$. Hence $f\bigcirc\eta = f$. Let η_1 be any function with this property, so that $f\bigcirc\eta_1 = f$. Set $f=\eta$. Then we find $\eta\bigcirc\eta_1=\eta$, whereas $\eta_1\bigcirc\eta=\eta_1$. Hence from the commutative law $\eta_1=\eta$ and η is unique.

The function η is called the *unit function*.

5. Inverse functions. Two functions f and f^{-1} are said to be mutually inverse in case $f\bigcirc f^{-1}=\eta$.

THEOREM 3. *If f has an inverse f^{-1}, then the equation $f\bigcirc g=h$ has a solution g for every h and conversely.*

If we are to have $h=f\bigcirc g$, then $f^{-1}\bigcirc h=f^{-1}\bigcirc f\bigcirc g=\eta\bigcirc g=g$. This exhibits a solution $g=f^{-1}\bigcirc h$. The converse is obvious by putting $h=\eta$.

6. Singular functions. The function which vanishes for all values of n is designated by 0. If the equation $f\bigcirc g=0$ has a solution $g\neq 0$, then f is called *singular*. In particular 0 is singular.

THEOREM 4. *A singular function has no inverse.*

Let f be singular and let $g\neq 0$ be a solution of $f\bigcirc g=0$ and suppose f has an inverse f^{-1}; then

$$g = \eta\bigcirc g = f^{-1}\bigcirc f\bigcirc g = f^{-1}\bigcirc 0 = 0,$$

but this contradicts $g\neq 0$. Hence the theorem.

THEOREM 5. *No function has more than one inverse.*

If possible let f have two inverses f_1 and f_2 such that* $f_1-f_2\neq 0$; then

$$f\bigcirc(f_1 - f_2) = f\bigcirc f_1 - f\bigcirc f_2 = \eta - \eta = 0.$$

Hence f is singular. But this is impossible because, by Theorem 4, f would have no inverse at all.

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* The function $h=f_1\pm f_2$ is defined by $h(n)=f_1(n)\pm f_2(n)$. The truth of the distributive law $f\bigcirc(f_1\pm f_2)=f\bigcirc f_1\pm f\bigcirc f_2$ is at once obvious.

7. **Regular functions.** The function f is called regular if it has an inverse f^{-1}. The class of regular functions is a subclass of non-singular functions. In fact by Theorem 4 every regular function is non-singular. Some non-singular functions, however, are irregular as we shall see. In order to characterize regular functions in a more definite way we introduce a new postulate.

8. **ψ-divisors of n.** If $\psi(x, y)=n$ has a solution x for some y, then x is called a ψ-divisor of n, and x and y are conjugate ψ-divisors of n. By Postulate I there exists a maximum ψ-divisor of n which we denote by $d(n)$. The divisors conjugate to $d(n)$ we designate by

$$\delta_1(n),\ \delta_2(n),\ \cdots,\ \delta_\tau(n).$$

We now introduce

POSTULATE V. *The equation $d(n)=m$, for each $m>0$, has one and only one solution n and $d(1)=1$. In other words $d(2), d(3), d(4), \cdots$ is a permutation* of $2, 3, 4, \cdots$.*

THEOREM 6. *A necessary and sufficient condition for f to be regular is that the sum*

$$\sum_{k=1}^{\tau(n)} f(\delta_k(n)) \tag{6}$$

be different from zero for every $n>0$.

Proof of necessity. Let f be regular and let f^{-1} be its inverse. Then the equation $f \bigcirc f^{-1}=\eta$ can be written in full as

$$f^{-1}(d(n)) \sum_{k=1}^{\tau(n)} f(\delta_k(n)) + \sum{}' f(i) f^{-1}(j) = [1/n], \tag{7}$$

where $\sum'$ extends over values of $j<d(n)$. Suppose that there is at least one value of n for which (6) vanishes. In fact let us choose n_0 the smallest value of such an n, and substitute it in (7). Then $f^{-1}(d(n_0))$ in (7) may have any finite value. By Postulate V, $d(n)$ never takes on the same value for different values of its argument. Hence a unique value of $f^{-1}(d(n_0))$ cannot be determined by other equations (7). That is, f has more than one inverse. This contradicts Theorem 5. Hence (6) never vanishes.

Proof of sufficiency. Suppose that f is such that the sum (6) never vanishes. Then we may write the equation (7) for all values of n ordering them according to increasing values of $d(n)$. The νth equation involves $f^{-1}(\nu)$ with a non-zero coefficient and thus enables us to solve uniquely for $f^{-1}(\nu)$ in terms of the given f and the values $f^{-1}(n)$ for $n<\nu$, the latter having been determined from previous equations. In short, f is regular.

* To be strictly logical this permutation should replace finite numbers by finite numbers.

9. Applications to special ψ's. To illustrate Theorem 6 choose $\psi(x, y) = xy$. Then $d(n) = n$, $\tau(n) = 1$, $\delta_1(n) = 1$. Hence f is regular in the divisor calculus if and only if $f(1) \neq 0$. It may be shown that in this calculus no function is singular except zero. Hence any function $f \neq 0$ for which $f(1) = 0$ is a non-singular irregular function. Exactly the same statements are true in the case $\psi(x, y) = x+y-1$. For $\psi(x, y) = [x, y]$, the L.C.M. of x and y, we have $d(n) = n$, $\tau(n)$ is the number of divisors of n, and $\delta_k(n)$ are the divisors of n. According to Theorem 6 a function f is regular if and only if its "numerical integral" $\sum_{\delta/n} f(\delta)$ never vanishes. The Möbius function μ is irregular in the L.C.M. calculus because $\sum_{\delta/n}\mu(\delta) = \eta(n) = 0$ for $n > 1$. As a matter of fact μ is also singular; we have shown* that

$$\sum \mu(i)g(j) = g(1)\mu(n) \qquad ([i, j] = n)$$

where g is an arbitrary function. To show that μ is singular we need only choose a g for which $g(1) = 0$; such a g is also singular.

It is possible to show by equation (7) that if ψ fails to satisfy Postulate V all functions except η are irregular. In the light of Theorem 3, in such a calculus unique division is impossible except by the unit η.

10. Inversion. The function u which is 1 for all values of $n > 0$ is regular by Theorem 6. Let its inverse be u^{-1}. If f is any function and if $f \bigcirc u = F$, then by multiplying by u^{-1} we get $f = F \bigcirc u^{-1}$. This inversion which is a generalization of that of Dedekind may be written out at length as follows:

$$F(n) = \sum_n f(i),$$
$$f(n) = \sum_n F(i)u^{-1}(j) \qquad (\psi(i, j) = n).$$

Following Bougaieff, F might be termed the ψ-integral of f. "Differentiation" is always possible since u^{-1} always exists uniquely. For

$$\psi(x, y) = xy,\ u^{-1} = \mu;$$

for

$$\psi(x, y) = x + y - 1,\ u^{-1}(1) = 1; \quad u^{-1}(2) = -1; \quad u^{-1}(k) = 0,\ k > 2;$$

for

$$\psi(x, y) = [x, y],\ u^{-1}(n) = \prod_\nu (-n_\nu^2 - n_\nu)^{-1},$$

where $n = \prod_\nu p_\nu^{n_\nu}$ is the decomposition of n into its constituent primes.

11. ψ-multiplicative functions. If the function $f \neq 0$, and is such that, for every pair of integers m, n,

$$f(m)f(n) = f(\psi(m, n)), \tag{8}$$

then f is called ψ-multiplicative. It follows that $f(1) = 1$.

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* American Journal of Mathematics, vol. 53.

THEOREM 7. *If f is multiplicative, it is regular. In fact it has the inverse $f^{-1}(n)=f(n)u^{-1}(n)$, u^{-1} being the inversion function of* §10.

By actual substitution

$$\begin{aligned}\sum_n f(i)f^{-1}(j) &= \sum_n f(i)f(j)u^{-1}(j)\\ &= f(n)\sum_n u^{-1}(j) = f(n)\eta(n) = \eta(n),\end{aligned}$$

hence $f\circ f^{-1}=\eta$, which is the theorem.

The functional equation (8) has been discussed by many writers in the particular case $\psi(m, n)=mn$. Familiar solutions in this case are $f(n)=n^s$, Liouville's function $\lambda(n)$, Legendre's symbol (n/p), etc. If $\psi(x, x)=x$ as in the case of the L.C.M. calculus, then $f^2(n)=f(\psi(n, n))=f(n)$. Hence $f(n)=0$, or 1. The values of n for which $f(n)=1$ belong to a set S which is such that $\psi(x, y)$ is in S if and only if both x and y are in S. This fact is helpful in constructing examples of solutions of (8). We may consider for example the set of all simple numbers, i.e., all numbers not divisible by a square >1. The L.C.M. of any two integers is in this set if and only if both integers belong to it. Hence the function $f(n)=\mu^2(n)$ which is characteristic* of this set is a solution of $f(m)f(n)=f([m, n])$. Even when $\psi(n, n)=n$ does not hold this method leads to a solution. Thus $f=\eta$ is a solution for every ψ. The logical product of two of the above sets is a set of the same sort. Only in case $\psi(n, n)=n$, however, does this method lead to all the solutions of (8). In this case there is but a denumerable infinity of solutions. Other examples of multiplicative functions are given in §17.

12. Factorable functions. If f is such that, for every pair of relatively prime integers m, n,

$$f(m)f(n) = f(mn) \text{ and } f(1) = 1,$$

f is called factorable. These functions comprise one of the most conspicuous classes of numerical functions. To this class belong most of the fundamental functions of the theory of numbers. It follows from the definition that a factorable function may be defined arbitrarily for prime power values of its argument. All other values of the function are then determined.

13. Matrix notation for integers. One reason for the importance of factorable functions is the fundamental theorem of arithmetic.

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* The function f is said to be characteristic of a set S if $f(n)$ is 0 or 1 according as n belongs to the set or not.

Every positive integer n may be written

$$n = 2^{n_1}3^{n_2}5^{n_3}7^{n_4}\cdots p_\nu^{n_\nu}\cdots,$$

where p_ν is the νth prime and where $n_\nu>0$ if and only if p_ν divides n. By the theorem just referred to there is a one-to-one correspondence between integers n and one-rowed matrices $\{n_1, n_2, \cdots\}$ whose elements are zero except for a finite number that are positive integers and have finite subscripts. This correspondence we indicate by $\sim$. Thus $n\sim\{n_1, n_2, n_3, \cdots\}$ or simply $n\sim\{n_\nu\}$. For example $\{[6/\nu]\}\sim 2^6\cdot 3^3\cdot 5^2\cdot 7\cdot 11\cdot 13 = 43243200$.

14. Factorable ψ-functions. The function $\psi(x, y)$ is said to be factorable if, for every $i\sim\{i_\nu\}$ and $j\sim\{j_\nu\}$, $\psi(i,j)\sim\{\theta(i_\nu, j_\nu)\}$, the function $\theta(x, y)$ depending on ψ alone. In the presence of Postulates I, II, III, and IV, $\theta(x, y)$ must satisfy the following conditions:

I. *For* $n\geqq 0$, $\theta(x, y)=n$ *has a finite number of solutions.*

II. $\theta(x, y)=\theta(y, x)$.

III. $\theta(x, \theta(y, z))=\theta(\theta(x, y), z)$.

IV. $\theta(0, x)=n$ *implies* $x=n$.

The effect of Postulate V on θ is not so easily expressed. Since $d(1)=1$ however, we may assert that $\theta(x, y)=0$ implies $x=y=0$.

THEOREM 8. *If ψ is factorable, the prime factors of $\psi(i, j)$ are those of ij.*

Let $i\sim\{i_\nu\}$, $j\sim\{j_\nu\}$ and $\theta(i_\nu, j_\nu)=k_\nu$. If either i_ν or $j_\nu>0$, then $k_\nu>0$, since $\theta(x, y)=0$ implies $x=y=0$. Hence every prime factor of ij divides $\psi(i, j)$. Conversely if $k_\nu>0$ then either i_ν or $j_\nu>0$ since $\theta(0, 0)=0$. Hence $\psi(i, j)$ contains only those prime factors which divide ij.

COROLLARY. *If ψ is factorable and if m and n are coprime, then every ψ-divisor of m is prime to every ψ-divisor of n.*

THEOREM 9. *If ψ is factorable and if i and j are coprime, then $\psi(i, j)=ij$.*

Since $i\sim\{i_\nu\}$ is prime to $j\sim\{j_\nu\}$, then $i_\nu j_\nu=0$. Hence by IV $\theta(i_\nu, j_\nu)=i_\nu+j_\nu$. That is $\psi(i, j)=ij$.

THEOREM 10. *If ψ is factorable, the ψ-product of factorable functions is factorable.*

It is sufficient to show that if f and g are factorable so is $h=f\bigcirc g$. Let m and n be coprime. Then

$$h(n)h(m) = \Big(\sum_n f(i)g(j)\Big)\Big(\sum_m f(k)g(l)\Big) = \sum f(i)f(k)g(j)g(l),$$

where the summation extends over all (i, j, k, l) for which $\psi(i, j) = n$ and $\psi(k, l) = m$. By the corollary of Theorem 8, $(i, k) = (j, l) = 1$. Since f and g are factorable we have

$$h(n)h(m) = \sum f(ik)g(jl). \tag{9}$$

We proceed to show that this sum is equal term by term to

$$\sum_{mn} f(a)g(b) = h(mn), \tag{10}$$

where a and b are all solutions of $\psi(a, b) = mn$. From this the theorem will follow at once.

To show that every term of (9) is in (10) we write

$$\begin{aligned} \psi(ik, jl) &= \psi(\psi(i, k), \psi(j, l)) = \psi(i, \psi(k, \psi(l, j))) \\ &= \psi(i, \psi(\psi(k, l), j)) = \psi(i, \psi(m, j)) \\ &= \psi(m, \psi(i, j)) = \psi(m, n) = mn. \end{aligned}$$

These equalities follow from Postulates II and III and Theorem 9.

To show that every term of (10) is in (9) we proceed as follows. Let $a \sim \{a_\nu\}$ and $b \sim \{b_\nu\}$ be a pair occurring in (10). We then define for each ν four numbers $i_\nu, j_\nu; k_\nu, l_\nu$:

$$i_\nu = \begin{cases} a_\nu, & p_\nu \mid n, \\ 0, & \text{otherwise}; \end{cases} \qquad j_\nu = \begin{cases} b_\nu, & p_\nu \mid n, \\ 0, & \text{otherwise}; \end{cases}$$

$$k_\nu = \begin{cases} a_\nu, & p_\nu \mid m, \\ 0, & \text{otherwise}; \end{cases} \qquad l_\nu = \begin{cases} b_\nu, & p_\nu \mid m, \\ 0, & \text{otherwise}. \end{cases}$$

Here p_ν is the νth prime >1. Finally let $i \sim \{i_\nu\}$, $j \sim \{j_\nu\}$, $k \sim \{k_\nu\}$, $l \sim \{l_\nu\}$ be the four integers defined by the four sequences. To exhibit the term $f(a)g(b)$ in (9) we must prove that

$$ik = a, \qquad jl = b, \tag{11}$$

$$\psi(i, j) = n, \qquad \psi(k, l) = m. \tag{12}$$

To prove (11) we write $ik \sim \{i_\nu + k_\nu\}$. By definition

$$i_\nu + k_\nu = \begin{cases} a_\nu, & p_\nu \mid mn, \\ 0, & \text{otherwise}. \end{cases}$$

But if p_ν does not divide mn it does not divide a by Theorem 8. Hence $a_\nu = 0$ in this case. For all values of ν, then $i_\nu + k_\nu = a_\nu$. Therefore $ik = a$. Similarly $jl = b$.

To prove (12) we use property IV of $\theta(x, y)$, and write $\psi(i, j) \sim \{\theta(i_\nu, j_\nu)\}$ By definition,

$$\theta(i_\nu, j_\nu) = \begin{cases} \theta(a_\nu, b_\nu), & p_\nu \mid n, \\ 0, & \text{otherwise}. \end{cases}$$

Hence the matrix $\{\theta(i_\nu, j_\nu)\}$ is the result of equating to zero those elements of the matrix $\{\theta(a_\nu, b_\nu)\}$ for which $p_\nu | m$. But $\{\theta(a_\nu, b_\nu)\} \sim \psi(a, b) = mn$. Hence $\psi(i, j) \sim \{n_\nu\} \sim n$. Similarly $\psi(k, l) = m$. This completes the proof of the theorem.

This theorem enables us to express the ψ-product of two factorable functions in terms of θ alone as follows.

Theorem 11. *Let ψ, f and g be factorable functions and let $h = f \bigcirc g$. Then*

$$h(n) = \prod_\nu \sum f(p_\nu^{r_\nu}) g(p_\nu^{s_\nu}),$$

where

$$n = \prod_\nu p_\nu^{n_\nu}$$

and where the sum extends over all solutions (r_ν, s_ν) of $\theta(r_\nu, s_\nu) = n_\nu$.

By the preceding theorem h is factorable so that

$$h(n) = \prod_\nu h(p_\nu^{n_\nu}).$$

But

$$h(p_\nu^{n_\nu}) = \sum_\nu f(p_\nu^{r_\nu}) g(p_\nu^{s_\nu})$$

where

$$\psi(p_\nu^{r_\nu}, p_\nu^{s_\nu}) = p_\nu^{n_\nu}, \text{ that is, } \theta(r_\nu, s_\nu) = n_\nu.$$

Hence the theorem.

15. Examples of Theorem 11. For $\psi(i, j) = ij$ we have $\theta(x, y) = x + y$. Applying Theorem 11 we have

$$h(n) = \prod_\nu \sum_{r=0}^{n_\nu} f(p_\nu^r) g(p_\nu^{n_\nu - r}).$$

This result is as useful as it is familiar.

If $\psi(i, j) = [i, j]$, then $\theta(x, y)$ is the greater of x and y. Applying the theorem we have

$$h(n) = \prod_\nu \left\{ f(p_\nu^{n_\nu}) \sum_{r=0}^{n_\nu} g(p_\nu^r) + g(p_\nu^{n_\nu}) \sum_{r=0}^{n_\nu - 1} f(p_\nu^r) \right\}.$$

The $\psi(i, j)$ for which $\theta(x, y) = x + y + xy$ is designated by $\Phi(x, y)$, and it is found to satisfy all the Postulates I–V. Hence for this calculus

$$h(n) = \prod_{\nu} \sum_{d\delta = n_\nu + 1} f(p_\nu^{d-1}) g(p_\nu^{\delta-1}).$$

Incidentally the inversion function u^{-1} for this calculus is

$$\rho(n) = \prod_{\nu} \mu(1 + n_\nu) = \pm 1 \text{ or } 0.$$

16. Infinite series. Returning to the subject of §1 let us consider the infinite series

$$F(z) = \sum_{n=1}^{\infty} f(n)\Omega(n, z), \tag{13}$$

where z is a complex variable and where the f's are mere coefficients.

THEOREM 12. *If $\Omega(n, z)$ is a ψ multiplicative function of n, then if the series*

$$\sum_{n=1}^{\infty} f(n)\Omega(n, z), \quad \sum_{n=1}^{\infty} g(n)\Omega(n, z)$$

converge absolutely, their product is given by

$$\sum_{n=1}^{\infty} h(n)\Omega(n, z),$$

where f and g are arbitrary and $h = f \bigcirc g$.

The product in question is

$$\sum_{i,j=1}^{\infty} \Omega(i, z)\Omega(j, z) f(i) g(j), \tag{14}$$

and since

$$\Omega(i, z)\Omega(j, z) = \Omega(\psi(i, j), z),$$

we may write (14)

$$\sum_{n=1}^{\infty} \Omega(n, z) \sum_{n} f(i) g(j) = \sum_{n=1}^{\infty} h(n)\Omega(n, z).$$

17. Examples. The power series

$$f(1) + f(2)z + f(3)z^2 + \cdots$$

corresponds to the case $\psi(x, y) = x+y-1$, while the Dirichlet series

$$\sum_{n=1}^{\infty} f(n) n^{-s}$$

corresponds to $\psi(x, y) = xy$.

To obtain a series corresponding to $\psi(x, y) = [x, y]$ we consider a set of L.C.M.-multiplicative functions depending on a complex variable z. For example we may take

$$\Omega(n, z) = \epsilon([\,|z|\,]/n),$$

where $\epsilon(x) = 1$ or 0 according as x is or is not an integer. Then Theorem 12 enables us to write

$$\sum_{n=1}^{\infty} f(n)\epsilon([\,|z|\,]/n) \cdot \sum_{n=1}^{\infty} g(n)\epsilon([\,|z|\,]/n) = \sum_{n=1}^{\infty} h(n)\epsilon([\,|z|\,]/n), \tag{15}$$

where h is the L.C.M. product of f and g. If we let $m = [\,|z|\,]$ this equation may be written

$$\sum f(\delta) \cdot \sum g(\delta) = \sum h(\delta)$$

where $\delta \mid m$. This important theorem was first stated by von Sterneck* and affords a simple way of calculating L.C.M. products. Of course f and g may be arbitrary since the series in (15) are actually finite.

Another choice for $\Omega(n, z)$ which gives infinite series is $\Omega(n, z) = 0$ or 1 according as n is or is not divisible by a power of a prime p^α with $\alpha \geqq |z|$. The function $\Omega(n, z)$ can never be continuous since it can have only two values 0 and 1. This is true for every calculus in which $\psi(n, n) = n$, as we have seen in §11.

In case $\psi(x, y) = \Phi(x, y)$ (§14) we may use the following numerical function defined for $n = \prod_\nu p_\nu^{n_\nu}$:

$$T(n) = \prod_\nu (1 + n_\nu)^\nu.$$

This function is Φ-multiplicative; in fact if $m = \prod_\nu p_\nu^{m_\nu}$,

$$\begin{aligned} T(m)T(n) &= \prod_\nu (1 + m_\nu)^\nu (1 + n_\nu)^\nu \\ &= \prod_\nu (1 + m_\nu + n_\nu + m_\nu n_\nu)^\nu = T(\Phi(m, n)). \end{aligned}$$

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* Monatshefte für Mathematik und Physik, vol. 5, p. 265. The proof is inadequate however.

We may take, then, $\Omega(n, z) = T^{-z}(n)$ and consider series of the type

$$\sum_{n=1}^{\infty} f(n)T^{-z}(n).$$

The fundamental series is that for which $f(n) \equiv 1$, namely

$$Z(z) = \sum_{n=1}^{\infty} T^{-z}(n) = \prod_{\nu} \sum_{\alpha=0}^{\infty} T^{-z}(p_\nu^{\alpha}),$$

since T is factorable. But

$$\sum_{\alpha=0}^{\infty} T^{-z}(p_\nu^{\alpha}) = \sum_{\alpha=0}^{\infty} (1 + \alpha)^{-z\nu} = \zeta(\nu z).$$

Hence

$$Z(z) = \prod_{\nu=1}^{\infty} \zeta(\nu z),$$

where ζ is Riemann's function.

The function Z takes the place of ζ in Dirichlet series. For example its reciprocal is given by

$$Z^{-1}(z) = \sum_{n=1}^{\infty} \rho(n)T^{-z}(n),$$

where ρ is the inversion function of §15. If $\tau(n)$ is the number of divisors of n, then

$$\sum_{n=1}^{\infty} \tau^k(n)T^{-z}(n) = \prod_{\nu=1}^{\infty} \zeta(z\nu - k).$$

As a final example consider the function $P_z(n)$ defined for $n = \prod_\nu p_\nu^{n_\nu}$ by

$$P_z(n) = \prod_{\nu} J_z(1 + n_\nu),$$

where $J_z(n)$ is Jordan's totient function.* We now show that

$$\sum_{n=1}^{\infty} P_{-z}(n)T^{-z}(n) = \zeta^{-1}(z). \tag{16}$$

This is a consequence of multiplying the series

$$\sum_{n=1}^{\infty} \rho(n)T^{-z}(n) \quad \text{by} \quad \sum_{n=1}^{\infty} \tau^{-z}(n)T^{-z}(n).$$

* Dickson, *History of the Theory of Numbers*, vol. 1, 147ff.

By §15 the nth coefficient of the product is

$$\prod_{\nu} \sum_{d\delta = n_\nu + 1} \rho(p_\nu^{d-1}) \tau^{-z} (p_\nu^{\delta-1}) = \prod_{\nu} \sum \mu(d) \delta^{-z} = \prod_{\nu} J_{-z}(1 + n_\nu) = F_{-z}(n).$$

On the other hand the two series represent for $\mathrm{Re}(z) > 1$ the functions $Z^{-1}(z)$ and $\prod_{\nu=1}^{\infty} \zeta(z\nu + z)$, whose product is $\zeta^{-1}(z)$. This proves (16). For $z = 10$, equation (16) becomes

$$1 - \frac{2^{10} - 1}{2^{20}} - \frac{2^{10} - 1}{2^{30}} - \frac{3^{10} - 1}{3^{20}} - + \cdots .$$

These first four terms give for

$$\zeta^{-1}(10) = \frac{93555}{\pi^{10}}$$

the value 0.9990065, which is correct to six decimals.

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ON THE EXISTENCE OF ACYCLIC CURVES SATISFYING CERTAIN CONDITIONS WITH RESPECT TO A GIVEN CONTINUOUS CURVE*

BY

C. M. CLEVELAND

Part I of this paper has to do with connected sets of cut points† of a given continuous curve. It is shown that, in the plane, any two points belonging to a connected set K of cut points of a given continuous curve M lie together in an arc which is common to K and a point set consisting of the boundaries of a finite number of complementary domains of M. G. T. Whyburn‡ calls attention to the fact that from his results it follows that K is arcwise connected. To show that any two points of K can be joined by an arc which is common to K and a point set consisting of the boundaries of a finite number of complementary domains of M, is the object of Part I. Part II has to do with a totally disconnected closed subset K of a given plane continuous curve M no subset of which disconnects M. R. L. Moore§ has shown that, in the plane, any two points not belonging to a bounded continuous curve can be joined by an arc which does not disconnect the continuous curve. The object of Part II of this paper is to show that if, in the plane, M is a bounded continuous curve which contains no domain, and K is a closed and totally disconnected subset of M, such that no subset of K disconnects M, then there exists an acyclic continuous curve T, containing K, such that (1) all the end points‖ of T belong to K and (2) the point set $M \cdot T$ is totally disconnected and $M-T$ is connected.

I wish to acknowledge my indebtedness to Professor R. L. Moore, not only for the suggestions of the problems treated, but also for his many helpful criticisms in their solutions. To him is due, in a large measure, the success

* Presented to the Society, in part, June 12, 1926, and September 9, 1930; received by the editors December 22, 1930.

† A point P is said to be a cut point of a continuum M if $M-P$ can be expressed as the sum of two mutually separated sets.

‡ *Concerning the structure of a continuous curve*, American Journal of Mathematics, vol. 50 (1928), p. 176.

§ *Concerning paths that do not separate a given continuous curve*, Proceedings of the National Academy of Sciences, vol. 12 (1926), pp. 745–753.

‖ The term *end point* will be used in the sense as defined by R. L. Wilder, *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7 (1925), p. 358, i.e., a point P of a continuous curve M is an end point of M provided it is true that if t is an arc of M having P as one of its extremities, then $M-(t-P)$ contains no connected subset which contains P.

of these investigations. His stimulating personality has been a source of constant encouragement to me in the study of mathematics.

PART I

LEMMA I. *If K is a connected set of points belonging to a continuous curve M, and J is the outer boundary of a complementary domain D of M, and M_1 is a component of $M-J$ which lies within J, then $K\cdot\overline{M}_1$ is connected.*

By a theorem of R. L. Moore,† J contains exactly one limit point P of M_1 and thus $\overline{M}_1=M_1+P$. The closed point sets M_1+P and $M-M_1$ have only the point P in common and their sum is M. Hence, unless $K=K\cdot\overline{M}_1$, then $K-P$ is the sum of the two mutually separated point sets $K\cdot M_1$ and $K(M-M_1)$. Therefore $K\cdot M_1+P$ is connected. But $K\cdot M_1+P=K\cdot\overline{M}_1$. Hence $K\cdot\overline{M}_1$ is connected.

LEMMA II. *If X is a point belonging to the boundary of a complementary domain D of a continuous curve M, then there do not exist infinitely many simple closed curves of M, each of which encloses X and is the outer boundary of a complementary domain of M.*

Lemma II follows from the fact that the complementary domains of M form a contracting sequence,‡ and the fact that no two complementary domains of M have the same outer boundary.§

LEMMA III. *Suppose* (1) *K is a connected set of points belonging to a continuous curve M,* (2) *$D_1, D_2, D_3, \cdots$ are all the bounded complementary domains of M such that for each i, $\overline{D}_i$ contains a point of K,* (3) *for each i, J_i is the outer boundary of D_i, and* (4) *$J_1^*, J_2^*, J_3^*, \cdots$ is a finite or infinite subsequence of the sequence $J_1, J_2, J_3, \cdots$. Then the set K^* consisting of all points X of K such that X is not interior to J_i^* for any value of i is connected, and if all but a countable number of the points of K are cut points of M, and no point of K^* is interior to J_i for any value of i, and K^* contains more than one point, then K^* is a subset of the boundary of the unbounded complementary domain of M.*

Suppose there exists a subsequence $J_1^*, J_2^*, J_3^*, \cdots$ of the sequence $J_1, J_2, J_3, \cdots$ for which the set K^* consisting of all points X of K, such that

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† *Concerning paths that do not separate a given continuous curve*, loc. cit., Theorem 6.

‡ If H is a sequence of point sets and for each positive number e only a finite number of point sets of the set H are of diameter greater than e, then H is said to be *a contracting sequence* of point sets. See R. L. Moore, *Concerning upper semi-continuous collections*, Monatshefte für Mathematik und Physik, vol. 36 (1929), pp. 81–88. See also R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, Bulletin of the American Mathematical Society, vol. 29 (1923).

§ See R. L. Moore, *Concerning paths that do not separate a given continuous curve*, loc. cit., Theorem 1.

X is not interior to J_i^* for any value of i, is not connected. The set K^* can be expressed as the sum of two mutually separated sets K_1 and K_2. It follows from Lemma II that there exists a subsequence $J_1^{**}, J_2^{**}, J_3^{**}, \cdots$ of the sequence $J_1^*, J_2^*, J_3^*, \cdots$ such that (1) if X is any point lying on or within some curve of the sequence $J_1^*, J_2^*, J_3^*, \cdots$, then X lies on or within some curve of the sequence $J_1^{**}, J_2^{**}, J_3^{**}, \cdots$, (2) for $m \neq n$, J_m^{**} contains no point interior to J_n^{**}. For each positive integer n, and for each point X of K_1 belonging to J_n^{**}, add to K_1 every point Y of K such that Y belongs to some component of $M-J_n^{**}$ which lies within J_n^{**} and has X as a limit point. Let K_1^* denote the resulting set. For each positive integer n and for each point X of K_2 belonging to J_n^{**} add to K_2 every point Y of K such that Y belongs to some component of $M-J_n^{**}$ which lies within J_n^{**} and has X as a limit point. Let K_2^* denote the resulting set. Since K is the sum of the sets K_1^* and K_2^*, one of these sets contains a limit point of the other. We shall consider the case where K_1^* contains a limit point of K_2^*. Let P denote one such limit point. The point P belongs to K_1, for suppose it does not. There exists a positive integer t such that J_t^{**} encloses P. Only a finite number of components of $M-J_t^{**}$ are of diameter greater than a positive number,† and all the points of K belonging to that component of $M-J_t^{**}$ which contains P belong also to K_1^*. It follows that P is not a limit point of K_2^* contrary to hypothesis. Hence P belongs to K_1. It follows then that P is the sequential limit point of a sequence $P_1, P_2, P_3, \cdots$ of points belonging to the set $K_2^*-K_2$ such that for $i \neq j$, P_i and P_j belong to different components of $M-(J_1^{**}+J_2^{**}+J_3^{**}+\cdots)$. For each positive integer n let Z_n denote the limit point of that component of $M-(J_1^{**}+J_2^{**}+J_3^{**}+\cdots)$ which contains P_n. Since (1) for each n the point Z_n belongs to K_2, (2) the curves J_1^{**}, $J_2^{**}, J_3^{**}, \cdots$ form a contracting sequence,‡ and (3) not more than a finite number of components of $M-J_i^{**}$, for any value of i, are of diameter greater than a positive number, it follows that P is a limit point of K_2 and belongs to K_1, contrary to the assumption that K_1 and K_2 are mutually separated. Hence K^* is connected and the supposition that a subsequence $J_1^*, J_2^*, J_3^*, \cdots$ of the sequence $J_1, J_2, J_3, \cdots$ exists for which the set of all points X of K such that X is not interior to J_i^* for any value of i is not connected, has led to a contradiction.

Let K^{**} denote the set of all points X of K such that X is not interior to J_i for any value of i. Suppose K^{**} contains more than one point. If Q denotes

† See W. L. Ayres, *Concerning continuous curves and correspondences*, Annals of Mathematics, (2), vol. 28 (1927), Theorem 1.

‡ See R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, loc. cit.

a point of K^{**} which does not belong to the boundary B^* of the unbounded complementary domain D^* of M, let C_q denote a circle with center at Q, such that C_q encloses no point of D^*. By a theorem of G. T. Whyburn,† the set of all cut points X of M such that X lies on some simple closed curve of M is a countable set. But every cut point of M is on the boundary of some complementary domain‡ of M, and therefore either belongs to B^* or lies on or within J_i for some value of i. Hence, since K^{**} is connected and therefore contains uncountably many points within C_q, there are points of K^{**} interior to C_q which belong also to $K-K^{**}$, which is impossible. Therefore K^{**} is a subset of B^*.

Theorem I. *If B is the boundary and J the outer boundary of a bounded complementary domain D of a continuous curve M, and K is a connected set of cut points of M such that K contains a point in common with J, any point of K which belongs to $B-J$ can be joined to some point of J by an arc common to K and to B.*

Let P denote a point of K belonging to $B-J$, and let M_1 denote that component of $M-J$ which contains P. Let Q denote the limit point§ of M_1 belonging to J and let K^* denote the set $K\cdot\overline{M}_1$. Since K is connected, Q belongs to K. By Lemma I, K^* is connected. Since M_1+Q is a continuous curve, and $B(M_1+Q)$ is the boundary of the unbounded complementary domain of M_1+Q, by Lemma III the set $B\cdot K^*$ is connected. Hence, by a theorem of R. L. Wilder,‖ P can be joined to Q by an arc belonging to B and to K^*.

Theorem II. *If J is the outer boundary of a bounded complementary domain D of a continuous curve M, and K is a connected set of cut points of M containing a point in common with J, then any point of K within J can be joined to some point of J by an arc which is common to K and to a point set consisting of the boundaries of a finite number of complementary domains of M.*

Let P denote a point of K within J. By Lemma II, if G denotes the collection of all simple closed curves of M each of which encloses P and is the outer boundary of some complementary domain of M, then G is finite. Let J_1, J_2, $J_3, \cdots, J_n$ denote the curves of the collection G, where for each positive integer $i(i<n)$, J_i is a subset of the point set consisting of J_{i+1} together with

† *Concerning continua in the plane*, these Transactions, vol. 29 (1927), pp. 369–400, Theorem 29.

‡ See R. L. Moore, *Concerning the common boundary of two domains*, Fundamenta Mathematicae, vol. 6 (1924), pp. 203–213.

§ See R. L. Moore, *Concerning paths that do not separate a given continuous curve*, loc. cit., Theorem 6.

‖ *Concerning continuous curves*, Fundamenta Mathematicae, vol. 7, pp. 340–377, Theorem 20.

its interior. Suppose J_m is J. For each positive integer $k(k<m)$, let D_k denote that complementary domain of M whose outer boundary is J_k, let M_k denote that component of $M-J_k$ which contains P, and let Q_k denote the point of J_k which is a limit point of M_k. Suppose there exists a positive integer $q(q<m)$ such that Q_{q+1} does not belong to the boundary of D_q. Let K_{q+1} denote the set of all points X of K such that (a) X belongs to $K\cdot\overline{M}_{q+1}$, (b) the point X is not interior to the outer boundary of any complementary domain of M (except D_{q+1}) whose boundary contains a point of $K\cdot\overline{M}_{q+1}$. Since J_{q+1} is the outer boundary of D_{q+1}, every cut point of M which belongs to M_{q+1} is also a cut point of M_{q+1}. By Lemmas I and III together with the fact that $\overline{M}_{q+1}$ is a continuous curve, K_{q+1} is connected. But it contains Q_q and Q_{q+1} and is a subset of the boundary of D_{q+1}. By Wilder's theorem† there exists for each positive integer $i(i<m)$ an arc from Q_i to Q_{i+1} which is common to K and the boundary of D_{i+1}. Hence there exists an arc from P to a point of J which is common to K and a point set consisting of the boundaries of a finite number of complementary domains of M.

Theorem III. *If K is a connected set of cut points of a continuous curve M, and X and Y are two points of K, then there exists an arc from X to Y which is common to K and a point set consisting of the sum of the boundaries of a finite number of complementary domains of M.*

Suppose X belongs to the boundary B^* of the unbounded complementary domain‡ D^* of M. If Y belongs to B^* the proof of the theorem follows from Lemma III together with Wilder's theorem.§ Suppose Y does not belong to B^*. By Lemma II if G denotes the collection of all simple closed curves of M each of which encloses Y and is the outer boundary of some complementary domain of M, then G is finite. There exists a simple closed curve J^* of the collection G such that the point set consisting of the sum of all the simple closed curves of G contains no point exterior to J^*. By Theorem II the point Y can be joined to a point Y_1 of J^* by an arc YY_1, and by Lemma III together with Wilder's theorem‖ Y_1 can be joined to X by an arc Y_1X such that each of the arcs YY_1 and Y_1X is common to K and a point set consisting of the sum of the boundaries of a finite number of complementary domains of M. From the sum of the arcs YY_1 and Y_1X there exists an arc from X to Y which satisfies the conditions of Theorem III.

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† *Concerning continuous curves*, loc. cit.

‡ Every other case may be reduced to this one by an inversion.

§ *Concerning continuous curves*, loc. cit.

‖ Ibid.

Part II

Definition. An arc XY will be said to have property α with respect to a continuous curve M provided it satisfies the following conditions: (1) the arc XY contains no cut point of M, (2) the common part of arc XY and M is totally disconnected, (3) if the boundary B of a complementary domain D of M contains two points U and V in common with arc XY, then segment UV is a subset of D, (4) segment XY of the arc XY contains no point common to the boundaries B_1 and B_2 of two distinct complementary domains D_1 and D_2 of M which is also a boundary point of some complementary domain of the point set $D_1+D_2+B_1+B_2$.

It is clear that if XY is an arc having property α with respect to a continuous curve M, and YZ is an arc which contains only the point Y in common with M or with arc XY, and if XZ, the sum of the arcs XY and YZ, satisfies properties (3) and (4) of the preceding definition, then arc XZ has property α with respect to M.

Theorem IV. *If M is a bounded continuous curve, and K is a closed and totally disconnected subset of M whose omission leaves M connected and such that K belongs to the boundary B of a complementary domain D of M, then there exists an acyclic continuous curve whose end points are identical with the set K and which is a subset of $D+K$.*

Since B is a continuous curve,† by a theorem of R. L. Moore‡ there exists a continuum N containing K and which is a subset of $D+K$. Let N_1 denote the continuum formed by adding to N all the bounded complementary domains of N. Since K is closed and totally disconnected, it follows that there exists a circle C which is a subset of $D-N_1$. Let T denote an inversion about C, let M^* and $N_1{}^*$ denote the images of M and N_1 respectively under T, and let $M_1{}^*$ denote the continuum formed by adding to M^* all the bounded complementary domains of M^*. It is clear that (a) M_1^*-K is connected, (b) the set of points common to M_1^* and N_1^* is K, (c) neither M_1^* nor N_1^* separates the plane. Therefore by a theorem of R. L. Moore§ there exists a simple closed curve J which encloses M_1^*-K but encloses no point of N_1^*-K and contains K but no point $(N_1^*+M_1^*)-K$. Let P denote the center of C. If J does not contain P let J^* denote J, and if J contains P let J^* denote a

† See Marie Torharst, *Über den Rand der einfach zusammenhängenden ebenen Gebiete*, Mathematische Zeitschrift, vol. 9 (1921), pp. 45–65.

‡ *Some separation theorems*, Proceedings of the National Academy of Sciences, vol. 13 (1927), pp. 711–716, Theorem I.

§ *Concerning the separation of point sets by curves*, Proceedings of the National Academy of Sciences, vol. 11 (1925), pp. 469–476, Theorem 2.

simple closed curve which contains K but not P and is a subset of J plus a circle which encloses P but which neither contains nor encloses any point of $N_1^* + M_1^*$. H. M. Gehman† has shown that if EF is an arc containing a totally disconnected closed point set L, then L is identical with the end points of an acyclic continuous curve which contains only the set L in common with the arc EF. From this it follows that there exists an acyclic continuous curve W whose end points are identical with K and which is a subset of K plus the exterior of J^*. The image W^* of W under the inverse of T satisfies the conditions of Theorem IV.

THEOREM V. *If M is a bounded continuous curve, and K is a closed point set, and K^* is the set of all points X such that X belongs either to K or to a point set containing a point of K and consisting of a complementary domain of M together with its boundary, then K^* is closed.*

Suppose P is a limit point of K^* which does not belong to K^*. The point P belongs to M and since P does not belong to K^* it is not a boundary point of any complementary domain D of M such that D is a subset of K^*. Since P does not belong to K, there exists a circle C with center at P such that C neither contains nor encloses any point of K. Let C^* denote a circle with center at P and of diameter one-half that of C. It follows that there are infinitely many complementary domains of M each of which contains a point exterior to C and a point interior to C^*. This is impossible since the complementary domains of M form a contracting sequence.‡ Hence the suppositon that P is a limit point of K^* which does not belong to K^* has led to a contradiction. Therefore K^* is closed.

THEOREM VI. *Suppose K is a closed and totally disconnected point set, and M is a closed and bounded point set containing K such that* (a) *the sum of all the components of M which are not single points can be expressed as the sum of a countable number of continuous curves $C_1, C_2, C_3, \cdots$, not more than a finite number of which are of diameter greater than a positive number,* (b) *the set of all points X such that X belongs to at least two curves of the sequence $C_1, C_2, C_3, \cdots$ is a subset of K. Then each component of M is a continuous curve and not more than a finite number of components of M are of diameter greater than a positive number.*

From the fact that a continuum which is not a continuous curve fails to be connected im kleinen at a continuum of points,‡ and the fact that K is

† *Concerning acyclic continuous curves*, these Transactions, vol. 29 (1927), pp. 553–568, Theorem 6.

‡ See R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, loc. cit.

closed and totally disconnected, it follows that each component of M is a continuous curve. Suppose there exists a positive number e and an infinite sequence $M_1, M_2, M_3, \cdots$ of components of M each of which is of diameter greater than e. There exists a subcontinuum M^* of M and a subsequence $M_1^*, M_2^*, M_3^*, \cdots$ of the sequence $M_1, M_2, M_3, \cdots$ having M^* as sequential limiting set.† There exists a point P belonging to M^*-K, and a positive integer n, such that, if C and C^* are circles with centers at P and of diameters e/n and $e/(2n)$ respectively, then neither C nor C^* contains or encloses any point of K. Since there are infinitely many components of M each containing a point exterior to C and a point interior to C^*, there are infinitely many curves of the sequence $C_1, C_2, C_3, \cdots$ each of which is of diameter greater than $e/(4n)$ contrary to hypothesis. Hence the supposition that there exists a positive number e and an infinite sequence $M_1, M_2, M_3, \cdots$ of components of M each of which is of diameter greater than e has led to a contradiction.

THEOREM VII. *If M is a bounded continuous curve which contains no domain, and T is an acyclic continuous curve having the properties* (a) *T contains no cut point of M,* (b) *the set of all points common to T and M is totally disconnected,* (c) *if D is any complementary domain of M, any two points of the set $T\cdot\overline{D}$ can be joined by an arc of T which except for end points is a subset of D, then $M-T$ is connected.*

Let D denote a complementary domain of the continuous curve $M+T$ whose boundary B contains a point of T. Let D_M denote the complementary domain of M which contains D, and let B_M denote its boundary. Let M_1 denote the set $B-T$. Suppose M_1 is not connected. If T contains only one point A in common with B_M the point A is a cut point of B_M, and by a theorem of R. L. Moore‡ A is a cut point of M contrary to hypothesis. Hence there exists by property (c) a complementary domain D_1 of $M+T$ distinct from D such that D_1 is a subset of D_M. Let J denote the outer boundary of D with respect to D_1. By a theorem of R. L. Moore§ J is a simple closed curve. The curve J contains a point of $T-M$ and a point of $M-T$. Suppose there exists a component L of M_1 which contains no point of J. By a theorem of R. L. Moore|| the component L^* of $B-J$ which contains L has exactly one limit point Q in J. Since T contains no cut point of M the point Q does not belong to $T\cdot M$. If Q belongs to T let Q_1 denote a point of L and let Q_2 de-

† See R. G. Lubben, *Concerning limiting sets in abstract spaces*, these Transactions, vol. 30, pp. 668–685.

‡ *Concerning the common boundary of two domains*, loc. cit.

§ *Concerning continuous curves in the plane*, Mathematische Zeitschrift, vol. 15 (1922).

|| *Concerning paths that do not separate a given continuous curve*, loc. cit., Theorem 6.

note a point of $M_1 \cdot J$. There exists an arc† of M from Q_1 to Q_2. If Q belongs to M_1, let Q_1 denote a point of T belonging to L^* and let Q_2 denote a point of T belonging to J. There exists an arc of T from Q_1 to Q_2. In either case the supposition that there exists a component of M_1 containing no point of J contradicts the fact that J is the outer boundary of D with respect to D_1. Hence there exist arcs a_1 with end points A_1 and A_2 and b_1 with end points B_1 and B_2 such that arcs a_1 and b_1 belong to different components of $J \cdot M_1$. Let s_1 and s_2 denote the two arcs of J from A_1 to B_1. Both s_1 and s_2 contain points of T. There exists an arc‡ t of T from a point in s_1 to a point in s_2. Let t_1 denote a subarc of t having one end point C_1 in s_1 and one end point C_2 in s_2 and containing only the points C_1 and C_2 in common with J. Let t_2 denote an arc with C_1 and C_2 as end points and which except for C_1 and C_2 is a subset of D. Let J_1 denote the simple closed curve formed from the sum of the arcs t_1 and t_2. The curves J and J_1 have only the points C_1 and C_2 in common. Since the segment t_2 is a subset of D and the segment t_1 contains no point in common with D, the point set $J-J_1$ is not a subset of a single complementary domain of J_1. Hence a_1 and a_2 belong to different complementary domains of J_1. Of the two points C_1 and C_2, if one belongs to $T-M$ the other is a cut point of M contrary to hypothesis. If C_1 and C_2 both belong to M, by property (c) they do not both belong to the boundary of any complementary domain of M except D_M. Hence $J-(C_1+C_2)$ is a subset of D_M. There exists an arc t_3 in D_M having one end point in t_1 and one end point in t_2 and which, except for end points, contains no point in common with J_1. The point set J_1+t_3 contains a simple closed curve enclosing a point of M and having in common with the set C_1+C_2 exactly one point. This is impossible since neither C_1 nor C_2 is a cut point of M. Thus the supposition that M_1 is not connected has led to a contradiction.

Suppose X and Y are points belonging to different components of $M-T$. There exists an arc XY which contains no point of T. Let Y_1 denote the first point in the order from X to Y which XY has in common with that component M_2 of $M-T$ which contains Y. Since M is a continuous curve there exists a first point X_1 in the order from Y_1 to X which the interval Y_1X of XY has in common with $M-M_2$. Hence X_1 and Y_1 belong to the boundary of the same complementary domain of $M+T$ but to different components of $M-T$, which is impossible. Therefore the supposition that X and Y belong to different components of $M-T$ has led to a contradiction and Theorem VII is established.

† See R. L. Moore, *A theorem concerning continuous curves*, Bulletin of the American Mathematical Society, vol. 23 (1917).

‡ Ibid.

THEOREM VIII. *Suppose* (a) *M is a continuous curve which contains no domain,* (b) *A, B and C are three distinct points,* (c) *a_1 and a_2 are two arcs with end points A, B and B, C respectively such that a_1 and a_2 each have property α with respect to M,* (d) *B does not belong to M. Then there exists an arc from A to C which is a subset of a point set consisting of the sum of the arcs a_1 and a_2 together with a single complementary domain of M, such that arc AC has property α with respect to M.*

The case where A and C both belong to a point set consisting of a single complementary domain of M together with its boundary, or where A belongs to arc a_2, is trivial. If A belongs to a complementary domain D of M let H denote $\bar{D}$. If A belongs to M let G_1 denote the collection of all complementary domains of M such that A belongs to the boundary of each domain of the collection G_1, and let H denote the point set consisting of the sum of all the domains of the collection G_1 together with their boundaries. Since the complementary domains of M form a contracting sequence,† H is closed. If a_2 contains a point of H let E denote the first such point in the order from C to B. Then E belongs to the boundary B_1 of a domain D_1 of the collection G_1. There exists an arc from A to E which, except for end points, is a subset of D_1. Arc AE plus the interval EC of a_2 is an arc from A to C which satisfies the conditions of the theorem. If a_2 contains no point of H, let G_2 denote the collection of all the complementary domains of M such that each domain of the collection G_2 either contains a point of a_2 or has a point of a_2 on its boundary. Let K denote the point set consisting of the sum of all the domains of the collection G_2 together with their boundaries. By Theorem V the set K is closed. Since a_2 contains no point in common with H, the point A does not belong to K. Let F denote the first point in the order from A to B which a_1 has in common with K. If F belongs to a_2 then the sum of the intervals AF and FC of a_1 and a_2 respectively is an arc from A to C satisfying the conditions of the theorem. If F does not belong to a_2 it belongs to the boundary B_2 of a domain D_2 of the collection G_2. Let V denote the first point in the order from C to B which a_2 has in common with B_2+D_2. There exists an arc from F to V which except for end points is a subset of D_2. The sum of the intervals AF and VC of a_1 and a_2 respectively together with arc FV is an arc from A to C satisfying the conditions of the theorem and the proof is complete.

THEOREM IX. *If P is not a cut point of a continuous curve M and e is a positive number, there exists a circle C enclosing P and of diameter less than e such that the set of all points of M, each at a distance from P greater than e, lie in a connected subset of $M-C$.*

† See R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, loc. cit.

Let M_1 denote the set of all points of M each at a distance from P greater than or equal to e. Let G denote a collection of connected open subsets of M no one of which contains P but such that if X is any point of M_1 then X belongs to some subset g_X of the collection G. There exists a finite subcollection G_1 of G which contains all the points of M_1. Since $M-P$ is connected and any two points belonging to a connected open subset of a continuous curve can be joined by an arc† lying wholly in the open subset it follows that there exists a finite set A of arcs, each lying within $M-P$, and such that the point set M_2 consisting of M_1 plus the arcs of the set A is connected. Since M_2 does not contain P there exists a circle enclosing P which neither contains nor encloses any point of M_2 and the theorem is established.

THEOREM X. *If P is not a cut point of a bounded continuous curve M which contains no domain and e is a positive number, there exists a positive number d_e such that if A and B are any two points, each at a distance from P less than d_e, and $M-(A+B)$ is connected, then A and B are the extremities of an arc which has property α with respect to M and which is of diameter less than e.*

By Theorem IX, if e is a positive number there exists a positive number d_e such that the set of all points of M each at a distance from P greater than e lie in a connected subset M_1 of M such that no point of M_1 is at a distance from P less than d_e. Let C and C_1 denote circles with P as center and of diameters e and d_e respectively. Let I and I_1 denote the interiors of C and C_1 respectively, and let M_1^* denote that component of $M-I_1$ which contains M_1. Let S denote the set of all points, let S_1 denote $S-I$, and let S_2 denote the set of all points X such that X belongs to a complementary domain of $M+C$ whose boundary is a subset of M_1^*+C. Let K denote the set $M_1^*+S_1+S_2$ and let D denote the complementary domain of K. Clearly P is in D. Suppose P_1 and P_2 are two points of D which belong to a complementary domain D_1 of M but which can not be joined by an arc lying in $I\cdot D_1$. Let a_1 denote an arc from P_1 to P_2 which is a subset of D and let a_2 denote an arc from P_1 to P_2 which is a subset of D_1. If a_2 is a subset of $C+I$ let a_n denote a_2. If a_2 is not a subset of $C+I$ let E_1 denote the first point in the order from P_1 to P_2 which a_2 has in common with a_1 such that interval P_1E_1 of a_2 contains a point exterior to C and let F_1 denote the last point in the order from P_1 to E_1 which $P_1E_1-E_1$ has in common with a_1. Let J_1 denote the simple closed curve which is the sum of the intervals E_1F_1 of a_1 and E_1F_1 of a_2. If M_1^* is interior to J_1 there exists a complementary domain of J_1+C common to I and the exterior of J_1 such that its boundary J_1^* contains the interval E_1F_1 of a_1. If M_1^* is a subset of the exterior of J_1 there exists a complementary domain of J_1+C common

† See R. L. Moore, *Concerning continuous curves in the plane*, loc. cit., Theorem 1.

to I and the interior of J_1 such that its boundary J_1^* contains the interval E_1F_1 of a_1. In either case let t_1 denote the arc from P_1 to E_1 which is the sum of the interval P_1F_1 of a_2 and the arc E_1F_1 of J_1^* which contains a point of C. Let W_1 denote the last point in the order from P_1 to P_2 which a_2 has in common with t_1. Let a_3 denote the sum of the intervals P_1W_1 and W_1P_2 of t_1 and a_2 respectively. Arc a_3 contains no point of M. If a_3 is a subset of $C+I$ let a_n denote a_3. If a_3 is not a subset of $C+I$ let E_2 denote the first point in the order from P_1 to P_2 which a_3 has in common with a_1 such that interval P_1E_2 of a_3 contains a point exterior to C, and let F_2 denote the last point in the order from P_1 to E_2 which $P_1E_2-E_2$ has in common with a_1. Let J_2 denote the simple closed curve which is the sum of the intervals E_2F_2 of a_1 and E_2F_2 of a_3. If M_1^* is interior to J_2 there exists a complementary domain of J_2-C common to I and the exterior of J_2 such that its boundary J_2^* contains the interval E_2F_2 of a_1. If M_1^* is exterior to J_2 there exists a complementary domain of J_2+C common to I and the interior of J_2 such that its boundary J_2^* contains the interval E_2F_2 of a_1. In either case let t_2 denote the arc from P_1 to E_2 which is the sum of the interval P_1F_2 of a_3 and the arc E_2F_2 of J_2^* which contains a point of C. Let W_2 denote the last point in the order from P_1 to P_2 which a_3 has in common with t_2. Let a_4 denote the sum of the intervals P_1W_2 and W_2P_2 of t_2 and a_3 respectively. Arc a_4 contains no point of M. If a_4 is a subset of $C+I$ let a_n denote a_4. If a_4 contains a point exterior to C it is clear that after a finite number of operations just described one may obtain an arc a_n from P_1 to P_2 which is a subset of $C+I$ and which contains no point of M. It follows that there exists an arc a_m from P_1 to P_2 which is a subset of $I\cdot D_1$ contrary to the assumption that P_1 and P_2 are points of D_1 which can not be joined by an arc of $I\cdot D_1$.

If K is designated by a point and each point of $S-K$ is designated by a point, the set of elements thus obtained is an upper semi-continuous collection† of elements filling up the plane and is in one-to-one continuous correspondence T with the surface H of a sphere. For each point set Q in S let $T(Q)$ designate its image under T. Since any two points of D belonging to a complementary domain D_i of M can be joined by an arc of $I\cdot D_i$, $T(D\cdot D_i)$

† See R. L. Moore, *Concerning upper semi-continuous collections of continua*, these Transactions, vol. 27 (1925), pp. 416–428. A collection G of continua is said to be an *upper semi-continuous* collection if for each element g of the collection G and each positive number e there exists a positive number d such that if x is any element of G at a lower distance from g less than d then the upper distance of x from g is less than e. If M is a point set and P is a point, then by $l(PM)$ is meant the lower bound of the distance from P to all the different points of M. If M and N are two point sets, then by $l(MN)$ is meant the lower bound of the values $l(PN)$ for all points P of M, while by $u(MN)$ is meant the upper bound of these values for all points P of M. The point set M is said to be at the upper distance $u(MN)$ from the point set N and is said to be at the lower distance $l(MN)$ from N.

is a complementary domain of $T(K+M)$. If AB is an arc in D such that $T(AB)$ has property α with respect to $T(K+M)$, then AB has property α with respect to M. By a slight modification of a theorem by R. L. Moore† it may easily be seen that any two points belonging to $H-T(K)$ whose omission leaves $T(K+M)$ connected may be joined by an arc lying in $H-T(K)$ and which has property α with respect to M. Therefore Theorem X is established.

THEOREM XI. *If M is a bounded continuous curve which contains no domain, and A and B are two points such that $M-(A+B)$ is connected, there exists an arc from A to B which has property α with respect to M.*

If neither A nor B belongs to M then R. L. Moore‡ has shown how to construct an arc from A to B which does not disconnect M. From the nature of his construction, this arc has property α with respect to M. If M contains no domain and $M-(A+B)$ is connected it follows from a slight modification of his construction that there exists an arc from A to B which has property α with respect to M.

THEOREM XII. *If M is a bounded continuous curve which contains no domain and K is a totally disconnected closed subset of M, there exists an acyclic continuous curve T containing K, and such that* (1) *all the end points of T belong to K, and* (2) *the point set $M\cdot T$ is totally disconnected and $M-T$ is connected.*

By Theorem IV, for each complementary domain D_i of M whose boundary δ_i contains two or more points of K, there exists an acyclic continuous curve C_i whose end points are identical with the points of the set $K\cdot\delta_i$ and lying except for end points wholly within D_i. Let K_1 denote the point set $K+C_1+C_2+C_3+\cdots$. That K_1 is closed may be proved with the use of the fact that the complementary domains of M form a contracting sequence.§ By Theorem VI each component of K_1 is a continuous curve and not more than a finite number of components of K_1 are of diameter greater than a positive number. Suppose K_1 contains a simple closed curve J. Since $J\cdot M=J\cdot K$ and $M-K$ is connected, then $M-J$ is a subset of one of the complementary domains of J and hence J is a subset of a point set consisting of a single complementary domain of M plus its boundary which is impossible. Hence K_1 contains no simple closed curve.

By Theorems IX and X together with the fact that a closed and bounded

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† *Concerning paths that do not separate a given continuous curve*, loc. cit.

‡ Ibid.

§ See R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, loc. cit.

point set has the Borel-Lebesgue property, there exists a sequence of sets of circular regions $G_1, G_2, G_3, \cdots$ having the properties that for each positive integer i, (1)G_i is finite and covers K and each region of G_i contains at least one point of K, (2) each region of the set G_i is of diameter less than $1/i$, (3) if g is a region of the set $G_i(i>1)$, $\bar{g}$ lies interior to some region of the set G_{i-1} and has the property that if X_1 and X_2 are any two points of g whose omission does not disconnect M then there exists an arc from X_1 to X_2 having property α with respect to M and lying interior to each region of the set G_{i-1} which contains $\bar{g}$.

For each positive integer i let N_{1i} denote the point set obtained by adding to K_1 all the regions of the set G_i together with their boundaries. If for all values of i the set N_{1i} is connected, let T denote the set of points common to the sets $N_{11}, N_{12}, N_{13}, \cdots$. If for some value of i the set N_{1i} is not connected let m denote the smallest positive integer such that N_{1m} is not connected. The point set N_{1m} has only a finite number n of components. Let L_1 denote one of them. If m is greater than 3 there exists a component L_2 of N_{1m} distinct from L_1 and two regions g_s and g_t of the set G_{m-2} having in common a point P not belonging to M such that g_s contains a point X_1 of the set $L_1 \cdot K_1$ and g_t contains a point X_2 of the set $L_2 \cdot K_1$. By property (3) of the sets of circular regions, there exist arcs PX_1 and PX_2 each having property α with respect to M and each lying in a single region of the set G_{m-3}. By Theorem VIII there exists an arc X_1X_2 having property α with respect to M and which is a subset of a point set consisting of the sum of the arcs PX_1 and PX_2 together with a single complementary domain of M. For $m=1, 2, 3$ there exists an arc X_1X_2 from a point X_1 of $L_1 \cdot K_1$ to a point X_2 of $K_1 - L_1$ which has property α with respect to M.

If k is a component of K_1 then for each point Z of k add to k all points W such that W lies within or on the boundary of some complementary domain of M which contains Z or has Z on its boundary. Let H_k denote the set thus obtained and let H denote the point set obtained by adding together all points of all the sets H_k for all components k of K_1. Since each point of H at a distance from every point of K greater than a positive number e lies either within or on the boundary of a complementary domain of M of diameter greater than e it follows that H is closed. From the fact that K is closed and totally disconnected and the fact that every infinite set of point sets whose sum is bounded has a limiting set,† it follows that the nondegenerate‡ sets H_k form a contracting sequence.

Let H^* denote the point set obtained by adding together all points of all

† See R. G. Lubben, *Concerning limiting sets in abstract spaces*, loc. cit.

‡ A point set containing but a single point is said to be degenerate.

the sets H_k for all components k of K_1 such that k is a subset of L_1. Since the components of K_1 and the complementary domains of M form contracting sequences and each component of K_1 contains a point of K, therefore H^* is closed. The point set $H^* \cdot K_1$ is a subset of L_1. Let E_1 denote the last point which the arc X_1X_2 has in common with H^*, and let E_2 denote the first point which the interval E_1X_2 of X_1X_2 has in common with the set $\overline{H-H^*}$. If $E_1=E_2$ then E_1 does not belong to K_1. For suppose it does. Then since E_1 belongs to M it belongs also to K. Hence E_1 does not belong to any set H_k for any component k of K_1 such that K is a subset of K_1-L_1. Since E_1 belongs to K it is interior to some region of the set G_m belonging to L_1 and there exists a positive number e such that the distance from E_1 to any point of K_1-L_1 is greater than e. Therefore since E_1 belongs to $\overline{H-H^*}$ there are infinitely many sets H_k, each of diameter greater than $e/2$, which is impossible. Hence the supposition that $E_1=E_2$ and E_1 belongs to K_1 has led to a contradiction. Thus if $E_1=E_2$, then E_1 belongs to the boundaries of two distinct complementary domains D_1 and D_2 of M such that the boundary of D_1 contains a point of K belonging to L_1 and the boundary of D_2 contains a point of K belonging to the set $N_{1m}-L_1$. There exists in $\overline{D}_1$ an arc Y_1E_1 and in $\overline{D}_2$ an arc Z_1E_2 such that (a) the point Y_1 belongs to K_1 and also to $K_1 \cdot D_1$ if D_1 contains a point of K_1, (b) the point Z_1 belongs to K_1 and also to $K_1 \cdot D_2$ if D_2 contains a point of K_1, (c) segments Y_1E_1 and Z_1E_2 have no point in common with K_1 or with M. If $E_1 \neq E_2$ and E_1 belongs to K_1, let Y_1 denote E_1. If $E_1 \neq E_2$ and E_1 does not belong to K_1, it belongs to the boundary B_1 of a complementary domain D_1 of M such that B_1 contains a point of $K_1 \cdot L_1$. There exists an arc Y_1E_1 such that (a) the point Y_1 belongs to K_1 and also to K_1-K if D_1 contains a point of K_1, (b) segment Y_1E_1 contains no point in common with K_1 or with M. If $E_1 \neq E_2$ and E_2 belongs to K_1, let Z_1 denote E_2. If $E_1 \neq E_2$ and E_2 does not belong to K_1, it belongs to the boundary B_2 of a complementary domain D_2 of M such that B_2 contains a point of K_1-L_1. There exists an arc Z_1E_2 such that (a) the point Z_1 belongs to K_1 and also to K_1-K if D_2 contains a point of K_1, (b) segment Z_1E_2 contains no point in common with K_1 or with M. Let a_{11} denote the arc which is the sum of the arcs Y_1E_1 and Z_1E_2 and the interval E_1E_2 of the arc X_1X_2. Then (1) the arc a_{11} has property α with respect to M, (2) for m greater than 3 the set of points common to arc a_{11} and M which does not belong to K_1 is a subset of a point set consisting of the sum of two regions of the set G_{m-3}, (3) the end points Y_1 and Z_1 of arc a_{11}, which are the only points common to a_{11} and to K_1, belong to different components of N_{1m}. The number of components of $N_{1m}+a_{11}$ is at least one less than the number of components of N_{1m}. If $N_{1m}+a_{11}$ is not connected, by treating the sets K_1+a_{11} and $N_{1m}+a_{11}$ in the same manner as

that in which the sets K_1 and N_{1m} respectively were treated one may obtain an arc a_{12} which satisfies with respect to M, G_{m-3}, K_1+a_{11}, and $N_{1m}+a_{11}$, the same properties that arc a_{11} satisfies with respect to M, G_{m-3}, K_1 and N_{1m} respectively. The number of components of the set $N_{1m}+a_{11}+a_{12}$ is at least two less than the number of components of the set N_{1m}. It is clear that by the addition to K_1 of a finite set A_1 of arcs $a_{11}, a_{12}, a_{13}, \cdots, a_{1k}$ $(k<n)$, each arc a_{1j} of the set A_1 satisfying with respect to M, G_{m-3}, $K_1+\sum_{i=1}^{j-1}a_{1i}$ and $N_{1m}+\sum_{i=1}^{j-1}a_{1i}$ the same properties that arc $a_{i(j-1)}$ satisfies with respect to M, G_{m-3}, $K_1+\sum_{i=1}^{j-2}a_{1i}$ and $N_{1m}+\sum_{i=1}^{j-2}a_{1i}$ respectively, one may obtain a point set K_2 which has the following properties: (1) K_2 is closed and contains no simple closed curve, (2) if P and Q are any two points of K_2 belonging to the boundary of a complementary domain D of M then P and Q are the extremities of an arc belonging to K_2 such that segment PQ is a subset of D, (3) the set K_2+N_{1m} is connected, (4) the set $M\cdot K_2$ is totally disconnected, (5) each component of K_2 is an acyclic continuous curve, and (6) K_2 contains no cut point of M. That K_2 has properties (1) and (2) is clear from the facts (a) A_1 is finite, (b) K_1+a_{11} has properties (1) and (2), (c) if $K_1+\sum_{i=1}^{j-1}a_{1i}(j<k-1)$ has properties (1) and (2) then $K_1+\sum_{i=1}^{j}a_{1i}$ has them also. The point set K_2 has property (3) since the number of components of N_{1m} is n and the number of components of $N_{1m}+\sum_{i=1}^{r}a_{1i}$ $(r<k-1)$ is at least one less than the number of components of $N_{1m}+\sum_{i=1}^{r-1}a_{1i}$. The set of points common to M and the set A_1 of arcs is closed and totally disconnected, hence K_2 has property (4). By Theorem VI the set K_2 has property (5), and since no arc of the set A_1 contains a cut point of M the set K_2 has the property (6). Add K_2 to each of the sets $N_{11}, N_{12}, N_{13}, \cdots$ and denote the resulting sets by $N_{21}, N_{22}, N_{23}, \cdots$ respectively. If for each value of i, N_{2i} is connected, let T denote the set of all points common to the sets $N_{21}, N_{22}, N_{23}, \cdots$. If, for some value of i, N_{2i} is not connected, let n denote the smallest positive integer such that N_{2n} is not connected. Let r denote the number of components of N_{2n}. By the addition to K_2 of a finite set A_2 of arcs $a_{21}, a_{22}, a_{23}, \cdots, a_{2j}(j<r)$, each arc a_{2k} of the set A_2 satisfying with respect to M, G_{n-3}, $K_2+\sum_{i=1}^{k-1}a_{2i}$ and $N_{2n}+\sum_{i=1}^{k-1}a_{2i}$ $(k<r)$ the same properties that arc a_{11} satisfies with respect to M, G_{m-3}, K_1 and N_{1m} respectively, one may obtain a point set K_3 which satisfies the same properties with respect to M, G_{n-3}, N_{2n} that K_2 satisfies with respect to M, G_{m-3} and N_{1m} respectively. Suppose the process is continued indefinitely. Let K^* denote the set $K_1+K_2+K_3+\cdots$. The set K^* is connected and the set $M\cdot K^*$ is closed. For suppose W denotes a limit point of $M\cdot K^*$ which does not belong to $M\cdot K^*$. The point W does not belong to K, and since for any positive number e there exists a positive integer t such that no arc of the set $A_j(j>t)$ contains a point of $M-K_{t-1}$ at a distance from every point of K

greater than e, it follows that W belongs to K_{t-1} and hence to $M \cdot K^*$ contrary to hypothesis. Therefore $M \cdot K^*$ is closed. Since for each positive integer i the set K_i contains in common with M only a totally disconnected set of points, the point set $M \cdot K^*$ is totally disconnected. From the fact that each arc of the set A_i for all values of i has property α with respect to M and the fact that $M-K$ is connected it follows that $M \cdot K^*$ contains no cut point of M.

If β_i is the boundary of a complementary domain D_i of M and the set of points $\overline{D}_i \cdot K^*$ is not an acyclic continuous curve whose end points are identical with the set $\beta_i \cdot K^*$, there exists by Theorem IV an acyclic continuous curve C_i whose end points are identical with the set $\beta_i \cdot K^*$ and which, except for end points, is a subset of D_i. Let T_1 denote the set formed from K^* by replacing the set $D_i \cdot K^*$ by C_i for each value of i. The point set T_1 is closed and connected and has the property that if X and Y are any two points of T_1 belonging to the boundary of a complementary domain D of M then X and Y are the extremities of an arc belonging to T_1 such that segment XY is a subset of D. That T_1 is a continuous curve may be seen from the fact that any point of T_1 not belonging to the totally disconnected set $M \cdot T_1$ lies in some complementary domain of M, together with the fact that a continuous curve can not fail to be connected im kleinen at only a totally disconnected set of points.†

Suppose there exists an end point P of T_1 which does not belong to K. Then P belongs to M and hence to K^* and there exists a positive integer r such that P belongs to K_r. By a theorem of G. T. Whyburn‡ and the fact that each nondegenerate component of K_r is an acyclic continuous curve, all of whose end points belong to K, it follows that there exists an arc c_r lying in $K_r - K$ whose end points belong to M and such that c_r contains P as an interior point. For each segment c_i of $c_r - M$ whose end points belong to M let b_i denote a segment of $T_1 - M$ such that the end points of b_i are the end points of c_i. The point set consisting of the sum of the segments b_i together with the set $M \cdot c_r$ is an arc lying in T_1 and containing P as interior point which contradicts the assumption that P is an end point of T_1. Hence the end points of T_1 all belong to K.

Suppose J is a simple closed curve lying in T_1. The curve J contains a point of $M-K$, otherwise J is a subset of a point set consisting of a single complementary domain D of M together with its boundary, contrary to the fact that $\overline{D} \cdot T_1$ contains no simple closed curve. The curve J contains a point of K. For suppose it does not. There exists a positive integer s such that $M \cdot J$

† See R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, loc. cit.

‡ *Concerning continua in the plane*, loc. cit.

is a subset of $M \cdot K_s$. Since J separates M there exists a point of $M-K_s$ interior to J and a point of $M-K_s$ exterior to J. For each segment c_i of $J-M$ whose end points belong to M let b_i denote a segment lying in K_s-M and such that the end points of b_i are the end points of c_i. The point set consisting of the sum of the segments b_i together with the set $M \cdot J$ is a connected set of points which is a subset of a single component k of K_s. Hence k separates M. Since K_s contains no cut point of M and has the properties (1) K_s contains no simple closed curve, (2) $M \cdot K_s$ is totally disconnected, (3) any two points of K_s belonging to the boundary B of a complementary domain D of M are the end points of a segment lying in $D \cdot K_s$, therefore any component of K_s is an acyclic continuous curve which satisfies with respect to M the same properties that T satisfies with respect to M in Theorem VII. Hence k does not separate M and the supposition that J contains no point of K has led to a contradiction.

For each positive integer i let S_i denote the sum of the regions of the set G_i together with their boundaries. That the set of all junction points† of T_1 not belonging to S_i is finite may be shown as follows. Since T_1 is a continuous curve, for $j>i$ there are not more than a finite number of components of T_1-S_j each containing a point of T_1-S_i. Since all the end points of T_1 belong to K and any simple closed curve of T_1 contains a point of K, then each component of T_1-S_j together with its limit point, which contains a point of T_1-S_i, is an acyclic continuous curve whose end points, finite in number, all belong to S_i. Let L denote the set of all junction points of T_1. For each positive integer i there are only a finite number of components of the set $T_1-(L+S_{i+1})$ each of which contains a point of T_1-S_i. Suppose there exist a positive integer m and a component r of $T_1-(L+S_{m+1})$ such that for each positive integer n there exists a simple closed curve of T_1 which contains r but which contains no point of $M-S_n$. Since each simple closed curve of T_1 contains a point of $M-K$ it follows that there exists a sequence $J_1, J_2, J_3, \cdots$ of simple closed curves of T_1 and a subsequence $S_1^*, S_2^*, S_3^*, \cdots$ of the sequence $S_1, S_2, S_3, \cdots$ such that for each i, J_i contains r and at least one point of $M-S_{i+1}^*$ but contains no point of $M-S_i^*$. Let $J_1^*, J_2^*, J_3^*, \cdots$ denote a subsequence of the sequence $J_1, J_2, J_3, \cdots$ having a sequential limiting set N^*. Since for each i the curve J_i^* contains r the set N^* separates the plane. Suppose N^* contains

† See R. L. Moore, *Concerning triods in the plane and the junction points of plane continua*, Proceedings of the National Academy of Sciences, vol. 14 (1928). If P is a point of a continuous curve N and K is a domain containing P such that P is a cut point of the component of $N \cdot K$ which contains P, and furthermore there exist three arcs PA_1, PA_2, and PA_3 which lie in N and have only the point P in common, then P is said to be a *junction point* of N. The continuum $PA_1+PA_2+PA_3$ is called a *triod* and the point P is its *emanation point*.

a point Q belonging to $M-K$. There exists a positive integer m such that Q does not belong to S_m. It follows that Q belongs to all but a finite number of the simple closed curves of the sequence $J_1, J_2, J_3, \cdots$. This is impossible, hence N^* contains no point of $M-K$. Since N^* is connected im kleinen at every point not belonging to K, then N^* is a continuous curve.† Therefore by a theorem of R. L. Moore,‡ N^* contains a simple closed curve. This is impossible since N^* contains no point of $M-K$ and the supposition that for each positive integer n there exists a simple closed curve of T_1 which contains r but which contains no point of $M-S_n$ has led to a contradiction. Hence if r is a component of $T_1-(L+S_m)$ there exists a positive integer n such that every simple closed curve of T_1 which contains r contains a point of $M-S_n$. It follows then that there exists a subsequence $S_1^{**}, S_2^{**}, S_3^{**}, \cdots$ of the sequence $S_1, S_2, S_3, \cdots$ such that for each positive integer i, if J is a simple closed curve of T_1 which is not a subset of S_i, then J contains a point of $M-S_i^{**}$. Suppose there exists a component of $T_1-(L+S_2)$ which is not a subset of S_1 but which is a subset of a simple closed curve J lying in T_1. Let P denote a point of J belonging to $M-S_1^{**}$. If P is not a junction point of T_1 there exists a component t of $J-(L+K)$ containing P. If P is a junction point of T_1 it is a limit point of the set $M\cdot J$. For suppose there exists a segment s_1 of $(J-M)+P$ which contains P. Since P does not belong to K there exists a positive integer e such that P belongs to K_e but not to K_{e-1}. Thus P is an interior point of some arc of the set A_{e-1}. Since for any complementary domain D of M the set $\overline{D}\cdot T_1$ is an acyclic continuous curve whose end points are identical with the set $T_1(\overline{D}-D)$ it follows that the two components of s_1-P belong to different complementary domains D_1 and D_2 of M and that P is both a junction point of T_1 and a boundary point of a complementary domain of the point set $\overline{D}_1+\overline{D}_2$. This is impossible since P is interior to some arc of the set A_{e-1}, each arc of which has property α with respect to M and the supposition that there exists a segment of $(J-M)+P$ which contains P has led to a contradiction. It follows then that since $L-S_1^{**}$ is a finite set of points there exists a point Q belonging to $M\cdot J-S_1^{**}$ which is not a junction point of T_1. Let t denote the component of $J-(L+K)$ which contains Q. The component t is either a segment of the curve J or the curve minus a single point. Subtract t from T_1. Let T_1^* denote the point set $T_1-\sum_{i=1}^{i_1}t_i$ where for each positive integer $k(k\leqq i_1)$, (1) t_k is either a segment of a simple closed curve or a simple closed curve minus a point such that the curve is a subset of $T_1-\sum_{i=1}^{k-1}t_i$ and contains a component of the set $T_1-(L+S_2)$, (2) t_k

† See R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, loc. cit.

‡ *Concerning continuous curves in the plane*, loc. cit.

contains a point of $M-S_1^{**}$, (3) if t_k is a segment its end points belong to the point set consisting of K together with the junction points of $T_1-\sum_{i=1}^{k-1}t_i$ and if t_k is not a segment then t_k plus a single point of K is a simple closed curve, (4) t_k contains no point of K nor any junction point of $T_1-\sum_{i=1}^{k-1}t_i$, and further such that the point set T_1^* contains no simple closed curve which is not a subset of S_1. In general let T_j^* denote the point set $T_{j-1}^*-\sum_{i=1}^{i_j}t_i$, where for each positive integer $k(k\leqq i_j)$, (1) t_k is either a segment of a simple closed curve or a simple closed curve minus a point such that the curve is a subset of $T_{j-1}^*-\sum_{i=1}^{k-1}t_i$ and contains a component of the set $T_1-(L+S_{j+1})$, (2) t_k contains a point of $M-S_j^{**}$, (3) if t_k is a segment its end points belong to the point set consisting of K together with the junction points of $T_{j-1}^*-\sum_{i=1}^{k-1}t_i$ and if t_k is not a segment then t_k plus a single point of K is a simple closed curve, (4) t_k contains no point of K nor any junction point of $T_{j-1}^*-\sum_{i=1}^{k-1}t_i$, and further such that the point set T_j^* contains no simple closed curve which is not a subset of S_j.

Let T denote the set of points common to the sets T_1^*, T_2^*, T_3^*, $\cdots$. The continuum T contains no simple closed curve. Since (a) for each positive integer i the set $T_i^*-T_{i+1}^*$ consists of a finite number of components, each of which is a subset of S_i, (b) the continuous curve T_1 has the property that every simple closed curve lying in T_1 contains a point of K, it follows that T is connected im kleinen at every point of $T-K$ and hence† T is a continuous curve. Since $M\cdot T$ is a subset of $M\cdot K^*$ the set $M\cdot T$ is totally disconnected and contains no cut point of M.

Suppose there exist two points X and Y which belong to the boundary B of a complementary domain D of M such that X and Y are not the extremities of any arc belonging to T and lying, except for end points, within D. There exists an arc XY lying within $\overline{D}\cdot T_1$ such that segment XY is a subset of D. There exist two positive integers d and e such that $T_d^*-\sum_{i=1}^{e}t_i$ contains segment XY and $T_d^*-\sum_{i=1}^{e+1}t_i$ does not. Hence $(T_d^*-\sum_{i=1}^{e}t_i)-(T_d^*-\sum_{i=1}^{e+1}t_i)$ contains a junction point of $T_d^*-\sum_{i=1}^{e}t_i$ contrary to the fact that for any two positive integers r and s $(T_r^*-\sum_{i=1}^{s}t_i)-(T_r^*-\sum_{i=1}^{s+1}t_i)$ contains no junction point of $T_r^*-\sum_{i=1}^{s}t_i$. Hence the supposition that X and Y are not the extremities of any arc belonging to T and lying, except for end points, within D has led to a contradiction. It follows then by Theorem VII that $M-T$ is connected.

Suppose there exists an end point P of T which does not belong to K. Since P is not an end point of T_1 and for any positive integer r there are only a finite number of components of the set $T_1-(L+S_{r+1})$ each of which con-

† See R. L. Moore, *Report on continuous curves from the viewpoint of analysis situs*, loc. cit.

tains a point of $T_1 - S_r$ it follows that there exist two positive integers d and e such that P is an end point of $T_d^* - \sum_{i=1}^{e+1} t_i$ but not an end point of $T_d^* - \sum_{i=1}^{e} t_i$. Since P does not belong to K, $(T_d^* - \sum_{i=1}^{e} t_i) - (T_d^* - \sum_{i=1}^{e+1} t_i)$ is a segment lying in a simple closed curve which is a subset of $T_d^* - \sum_{i=1}^{e} t_i$. Hence the point P is a junction point of $T_d^* - \sum_{i=1}^{e} t_i$ and an end point of $T_d^* - \sum_{i=1}^{e+1} t_i$. This contradicts the fact that no junction point of a continuous curve N is an end point of $N - t$, where t is a segment lying in N. Hence all the end points of T belong to K and Theorem XII has been established.

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PROBLEMS OF CLOSEST APPROXIMATION CONNECTED WITH THE SOLUTION OF LINEAR DIFFERENTIAL EQUATIONS*

BY

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1. Introduction. Consider the differential system consisting of the mth-order ordinary linear differential equation

$$L(y) \equiv \frac{d^m y}{dx^m} + Q_1(x)\frac{d^{m-1}y}{dx^{m-1}} + \cdots + Q_m(x)y = R(x),$$

in which the functions $Q_1(x), \cdots, Q_m(x), R(x)$ are defined and continuous on $a \leqq x \leqq b$, and the m linearly independent two-point boundary conditions

$$U_i(y) \equiv \sum_{j=1}^{m}\left\{\alpha_i^{(j-1)} y^{(j-1)}(a) + \beta_i^{(j-1)} y^{(j-1)}(b)\right\} = h_i \qquad (i = 1, 2, \cdots, m).$$

Let it be assumed that the system has a unique solution $y(x)$, or in other words that the corresponding reduced system is incompatible. Let $\phi_1(x), \phi_2(x), \cdots$ be an infinite sequence of functions, defined and continuous and linearly independent (in finite subsets) and having continuous derivatives of all orders up to and including the mth on the interval, and let

$$y_n(x) = c_1\phi_1(x) + c_2\phi_2(x) + \cdots + c_n\phi_n(x)$$

be a linear combination of the first n of these functions, subject to the restriction that $y_n(x)$ shall satisfy the boundary conditions, but otherwise arbitrary. Then it is a problem of minima to determine a sum of this type for which the integral $\int_a^b |L(y_n) - R|^r dx$ is as small as possible, where r is any given positive real number. Such a function, when it exists, will be called "a minimizing function of order n."

The aim of this paper is to investigate the questions of the existence of such a function and the convergence of it and certain of its derivatives as n becomes infinite, the convergence problem, however, being treated only for the cases when $y_n(x)$ is a trigonometric sum or a polynomial. Kryloff† and

* Presented to the Society, September 12, 1930; received by the editors June 6, 1931.

† See, for example, N. Kryloff, *Sur une méthode, basée sur le principe de minimum, pour l'intégration approchée des équations différentielles*, Comptes Rendus, vol. 181 (1925), pp. 86–88. This is just one of a long series of papers by Kryloff dealing with more or less closely related problems; see also a volume by him in the Mémorial des Sciences Mathématiques, fascicule 49, *Les Méthodes de Solution Approchée des Problèmes de la Physique Mathématique*, Paris, Gauthier-Villars, 1931.

Krawtchouk* have considered the convergence question for a second-order system with the simple boundary conditions $y(a)=y(b)=0$, but only for the case when $r=2$. We shall treat the problem in its more general aspect by methods which are essentially different from those of the authors cited. In this connection mention should be made also of a paper by Picone† which deals with a different but somewhat analogous problem of minima, relating mainly to a second-order differential system, but involving an arbitrary power of the error.

2. **Existence of a minimizing function. Uniqueness.** In discussing the question of existence we must distinguish three cases in respect to the boundary conditions:

(a) when they are homogeneous and are satisfied by the ϕ's individually,
(b) when they are homogeneous but are not satisfied by the ϕ's individually,
(c) when they are non-homogeneous.

The first case can be disposed of immediately. When the boundary conditions are homogeneous and are satisfied by the ϕ's individually the problem reduces to that of approximating $R(x)$ by means of a linear combination of the n functions $L(\phi_1), L(\phi_2), \cdots, L(\phi_n)$ to give the best approximation in the sense of the method of least rth powers, and it is well known that this problem has a solution when $r>0$, and indeed a unique solution when $r>1$ and the functions $L(\phi_1), \cdots, L(\phi_n)$ are linearly independent.‡ This latter condition, of course, means that the homogeneous differential equation $L(y)=0$ has no non-trivial solution which is a linear combination of $\phi_1, \cdots, \phi_n$, and this requirement in turn is met, by reason of the boundary conditions satisfied by the ϕ's, if the homogeneous system $L(y)=0$, $U_i(y)=0$, $i=1, 2, \cdots, m$, is incompatible.

When, on the other hand, the boundary conditions are non-homogeneous, or are homogeneous but are not satisfied by the ϕ's individually, the additional question arises whether it is possible to satisfy these conditions by sums of the form $y_n(x)$ at all. The requirement is that the n coefficients c_k satisfy the m linear equations

$$\sum_{k=1}^{n} c_k U_i(\phi_k) = h_i \qquad (i = 1, 2, \cdots, m).$$

* See, for example, M. Krawtchouk, *Sur les dérivées des intégrales approchées de certaines équations différentielles*, Rendiconti del Circolo Matematico di Palermo, vol. 54 (1930), pp. 194–198.

† M. Picone, *Sul metodo delle minime potenze ponderate e sul metodo di Ritz*, etc., Rendiconti del Circolo Matematico di Palermo, vol. 52 (1928), pp. 225–253.

‡ For a proof, see, for example, D. Jackson, *On functions of closest approximation*, these Transactions, vol. 22 (1921), pp. 117–128, pp. 118–122; *A generalized problem in weighted approximation*, these Transactions, vol. 26 (1924), pp. 133–154; pp. 133–138.

If the numbers h_i are all zero there will certainly be infinitely many sets of the c's satisfying these equations for values of $n>m$. If they are not all zero, but if the equations can be satisfied for a particular value of n, then they can be satisfied by infinitely many choices of the c's for larger values of n, since the sum of a solution of the non-homogeneous equations and an arbitrary solution of the homogeneous equations will satisfy the non-homogeneous equations. Incidentally, it will be shown in §3 (in connection with the proof of Theorem E) that this condition can actually be met in the one case of immediate importance, that of polynomials.

Let it be assumed that linear combinations of $\phi_1, \cdots, \phi_n$ satisfying the boundary conditions do exist; it is assumed also, as already stated, that the reduced system $L(y)=0$, $U_i(y)=0$, $i=1, 2, \cdots, m$, is incompatible. Then, in particular, a linear combination of the ϕ's satisfying the homogeneous boundary conditions will not satisfy the reduced differential equation unless it vanishes identically.

With the qualifications just cited the minimizing problem in cases (b) and (c) also has a solution when $r>0$, and a unique solution when $r>1$. If the functions $L(\phi_1), \cdots, L(\phi_n)$ are linearly independent, the existence of a solution can be inferred almost immediately from the second paper referred to in the last footnote. For the essential part of the argument there depended on showing that the c's for which the integral to be minimized has values not greater than a specified upper bound are to be sought in a closed region of n-dimensional space; and the addition of the auxiliary conditions has the effect merely of narrowing the consideration to a subset of this closed region, which is likewise closed. If the equation $L(y)=0$, while not satisfied by any non-trivial combination of the ϕ's subject to the boundary conditions, is satisfied by other linear combinations of the ϕ's, the argument has to be re-examined with a little more care. In the case of homogeneous boundary conditions all linear combinations of the ϕ's satisfying the conditions can be expressed linearly in terms of a fundamental system of such combinations, and the proof can be carried through for approximation in terms of the functions of this fundamental system. For the case of non-homogeneous conditions it is to be noted that if Φ is a particular combination of the ϕ's satisfying the conditions the problem of approximating $R(x)$ by a combination satisfying the non-homogeneous conditions is the same as that of approximating $R-L(\Phi)$ by a combination satisfying the corresponding homogeneous conditions.

As to the uniqueness proof, the fact that the arithmetical average of two solutions of the auxiliary conditions is itself a solution, even in the non-

homogeneous case, is sufficient to insure the applicability of the method used in the passage to which reference has been made.

3. **Preliminary theorems on approximation.** In this section we shall introduce certain theorems on approximation which in part at least are well known in substance, if not in all cases in the exact form in which they are stated here. The letter m will be used throughout the section to denote any arbitrarily chosen positive integer, not necessarily the order of the differential system, although in the applications to be made later the particular m of the differential system will be the one required. The first theorem is an extension of the Weierstrass theorem for trigonometric sums:

THEOREM A. *If $f(x)$ is a given function which is continuous and periodic with the period 2π, and if $f'(x), \cdots, f^{(m)}(x)$ exist and are continuous, then for any given positive number ϵ there exists a trigonometric sum $T_n(x)$ of some order n such that*

$$\left| f^{(k)}(x) - T_n^{(k)}(x) \right| \leqq \epsilon,$$

for $k=0, 1, \cdots, m$, and for all values of x.

The truth of this statement is apparent at once from a consideration of the Fejér mean

$$\sigma_n(x) = \frac{1}{n\pi}\int_{-\pi}^{\pi} f(\xi)\phi(\xi - x)d\xi,$$

where

$$\phi(\xi - x) = \frac{\sin^2\left(\frac{n}{2}(\xi - x)\right)}{2\sin^2\left(\frac{1}{2}(\xi - x)\right)}\cdot$$

It is well known that this function, which is a trigonometric sum of order $n-1$ in x, converges uniformly to $f(x)$ as n becomes infinite. Hence by taking n sufficiently large it can be made to approximate $f(x)$ with an error less than any preassigned positive number ϵ. If this function is differentiated with respect to x and $\partial\phi/\partial x$ replaced by its equal $-\partial\phi/\partial\xi$ and the resulting expression integrated by parts, it will be seen that

$$\sigma_n'(x) = \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(\xi)\phi(\xi - x)d\xi.$$

But this is the Fejér mean of $f'(x)$ and it therefore converges to $f'(x)$ as n becomes infinite. By repetitions of this process it can be shown that the higher derivatives of $\sigma_n(x)$ converge to the respective derivatives of $f(x)$. Hence the theorem is established.

The extended form of the Weierstrass theorem for polynomials we shall call

THEOREM B. *If $f(x)$ is a given function which is defined and continuous on $a \leqq x \leqq b$, and if $f'(x), \cdots, f^{(m)}(x)$ exist and are continuous there, then for any given positive number ϵ there exists a polynomial $P_n(x)$ of some degree n such that $|f^{(k)}(x) - P_n^{(k)}(x)| \leqq \epsilon$, for $k = 0, 1, \cdots, m$, and for all values of x on the interval.*

Consider the function $f^{(m)}(x)$. It is continuous on (a, b), and therefore by Weierstrass' theorem a polynomial $Q_1(x)$ of some degree N can be found, such that

$$|f^{(m)}(x) - Q_1(x)| \leqq \eta, \quad a \leqq x \leqq b,$$

where η is any preassigned positive number. Let

$$Q_2(x) = f^{(m-1)}(a) + \int_a^x Q_1(x)dx.$$

Then

$$|f^{(m-1)}(x) - Q_2(x)| = \left|\int_a^x [f^{(m)}(x) - Q_1(x)]dx\right| \leqq (b - a)\eta.$$

By successive integrations, with suitable determination of the constant of integration at each stage, polynomials $Q_3(x), \cdots, Q_{m+1}(x)$ can be obtained, of degrees $N+2, \cdots, N+m$, respectively, so that each is the derivative of the following, and so that

$$|f^{(m-2)}(x) - Q_3(x)| \leqq (b - a)^2\eta, \cdots, |f(x) - Q_{m+1}(x)| \leqq (b - a)^m\eta.$$

The positive number ϵ being given, let $\eta = \epsilon$ or $\epsilon/(b-a)^m$, according as $b-a$ is $\leqq 1$ or >1, and let $P_n(x) = Q_{m+1}(x)$. Then $P_n(x)$ is a polynomial of degree $n = N+m$, and $|f^{(k)}(x) - P_n^{(k)}(x)| \leqq \epsilon$ for $k = 0, 1, \cdots, m$.

For the discussion of rapidity of convergence we shall need to know what degree of approximation can be obtained for a specified value of n. In this connection we have

THEOREM C. *If, in addition to the hypotheses of Theorem* A, *it is assumed that $f^{(m)}(x)$ satisfies a Lipschitz condition*

$$|f^{(m)}(x_1) - f^{(m)}(x_2)| \leqq \lambda |x_1 - x_2|,$$

then for each positive integral value of n there exists a trigonometric sum $T_n(x)$ of the nth order such that $|f^{(k)}(x) - T_n^{(k)}(x)| \leqq A/n$ for $k = 0, 1, \cdots, m$, and for all values of x, where A is a constant independent of n.

Under the hypothesis that $f^{(m)}(x)$ satisfies a Lipschitz condition it is possible to find for each positive integral value of n a trigonometric sum $T_n(x)$ of order n such that

$$|f(x) - T_n(x)| \leqq \frac{K_{m+1}\lambda}{n^{m+1}},$$

where K_{m+1} is a constant independent of n.* Such a sum, for example, is the function†

$$I_s(x) = h_s \int_{-\pi/2}^{\pi/2} \left[\pm f(x + 2(m+1)u) \mp \binom{m+1}{1} f(x + 2mu) + \cdots \right.$$
$$\left. + \binom{m+1}{m} f(x + 2u) \right] \left[\frac{\sin su}{s \sin u} \right]^{2\kappa} du,$$

in which the number κ is the smallest integer such that $2\kappa - (m+1) > 1$, h_s satisfies the equation

$$\frac{1}{h_s} = \int_{-\pi/2}^{\pi/2} \left[\frac{\sin su}{s \sin u} \right]^{2\kappa} du,$$

and s bears a determinate relationship to n. The discussion here will be based on the properties of the function $I_s(x)$, as set forth in the passage cited, rather than on the somewhat different treatment in the Colloquium.

Although the form of $I_s(x)$ has to be readjusted if it is desired to pass from a given value of m to a higher value, the form which corresponds to any given m can be used to approximate $f(x)$ with a lower order of approximation than $1/n^{m+1}$ when it happens that $f(x)$ exhibits a correspondingly lower order of continuity. For example, if $f^{(m-1)}(x)$ satisfies a Lipschitz condition the $I_s(x)$ corresponding to the value m will approximate $f(x)$ with an error not greater than a constant multiple of $1/n^m$.

By writing this integral as the sum of the $m+1$ integrals corresponding to the $m+1$ terms in the first factor of the integrand, suitably changing the variable of integration and the limits of integration in each part, and recombining again into a single integral, we obtain an expression of the form

$$I_s(x) = h_s \int_{-\pi}^{\pi} f(\xi)\Phi(\xi - x) d\xi,$$

* D. Jackson, *The Theory of Approximation*, American Mathematical Society Colloquium Publications, vol. XI, New York, 1930 (referred to hereafter as Colloquium), pp. 10–12.

† For an account of this formula see D. Jackson, *On approximation by trigonometric sums and polynomials*, these Transactions, vol. 13 (1912), pp. 491–515; pp. 496–500.

where Φ is given by a somewhat complicated formula when written out at full length, but has the essential properties that it depends on x and ξ only through the difference $\xi-x$ and is of period 2π.

If we differentiate $I_s(x)$ with respect to x, and replace $\partial\Phi/\partial x$ by its equal $-\partial\Phi/\partial\xi$ and integrate by parts, we get

$$I_s'(x) = h_s\int_{-\pi}^{\pi} f'(\xi)\Phi(\xi - x)d\xi,$$

which is the I_s-function for $f'(x)$ corresponding to the value m. But $f'(x)$ is a function whose $(m-1)$th derivative $f^{(m)}(x)$ satisfies a Lipschitz condition. Hence there will exist some constant K_m such that

$$|f'(x) - I_s'(x)| \leqq \frac{K_m\lambda}{n^m}\cdot$$

Likewise we can obtain the inequalities

$$|f''(x) - I_s''(x)| \leqq \frac{K_{m-1}\lambda}{n^{m-1}}, \cdots, |f^{(m)}(x) - I_s^{(m)}(x)| \leqq \frac{K_1\lambda}{n},$$

where $K_{m-1}, \cdots, K_1$ are constants independent of n. If A stands for the greatest of the numbers $K_{m+1}, \cdots, K_1$, then

$$|f^{(k)}(x) - I_s^{(k)}(x)| \leqq \frac{A}{n} \text{ for } k = 0, 1, \cdots, m.$$

Thus the function $I_s(x)$, with s properly adjusted to correspond to n, can be made to serve the purpose of the $T_n(x)$ demanded by the theorem.

The corresponding theorem for polynomial approximation we shall call

THEOREM D. *If, in addition to the hypotheses of Theorem* B, *it is assumed that $f^{(m)}(x)$ satisfies a Lipschitz condition on the interval $a\leqq x\leqq b$, then for each positive integral value of n there exists a polynomial $P_n(x)$ of the nth degree such that*

$$|f^{(k)}(x) - P_n^{(k)}(x)| \leqq \frac{B}{n} \textit{ for } k = 0, 1, \cdots, m,$$

and for all values of x on the interval, where B is a constant independent of n.

In the proof of Theorem B the polynomial $Q_1(x)$ can be chosen so that $|f^{(m)}(x)-Q_1(x)|$ does not exceed a constant multiple of $1/n$,* and then

* Colloquium, pp. 13–14.

the error in the approximation of each of the lower derivatives does not exceed this quantity multiplied by a factor independent of n.

It is worthy of special note that Theorems C and D can be generalized further. Thus if $f^{(m+\alpha)}(x)$ satisfies a Lipschitz condition, α being a positive integer, it can be shown by appropriate modification of the reasoning that the error in either the trigonometric or polynomial case does not exceed a quantity of the order of $1/n^{1+\alpha}$.

For the treatment of polynomial convergence certain further results related to Theorems B and D will be required.

THEOREM E. *If, in addition to the hypotheses of Theorem* B, *it is assumed that $f(x)$ satisfies the m linearly independent two-point boundary conditions*

$$U_i(f) = h_i \qquad (i = 1, 2, \cdots, m),$$

then for any positive number ϵ there exists a polynomial $P_n(x)$ of some degree n such that $U_i(P_n)=h_i$, $i=1, 2, \cdots, m$, and $|f^{(k)}(x)-P_n^{(k)}(x)| \leqq \epsilon$ for $k=0, 1, 2, \cdots, m$ and for all values of x on the interval (a, b).

Let ϵ' be any positive number. Then, by Theorem B, there exists a polynomial $p_n(x)$ of some degree n such that $|f^{(k)}(x)-p_n^{(k)}(x)| \leqq \epsilon'$ for $k=0, 1, 2, \cdots, m$ and for $a \leqq x \leqq b$. Let the quantities $g_1, g_2, \cdots, g_m$ be defined by the equations $U_i(p_n)=h_i-g_i$, $i=1, 2, \cdots, m$. Then it is clear that

$$|g_i| = |U_i(f) - U_i(p_n)| = |U_i(f - p_n)| \leqq C\epsilon' \qquad (i = 1, 2, \cdots, m),$$

where C is a constant independent of n.

Let $q(x)$ denote a polynomial of the $(2m-1)$th degree, the coefficients of which are determined as follows. Set up the $2m$ equations

$$U_1(q) = g_1, \cdots, U_m(q) = g_m, U_{m+1}(q) = 0, \cdots, U_{2m}(q) = 0,$$

in which $U_{m+1}, \cdots, U_{2m}$ are any linear combinations of the $2m$ arguments $q(a), q'(a), \cdots, q^{(m-1)}(a), q(b), \cdots, q^{(m-1)}(b)$, so chosen that all $2m$ of the U's are linearly independent. Since the U's are linearly independent, the $2m$ equations in the $2m$ arguments will have a unique solution, and so will determine uniquely the values of these quantities. Hence if $q(x)$ has the form

$$\begin{aligned} q(x) = {} & A_0 + A_1(x - a) + A_2(x - a)^2 + \cdots + A_{m-1}(x - a)^{m-1} \\ & + A_m(x - a)^m + A_{m+1}(x - a)^m(x-b) + \cdots + A_{2m-1}(x - a)^m(x - b)^{m-1}, \end{aligned}$$

the coefficients $A_0, \cdots, A_{2m-1}$ can be determined successively to fit the values $q(a), \cdots, q^{(m-1)}(a), q(b), \cdots, q^{(m-1)}(b)$.

With this definition of the polynomial $q(x)$ it is not difficult to see that an upper bound placed on the absolute values of the g's in the $2m$ equations

above sets an upper bound for $|q(a)|, \cdots, |q^{(m-1)}(b)|$, and this in turn sets an upper bound for the absolute values of the coefficients in $q(x)$ and so for the absolute values of $q(x)$ itself and all its derivatives throughout the interval. Hence, since $C\epsilon'$ is an upper bound for the g's, we can write $|q^{(k)}(x)| \leqq C'\epsilon'$ for $k=0, 1, 2, \cdots, m$ and $a \leqq x \leqq b$, where C' is a constant independent of n.

Let us now consider the function $P_n(x) = p_n(x) + q(x)$. This is a polynomial of the nth degree (if $n \geqq 2m-1$*) which satisfies the m boundary equations $U_i(P_n) = h_i$, since $p_n(x)$ and $q(x)$ satisfy the conditions $U_i(p_n) = h_i - g_i$, $U_i(q) = g_i$ respectively, and furthermore,

$$| f^{(k)}(x) - P_n^{(k)}(x) | = | f^{(k)}(x) - p_n^{(k)}(x) - q^{(k)}(x) | \leqq (1 + C')\epsilon',$$

for $k=0, 1, 2, \cdots, m$, and $a \leqq x \leqq b$. If ϵ is arbitrary, and if ϵ' is taken equal to $\epsilon/(1+C')$, the polynomial $P_n(x)$ will then fulfill the requirements of the theorem.

The above proof shows in substance that if $f(x)$ satisfies the boundary conditions and if polynomials $p_n(x)$ can be determined so that $|f^{(k)}(x) - p_n^{(k)}(x)| \leqq \epsilon_n$ then there exist polynomials $P_n(x)$ satisfying the boundary conditions and such that $|f^{(k)}(x) - P_n^{(k)}(x)| \leqq (1+C')\epsilon_n$, where C' is a constant independent of n. Combined with Theorem D, this yields

THEOREM F. *Under the hypotheses of Theorem* E *and the additional hypothesis that $f^{(m)}(x)$ satisfies a Lipschitz condition on the interval, there exists for each positive integral value of $n \geqq 2m-1$ a polynomial $P_n(x)$ of the nth degree such that $U_i(P_n) = h_i$, $i=1, 2, \cdots, m$, and such that*

$$| f^{(k)}(x) - P_n^{(k)}(x) | \leqq D/n \qquad (k = 0, 1, \cdots, m),$$

for all values of x on the interval, where D is a constant independent of n.

The generalization of Theorem F when $f(x)$ exhibits higher properties of continuity is obvious.

4. Convergence in the trigonometric case when $r \geqq 1$. In this section we shall discuss the questions of convergence and degree of approximation relative to a periodic differential system for which the minimizing function is a trigonometric sum defined for some value of $r \geqq 1$.

Suppose we have given the differential system

* The form of statement of the conclusion is not affected by the fact that a finite number of values of n have to be set aside as exceptional in the course of the proof, to insure the possibility of satisfying the boundary conditions.

$$
(1)\qquad
\begin{aligned}
L(y) &\equiv \frac{d^m y}{dx^m} + Q_1(x)\frac{d^{m-1}y}{dx^{m-1}} + \cdots + Q_m(x)y = R(x),\\
U_i(y) &\equiv y^{(i-1)}(0) - y^{(i-1)}(2\pi) = 0 \qquad (i = 1, 2, \cdots, m),
\end{aligned}
$$

in which the functions $Q_1(x), \cdots, Q_m(x), R(x)$ are continuous and periodic with the period 2π, and suppose further that the reduced system

$$
(2)\qquad
\begin{aligned}
L(y) &= 0,\\
U_i(y) &= 0 \qquad (i = 1, 2, \cdots, m)
\end{aligned}
$$

is incompatible, so that (1) has a unique solution $y(x)$. Then $y(x)$ and its first m derivatives will necessarily be continuous and periodic with the period 2π.

As functions $\phi_i(x)$ we shall use the periodic trigonometric functions $1, \cos x, \sin x, \cos 2x, \sin 2x, \cdots$. The minimizing function of order $2n+1$ is then a trigonometric sum $T_n(x)$ of order n,

$$
\begin{aligned}
y_{2n+1}(x) = T_n(x) = a_0 + a_1 \cos x + \cdots + a_n \cos nx\\
+ b_1 \sin x + \cdots + b_n \sin nx,
\end{aligned}
$$

in which the coefficients are so determined that the integral

$$\int_0^{2\pi} |L(T_n) - R|^r dx = \int_0^{2\pi} |L(y - T_n)|^r dx, \quad r \geqq 1,$$

is a minimum. The function $T_n(x)$ may be described as "the minimizing trigonometric sum of order n for $y(x)$ corresponding to the exponent r."

The function $y(x)$ satisfies the hypotheses of Theorem A, and hence, if ϵ is any positive number, there exists a trigonometric sum $t_n(x)$ of some order n such that $|y^{(k)}(x) - t_n^{(k)}(x)| \leqq \epsilon$, for $k = 0, 1, \cdots, m$. Let $F(x) = y(x) - t_n(x)$ and let $\pi_n(x)$ be the minimizing sum of order n for $F(x)$. Then the integral

$$\gamma = \int_0^{2\pi} |L(F - \pi_n)|^r dx$$

is a minimum, and therefore, since zero is itself a trigonometric sum of order n (in which the coefficients are all zero), we can write

$$\gamma \leqq \int_0^{2\pi} |L(F)|^r dx.$$

But

$$|L(F)| = |F^{(m)} + Q_1 F^{(m-1)} + \cdots + Q_m F| \leqq (m+1)M\epsilon,$$

where M is an upper bound for the bounded functions $1, |Q_1|, \cdots, |Q_m|$. Hence

$$\gamma \leqq 2\pi\{(m+1)M\epsilon\}^r. \tag{3}$$

Let $u(x)=F(x)-\pi_n(x)$ and let $Z(x)=L(u)$. Then $u(x)$ satisfies the differential system

$$L(u) = Z(x),$$
$$U_i(u) = 0 \qquad (i = 1, 2, \cdots, m),$$

and indeed is the unique solution, since (2) is incompatible. Let $G(x, \xi)$ be the Green's function associated with (2).* Then

$$u(x) = \int_0^{2\pi} G(x, \xi)Z(\xi)d\xi.$$

Likewise

$$u'(x) = \int_0^{2\pi} \frac{\partial}{\partial x} G(x, \xi)Z(\xi)d\xi,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$u^{(m-1)}(x) = \int_0^{2\pi} \frac{\partial^{m-1}}{\partial x^{m-1}} G(x, \xi)Z(\xi)d\xi.$$

The differentiation under the integral sign might be questioned in the case of the last equation on the ground that $(\partial^{m-1}/\partial x^{m-1})G(x, \xi)$ is discontinuous on the diagonal $x=\xi$. By considering separately the integrals from 0 to x and from x to 2π, however, it can be shown that the formula is valid in this case also.†

Let $\overline{G}$ be an upper bound for the bounded functions $|G(x, \xi)|, \cdots, |(\partial^{m-1}/\partial x^{m-1})G(x, \xi)|$. Then

$$|u^{(k)}(x)| \leqq \overline{G} \int_0^{2\pi} |Z(\xi)|\, d\xi \qquad (k = 0, 1, \cdots, m-1).$$

We make use of Hölder's inequality in the form which states that if $\phi(x) \geqq 0$ and r is any real number $\geqq 1$ then

$$\int_a^b \phi(x)dx \leqq (b-a)^{(r-1)/r}\left[\int_a^b \{\phi(x)\}^r dx\right]^{1/r}.$$

In the present case this gives‡

$$|u^{(k)}(x)| \leqq (2\pi)^{(r-1)/r}\, \overline{G}\left[\int_0^{2\pi} |Z(\xi)|^r d\xi\right]^{1/r} = (2\pi)^{(r-1)/r}\, \overline{G}\gamma^{1/r},$$

* The writer is indebted to Professor D. V. Widder for a valuable suggestion in connection with this stage of the work.

† See, for example, Bôcher, *Leçons sur les Méthodes de Sturm*, Paris, 1917, pp. 102–103.

‡ It is at this point that we use the hypothesis in the definition of the minimizing sums that $r \geqq 1$.

and by (3), if N denotes the quantity $2\pi\overline{G}M$, independent of n,

$$|u^{(k)}(x)| \leqq N\epsilon \qquad (k = 0, 1, \cdots, m-1).$$

But since $\pi_n(x)$ and $T_n(x)$ are the respective minimizing trigonometric sums for $F(x)$ and $y(x)$, two functions whose difference is the trigonometric sum $t_n(x)$, it is readily seen that $\pi_n(x)$ is identical with $T_n(x)-t_n(x)$. Hence

$$u(x) = F(x) - \pi_n(x) = y(x) - T_n(x),$$

and consequently

$$|y^{(k)}(x) - T_n^{(k)}(x)| \leqq N\epsilon \qquad (k = 0, 1, \cdots, m-1).$$

This is true for any value of n for which a trigonometric sum $t_n(x)$ of the nth order exists, satisfying the relations $|y^{(k)}(x)-t_n^{(k)}(x)| \leqq \epsilon$, $k=0, 1, \cdots, m-1$, and so, in the case of any specified positive ϵ, is true for all values of n from a certain point on. Thus it is possible to state

THEOREM I. *If the periodic system* (1) *has a unique solution* $y(x)$, *the minimizing trigonometric sum* $T_n(x)$ *of order n corresponding to any given real number* $r \geqq 1$ *converges uniformly to the value of* $y(x)$ *as n becomes infinite, and furthermore, the first* $(m-1)$ *derivatives of* $T_n(x)$ *converge uniformly to the respective derivatives of* $y(x)$.

If $y^{(m)}(x)$ satisfies a Lipschitz condition, which will necessarily be the case if the functions $Q_1(x), \cdots, Q_m(x)$, $R(x)$ satisfy such a condition, then, by Theorem C, sums $t_n(x)$ can be determined so that $|y^{(k)}(x)-t_n^{(k)}(x)|$ has an upper bound of the order of $1/n$. Hence we can state

THEOREM II. *If, in addition to the hypotheses of Theorem* I, *it is assumed that* $y^{(m)}(x)$ *satisfies a Lipschitz condition, then for all positive integral values of n*

$$|y^{(k)}(x) - T_n^{(k)}(x)| \leqq \frac{E}{n} \qquad (k = 0, 1, \cdots, m-1),$$

where E is a constant independent of n.

5. **Convergence in the polynomial case when $r \geqq 1$.** Let us consider again the differential system with non-homogeneous boundary conditions described in §1. It has the form

$$(4) \qquad \begin{aligned} L(y) &= R(x), \\ U_i(y) &= h_i \qquad (i = 1, 2, \cdots, m). \end{aligned}$$

The coefficients in the differential equation are defined and continuous on $a \leqq x \leqq b$, the boundary conditions are linearly independent, and the reduced system

$$(5) \qquad \begin{aligned} L(y) &= 0, \\ U_i(y) &= 0 \qquad (i = 1, 2, \cdots, m) \end{aligned}$$

is incompatible. If $y(x)$ is the solution, it is unique, and it and its first m derivatives are continuous on the interval.

As functions $\phi_i(x)$ we shall use $1, x, x^2, \cdots$. The minimizing function of order $n+1$ corresponding to a given value of $r \geqq 1$ is then a polynomial $P_n(x)$ of the nth degree:

$$y_{n+1} = P_n(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_n x^n$$

which satisfies the m boundary equations $U_i(P_n) = h_i$, and is such that the integral

$$\int_a^b | L(y - P_n) |^r dx, \quad r \geqq 1,$$

is a minimum. The function $P_n(x)$ may be called "the minimizing polynomial of the nth degree for $y(x)$ corresponding to the exponent r."

The function $y(x)$ satisfies the hypotheses of Theorem E, and hence if ϵ is any positive number, there exists a polynomial $p_n(x)$ of some degree n which satisfies the boundary conditions $U_i(p_n) = h_i$, $i = 1, 2, \cdots, m$, and is such that $| y^{(k)}(x) - p_n^{(k)}(x) | \leqq \epsilon$ for $k = 0, 1, \cdots, m$. Let $F(x) = y(x) - p_n(x)$. Then $F(x)$ satisfies the m homogeneous boundary conditions $U_i(F) = 0$, $i = 1, 2, \cdots, m$. Let $\pi_n(x)$ be the minimizing polynomial of the nth degree for $F(x)$ which satisfies the same homogeneous equations $U_i(\pi_n) = 0$. Then the integral

$$\gamma = \int_a^b | L(F - \pi_n) |^r dx$$

is a minimum for polynomials of this type.

But zero is itself a polynomial of the nth degree, in which the coefficients are all zero, and it satisfies the homogeneous boundary equations. Hence

$$\gamma \leqq \int_a^b | L(F) |^r dx,$$

and consequently, if M is an upper bound of the bounded functions 1, $|Q_1|, \cdots, |Q_m|$,

$$\gamma \leqq (b - a) \{ (m + 1) M \epsilon \}^r.$$

Let $u(x) = F(x) - \pi_n(x)$ and let $Z(x) = L(u)$. Then $u(x)$ is the unique solution of the differential system

$$L(u) = Z(x),$$
$$U_i(u) = 0 \qquad (i = 1, 2, \cdots, m),$$

and therefore, if $G(x, \xi)$ is the Green's function associated with the reduced ystem (5),

$$u^{(k)}(x) = \int_a^b \frac{\partial^k}{\partial x^k} G(x, \xi) Z(\xi) d\xi \quad (k = 0, 1, \cdots, m-1).$$

Without detailed repetition of the argument used in the last section it is clear that $u(x) = y(x) - P_n(x)$, and that there is a number N, independent of n, such that

$$| y^{(k)}(x) - P_n^{(k)}(x) | \leqq N\epsilon \quad (k = 0, 1, \cdots, m-1),$$

for all values of x on (a, b). To cover the conclusions in this case we can state the following two theorems:

THEOREM III. *If the differential system* (4) *has a unique solution* $y(x)$, *then the minimizing polynomial* $P_n(x)$ *of the nth degree, corresponding to a given value of* $r \geqq 1$, *converges uniformly on the interval* $a \leqq x \leqq b$ *to the value of* $y(x)$ *as n becomes infinite, and the first* $m-1$ *derivatives of* $P_n(x)$ *converge uniformly to the respective derivatives of* $y(x)$.

THEOREM IV. *If, in addition to the hypotheses of Theorem* III, *it is assumed that* $y^{(m)}(x)$ *satisfies a Lipschitz condition on the interval, then the errors* $|y^{(k)}(x) - P_n^{(k)}(x)|$ *have an upper bound of the order of* $1/n$.

6. Preliminary theorems for the treatment of convergence when $r < 1$. The foregoing proofs of convergence were based on a direct application of Hölder's inequality, a relation which holds only when $r \geqq 1$. Consequently when the minimizing sum is defined for a value of $r < 1$ a different method must be used. In this section we shall derive two theorems in preparation for the discussion of this case; one is an extension of Bernstein's theorem on the derivative of a trigonometric sum, and the other an extension of Markoff's heorem on the derivative of a polynomial.

THEOREM G. *If* $T_n(x)$ *is an arbitrary trigonometric sum of the nth order, and if*

$$L(y) = R(x),$$
$$U_i(y) \equiv y^{(i-1)}(0) - y^{(i-1)}(2\pi) = 0 \quad (i = 1, 2, \cdots, m),$$

is the differential system (1) *described in* §4, *of which the reduced system is incompatible, and if* $\delta = \max |L(T_n)|$, *then*

(a) $$| T_n^{(k)}(x) | \leqq A\delta \qquad (k = 0, 1, \cdots, m),$$

and if $Q_1(x), \cdots, Q_m(x)$ *have bounded first derivatives,*

(b) $$\left| \frac{d}{dx} L(T_n) \right| \leqq Bn\delta,$$

where A *and* B *are constants independent of* n *and independent of the coefficients in* $T_n(x)$.

Let $Z(x) = L(T_n)$. Then $T_n(x)$, being periodic with the period 2π, will satisfy the differential system

$$L(y) = Z(x), \qquad U_i(y) = 0 \qquad (i = 1, 2, \cdots, m).$$

But the reduced system is incompatible, and so, if $G(x, \xi)$ is its Green's function, we can write

$$T_n^{(k)}(x) = \int_0^{2\pi} \frac{\partial^k}{\partial x^k} G(x, \xi) Z(\xi) d\xi \qquad (k = 0, 1, \cdots, m-1).$$

Let $\overline{G}$ be an upper bound of the bounded functions $|G(x, \xi)|, \cdots,$ $|(\partial^{m-1}/\partial x^{m-1}) G(x, \xi)|$. Then since δ is an upper bound of $|Z(\xi)|$,

$$| T_n^{(k)}(x) | \leqq 2\pi \overline{G} \delta,$$

for $k = 0, 1, \cdots, m-1$. By virtue of these relations and the identity $T_n^{(m)}(x) = L(T_n) - Q_1 T_n^{(m-1)} - \cdots - Q_m T_n$ it follows that $| T_n^{(m)}(x) |$ also has an upper bound of the same form:

$$| T_n^{(m)}(x) | \leqq \delta + mM(2\pi \overline{G} \delta) = (1 + 2\pi m M \overline{G})\delta,$$

where M is an upper bound of $1, |Q_1|, \cdots, |Q_m|$. By letting A stand for the greater of the two numbers $2\pi\overline{G}$, $(1 + 2\pi m M \overline{G})$ we obtain conclusion (a).

The function $T_n^{(m)}(x)$ is a trigonometric sum of the nth order of which $A\delta$ is an upper bound. Hence, by Bernstein's theorem on the derivative of a trigonometric sum,

$$| T_n^{(m+1)}(x) | \leqq An\delta,$$

and therefore,

$$\begin{aligned} \left| \frac{d}{dx} L(T_n) \right| &= | T_n^{(m+1)} + (Q_1 T_n^{(m)} + \cdots + Q_m T_n') + (Q_1' T_n^{(m-1)} + \cdots \\ &\quad + Q_m' T_n) | \\ &\leqq An\delta + mMA\delta + mM'A\delta \leqq Bn\delta, \end{aligned}$$

where M' is an upper bound of $|Q_1'|, \cdots, |Q_m'|$ and $B=A(1+mM+mM')$. This is conclusion (b). (It is readily seen from the details of the calculation, or can evidently be assumed outright without affecting the truth of the conclusion, that $B \geqq 1$.)

THEOREM H. *If*

$$L(y) = R(x),$$
$$U_i(y) = h_i \qquad (i = 1, 2, \cdots, m),$$

is the differential system (4) *with general non-homogeneous two-point boundary conditions, of which the reduced system is incompatible, and if* $P_n(x)$ *is any polynomial of the nth degree which satisfies the homogeneous boundary equations*

$$U_i(P_n) = 0 \qquad (i = 1, 2, \cdots, m),$$

and if $\delta = \max |L(P_n)|$ *on* $a \leqq x \leqq b$, *then*

$$\text{(a)} \qquad |P_n^{(k)}(x)| \leqq C\delta \qquad (k = 0, 1, \cdots, m),$$

and if $Q_1(x), \cdots, Q_m(x)$ *have bounded first derivatives,*

$$\text{(b)} \qquad \left| \frac{d}{dx} L(P_n) \right| \leqq Dn^2\delta,$$

for all values of x on (a, b), *where C and D are constants independent of n and independent of the coefficients in* $P_n(x)$.

The method of proof here is practically the same as in Theorem G, except that Markoff's theorem* is used instead of Bernstein's theorem in obtaining conclusion (b).

7. **Convergence when $r<1$.** In this section the convergence problem when $r<1$ will be discussed in detail for the trigonometric case and only the results stated for the polynomial case. It will be seen that the method used here† is not restricted in application to the cases when $r<1$, but is generally applicable when r is any positive real number. Moreover, although the bound which it assigns to the errors is not as good as that obtained by the first method under given hypotheses when $r \geqq 1$, it proves the convergence of m derivatives, as compared with $m-1$ in the earlier treatment.

---

* See, for example, D. Jackson, these Transactions, vol. 22, loc. cit., p. 163; M. Riesz, *Eine trigonometrische Interpolationsformel und einige Ungleichungen für Polynome*, Jahresbericht der Deutschen Mathematiker-Vereinigung, vol. 23 (1914), pp. 354–368.

† The general scheme was suggested by convergence proofs given in other connections by D. Jackson; see, for example, *On the convergence of certain trigonometric and polynomial approximations*, these Transactions, vol. 22 (1921), pp. 158–166; Colloquium, Chapter III.

Let it be assumed that the periodic system

$$L(y) = R(x),$$
$$U_i(y) \equiv y^{(i-1)}(0) - y^{(i-1)}(2\pi) = 0 \qquad (i = 1, 2, \cdots, m),$$

has the unique solution $y(x)$, and that the coefficients in the differential equation are provided with the derivatives called for. Let $t_n(x)$ be an arbitrary trigonometric sum of the nth order, and let ϵ be a number such that

$$(6) \qquad | y^{(k)}(x) - t_n^{(k)}(x) | \leqq \epsilon \qquad (k = 0, 1, \cdots, m)$$

for all values of x. Let $F(x) = y(x) - t_n(x)$, and let $\pi_n(x)$ be a minimizing sum of order n for $F(x)$. Then the integral

$$\gamma = \int_0^{2\pi} | L(F - \pi_n) |^r dx$$

is a minimum, and since zero is itself a trigonometric sum of the nth order,

$$(7) \qquad \gamma \leqq \int_0^{2\pi} | L(F) |^r dx \leqq 2\pi\{(m + 1)M\epsilon\}^m,$$

where M is again an upper bound for $1, |Q_1|, \cdots, |Q_m|$.

Let x_1 be a point of the interval $(0, 2\pi)$ at which $|L(\pi_n)|$ takes on its maximum value δ. Then, by virtue of the mean value theorem and conclusion (b) of Theorem G,

$$| L(\pi_n(x)) - L(\pi_n(x_1)) | \leqq | x - x_1 | Bn\delta.$$

Therefore for values of x in the interval $|x - x_1| \leqq 1/[2Bn]$,

$$| L(\pi_n) | \geqq \delta/2.$$

Assume for the moment that $\epsilon < \delta/[4(1+m)M]$. Then

$$| L(F) | = | F^{(m)} + Q_1F^{(m-1)} + \cdots + Q_mF | \leqq (m + 1)M\epsilon < \delta/4$$

and therefore, when x belongs to the interval $|x - x_1| \leqq 1/[2Bn]$, $|L(\pi_n) - L(F)| \geqq \delta/4$, and hence

$$\gamma \geqq \int_{x_1 - 1/(2Bn)}^{x_1 + 1/(2Bn)} | L(\pi_n) - L(F) |^r dx \geqq \frac{1}{Bn}\left(\frac{\delta}{4}\right)^r.$$

Combining this with (7) we get

$$(8) \qquad \delta \leqq 4(2\pi B)^{1/r}(m + 1)Mn^{1/r}\epsilon.$$

But this relation is true also when $\epsilon \geqq \delta/[4(m+1)M]$ and so it holds universally.

Consider the errors

$$|F^{(k)}(x) - \pi_n^{(k)}(x)| \qquad (k = 0, 1, \cdots, m).$$

By virtue of conclusion (a) of Theorem G and relation (8),

$$|F^{(k)}(x) - \pi_n^{(k)}(x)| \leqq \epsilon + A\delta \leqq \epsilon + 4A(2\pi B)^{1/r}(m+1)Mn^{1/r}\epsilon$$
$$\leqq En^{1/r}\epsilon,$$

where E is a constant independent of n. But, since $F(x)$ and $y(x)$ differ only by the terms of the trigonometric sum $t_n(x)$, their respective minimizing sums will differ only by $t_n(x)$. Hence if $T_n(x)$ is the minimizing sum for $y(x)$,

$$|y^{(k)}(x) - T_n^{(k)}(x)| = |F^{(k)}(x) - \pi_n^{(k)}(x)| \leqq En^{1/r}\epsilon$$

for $k=0, 1, \cdots, m$ and for all values of x.

The questions of convergence and degree of approximation are thus directly connected with the approximations in (6). Any hypotheses that will make it possible to define sums $t_n(x)$ so that $n^{1/r}\epsilon$ approaches zero as n becomes infinite will insure the convergence of $T_n(x)$ and its first m derivatives to the respective values of $y(x)$ and its derivatives. For example, when $r=\frac{1}{2}$ it is sufficient to assume that $y^{(m+2)}(x)$ satisfies a Lipschitz condition, for then ϵ can be made to have the order of $1/n^3$ and $n^{1/r}\epsilon$ the order of $1/n$.

The conclusion, and the result obtained by the same method for the case of polynomial approximation, are expressed by the following two theorems:

THEOREM V. *If the periodic system*

$$L(y) = R(x),$$
$$U_i(y) = 0 \qquad (i = 1, 2, \cdots, m),$$

has a unique solution $y(x)$ of such regularity that it and its first m derivatives can be simultaneously approximated by a trigonometric sum of the nth order and its corresponding derivatives with an upper bound of error ϵ satisfying the condition that

$$\lim_{n\to\infty} n^{1/r}\epsilon = 0,$$

and if the coefficients in $L(y)$ have bounded first derivatives, then the minimizing trigonometric sum and its first m derivatives converge uniformly to $y(x)$ and its corresponding derivatives respectively as n becomes infinite.

THEOREM VI. *If the system*

$$L(y) = R(x),$$
$$U_i(y) = h_i \qquad (i = 1, 2, \cdots, m),$$

with general non-homogeneous two-point boundary conditions, has a unique solution $y(x)$ of such regularity that it and its first m derivatives can be simultaneously approximated on the interval (a, b) by means of a polynomial of the nth degree, satisfying the boundary conditions, and its derivatives, with an upper bound of error ϵ for which

$$\lim_{n\to\infty} n^{2/r}\epsilon = 0,$$

and if the coefficients in $L(y)$ have bounded first derivatives, then the minimizing polynomial satisfying the boundary conditions, and its first m derivatives, converge uniformly on the interval to the respective values of $y(x)$ and its corresponding derivatives as n becomes infinite.

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ERRATA, VOLUME 31

H. S. VANDIVER, *On Fermat's last theorem.*

Page 618. In left hand member of relation (3b) insert product sign in lieu of summation sign. In lieu of (3c) read

$$\prod_{i=1}^{l_1} r^{(l-1)a_i\rho(1+r^{-2i}+\cdots+r^{-(l-3)i})} \equiv c^{l\rho(l-1)t} + bl \qquad (\text{mod } l^2).$$

Page 620. In the right hand member of (4a) the exponent of (-1) is $(i+1)$ in lieu of $(i+n)$. In fifth line below (5) read (5) in lieu of (4).

Page 626. The argument given for the proof of Theorem II in the case where the prime ideal $\mathfrak{q}$ is of the first degree in $k(\zeta)$ and belongs to the c classes is not covered. It appears necessary to add the assumption that the second factor of the class number of $k(\zeta)$ is prime to l in order to obtain the three relations immediately preceding (12). With this assumption, however, Theorem II is included in Theorem I.

Page 627. Add to the statement of Theorem III the assumption that the second factor of the class number of $k(\zeta)$ is prime to l for the same reasons specified in the above correction to Theorem II. These corrections to Theo rems II and III do not affect any of the other results in the paper except that Theorem VII is not proved in so many different ways.

Page 631. In the statement of Theorem IV second line, read a in lieu of k.

Page 633. In the second line above relation (27a) add "or (-1)" after the word unity.

Page 634. In the last line read "$k=1$ or (-1)," in lieu of "$k=1$."

Page 641. In the second line under the first congruence read $B'_{29\cdot 67}$ in lieu of $B'_{9\cdot 67}$. In the fractional expression written separately near the middle of the page read $58\cdot 67$ as the exponent in each numerator in lieu of $32\cdot 37$ and read 65 in lieu of 35 in the last denominator.

Page 642. In fourteenth line read 1193 in lieu of 1093.

ERRATA, VOLUME 32

WILHELM MAIER, *Elementary properties of the t_r-functions.*

Page 905, last line. For "Mathematische Zeitschrift, 1930" read "Mathematische Annalen, 1931."

Page 906, line 9, for "$F(\omega_2/\omega_1)>0$" read "$\mathfrak{J}(\omega_2/\omega_1)>0$."

Page 909, line 4, for "$-V^{1/2}$" read "$iV^{1/2}$."

Page 911, line 11, for "$|\omega_2|\to\infty$" read "$\mathfrak{J}(\omega_2/\omega_1)\to\infty$."

ERRATA, VOLUME 33

A. A. ALBERT, *On direct products, cyclic division algebras, and pure Riemann matrices.*

The author wishes to correct a theorem on page 230 of this paper. The corrected theorem is to read

THEOREM 21. *A direct product $A = B \times C$ of a cyclic division algebra B of order p^2, p a prime, and a normal division algebra C of order p^2 is a total matric algebra if and only if C is reciprocal to B.*

The change is thus the change of *equivalent* to *reciprocal* both in the theorem and throughout the proof. The only other change in the proof of the above theorem is the change of $X_i = \theta^i(X)$ to $X_i = \theta^{n+1-i}(X)$, this latter error being the cause of the incorrect statement of the theorem. The result is an independent one and in no way affects the remainder of the paper.

M. FRÉCHET AND J. SHOHAT, *A proof of the second limit-theorem in the theory of probability.*

The title should read as above, with "second limit-theorem" in place of "second-limit theorem" as printed.

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